Moduli spaces of curves on low degree hypersurfaces and the circle method

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1 A motivating question and a few results

Let $X \subset \mathbb{P}^n_{\mathbb{C}}$ be a smooth hypersurface of degree d defined by $F(x_0, \ldots, x_n) = 0$.

The "naive" moduli space of rational curves of degree e on X, denoted by $\mathrm{Mor}_e\left(\mathbb{P}^1, X\right)$, can be thought of as tuples $\left(f_0(u,v), \ldots, f_n(u,v)\right)$ of homogeneous polynomials up to scaling with no common roots, with each f_i of degree e, and such that $F\left(f_0, \ldots, f_n\right)$ vanishes identically.

Its expected dimension is given by

$$\bar{\mu} = \underbrace{(n+1)(e+1)}_{\text{coefficients for } f_i} - \underbrace{(de+1)}_{\text{vanishing of coefficients of } F(f_0,\ldots,f_n)} - \underbrace{1}_{\text{scaling}} .$$

Related spaces: $\mathcal{M}_{0,0}(X,e)$ (morphisms up to automorphisms of $\mathbb{P}^1$) and $\overline{\mathcal{M}}_{0,0}(X,e)$ (Kontsevich compactification), which have expected dimension $\overline{\mu} - 3$.

Question 1. What can we say about the geometry of these moduli spaces and how can we use this to better understand X?

1.1 Irreducibility and expected dimension

Conjecture 2 (Coşkun–Harris–Starr). If X is general, $d \geq 3$, and $n \geq d+1$, then $\overline{\mathcal{M}}_{0,0}(X,e)$ is irreducible of the expected dimension.

Some known results about irreducibility and expected dimension ([9, 4, 3, 1]):

- Riedl–Yang: Proved for $\overline{\mathcal{M}}_{0,0}(X,e)$ with X general and $n \geq d+2$.
- Coşkun–Starr: Proved for $\overline{\mathcal{M}}_{0,0}(X,e)$ with X any smooth cubic hypersurface and $n \geq 5$.
- Browning–Vishe/Browning–Bilu: Proved for $\mathcal{M}_{0,0}(X,e)$ with X any smooth hypersurface and $n \geq 2^d(d-1)$.

We prove the following generalization to higher genus curves ([7]):

Theorem 3 (HL). Let C be a smooth projective curve of genus g and $X \subset \mathbb{P}^n_{\mathbb{C}}$ a smooth hypersurface of degree $d \geq 2$. If $n \geq 2^d(d-1)+1$ and $e \gg_{g,d} 1$, then $\mathrm{Mor}_e(C,X)$ is irreducible of the expected dimension $(n+1)(e-g+1) - (de-g+1) - 1 + g$.

1.2 Singularities

Some known results about singularities ([6, 2, 10]):

- Harris–Roth–Starr: $\overline{\mathcal{M}}_{0,0}(X,e)$ is generically smooth for X general and $n \geq 2d$.
- Browning–Sawin: Lower bound on the codimension of the singular locus of $\mathcal{M}_{0,0}(X,e)$ for X any smooth hypersurface and $n \geq 3(d-1)2^d$.
- Starr: $\overline{\mathcal{M}}_{0,0}(X,e)$ has at worst canonical singularities for X general and $n \geq d+e$.

We prove that these moduli spaces have mild singularities for n sufficiently large compared to d ([5]):

Theorem 4 (Glas–HL). Let C be a smooth projective curve of genus g and $X \subset \mathbb{P}^n_{\mathbb{C}}$ a smooth hypersurface of degree $d \geq 2$. If $n \geq (d-1)(4d^2-4d+3)2^{d-2}$ (resp. $n \geq (d-1)(d^2-d+1)2^{d-1}$) and $e \gg_{g,d} 1$, then $\mathrm{Mor}_e(C,X)$ has at worst terminal (resp. canonical) singularities.

Remark 5. One reason to care about mild singularities is that $\mathrm{Mor}_e\left(\mathbb{P}^1, X\right)$ (and $\mathcal{M}_{0,0}(X,e)$) is of general type when n is close to d if it is irreducible of the expected dimension and has at worst canonical singularities.

1.3 Applications to hypersurfaces

The geometry of $\mathrm{Mor}_e(C,X)$ for higher genus curves C can be used to extract information about X. Recall the Fujita invariant:

Definition 6. Let X be a complex projective variety, $\overline{X} \to X$ a resolution of singularities, and L a big and nef $\mathbb{Q}$-divisor on X. Then,

$$a(X,L) \coloneqq \min\left\{t \in \mathbb{R} : t[L] + [K_{\overline{X}}] \text{ is pseudo-effective}\right\}.$$

For instance, if X is a smooth Fano variety, then $a(X, -K_X) = 1$.

Definition 7. If X is a smooth Fano variety, an accumulating map $f : Y \to X$ is a morphism f that is generically finite, non-birational, and satisfies $a(Y, -f^*K_X) \geq 1$.

We show the following ([7, 8]):

Theorem 8 (HL). If $X \subset \mathbb{P}^n_{\mathbb{C}}$ is either

- a smooth hypersurface of degree $d \geq 2$ with $n \geq 2^d(d-1)+1$, or
- a general or Fermat hypersurface of degree $d \geq 5$ with $n \geq 4d-6$,

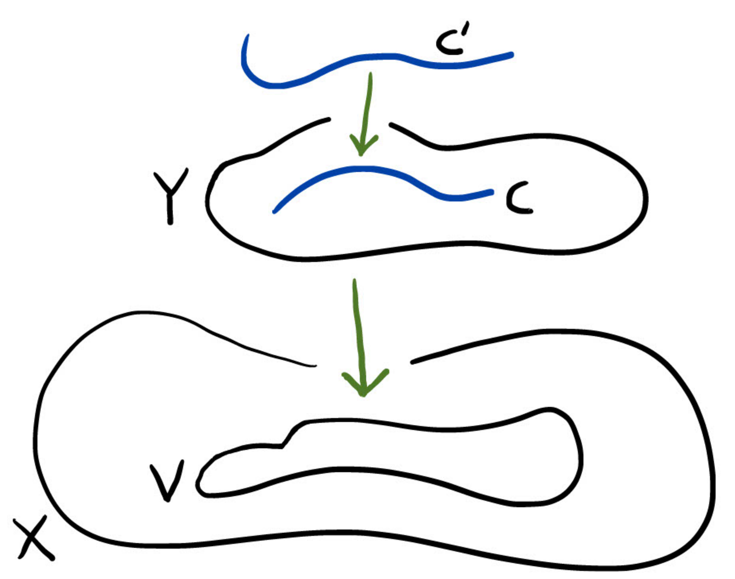
then there are no accumulating maps to X.

Remark 9. This can be thought of as a kind of converse to geometric Manin's conjecture, which predicts that poorly-behaved components of moduli spaces of curves on X are controlled by the Fujita invariant.

2 Ideas behind Theorem 8

Theorem 8 follows from our dimension results on moduli spaces of curves on hypersurfaces. For simplicity, we only consider accumulating maps given by inclusions of proper subvarieties.

The key point is to assume the existence of a proper subvariety V with large Fujita invariant. This produces a curve C such that the space of maps $C \to V$ is larger than the space of maps $C \to X$, which is impossible. To control the dimensions of these moduli spaces, we use lower bounds from deformation theory and upper bounds from Theorem 3.



Proof outline:

1. Assume for contradiction that there exists a proper subvariety $V \subset X$ for which $a(V, -K_X|_V) \geq 1$.
2. Pass to a resolution of singularities $Y \to V$ and find a curve C on Y so that $\left(K_Y - \frac{m-1}{m}K_X|_Y\right) \cdot C < 0$ for some large m.
3. Observe that $\dim \mathrm{Mor}_{[h]}(C,V) - \dim \mathrm{Mor}_{[h]}(C,X)$ is at least $(-K_Y + K_X|_Y) \cdot C + (g(C)-1)(\dim X - \dim V)$ for suitable h.
4. Spread out to work over characteristic p and replace C with Artin–Schreier covers and Frobenius twists to increase both genus and degree to make the right-hand side positive to achieve contradiction.

3 Ideas behind Theorems 3 and 4

We prove Theorems 3 and 4 by generalizing Browning–Vishe's work using the circle method from analytic number theory. For simplicity, we only consider $C = \mathbb{P}^1$.

Proof outline:

1. Use Mustaţă's criterion to relate $\mathrm{Mor}_e(\mathbb{P}^1, X)$ having mild singularities to another variety (encoding information about singularities) being irreducible of the expected dimension.
2. Pass to positive characteristic and point-counting via spreading out and the Lang–Weil bounds.
3. Express point-counts in terms of exponential sums and apply a function-field version of the circle method.

3.1 Mustaţă's criterion and jet schemes

Definition 10. The mth jet scheme $J_m(Y)$ of a variety Y is a variety whose A-points are the $A[t]/t^{m+1}$-points of Y, for any algebra A.

The following criterion relates the quality of singularities of a variety to simpler geometric properties of its jet schemes.

Theorem 11 (Mustaţă). Let Y be a local complete intersection variety. For all $m \geq 0$, if $J_m(Y)$ is irreducible of the expected dimension $(m+1)\dim Y$, then Y has at worst canonical singularities. Moreover, if $J_1(J_m(Y))$ is irreducible of the expected dimension $2(m+1)\dim Y$, then Y has at worst terminal singularities.

Goal: It suffices to show $J_m\left(\mathrm{Mor}_e\left(\mathbb{P}^1, X\right)\right)$ and $J_1\left(J_m\left(\mathrm{Mor}_e\left(\mathbb{P}^1, X\right)\right)\right)$ are irreducible of the expected dimension.

3.2 Passing to positive characteristic and point-counting

For further simplicity, we only consider $m = 0$, i.e. $J_0\left(\mathrm{Mor}_e\left(\mathbb{P}^1, X\right)\right) = \mathrm{Mor}_e\left(\mathbb{P}^1, X\right)$.

Spreading out procedure: $\mathrm{Mor}_e\left(\mathbb{P}^1, X\right)$ can be expressed as the base change of the generic fiber of a family $\mathcal{M} \to \operatorname{Spec}\Lambda$, where Λ is a finitely-generated $\mathbb{Z}$-algebra obtained by adjoining coefficients of X. Closed points of $\operatorname{Spec}\Lambda$ are dense and have finite residue field, and one can show it suffices to prove most fibers above a closed point are irreducible of the expected dimension.

Moreover, the Lang–Weil bounds give an asymptotic expression for the number of $\mathbb{F}_q$-points of a variety in terms of the number of its components and its dimension.

New (simplified) goal: Let $X \subset \mathbb{P}^n_{\mathbb{F}_q}$ be a smooth hypersurface with $\mathrm{char}(\mathbb{F}_q) > d$. It suffices to show

$$\lim_{q \to \infty} q^{-\overline{\mu}} \# \mathrm{Mor}_e(\mathbb{P}^1, X)(\mathbb{F}_q) \leq 1.$$

3.3 Points-counts in terms of exponential sums

Let Poly_e denote the vector space of homogeneous polynomials in $\mathbb{F}_q[u,v]$ of degree e.

The circle is defined as the space of linear functionals $\mathrm{Poly}_{de}^\vee$ (linear maps $\mathrm{Poly}_{de} \to \mathbb{F}_q$).

For $\alpha \in \mathrm{Poly}_{de}^\vee$, we define the exponential sum associated to α as follows:

$$S(\alpha) \coloneqq \sum_{\vec{x} \in \mathrm{Poly}_e^{n+1}} \psi\left(\alpha\left(F\left(\vec{x}\right)\right)\right),$$

where ψ is a non-trivial additive character $\mathbb{F}_q \to \mathbb{C}^\times$.

Then the "igniting spark" of the circle method is the observation that

$$\begin{aligned}\sum_{\alpha \in \mathrm{Poly}_{de}^\vee} S(\alpha) &= \sum_{\vec{x} \in \mathrm{Poly}_e^{n+1}} \sum_{\alpha \in \mathrm{Poly}_{de}^\vee} \psi\left(\alpha\left(F\left(\vec{x}\right)\right)\right) \\ &= \sum_{\vec{x} \in \mathrm{Poly}_e^{n+1}} \begin{cases} 0 & \text{if } F\left(\vec{x}\right) \neq 0, \\ \#\,\mathrm{Poly}_{de}^\vee & \text{else} \end{cases} \\ &= \underbrace{\#\,\mathrm{Poly}_{de}^\vee}_{q^{de+1}} \underbrace{\#\left\{\left(x_0, \ldots, x_n\right) \in \mathrm{Poly}_e^{n+1} : F\left(x_0, \ldots, x_n\right) = 0\right\}}_{\text{almost } \mathrm{Mor}_e(\mathbb{P}^1, X)}.\end{aligned}$$

"Almost $\mathrm{Mor}_e\left(\mathbb{P}^1, X\right)$" is basically $\mathrm{Mor}_e\left(\mathbb{P}^1, X\right)$, except we forget the condition that $x_0, \ldots, x_n$ don't have a common root.

New goal: Show

$$\sum_{\alpha \in \mathrm{Poly}_{de}^\vee} S(\alpha) \sim q^{(n+1)(e+1)} \text{ as } q \to \infty.$$

3.4 Analysis of $\sum_{\alpha \in \mathrm{Poly}_{de}^\vee} S(\alpha)$

Observe that for $\alpha = 0$, we have

$$S(0) = \sum_{\vec{x} \in \mathrm{Poly}_e^{n+1}} \psi(0) = \#\,\mathrm{Poly}_e^{n+1} = q^{(n+1)(e+1)},$$

so we need $\sum_{\alpha \neq 0} S(\alpha)$ to be small! To make this work, we stratify the circle by degree. Say α factors through a closed subscheme $Z \subset \mathbb{P}^1$ if α is determined by its restriction to Z, and define $\deg(\alpha)$ to be the minimum over all degrees of Z that α factors through.

Idea of stratification:

- For $\deg(\alpha) \leq e+1$, $\sum_{\alpha \neq 0, \deg(\alpha) \leq e+1} S(\alpha) = o\left(q^{(n+1)(e+1)}\right)$. This is proved by a direct point-counting argument using the principle of inclusion-exclusion.
- For $\deg(\alpha) > e+1$, each individual $S(\alpha)$ is sufficiently small. This is proved by relating $S(\alpha)$ to a more linearized quantity $N(\alpha)$ (Weyl differencing), which is deformed to a slightly different counting expression $N_s(\alpha)$ (Davenport shrinking), which is finally bounded by an easier point-counting problem.

Remark 12. This strategy from analytic number theory can be interpreted geometrically. In particular, there is a vector bundle E on $\mathbb{P}^1$ constructed from Beauville–Laszlo glueing such that $N(\alpha)$ can be expressed in terms of $h^0(E)$ and $N_s(\alpha)$ in terms of $h^0(E(-s))$. Davenport shrinking then amounts to observing how E and $E(-s)$ split into line bundles. For higher genus curves, a natural generalization is to use the slopes coming from the Harder–Narasimhan filtration.

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