1 04/17 (Matthew): Convolution and applications to Evans and Rudnick sums

Intro to Gauss, Evans, and Rudnick sums

Let k be the finite field $\mathbb{F}_q$, $\psi: k \to \overline{\mathbb{Q}_\ell}^\times$ a non-trivial additive character, and $\chi: k^\times \to \overline{\mathbb{Q}_\ell}^\times$ a multiplicative character. Consider the following sums:

- (Gauss sums)
$$g(\psi, \chi) = -\frac{1}{\sqrt{q}} \sum_{t \in k^\times} \chi(t)\psi(t).$$
- (Evans sums)
$$E(\chi) = -\frac{1}{\sqrt{q}} \sum_{t \in k^\times} \chi(t)\psi\left(t - \frac{1}{t}\right).$$
- (Rudnick sums)
$$R(\chi) = -\frac{1}{\sqrt{q}} \sum_{t \in k^\times, \neq 1} \chi(t)\psi\left(\frac{t+1}{t-1}\right).$$

As χ varies over all multiplicative characters of k, numerical data suggests that these sums are approximately equidistributed according to the "Sato–Tate" measure. We'll see how new equidistribution results proved by Katz allows us to prove statements like these.

Let us first recall some notation and observe how the sums above can be viewed cohomologically. Let $\mathcal{L}_\psi$ be the Artin–Schreier sheaf on $\mathbb{A}^1/k$, which is a lisse sheaf of rank one of pure weight zero (i.e. the eigenvalues of Frobenius all have absolute value one). We also have the Kummer sheaf $\mathcal{L}_\chi$ on $\mathbb{G}_m/k$, which is also a lisse sheaf of rank one of pure weight zero.

Example 1 (Gauss sums)**.** Let $j_0: \mathbb{G}_m \hookrightarrow \mathbb{A}^1$ be the inclusion and consider $M = j_0^* \mathcal{L}_\psi$. Then, observe that
$$g(\psi, \chi) = -\frac{1}{\sqrt{q}} \sum_{t \in \mathbb{G}_m(k) = k^\times} \mathrm{Tr}\left(\mathrm{Fr}_t : \mathcal{L}_\chi \otimes M\right)$$
by definition (Fr_t acts on $\mathcal{L}_\chi$ by $\chi(t)$ and on M by $\psi(t)$, and the $\otimes$ means we multiply). If we replace M with $M(1/2)[1]$, then we obtain
$$g(\psi, \chi) = \sum_{t \in \mathbb{G}_m(k) = k^\times} \mathrm{Tr}\left(\mathrm{Fr}_t : \mathcal{L}_\chi \otimes M(1/2)[1]\right).$$

Example 2 (Evans sums)**.** We can view $x - 1/x$ as a morphism from $\mathbb{G}_m$ to $\mathbb{A}^1$, and write $\mathcal{L}_{\psi(x-1/x)}$ to mean the pullback of $\mathcal{L}_\psi$ (which is on $\mathbb{A}^1$) to $\mathbb{G}_m$ along the map $x - 1/x$. Let $M = \mathcal{L}_{\psi(x-1/x)}$. Then, we have
$$E(\chi) = -\frac{1}{\sqrt{q}} \sum_{t \in \mathbb{G}_m(k) = k^\times} \mathrm{Tr}\left(\mathrm{Fr}_t : \mathcal{L}_\chi \otimes M\right) = \sum_{t \in \mathbb{G}_m(k) = k^\times} \mathrm{Tr}\left(\mathrm{Fr}_t : \mathcal{L}_\chi \otimes M(1/2)[1]\right).$$

Example 3 (Rudnick sums)**.** We can view $(x+1)/(x-1)$ as a function from $\mathbb{G}_m \backslash \{1\}$ to $\mathbb{A}^1$. Similarly define $\mathcal{L}_{\psi((x+1)/(x-1))}$, which is a sheaf on $\mathbb{G}_m \backslash \{1\}$, and extend it by zero to $\mathbb{G}_m$. Call this sheaf on $\mathbb{G}_m$ M. Then, we have

$$R(\chi) = -\frac{1}{\sqrt{q}} \sum_{t \in \mathbb{G}_m(k) = k^\times} \mathrm{Tr}\left(\mathrm{Fr}_t : \mathcal{L}_\chi \otimes M\right) = \sum_{t \in \mathbb{G}_m(k) = k^\times} \mathrm{Tr}\left(\mathrm{Fr}_t : \mathcal{L}_\chi \otimes M(1/2)[1]\right).$$

In all three cases, $M(1/2)[1]$ is a perverse sheaf on $\mathbb{G}_m$ of pure weight zero!

A quick interlude on the derived category of ℓ-adic sheaves and perverse sheaves

Let X be a variety over k, where k is either a finite field or its algebraic closure. Grothendieck defined the bounded constructible derived category $D_c^b\left(X, \overline{\mathbb{Q}_\ell}\right)$ of bounded complexes of $\overline{\mathbb{Q}_\ell}$-sheaves on X with constructible cohomology and endowed it with the six functor formalism:

- For any $f: X \to Y$, there are derived pushforward and derived proper pushforward functors $Rf_*, Rf_!: D_c^b\left(X, \overline{\mathbb{Q}_\ell}\right) \to D_c^b\left(Y, \overline{\mathbb{Q}_\ell}\right)$ and derived pullback and derived upper shriek functors $f^*, f^!: D_c^b\left(X, \overline{\mathbb{Q}_\ell}\right) \to D_c^b\left(Y, \overline{\mathbb{Q}_\ell}\right)$.
- In particular, if $\pi: X \to \operatorname{Spec} k$ is the structure map, then for any $M \in D_c^b\left(X, \overline{\mathbb{Q}_\ell}\right)$, the derived global sections $R\Gamma\left(X, M\right)$ and derived compactly supported global sections $R\Gamma_c\left(X, M\right)$ are defined as $R\pi_* M$ and $R\pi_! M$.
- In fact, cohomology $H^i\left(X, M\right)$ is defined as $H^i R\Gamma\left(X, M\right)$ and $H_c^i\left(X, M\right)$ is defined as $H^i R\Gamma_c\left(X, M\right)$.
- There is a tensor product $- \otimes -: D_c^b\left(X, \overline{\mathbb{Q}_\ell}\right) \times D_c^b\left(X, \overline{\mathbb{Q}_\ell}\right) \to D_c^b\left(X, \overline{\mathbb{Q}_\ell}\right)$ and an internal Hom $R\operatorname{Hom}\left(-, -\right): D_c^b\left(X, \overline{\mathbb{Q}_\ell}\right)^{\mathrm{op}} \times D_c^b\left(X, \overline{\mathbb{Q}_\ell}\right) \to D_c^b\left(X, \overline{\mathbb{Q}_\ell}\right)$.
- The dualizing complex is defined as $\omega_X = \pi^! \overline{\mathbb{Q}_\ell}$, and the Verdier duality functor $\mathbb{D}: D_c^b\left(X, \overline{\mathbb{Q}_\ell}\right)^{\mathrm{op}} \to D_c^b\left(X, \overline{\mathbb{Q}_\ell}\right)$ is defined as $\mathbb{D}(M) = R\operatorname{Hom}\left(M, \pi^! \overline{\mathbb{Q}_\ell}\right)$. This interchanges Rf_* and $Rf_!$, i.e. $Rf_* \circ \mathbb{D} = \mathbb{D} \circ Rf_!$, and similarly for f^* and $f^!$.
- If X is smooth and of pure dimension d, then $\omega_X = \overline{\mathbb{Q}_\ell}(d)[2d]$. By taking cohomology of the equation $Rf_* \circ \mathbb{D} = \mathbb{D} \circ Rf_!$ applied to $\overline{\mathbb{Q}_\ell}$, we recover Poincare duality:

$$H^i\left(X, \overline{\mathbb{Q}_\ell}\right) = H_c^{2d-i}\left(X, \overline{\mathbb{Q}_\ell}\right)^\vee(-d).$$

Definition 4. $M \in D_c^b\left(X, \overline{\mathbb{Q}_\ell}\right)$ is **semiperverse** if all of its cohomology sheaves $\mathcal{H}^i(M)$ satisfy the condition $\dim \operatorname{Supp} \mathcal{H}^i(M) \leq -i$. If moreover $\mathbb{D}(M)$ is semiperverse, then we call M **perverse**.

Fact 5. *The full subcategory of perverse sheaves* $\operatorname{Perv}\left(X, \overline{\mathbb{Q}_\ell}\right) \subset D_c^b\left(X, \overline{\mathbb{Q}_\ell}\right)$ *is an abelian category that is Noetherian and Artinian! If we are working over an algebraically closed field, then the category is even semisimple.*

Example 6. If X is smooth of dimension d and L is a locally constant sheaf on X, then $L[d]$ is a perverse sheaf on X. This is why we need to shift M by 1 earlier.

Stating Katz's equidistribution theorem

In previous talks, we typically had a local system (aka lisse sheaf), which is equivalent to a representation of the fundamental group of some variety, and we defined the associated (arithmetic or geometric) monodromy group to be the closure of the image of the fundamental group (in its representation).

In the above examples, we now have a perverse sheaf instead, and we'll next explain what the notion of monodromy group should be in this setting.

Consider the subcategory $\mathrm{Neg}(\mathbb{G}_m) \subset \mathrm{Perv}\left(\mathbb{G}_m, \overline{\mathbb{Q}_\ell}\right)$ to be the "negligible" objects, i.e. those such that the Euler characteristic is zero. It is possible to take the quotient $\mathrm{Perv}\left(\mathbb{G}_m, \overline{\mathbb{Q}_\ell}\right) / \mathrm{Neg}(\mathbb{G}_m)$ to obtain another abelian category.

Fact 7. *There is a subcategory $P(\mathbb{G}_m) \subset \mathrm{Perv}\left(\mathbb{G}_m/\overline{k}, \overline{\mathbb{Q}_\ell}\right)$ that is equivalent to $\mathrm{Perv}\left(\mathbb{G}_m, \overline{\mathbb{Q}_\ell}\right) / \mathrm{Neg}(\mathbb{G}_m)$. In fact, this subcategory is given by perverse sheaves that have no subobjects or quotients that are given by $\mathcal{L}_\chi[1]$.*

Let $P_{\text{arith}}(\mathbb{G}_m)$ be the perverse sheaves M on $\mathbb{G}_m/k$ such that $M \otimes_k \overline{k}$ on $\mathbb{G}_m/\overline{k}$ is in $P(\mathbb{G}_m)$.

Fact 8.

- *$P(\mathbb{G}_m)$ and $P_{arith}(\mathbb{G}_m)$ can be endowed with the structure of a neutral Tannakian category such that the "dimension" of an object M is its Euler characteristic.*
- *Let $j_0 \colon \mathbb{G}_m/\overline{k} \hookrightarrow \mathbb{A}^1/\overline{k}$ and ρ be any multiplicative character from $E^\times$ to $\overline{\mathbb{Q}_\ell}^\times$, where E/k is a finite extension. Then, $\omega_\rho \colon M \mapsto H^0\left(\mathbb{A}^1/\overline{k}, j_{0!}\left(M \otimes \mathcal{L}_\rho\right)\right)$ is a fiber functor.*
- *The point is that both $P(\mathbb{G}_m)$ and $P_{arith}(\mathbb{G}_m)$ can be understood to be equivalent to the category of finite-dimensional $\overline{\mathbb{Q}_\ell}$-representations of some algebraic group.*

Suppose N is a perverse sheaf of pure weight zero in $P_{\text{arith}}(\mathbb{G}_m)$. Denote by $\langle N \rangle_{\text{arith}}$ the Tannakian subcategory generated by N, i.e. subquotients of tensor products of copies of N and $N^\vee$ (which is defined as some pullback of $\mathbb{D}N$). With respect to a choice of fiber functor ω, we have that $\langle N \rangle_{\text{arith}}$ is equivalent to the finite-dimensional representations of a group we denote by

$$G_{\text{arith},N,\omega} \subset \mathrm{GL}\left(\omega(N)\right).$$

Concretely, these linear automorphisms are those that stabilize the image of subquotients of $M^{\otimes r} \otimes M^{\vee \otimes s}$ (although here I haven't said what $\otimes$ for this Tannakian category is—it's "middle convolution"). We similarly define $G_{\text{geom},N,\omega}$, which is contained in $G_{\text{arith},N,\omega}$ since there may be more subquotients to stabilize for the former.

Fact 9. *Fr_E is an automorphism of the fiber functor ω_ρ, which defines an element $\mathrm{Fr}_{E,\rho} \in G_{arith,N,\omega}$. Since different $G_{arith,N,\omega}$ are known to all be pairwise isomorphic, unique up to conjugation, i.e. if we fix one fiber functor to get some $G_{arith,N}$, we may view $\mathrm{Fr}_{E,\rho}$ as a conjugacy class in $G_{arith,N}$.*

Since perverse sheaves are geometrically semisimple, viewing N as a faithful representation of $G_{\text{geom},N}$, it follows that $G_{\text{geom},N}$ is a reductive group.

Let us suppose that N is moreover arithmetically semisimple, i.e. semisimple as an object of $P_{\text{arith}}(\mathbb{G}_m)$. Then similarly $G_{\text{arith},N}$ is a reductive group. Let K be a maximal compact subgroup. For certain ρ and E, we may construct conjugacy classes $\theta_{E,\rho}$ in K.

Definition 10. $\rho \colon E^\times \to \overline{\mathbb{Q}_\ell}^\times$ is **good** for N if the canonical "forget supports" map $Rj_!\left(N \otimes \mathcal{L}_\rho\right) \to Rj_*\left(N \otimes \mathcal{L}_\rho\right)$ is an isomorphism.

Fact 11. *For good ρ, $\mathrm{Fr}_{E,\rho}$ has unitary (magnitude one) eigenvalues acting on $\omega_\rho(N)$.*

Let $\mathrm{Fr}^{\mathrm{ss}}_{E,\rho}$ be the semisimplification in the sense of the Jordan decomposition, which by the fact above has unitary eigenvalues, hence lives in a compact subgroup of $G_{\mathrm{arith},N}$, hence lives in a maximal compact subgroup, which is necessarily conjugate to K. Thand us we obtain a conjugacy class $\theta_{E,\rho}$ in K, which is well-defined by Peter–Weyl and Weyl's unitarian trick.

We can now finally state the main theorem.

Theorem 12. Let N be an arithmetically semisimple perverse sheaf on $\mathbb{G}_m/k$ that is pure of weight zero and is in $P_{\mathrm{arith}}(\mathbb{G}_m)$. Fix a maximal compact subgroup K of $G_{\mathrm{arith},N}$. Suppose we have an equality of groups $G_{\mathrm{arith},N} = G_{\mathrm{geom},N}$. Then, as E/k runs over larger and larger finite extension fields, the conjugacy classes $\{\theta_{E,\rho}\}_{\text{good }\rho}$ become equidistributed in the space $K^\#$ of conjugacy classes in K.

Proof. I won't do it here, but it's the same strategy as the proof of Deligne's equidistribution theorem. ☺

Corollary 13. *As E/k runs over larger and larger finite extensions, the exponential sums*

$$\left\{ S(N,E,\rho) = \sum_{t\in E^\times} \mathrm{Tr}\left(\mathrm{Fr}_{E,t}\colon \mathcal{L}_\rho \otimes N\right) \right\}_{\rho\ good}$$

become equidistributed in $\mathbb{C}$ for the measure given by the direct image of the trace map $\mathrm{Tr}\colon K \to \mathbb{C}$.

Applications

Example 14 (Gauss sums)**.** Recall that

$$g(\psi,\chi) = \sum_{t\in\mathbb{G}_m(k)=k^{\times s}} \mathrm{Tr}\left(\mathrm{Fr}_t\colon \mathcal{L}_\chi \otimes M(1/2)[1]\right)$$

for $M = j_0^*\mathcal{L}_\psi$ and $j_0\colon \mathbb{G}_m \hookrightarrow \mathbb{A}^1$. One can check that $N = M(1/2)[1]$ satisfies the conditions of the theorem and its "dimension" is

$$\chi\left(\mathbb{G}_m, M(1/2)[1]\right) = -\chi\left(\mathbb{G}_m, j_0^*\mathcal{L}_\psi\right) = \mathrm{Sw}_\infty\left(\mathcal{L}_\psi\right) = 1.$$

In this case, it turns out that $G_{\mathrm{geom},M} = G_{\mathrm{arith},M} = \mathrm{GL}(1)$, which has maximal compact subgroup S^1. The good characters χ are those for which $\omega_\chi\left(j_0^*\mathcal{L}_\psi(1/2)[1]\right)$ is pure of weight zero, which is the case when $H^0\left(\mathbb{A}^1/\overline{k}, j_{0!}\left(j_0^*\mathcal{L}_\psi(1/2)[1]\otimes\mathcal{L}_\chi\right)\right) = H^0_c\left(\mathbb{G}_m,\mathcal{L}_\chi\right)$ is. This is precisely when χ is non-trivial.

We then see that the normalized Gauss sums (ranging over non-trivial χ) are equidistributed in the unit circle S^1 with respect to the Haar measure.

Example 15 (Evans and Rudnick sums)**.** Recall that

$$E(\chi) = \sum_{t\in\mathbb{G}_m(k)=k^\times} \mathrm{Tr}\left(\mathrm{Fr}_t\colon \mathcal{L}_\chi \otimes M(1/2)[1]\right)$$

and

$$R(\chi) = \sum_{t \in \mathbb{G}_m(k) = k^\times} \mathrm{Tr}\left(\mathrm{Fr}_t : \mathcal{L}_\chi \otimes M'(1/2)[1]\right),$$

where $M = \mathcal{L}_{\psi(x-1/x)}$ and $M' = j_! \mathcal{L}_{\psi((x+1)/(x-1))}$, where $j\colon \mathbb{G}_m \backslash \{1\} \hookrightarrow \mathbb{G}_m$. One can check again that $M(1/2)[1]$ and $M'(1/2)[1]$ satisfy the conditions of the theorem and its "dimension" by Grothendieck–Ogg–Shafarevich is 2.

Katz computes that $G_{\text{geom}} = G_{\text{arith}} = \mathrm{SL}(2)$ in both cases, with corresponding maximal compact subgroup $\mathrm{SU}(2)$. The trace map sends this to $[-2, 2]$.

One can check that M is totally wildly ramified at 0 and ∞, which turns out to imply that every χ is good. M' has trivial monodromy at 0 and ∞, and this turns out to imply every non-trivial χ is good for M'. Applying Katz's theorem then gives the corresponding equidistribution statements.