A very informal intro to sheaves, local systems, and the Riemann-Hilbert correspondence

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We begin with a motivating example. Consider continuous functions f (valued in $\mathbb{R}$) on some open subset $U \subset \mathbb{R}$, i.e. $f\colon U \to \mathbb{R}$). For any open subset V of U, we can restrict f to get a continuous function on V, i.e. $f|_V\colon V \to \mathbb{R}$, given by the composition $V \subset U \to \mathbb{R}$. If $U = U_1 \cup U_2$ (cover by two open subsets), then if two continuous functions f, g on U agree on both U_1 and U_2, then $f = g$ (if this seems straightforward then you're not overthinking it). For any open subsets U_1 and U_2 and continuous functions $f_1\colon U_1 \to \mathbb{R}$ and $f_2\colon U_2 \to \mathbb{R}$ that agree on the overlap $U_1 \cap U_2$, we can "glue" f_1 and f_2 to get a continuous function f on $U_1 \cup U_2$. Indeed, for any $x \in U_1$, we let $f(x) = f_1(x)$ and for any $y \in U_2$, we let $f(y) = f_2(y)$. This is well-defined because for $x \in U_1 \cap U_2$, we have $f(x) = f_1(x) = f_2(x)$ by assumption. As an exercise, check this indeed gives a continuous function!

This example encodes the data of the "sheaf of continuous functions on $\mathbb{R}$," which, more precisely, is a collection of continuous functions for each open subset $U \subset \mathbb{R}$ that satisfies 1. restriction (i.e. that you can another continuous function by restricting to a smaller open subset V), 2. locality (i.e. if two functions agree on restrictions that cover the domain, then they are the same), and 3. glueing (i.e. that you can can construct a new continuous function from two that agree on the overlap of their domains).

Definition 1. Let X be a topological space. A **presheaf of sets F on X** is an assignment that associates to each open subset $U \subset X$ a set $F(U)$ along with maps $\rho_{UV}\colon F(U) \to F(V)$ (we will also write $\rho_{UV}(s) = s|_V$ as motivated by the example above) for any $V \subset U$ s.t. $\rho_{UU}\colon F(U) \to F(U)$ is the identity and for any chain $W \subset V \subset U$, we have $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$. Elements of $F(U)$ are called **sections of F over U**. Alternatively, one can think of F as a functor from the opposite category of open subsets (the morphisms given by inclusion) to the category Set (the conditions on ρ_{UV} are just what you need to make this a functor).

A presheaf F is moreover a **sheaf** if

(i) (Locality) Given U and a covering $\bigcup_{i \in I} U_i = U$ by open subsets, if $s, t \in F(U)$ satisfy $s|_{U_i} = t|_{U_i}$ for all $i \in I$, then $s = t$.

(ii) (Glueing) Given U and a covering $\bigcup_{i \in I} U_i = U$ by open subsets, along with $s_i \in F(U_i)$ s.t. $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all i, j, then there exists (and is unique by locality!) $s \in F(U)$ s.t. $s|_{U_i} = s_i$ for all $i \in I$.

Example 2. Let S be a discrete topological space. Then we can define a sheaf F on a topological space X by letting $F(U)$ be the set of continuous functions $U \to S$, which we call the **constant sheaf**. This name comes from the fact that for any connected open subset U, we have that $F(U)$ is just the constant functions, i.e. $F(U) = S$.

Example 3. If F is a sheaf on X and U is an open subset, then we can define the **restriction of F to U**, denoted $F|_U$, as the presheaf such that for any open subset $V \subset U$ (which is necessarily an open subset of X as well) we have $F|_U(V) = F(U)$ (along with the same ρ maps). Check that this is moreover a sheaf!

Combining the two examples above, we say that a sheaf F on X is **locally constant** if there exists an open cover $\bigcup_{i \in I} U_i = X$ s.t. $F|_{U_i}$ is constant.

We'll now try to relate this back to the earlier talks on covering spaces:

Definition 4. Let $p: Y \to X$ be a continuous map of spaces and $U \subset X$ an open subset. Then, a **section of p over U** is a continuous map $s: U \to Y$ s.t. $p \circ s$ is the identity on U.

Given a $p: Y \to X$, we can define the **sheaf of local sections** F_Y (which a priori is just a presheaf): for any $U \subset X$ an open subset, let $F_Y(U)$ be the set of sections of p over U and let the restriction maps ρ be given by restriction of domains of functions (i.e. if $f \in F_Y(U)$ is a map $f: U \to Y$, then by restricting to V, i.e. the composition $f|_V: V \subset U \to Y$, we get $f|_V \in F_Y(V)$).

Proposition 5. F_Y is actually a sheaf. If $p: Y \to X$ is moreover a covering space, then F_Y is locally constant. If $p: Y \to X$ is the trivial covering space, then F_Y is the constant sheaf.

Proof. The basic idea for the second statement is to take a trivializing cover of the base so that locally we have $V \times p^{-1}(x) \to V$ (recall that was the definition of a covering space, namely that locally on the base it is the trivial cover), where $p^{-1}(x)$ is the fiber over some point $x \in V \subset X$. Any section maps homeomorphically onto one of the connected components of $V \times p^{-1}(x)$, and so the sections over V are in bijection with the points of the fiber $p^{-1}(x)$. It follows that F_Y is locally constant (c.f. the examples above on locally constant and constant sheaves). It's also clear that if we start with a trivial covering space, then we already have a constant sheaf (can take V to be all of X). ☺

Now, we can define a functor $(p: Y \to X) \mapsto F_Y$ from covering spaces over X to locally constant sheaves on X. For a morphism $Y \to Z$ of covers over X, we get a map $F_Y \to F_Z$ by sending a local section $U \to Y$ of $Y \to X$ to the local section $U \to Y \to Z$ of $Z \to X$ by post-composition (I haven't defined what a map of sheaves is, but if you accept the definition as a functor, then it is just a natural transformation).

Theorem 6. The functor above induces an equivalence of categories between covering spaces over X and locally constant sheaves on X.

Using the Galois correspondence from earlier, we have the following corollary:

Corollary 7. Fix $x \in X$. The category of locally constant sheaves on X is equivalent to the category of sets with a left action of $\pi_1(X, x)$.

More precisely, this equivalence is obtained by sending a sheaf F to its stalk F_x at a point x, which is defined as the $\varinjlim_{x \in U \subset X \text{ open subset}} F(U)$. Since I haven't defined what a colimit is, I won't say too much about this in general, but in our case, for connected U, we know that $F(U)$ is some fixed set, so we we can just take F_x to be this fixed set.

From this point onwards, I will be even more imprecise and vague.

The story above works verbatim if we replace sheaf of sets with sheaf of abelian groups or sheaf of vector spaces. We will also say **complex local systems on** X instead of locally constant sheaf on X of finite-dimensional complex vector spaces. Then, the corollary above can be restated as follows:

Theorem 8. Let X be a connected and locally simply connected space and $x \in X$. The category of complex local systems on X is equivalent to the category of finite-dimensional left representations of $\pi_1(X, x)$.

In other words, a local system on X is the same data as a representation $\pi_1(X, x) \to \mathrm{GL}_n(\mathbb{C})$, which is called the **monodromy representation**.

Finally, we'll touch a bit on a very simple case of the Riemann-Hilbert correspondence, which relates monodromy representations to solutions of differential equations.

Fix an open connected subset $D \subset \mathbb{C}$.

First, note that the sheaf $\mathcal{O}$ of holomorphic functions on D is actually valued in rings (we can multiply functions together). We can sheafify the notion of a module: a **sheaf of $\mathcal{O}$-modules** F is a sheaf of abelian groups on D s.t. for any open subset $U \subset D$, the group $F(U)$ has an $\mathcal{O}(U)$-module structure, i.e. $\mathcal{O}(U) \times F(U) \to F(U)$ that is natural (i.e. the canonical maps to $\mathcal{O}(V) \times F(V) \to F(V)$ commute for any inclusion $V \subset U$).

We say that F is **locally free** if we can find an open cover of $D = \bigcup_i V_i$ s.t. $F|_{V_i} \cong \mathcal{O}^n|_{V_i}$ for some fixed $n > 0$, which we call the **rank of** F.

Recall that a holomorphic 1-form on an open subset $V \subset \mathbb{C}$ is an expression of the form $\omega = f(z)dz$, where f is holomorphic on V. We can define a **sheaf of holomorphic 1-forms on** D, denoted by Ω^1_D by letting $\Omega^1_D(U)$ be the holomorphic 1-forms on $U \subset D$.

Definition 9. A **holomorphic connection on** D is a pair (E, ∇), where E is a locally free sheaf on D and $\nabla \colon E \to E \otimes_{\mathcal{O}} \Omega^1_D$ is a morphism of sheaves of $\mathbb{C}$-vector spaces s.t.

$$\nabla(fs) = df \otimes s + f\nabla(s)$$

(this is also called the Leibnitz rule). ∇ is called the **connection map**.

Remark 10. We haven't defined what a tensor product of sheaves is, but roughly on sections you take the tensor product (you have to be careful because this doesn't necessarily give you a sheaf, so you have to "sheafify" this).

Example 11. Let $E = \mathcal{O}^n$. Then, we can define a connection map $d \colon \mathcal{O}^n \to (\Omega^1_D)^n$ by sending $(f_1, \dots, f_n)$ to $(df_1, \dots, df_n)$.

For any other connection map ∇, we can check that $(\nabla - d)(fs) = f(\nabla - d)(s)$ for any $f \in \mathcal{O}(D)$ and $s \in E(D)$. So it follows that $\nabla - d$ is given by some matrix of holomorphic 1-forms, which we call the **connection matrix of** ∇. Writing each entry as $f_{ij}dz$, it follows that setting

$f = (f_1, \ldots, f_n)$ and $A_{ij} = -f_{ij}$ (A is a matrix) means that $\nabla f = 0$ iff f satisfies the differential equation $y' = Ay$ (this is where differential equations come in!).

Definition 12. $s \in E(U)$ is called **horizontal** if $\nabla(s) = 0$. The subsheaf of horizontal sections is denoted by E^∇ (check this!).

Lemma 13. E^∇ is a local system of the same dimension as E.

The following proposition is one of the simplest ways of expressing the Riemann-Hilbert correspondence:

Proposition 14. The functor $(E, \nabla) \mapsto E^\nabla$ induces an equivalence between holomorphic connections on D and complex local systems on D.

Now, combining these proposition with the earlier equivalence of categories related to representations of π_1, it follows that every finite-dimensional representation of $\pi_1(D, x)$ is the monodromy representation corresponding to some system of holomorphic differential equations.

Exercise 15. For a sanity check, convince yourself this isn't interesting (i.e. is trivial) for the case where D is just some open ball around the origin.

In general, we can extend this to higher-dimensional complex manifolds, but we need to furthermore impose the condition that the connections are **integrable**, namely that composition $E \to E \otimes \Omega^1 \to E \otimes \wedge^2 \Omega^1$, given by the connection map followed by $s \otimes \omega \mapsto \nabla(s) \wedge \omega + s \otimes d\omega$ is 0.