Covers and the Fundamental Group

Jacob Clay

1 Covers

Definition 1. Throughout these notes, take X to be a topological space. A **space over** X is another topological space Y equipped with a continuous projection $p : Y \to X$. Spaces over X form a category in which a morphism between elements $p_1 : Y_1 \to X$ and $p_2 : Y_2 \to X$ is given by a continuous map f such that $p_1 = f \circ p_2$ or $p_2 = f \circ p_1$.

Definition 2. A **cover of** X is a space $p : Y \to X$ over X such that for all $x \in X$, there is an open subset V containing X such that $p^{-1}(X) = \bigsqcup_i U_i$ for open subsets U_i. Moreover, $p|_{U_i}$ must be a homeomorphism for all i.

Example 3. A helix embedded in $\mathbb{R}^3$ is a cover of the circle S^1.

Example 4. Take the product $X \times I$ of X with some discrete space I is a cover of X with the projection $(x, i) \mapsto x$. An open subset of $x \in X$ maps by p^{-1} to a set of open subsets, indexed by the elements of I, in $X \times I$. This is the "trivial cover" of X.

Proposition 5. Any cover of X is locally a trivial cover and vice versa. Precisely, Y is a cover of X if and only if for any $x \in X$, there is an open V containing X such that $p|_{p^{-1}(V)}$ is isomorphic to a trivial cover when considered as a space over V.

Proof. Example 4 leaves us with only the forward implication to prove. For this, let Y be a cover of X and fix an arbitrary $x \in X$. Then there is a $V \ni x$ such that $p^{-1}(V) = \bigsqcup_{i \in I} U_i$ for an indexed set I. We can define a homeomorphism from $\bigsqcup_{i \in I} U_i$ to $V \times I$ by mapping $u_i \in U_i$ to $(p(u_i), i)$. Since $V \times I$ is a cover of V, this is an isomorphism of covers. ☺

Remark 6. Fix $x \in X$ and consider $p^{-1}(x) = I$. By Proposition 5, the largest set containing x for which $p^{-1}(z) = I$ for all $z \in V$ is open. Thus, we can take the set A of all preimages of points of x, and iterating through the elements of A decomposes X into a disjoint union of open subsets. It follows that if X is connected, the preimages of p are all homeomorphic to the same discrete space.

Example 7. The boundary circle of a Mobius band is cover of the core circle; that it is locally a cover will be shown by the picture I draw in class.

How can we exploit this equivalence? One way is to take a group, endow it with the discrete topology, and determine the conditions under which it can play the role of I in our previous theorem.

Definition 8. The continuous action of a group G on a topological space Y is **even** if for all $y \in Y$, there is an open neighborhood U such that the open sets gU are disjoint from one another.

Proposition 9. If G acts evenly on a connected space Y, the projection $p_G : Y \to Y/G$ turns Y into a cover of Y/G.

Proof. Clearly, p_G is surjective. For any $x \in Y/G$, we can take an open subset $V = p_G(U)$ containing x. Then the open sets gU are disjoint, and p_G restricted to any of them is a homeomorphism. Thus, Y is a cover of Y/G. ☺

Example 10. $\mathbb{Z}^n$ acting on $\mathbb{R}^n$ by translations yields a cover of a linear torus.

Definition 11. A **path** in X is a continuous map $f : [0,1] \to X$. A **loop** is a path for which the endpoints are equal. Two loops f and g are homotopic if there exists an $h : [0,1] \times [0,1] \to X$, the first argument of which continuously takes f to g, and the second argument of which represents time. Homotopy of loops is an equivalence relation (exercise!).

Remark 12. We can compose two loops f, g by taking $(f \circ g)(x) = g(2x), 0 \leq x \leq 1/2$ and $(f \circ g)(x) = f(2x-1)$ for all other x. This defines a multiplication map on the set $\pi_1(X, x)$ of homotopy classes of loops in X based at x. In fact this is a group: the identity is self-evident, and $f(1-x)$ gives the inverse of x.

Definition 13. The **fundamental group** of X based at x, denoted by $\pi_1(X, x)$, is the group of homotopy classes of loops based at X, with loop composition as its operation.

Remark 14. Imagine a path-connected X. If $y \neq x$ is also a point in X. then there is a path f from y to x. If g is a path in $\pi_1(X, x)$, then we can transform it into a path in $\pi_1(X, y)$ by conjugation: $g \mapsto fgf^{-1}$. Thus, $\pi_1(X)$ for a path-connected X is independent of the choice of basepoint, and the common notation used at the beginning of this sentence is justified.