Overview and fundamental groups

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Today's talk will be split into two parts: 1. an overview of the course and its expectations and 2. a crash course on fundamental groups.

1 Overview

For grading and expections, please refer to the website: `http://www.math.columbia.edu/~mhaseliu/FunGrp.html`

As covered in the course video, the basic premise is understanding the relationship between fundamental groups in topology and Galois groups in algebra. In doing so, we will also see some complex analysis and differential equations. Here is a brief sketch of how the course should go (but don't expect to know many of these terms yet):

Starting from the usual formulation of Galois theory, we can recast it to be a statement about an equivalence (of categories) between finite separable extensions of a given field k and finite discrete transitive $\mathrm{Gal}(\overline{k}/k)$-sets, which suggests a connection with covering spaces. In particular, there is a similar relationship between covering spaces of a nice topological space X and $\pi_1(X, x)$-sets. The right-hand side is also equivalent to locally constant sheaves on X (a "sheaf" is roughly a natural way of assigning values to subsets of X that are compatible and "locally constant" means that for small enough subsets of X, the values are constant), and an analytic result (called the Riemann-Hilbert correspondence) that relates such sheaves with holomorphic connections will let us understand this equiavlence through differential equations. Next, we restrict to the case of Riemann surfaces X (which are 1-dimensional complex manifolds), and see how branched covers (covering spaces outside some exceptional set) are related to finite etale algebras over the field of meromorphic functions of X. As a consequence, we will see that every finite group is the Galois group of some finite Galois extension of $\mathbb{C}(t)$ (this is proven topologically!). The Riemann existence theorem relates compact connected Riemann surfaces with finite extensions of $\mathbb{C}(t)$, which hints that we can develop a theory of fundamental groups purely algebraically. Using some algebraic geometry, one can show that for a nice curve, one can algebraically define covering spaces ("finite etale morphisms") and fundamental groups borrowing ideas from the case of Riemann surfaces. Then, we will see a "Galois correspondence" for algebraic curves. This covers roughly the first four chapters of Szamuely's book.

If time permits, we can also see how Grothendieck generalized the case of algebraic curves to a vast generalization called "schemes," which are themselves generalizations of algebraic varieties (which can be thought of as algebraic analogues of manifolds). It is precisely in this setting that we can finally view $\mathrm{Gal}(\overline{k}/k)$ as an actual fundamental group, not just something that shares many properties. In particular, in Grothendieck's language, if we let $X = \operatorname{Spec} k$ (which is a point!), then $\mathrm{Gal}(\overline{k}/k)$ is equal to the algebraic fundamental group $\pi_1(X, \overline{x})$.

2 Fundamental groups

Recall the definition of a topological space:

Definition 1. A **topological space** is a set X with a specified collection of subsets called **open subsets**:

(i) The empty subset and X are open subsets.

(ii) Any (possibly infinite) union of open subsets is an open subset.

(iii) Any finite intersection of open subsets is an open subset.

A **pointed topological space** (X, x) is just a topological space X with some chosen point $x \in X$.

Example 2. $\mathbb{R}^n$ is a topological space (but there is more than one way to give it the structure of a topological space: silly one is to say **every** subset is open): the standard (or **Euclidean**) topological structure is declaring that every open subset is a union of open balls centered at points (open ball means a set of the form $\{x : |x - a| < b\}$). Exercise: check this is a topology!

Definition 3. If $X \subset Y$ with X a set and Y a topological space, then the **induced topology on** X is the topology on X whose open subsets are those of the form $X \cap U$ for $U \subset Y$. Exercise: check this is a topology!

Example 4. $I = [0, 1]$ (the unit interval) is a subset of $\mathbb{R}^1$, so it has an induced topology.

Definition 5. A **continuous map** of topological spaces $f\colon X \to Y$ is a map of sets such that for any open subset $U \subset Y$, the preimage $f^{-1}(U) \subset X$ is open. A **continuous map** of pointed topological spaces $f\colon (X, x) \to (Y, y)$ is a continuous map $f\colon X \to Y$ such that $f(x) = y$.

Example 6. If i is the inclusion $X \subset Y$ with X given the induced topology, then i is a continuous map. Exercise: check this!

Example 7. Say $\mathbb{R}^m$ and $\mathbb{R}^n$ have the standard topology, and let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ have the induced topologies. Show that $f\colon X \to Y$ is continuous iff for any sequence of points $\{x_i\}$ in X convering to $x \in X$, the sequence $\{f(x_i)\}$ in Y converges to $f(x) \in Y$.

Example 8. $f\colon [0, 2\pi] \to \mathbb{R}^2$ sending x to $(\cos x, \sin x)$ is continuous (use the previous example and the fact that $x \mapsto \sin x$ and $x \mapsto \cos x$ are continuous). Its image is the **unit circle**, which we denote by S^1.

Definition 9. A **path** is a continuous map $I \to X$. A **loop** in a pointed space (X, x_0) is a path $f\colon I \to X$ such that $f(0) = f(1) = x_0$. Sometimes we will abuse notation and say $f(I)$ is a loop (because it looks like one!).

Example 10. Silly example! The constant loop is the function that sends everything to x_0.

Example 11. The map $I \to S^1 \subset \mathbb{R}^2$ sending t to $(\cos 2\pi t, \sin 2\pi t)$ is a loop in $(S^1, (1,0))$ (the point $(1,0) \in S^1 \subset \mathbb{R}^2$).

Definition 12. Let f, g be two continuous maps $X \to Y$. Then, a **homotopy** between f and g is a continuous map $H\colon X \times I \to Y$ such that $H(-,0)$ is f and $H(-,1)$ is g (a way to deform one map into another).

In particular, a homotopy of paths is a continuous map $H\colon I \times I \to X$ with $H(0,t) = H(0,0)$ and $H(1,t) = H(1,0)$ (same starting and end points for each path). A homotopy of loops in (X, x_0) is similarly required to preserve basepoints: $H(0,t) = H(1,t) = x_0$.

If there exists an H between f and g, we say f and g are **homotopic**. If f is homotopic to the constant map (i.e. suppose g sends everything to a single point), then f is **null-homotopic**.

Example 13. Consider the pointed space $(\mathbb{R}^2, (1,0))$ and two loops $f, g\colon I \to \mathbb{R}^2$ such that

$$f(s) = (\cos 2\pi s, \sin 2\pi s) \text{ and } g(s) = (1,0).$$

Then, f and g are homotopic because we can intuitively imagine the circle S^1 getting squashed continuously to a point. More formally, we have the homotopy $H\colon I \times I \to \mathbb{R}^2$ such that

$$H(s,t) = (t\cos 2\pi s, t \sin 2\pi s).$$

Example 14. The ambient space matters! Consider the same example above but replace the pointed space $(\mathbb{R}^2, (1,0))$ with $(\mathbb{R}^2 - \{0\}, (1,0))$. Now, it turns out that $f, g\colon I \to \mathbb{R}^2 - \{0\}$ are no longer homotopic. Intuitively, this is because we can't collapse the circle down without going through the origin, which isn't there anymore. This is a bit hard to prove, and we'll be able to do this once we have fundamental groups.

Proposition 15. Let (X, x_0) be a pointed space. Then homotopy defines an equivalence relation on the set of loops.

Proof. Recall the definition of an equivalence definition: it is a way of saying things are "the same." More precisely, on a set Y, an equivalence is a relation $\sim$ such that:

(i) $y \sim y$ for all $y \in Y$.

(ii) $x \sim y \implies y \sim x$.

(iii) $x \sim y$ and $y \sim z$ imply $x \sim z$.

I'll leave the first two as exercises (do them!). For transitivity, we have $f \sim g$ and $g \sim h$ (intuitively, we can deform f into g and g into h, so we can deform f into h), which means we have homotopies $H, H'\colon I \times I \to X$ such that $H(-,0) = f$, $H(-,1) = g$, $H'(-,0) = g$, and $H'(-,1) = h$. Then define $H''\colon I \times I \to X$ by

$$H''(s,t) = \begin{cases} H(s,2t) & \text{if } 0 \le t \le \frac{1}{2} \\ H'(s,2t-1) & \text{if } \frac{1}{2} \le t \le 1 \end{cases}$$

(check this is continuous!). ☺

Remark 16. For a pointed space (X, x_0), we can then define the set of loops up to homotopy, i.e. take the set of loops and then mod out by any two loops that are homotopic—this is well-defined because we just showed that homotopy is an equivalence relation.

We're almost at the definition of the fundamental group. We just need to know what the group operation is.

Definition 17. Let $f, g\colon I \to X$ be two loops. Define **composition** of f, g, denoted by $f * g\colon I \to X$, to be

$$(f * g)(t) = \begin{cases} f(2t) & \text{if } 0 \le t \le \frac{1}{2} \\ g(2t-1) & \text{if } \frac{1}{2} \le t \le 1 \end{cases}$$

Intuitively, we are just doing the first loop and then the second loop, and calling this the composition.

Definition 18. The **fundamental group** of (X, x_0), denoted by $\pi_1(X, x_0)$, is a group whose underlying set is the set of loops up to homotopy (c.f. the remark above), where the group operation is given by composition.

Proposition 19. $\pi_1(X, x_0)$, as a group, is well-defined.

Proof. We need to check several things: well-definedness of the composition, existence of the identity element and inverses, and associativity of the group operation.

For composition, note that we are picking representative loops f, g of $[f], [g]$ and defining the group operation (let us also use $*$ for this) $[f] * [g]$ by $[f * g]$. But we need to make sure that if pick other representatives $f' \sim f$ and $g' \sim g$, then $f * g$ is still $\sim$ to $f' * g'$. Do this as an exercise!

The identity element is the class of the constant loop that sends everything to x_0. Intuitively, this makes sense because composing with the constant loop is just doing the loop at double speed then stopping, which is homotopic to just doing the loop normally. Inversion is just reversing: $f^{-1}(t) = f(1-t)$, which makes sense because doing a path and then going backwards along the same path is homotopic to not moving at all (draw a picture!). Associativity just means that $f * (g * h)$ and $(f * g) * h$ are homotopic, which is true because we're just doing loop f, then loop g, then loop h, just at different speeds.

This is all a bit handwavy, but the intuition should be clear! If you haven't seen this before, check all of these until you are sufficiently convinced. ☺

Example 20. $(\mathbb{R}^n, 0)$ has trivial fundamental group, i.e. $\pi_1(\mathbb{R}^n, 0)$ is the group with one element. Indeed, any loop $f\colon I \to \mathbb{R}^n$ is homotopic to the constant loop; consider the homotopy $H(s,t) = tf(s)$.

Theorem 21. Let $x_0 \in S^1$ and $f\colon I \to S^1$ be the loop that goes counterclockwise around the circle once. Then, the homomorphism $\mathbb{Z} \to \pi_1(S^1, x_0)$ sending 1 to $[f]$ is an isomorphism.

Remark 22. We won't prove this yet, but the idea is that $p\colon \mathbb{R} \to S^1$ is the "universal" covering space of S^1 (we'll define this some other time). Here, p sends t to $(\cos 2\pi t, \sin 2\pi t)$ (imagine a bunch of loops over S^1). One can show that $p^{-1}(x_0)$ is isomorphic to $\pi_1(S^1, x_0)$, where a point in

$p^{-1}(x_0)$ can be viewed as a loop in S^1 (this comes from the construction of the universal cover). but $p^{-1}(x_0)$ is also just $\mathbb{Z}$, and one can check that the group structures agree.

The fundamental group also interacts well with maps of spaces (in fancy language, $\pi_1(-)$ defines a functor from pointed topological spaces to groups):

Proposition 23. Let $f\colon (X, x_0) \to (Y, y_0)$ and $g\colon (Y, y_0) \to (Z, z_0)$ be continuous maps of pointed topological spaces. Then,

(i) f induces a homomorphism $\pi_1(X, x_0) \to \pi_1(Y, y_0)$ (written f_* or $\pi_1(f)$).

(ii) $\pi_1(g) \circ \pi_1(f) = \pi_1(g \circ f)$ (this might be a little confusing, but the two $\circ$'s are different—the former is a composition of group homomorphisms, the latter is a composition of continuous maps).

(iii) $\pi_1(\mathrm{Id}) = \mathrm{Id}$ (the first Id is the identity map on a pointed topological space, the second is the identity map on groups).

Proof. For (i), given a homotopy class of loop $[\gamma]$ in (X, x_0), map it to $[f \circ \gamma]$, which is a homotopy class of loop in (Y, y_0). There is something to check here (it is well-defined and is it a group homomorphism?)!

For (ii), assuming (i) makes sense, note that for any $\gamma\colon I \to X$, we have the following computation:

$$\pi_1(g \circ f)[\gamma] = [(g \circ f) \circ \gamma] = [g \circ (f \circ \gamma)] = \pi_1(g)([f \circ \gamma]) = \pi_1(g) \circ \pi_1(f).$$

For (iii), note that $\mathrm{Id} \circ \gamma = \gamma$, so $\pi_1(\mathrm{Id}) = \mathrm{Id}$. ☺

Since I wanted to talk about some algebra but didn't get to, let's end with a very nice application of the fundamental group (the fundamental theorem of algebra!):

Theorem 24. Let $f(z)\colon \mathbb{C} \to \mathbb{C}$ be a polynomial $z^n + a_{n-1}z^{n-1} \cdots + a_0$ with $n \geq 1$. Then f has a root.

Proof. For the sake of contradiction, suppose f has no root. Then f defines a continuous map $\mathbb{C} \to \mathbb{C} - \{0\}$. There is a continuous map $\pi\colon \mathbb{C} - \{0\} \to S^1$ given by sending every element to its norm, i.e. $x \mapsto x/|x|$ (check this is continuous!).

Compose these two maps gives a continuous map $g = \pi \circ f\colon \mathbb{C} \to S^1$. Now, let S^1_r be the circle of radius r around the origin (so scale S^1 by r). Then, consider the restriction of g to S^1_r, which is a map

$$g|_{S^1_r}\colon S^1_r \to S^1$$

between two circles.

By the proposition above, this induces a group homomorphism $\pi_1(S^1_r, x) \to \pi_1(S^1, g(x))$. Note that for r sufficiently large, the loop that goes once around counterclockwise in S^1_r is sent to the loop that goes around n times counterclockwise in S^1: for large z, the map f asymptotically looks like $z \mapsto z^n$. So, for large r, the map $\pi_1(S^1_r, x) \to \pi_1(S^1, g(x))$, identified as $\mathbb{Z} \to \mathbb{Z}$ by the theorem earlier, is just multiplication by n.

On the other hand, note that $S^1_r \to S^1$ factors through $\mathbb{C}$ by definition. It is the map $S^1_r \to \mathbb{C} \to \mathbb{C} - \{0\} \to S^1$, and this induces the composition of group homomorphisms

$$\pi_1(S^1_r, x) \to \pi_1(\mathbb{C}, x) \to \pi_1(\mathbb{C} - \{0\}, f(x)) \to \pi_1(S^1, g(x)),$$

which can be identified with

$$\mathbb{Z} \to 0 \to \pi_1(\mathbb{C} - \{0\}, f(x)) \to \mathbb{Z},$$

which is the zero map!

So $\pi_1(S^1_r, x) \to \pi_1(S^1, g(x))$ is both the zero map and multiplication by n, which means $n = 0$, which contradicts the assumption that $n \geq 1$ (also, if $n = 0$, note that f doesn't always have root; take any complex number that isn't 0 for instance!). ☺

Remark 25. This shows that $\mathbb{C}$ is the algebraic closure of $\mathbb{R}$, i.e. the smallest field L containing $\mathbb{R}$ such that any polynomial with coefficients in L of degree at least 1 has a root in L. Indeed, $\mathbb{R}$ is not algebraically closed because $x^2 + 1$ has no root in $\mathbb{R}$. So any algebraic closure of $\mathbb{R}$ must have degree at least 2 over $\mathbb{R}$ (the degree is the dimension as an $\mathbb{R}$-vector space). Since $\mathbb{C}$ has dimension 2 over $\mathbb{R}$ (since $\mathbb{C} = \mathbb{R} + \mathbb{R}i$), the result above implies this fact.

Next week, we'll start reviewing facts about fields and at least state the Galois correspondence.