1. 2.7

101. Let $a_1, \ldots, a_k \in \mathbf{R}$ such that

$$a_1v_1 + \ldots + a_kv_k = 0.$$

It follows that

$$T(0) = T(a_1v_1+\ldots+a_kv_k) = T(a_1v_1)+\ldots+T(a_kv_k) = a_1T(v_1)+\ldots+a_kT(v_k).$$

Since T is linear, $T(0) = 0$ and consequently

$$a_1T(v_1) + \ldots + a_kT(v_k) = 0.$$

Using the assumption that $T(v_1), \ldots, T(v_k)$ are linearly independent, it follows that $a_1 = \ldots = a_k = 0$.

So, we have seen that $a_1v_1+\ldots+a_kv_k = 0$ implies that $a_1 = \ldots = a_k = 0$ which is equivalent to saying that $v_1, \ldots, v_k$ are linearly independent.

2. 2.8

97(a). Let $a_1, \ldots, a_k \in \mathbf{R}$ such that

$$a_1T(v_1) + \ldots + a_kT(v_k) = 0$$

Since T is linear, we have that $T(0) = 0$ and

$$0 = a_1T(v_1)+\ldots+a_kT(v_k) = T(a_1v_1)+\ldots+T(a_kv_k) = T(a_1v_1+\ldots+a_kv_k).$$

So, we can see that

$$T(a_1v_1 + \ldots + a_kv_k) = T(0).$$

Since T is one-to-one, it follows that

$$a_1v_1 + \ldots + a_kv_k = 0.$$

Unsing the assumption that $v_1, \ldots, v_k$ are linearly independent, it follows that $a_1 = \ldots = a_k = 0$.

So, we have seen that $a_1T(v_1) + \ldots + a_kT(v_k) = 0$ implies that $a_1 = \ldots = a_k = 0$ which is equivalent to saying that $T(v_1), \ldots, T(v_k)$ are linearly independent.

3. 3.2

69. Since $\det(AA^{-1}) = \det(I) = 1$ and $\det(AA^{-1}) = \det(A)\det(A^{-1})$, we have $\det(A)\det(A^{-1}) = 1$. We can divide by the determinant of A because we know that it is non-zero since A is invertible. Hence

$$\det(A^{-1}) = \frac{1}{\det(A)}.$$

71. Using 69, we have

$$\begin{aligned}\det(B^{-1}AB) &= \det(B^{-1})\det(A)\det(B)\\ &= \frac{1}{\det(B)}\det(A)\det(B)\\ &= \det(A).\end{aligned}$$

4. 4.1

77. Let U and V be subspaces of $\mathbb{R}^n$.

For the set $U \cap V$, we have

1. $0 \in U \cap V$, since $0 \in U$ and $0 \in V$,

2. Whenever $\mathbf{u}$ and $\mathbf{v}$ in $U \cap V$, $\mathbf{u}+\mathbf{v}$ is in $U \bigcap V$ since $\mathbf{u}+\mathbf{v} \in U$ and $\mathbf{u}+\mathbf{v} \in V$,

3. Whenever $\mathbf{u} \in U \cap V$ and c is a scalar, then $c\mathbf{u} \in U \bigcap V$ since $c\mathbf{u} \in U$ and $c\mathbf{u} \in V$.

Hence $U \cap V$ is also a subspace.

5. 4.2

75. Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation and $\{\mathbf{u_1}, \ldots, \mathbf{u_n}\}$ be a basis of $\mathbb{R}^n$.

For any $\mathbf{v} \in R(T)$, the range of T, $\mathbf{v} = T(\mathbf{u})$ for some $\mathbf{u}$ in $\mathbb{R}^n$. Since $\{\mathbf{u_1}, \ldots, \mathbf{u_n}\}$ is a basis of $\mathbb{R}^n$, $\mathbf{u} = \sum_{i=1}^{n} c_i\mathbf{u_i}$ for some scalars c_i, $i = 1, \ldots, n$. Moreover, we know that T is linear. Hence

$$\begin{aligned}\mathbf{v} &= T(\mathbf{u}) = T(\sum_{i=1}^{n} c_i\mathbf{u_i})\\ &= c_i\sum_{i=1}^{n} T(\mathbf{u_i})\end{aligned}$$

Therefore $\{T(\mathbf{u_1}), \ldots, T(\mathbf{u_n})\}$ is a generating set for the range of T.

6. 5.1

66. An $n \times n$ matrix $\mathbf{A}$ is invertible if and only if the only solution of $\mathbf{Ax} = \mathbf{0}$ is $\mathbf{0}$. Since $\mathbf{A0} = \mathbf{0}$, it follows that $\mathbf{A}$ is invertible if and only if $\mathbf{Ax} = \mathbf{0}$ has no non-zero solution.

We know that $0\mathbf{x} = \mathbf{0}$ for any $\mathbf{x} \in \mathbb{R}^n$ and hence

$$\mathbf{Ax} = \mathbf{0} \qquad \text{if and only if} \qquad \mathbf{Ax} = 0\mathbf{x}.$$

Moreover, $\mathbf{Ax} = 0\mathbf{x}$ has no non-zero solution if and only if 0 is not an eigenvalue of $\mathbf{A}$.

Hence, $\mathbf{A}$ is invertible if and only if 0 is not an eigenvalue of $\mathbf{A}$.

69. Let $\mathbf{v}$ be an eigenvector of $\mathbf{A}$ with eigenvalue λ. Hence we have that $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$ and $\mathbf{v} \neq 0$. Observe that

$$\mathbf{A}^2(\mathbf{v}) = \mathbf{A}(\lambda\mathbf{v}) = \lambda\mathbf{A}\mathbf{v} = \lambda^2\mathbf{v}.$$

Since we have shown that $\mathbf{A}^2\mathbf{v} = \lambda^2\mathbf{v}$ for a non-zero vector $\mathbf{v}$, it follows that λ^2 is an eigenvector of $\mathbf{A}^2$.

73. Let $\mathbf{v_1}$ and $\mathbf{v_2}$ be eigenvectors of a linear operator T on $\mathbb{R}^n$, and let λ_1 and λ_2, respectively, be the corresponding eigenvalues. Prove that if $\lambda_1 \neq \lambda_2$, then $\{\mathbf{v_1}, \mathbf{v_2}\}$ is linearly independent.

If c_1 and c_2 are scalars such that $c_1\mathbf{v_1} + c_2\mathbf{v_2} = \mathbf{0}$, using the linearity of T we deduce that

$$\begin{aligned} T(c_1\mathbf{v_1} + c_2\mathbf{v_2}) &= c_1T(\mathbf{v_1}) + c_2T(\mathbf{v_2}) \\ &= c_1\lambda_1\mathbf{v_1} + c_2\lambda_2\mathbf{v_2} \\ &= c_1\lambda_1\mathbf{v_1} - c_1\lambda_2\mathbf{v_1} \\ &= c_1(\lambda_1 - \lambda_2)\mathbf{v_1} \end{aligned}$$

Moreover, $T(\mathbf{0}) = \mathbf{0}$ and hence

$$c_1(\lambda_1 - \lambda_2)\mathbf{v_1} = \mathbf{0}$$

Since $\lambda_1 \neq \lambda_2$ and $\mathbf{v_1} \neq \mathbf{0}$ (because $\mathbf{v_1}$ is an eigenvector), it follows that $c_1 = 0$.

Similarly,

$$\begin{aligned} T(c_1\mathbf{v_1} + c_2\mathbf{v_2}) &= c_1T(\mathbf{v_1}) + c_2T(\mathbf{v_2}) \\ &= c_1\lambda_1\mathbf{v_1} + c_2\lambda_2\mathbf{v_2} \\ &= -c_2\lambda_1\mathbf{v_2} + c_2\lambda_2\mathbf{v_2} \\ &= -c_2(\lambda_1 - \lambda_2)\mathbf{v_2} \\ &= T(\mathbf{0}) = \mathbf{0}. \end{aligned}$$

Hence, $-c_2(\lambda_1 - \lambda_2)\mathbf{v_2} = \mathbf{0}$ and consequently $c_2 = 0$ too.

Therefore $\{\mathbf{v_1}, \mathbf{v_2}\}$ is linearly independent.

7. 5.3

84. If $\mathbf{A}$ is a diagonalizable matrix, prove that $\mathbf{A}^k$ is diagonalizable for any positive integer k.

Since $\mathbf{A}$ is diagonalizable, there exist some diagonal matrix $\mathbf{D}$ and some invertible matrix $\mathbf{P}$ such that $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$. Then we have

$$\begin{aligned} \mathbf{A}^k &= (\mathbf{P}\mathbf{D}\mathbf{P}^{-1})(\mathbf{P}\mathbf{D}\mathbf{P}^{-1})\ldots(\mathbf{P}\mathbf{D}\mathbf{P}^{-1}) \\ &= \mathbf{P}\mathbf{D}(\mathbf{P}^{-1}\mathbf{P})\mathbf{D}(\mathbf{P}^{-1}\mathbf{P})\ldots(\mathbf{P}^{-1}\mathbf{P})\mathbf{D}\mathbf{P}^{-1} \\ &= \mathbf{P}\mathbf{D}^k\mathbf{P}^{-1}. \end{aligned}$$

Since $\mathbf{D}^k$ is a diagonal matrix, $\mathbf{A}^k$ is a diagonalizable matrix.

8. 6.1

87. $\mathbf{v} \cdot \mathbf{z} = \mathbf{v} \cdot \mathbf{w} - \mathbf{v} \cdot \mathbf{w} = \mathbf{0}$. Hence $\mathbf{v}$ and $\mathbf{z}$ are orthogonal. We now want to show that $\{\mathbf{v}, \mathbf{z}\}$ is linearly independent.

Let $c_1\mathbf{v} + c_2\mathbf{z} = 0$. Then $(c_1\mathbf{v} + c_2\mathbf{z}) \cdot \mathbf{v} = \mathbf{0} \cdot \mathbf{v} = 0$. Moreover

$$(c_1\mathbf{v} + c_2\mathbf{z}) \cdot \mathbf{v} = c_1(\mathbf{v} \cdot \mathbf{v}) + c_2(\mathbf{z} \cdot \mathbf{v}) = c_1(\mathbf{v} \cdot \mathbf{v})$$

as $\mathbf{z} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{z} = 0$. Hence $c_1(\mathbf{v} \cdot \mathbf{v}) = 0$. Since $\mathbf{v} \neq \mathbf{0}$ it follows that $\mathbf{v} \cdot \mathbf{v} \neq 0$ and consequently $c_1 = 0$.

Similarly, $(c_1\mathbf{v} + c_2\mathbf{z}) \cdot \mathbf{z} = \mathbf{0} \cdot \mathbf{z} = 0$ and

$$(c_1\mathbf{v} + c_2\mathbf{z}) \cdot \mathbf{z} = c_1(\mathbf{v} \cdot \mathbf{z}) + c_2(\mathbf{z} \cdot \mathbf{z})c_2(\mathbf{z} \cdot \mathbf{z}).$$

This implies that $c_2(\mathbf{z} \cdot \mathbf{z}) = 0$. Since $\mathbf{v}$ and $\mathbf{w}$ are linearly independent, we know that $\mathbf{z} \neq \mathbf{0}$ it follows that $\mathbf{z} \cdot \mathbf{z} \neq 0$ and hence $c_2 = 0$.

Consequently $\{\mathbf{v}, \mathbf{z}\}$ is linearly independent and orthogonal subset of W. Since $\{\mathbf{v}, \mathbf{w}\}$ is a basis of W, it follows that $\dim W = 2$. Therefore the number of vectors in $\{\mathbf{v}, \mathbf{z}\}$ is equal to the dimension of W. Consequently, $\{\mathbf{v}, \mathbf{z}\}$ is a orthogonal basis for W.

90. We know that

$$\|\mathbf{v} - \mathbf{w}\|^2 = (\mathbf{v} - \mathbf{w}) \cdot (\mathbf{v} - \mathbf{w}) = \mathbf{v} \cdot \mathbf{v} - 2\mathbf{v} \cdot \mathbf{w} + \mathbf{w} \cdot \mathbf{w}$$

Since $\mathbf{x} \cdot \mathbf{x} = \|\mathbf{x}\|^2$ for any vector $\mathbf{x}$ and $\mathbf{v} \cdot \mathbf{w} \leq \|\mathbf{u}\|\|\mathbf{w}\|$, it follows that

$$\mathbf{v} \cdot \mathbf{v} - 2\mathbf{v} \cdot \mathbf{w} + \mathbf{w} \cdot \mathbf{w} \geq \|\mathbf{v}\|^2 - 2\|\mathbf{u}\|\|\mathbf{w}\| + \|\mathbf{w}\|^2 = (\|\mathbf{v}\| - \|\mathbf{w}\|)^2.$$

We can then deduce that $\|\mathbf{v} - \mathbf{w}\| \geq |\|\mathbf{v}\| - \|\mathbf{w}\||$.