An introduction to Artin-Verdier duality

Matthew Hase-Liu

Contents

1 Introduction

In the late sixties, John Tate and Georges Poitou proved an important duality theorem for Galois cohomology groups of modules over global fields and local fields. In addition to being interesting in its own right, Poitou-Tate duality is an indispensable tool with numerous applications, such as in Tate's work on the Birch and Swinnerton-Dyer conjecture.

The goal of this paper is present an exposition of a variant of a more general version of Poitou-Tate duality, known as Artin-Verdier duality, which suggests that the ring of integers in a number field acts like a 3-manifold.

More precisely, Artin and Verdier showed that for a constructible sheaf F on the étale site of $X = \mathrm{Spec}(\mathcal{O}_K)$ for some K a totally imaginary number field, there is a canonical trace isomorphism

$$H^3(X, \mathbb{G}_m) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z},$$

and a non-degenerate Yoneda pairing of finite groups

$$H^i(X, F) \times \mathrm{Ext}_X^{3-i}(F, \mathbb{G}_m) \to H^3(X, \mathbb{G}_m) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}$$

for all i.

In this expository paper, we will discuss Deninger's variant of Artin-Verdier duality for function fields, i.e for X replaced with a smooth proper curve over a finite field with characteristic p. The duality theorem for function fields is of particular interest (at least to us) because of the similarity to the usual Poincaré duality in étale cohomology. It turns out that (by a sort of descent argument) the case of constructible sheaves with torsion prime to p is essentially Poincaré duality, but the case of constructible sheaves with p-torsion genuinely requires some arithmetic input.

We mostly follow a combination of Deninger's paper on Artin-Verdier duality for function fields ([1]), Mazur's paper on the étale cohomology of number fields ([2]), and Milne's book on arithmetic duality theorems ([3]).

We adopt the following conventions (unless otherwise specified):

(i) p is a prime and q is a power of p.

(ii) $k = \mathbb{F}_q$ and $\overline{k}$ is some algebraic closure.

(iii) $G = \mathrm{Gal}(\overline{k}/k)$ is the absolute Galois group of k.

(iv) A curve (over k) is a geometrically integral one-dimensional scheme of finite type over k.

(v) $\mathsf{Ab}(X_{\text{ét}})$ is the category of abelian sheaves on the étale site of X.

(vi) $\overline{X}$ denotes the base change of X to $\overline{k}$ and $\overline{F}$ is the restriction of a sheaf F to $\overline{X}$.

Remark 1. Unfortunately, we will need to assume that the reader is familiar with étale cohomology (and even the language of schemes) to fully understand the proofs; in particular, Artin-Verdier duality is fundamentally a statement about étale cohomology. On the other hand, if one is willing to blackbox étale cohomology as some cohomology theory for schemes with nice properties (e.g. Poincaré duality and vanishing in high degree), then the broad strokes of the argument will hopefully be clear—perhaps it is worth noting that étale cohomology should be viewed (very roughly, because we really want to consider ℓ-adic cohomology) as a sort of analogue to singular cohomology. Indeed, the primary motivation for the invention of étale cohomology was to develop a more topological cohomology theory for varieties in Weil's suggested approach (via the Lefschetz fixed point formula) to the Weil conjectures.

At the same time, étale cohomology can also be viewed as a generalization of Galois cohomology in the following sense. Let us take for granted the notion of a site (which is a structure on a category that makes its objects behave like open subsets of a topological space). Then, the étale site of a point (i.e. say $\operatorname{Spec} k$ for some field k) is equivalent to the category of $\mathrm{Gal}(\overline{k}/k)$-sets (endowed with the canonical topology), given by sending X in the étale site to the set of $\overline{k}$-points $X(\overline{k})$, which evidentally has a natural (continuous) $\mathrm{Gal}(\overline{k}/k)$-action. Under this equivalence, one can show that the categories of abelian sheaves on these sites are also equivalent, and the latter (quite remarkably) also with the category of $\mathrm{Gal}(\overline{k}/k)$-modules. As a consequence, étale cohomology of $\operatorname{Spec} k$ is the same as the Galois cohomology.

We have tried to make the second section of this paper especially detailed, which is mainly concerned with a few necessary computations for Artin-Verdier duality. To this end, we have also added precise references to facts that we blackbox.

2 Precise statement of the main result and a few computations

For the rest of the paper, let us fix X to be a smooth proper curve over k and $i: Y \hookrightarrow X$ be a closed immersion with non-trivial complement $j: U \subset X$ (i.e. Z is a finite collection of closed points).

Lemma 2. $H^i(X, \mathbb{G}_m) = \begin{cases} k^\times & i = 0 \\ \operatorname{Pic}(X) & i = 1 \\ 0 & i = 2 \\ \mathbb{Q}/\mathbb{Z} & i = 3 \\ 0 & i > 3 \end{cases}$

Proof. First, note that $H^0(\overline{X}, \mathbb{G}_m) = \overline{k}^\times$ (because $\overline{X}$ is integral, proper, and complete) and hence $H^0(X, \mathbb{G}_m) = \left(\overline{k}^\times\right)^G = k^\times$.

Next, $H^1(X, \mathbb{G}_m) = \operatorname{Pic}(X)$ is a standard result from descent theory (proposition 5.7.7 in [4]).

Recall the Hochschild spectral sequence (also called the Artin spectral sequence, theorem III.2.20 in [5]):

$$H^i(G, H^j(\overline{X}, \mathbb{G}_m)) \Rightarrow H^{i+j}(X, \mathbb{G}_m)$$

Since $H^i(\overline{X}, \mathbb{G}_m) = 0$ for $i \geq 2$ (theorem 7.2.7 in [4]), the strict cohomological dimension of G is 2 (proof of theorem 4.5.5 in [4]), and the Brauer group of finite field is trivial (Wedderburn's theorem), the spectral sequence induces an isomorphism

$$H^i(X, \mathbb{G}_m) \xrightarrow{\sim} H^{i-1}(G, \operatorname{Pic}(\overline{X})),$$

for $i \geq 2$.

Moreover, we have an exact sequence $0 \to \operatorname{Pic}^0(\overline{X}) \to \operatorname{Pic}(\overline{X}) \to \mathbb{Z} \to 0$ and Lang's lemma (theorem 6.1 in [6]), which states that $H^i(G, \operatorname{Pic}^0(\overline{X})) = 0$ for $i \geq 1$. The induced long exact sequence is then as follows:

$$\begin{tikzcd}
\cdots \arrow[r] & H^0(G, \operatorname{Pic}^0(\overline{X})) \arrow[r] & H^0(G, \operatorname{Pic}(\overline{X})) \arrow[r] & H^0(G, \mathbb{Z}) \arrow[dll] \\
& H^1(G, \operatorname{Pic}^0(\overline{X})) = 0 \arrow[r] & H^1(G, \operatorname{Pic}(\overline{X})) \arrow[r] & H^1(G, \mathbb{Z}) \arrow[dll] \\
& H^2(G, \operatorname{Pic}^0(\overline{X})) = 0 \arrow[r] & H^2(G, \operatorname{Pic}(\overline{X})) \arrow[r] & H^2(G, \mathbb{Z}) \arrow[r] & \cdots
\end{tikzcd}$$

We then have $H^i(X, \mathbb{G}_m) = 0$ for $i = 2$ (since $H^1(G, \mathbb{Z}) = \operatorname{Hom}(G, \mathbb{Z}) = 0$) and an isomorphism (from the degree map)

$$H^2(G, \operatorname{Pic}(\overline{X})) \xrightarrow{\sim} H^2(G, \mathbb{Z}).$$

The latter is $\mathbb{Q}/\mathbb{Z}$, so we obtain a canonical trace isomorphism

$$\mathrm{Tr}\colon H^3(X,\mathbb{G}_m) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}.$$

For $i > 3$, the result follows from the strict cohomological dimension of G and the long exact sequence. ☺

Lemma 3. *There is a natural trace isomorphism* $\mathrm{Tr}\colon H^3_c(U,\mathbb{G}_m) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}$.

Proof. We use the canonical trace isomorphism constructed in the proof of the previous lemma. Consider the composition

$$\mathrm{Tr}\colon H^3_c(U,\mathbb{G}_m) \coloneqq H^3(X, j_!\mathbb{G}_m) \to H^3(X,\mathbb{G}_m) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z},$$

which is an isomorphism by considering the exact sequence $0 \to j_!\mathbb{G}_m \to \mathbb{G}_m \to i_*\mathbb{G}_m \to 0$ (note that Y is a finite collection of closed points). ☺

Lemma 4. $\mathrm{Ext}^i_X(\mathbb{Z}/p,\mathbb{G}_m) = \begin{cases} 0 & i = 0 \\ \mathrm{Pic}(X)[p] & i = 1 \\ \mathrm{Pic}(X)/p\,\mathrm{Pic}(X) & i = 2 \\ \mathbb{Z}/p\mathbb{Z} & i = 3 \\ 0 & i > 3 \end{cases}$

Proof. Consider the short exact sequence $0 \to \mathbb{Z} \xrightarrow{\cdot p} \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \to 0$. Applying Ext, taking the long exact sequence, and using the first lemma gives

$$\begin{aligned} \cdots &\longrightarrow \mathrm{Ext}^0_X(\mathbb{Z}/p\mathbb{Z},\mathbb{G}_m) \longrightarrow \mathrm{Ext}^0_X(\mathbb{Z},\mathbb{G}_m) = k^* \longrightarrow \mathrm{Ext}^0_X(\mathbb{Z},\mathbb{G}_m) = k^* \\ &\longrightarrow \mathrm{Ext}^1_X(\mathbb{Z}/p\mathbb{Z},\mathbb{G}_m) \longrightarrow \mathrm{Ext}^1_X(\mathbb{Z},\mathbb{G}_m) = \mathrm{Pic}(X) \longrightarrow \mathrm{Ext}^1_X(\mathbb{Z},\mathbb{G}_m) = \mathrm{Pic}(X) \\ &\longrightarrow \mathrm{Ext}^2_X(\mathbb{Z}/p\mathbb{Z},\mathbb{G}_m) \longrightarrow \mathrm{Ext}^2_X(\mathbb{Z},\mathbb{G}_m) = 0 \longrightarrow \mathrm{Ext}^2_X(\mathbb{Z},\mathbb{G}_m) = 0 \\ &\longrightarrow \mathrm{Ext}^3_X(\mathbb{Z}/p\mathbb{Z},\mathbb{G}_m) \longrightarrow \mathrm{Ext}^3_X(\mathbb{Z},\mathbb{G}_m) = \mathbb{Q}/\mathbb{Z} \longrightarrow \mathrm{Ext}^3_X(\mathbb{Z},\mathbb{G}_m) = \mathbb{Q}/\mathbb{Z} \longrightarrow \cdots \end{aligned}$$

The result then follows. ☺

As stated in the introduction, the Yoneda pairing induces a pairing as follows:

$$\begin{tikzcd} H^i_c(U,F) \times \mathrm{Ext}^{3-i}_U(F,\mathbb{G}_m) \arrow[r, dashed] \arrow[d, "\cong"'] & H^3_c(U,\mathbb{G}_m) \arrow[r, "\mathrm{Tr}"] \arrow[d, "\cong"'] & \mathbb{Q}/\mathbb{Z} \arrow[d, "="'] \\ H^i(X, j_!F) \times \mathrm{Ext}^{3-i}_X(j_!F,\mathbb{G}_m) \arrow[r] & H^3(X,\mathbb{G}_m) \arrow[r, "\mathrm{Tr}"] & \mathbb{Q}/\mathbb{Z} \end{tikzcd}$$

Theorem 5 (Artin-Verdier duality). *Let X be a smooth proper curve over k. If $U \subset X$ is a non-empty open subscheme of X and $F \in \mathsf{Ab}(U_{\text{ét}})$ is constructible, then $\forall i \in \mathbb{Z}$, the pairing*

$$H^i_c(U, F) \times \operatorname{Ext}^{3-i}_U(F, \mathbb{G}_m) \to H^3_c(U, \mathbb{G}_m) \xrightarrow{\operatorname{Tr}} \mathbb{Q}/\mathbb{Z}$$

is non-degenerate. Also, $H^i_c(U, F)$ and $\operatorname{Ext}^{3-i}_U(F, \mathbb{G}_m)$ are finite groups.

By how this pairing is constructed, it's clear that it suffices to prove the theorem for the case $U = X$, which we will assume in the sequel. Denote by $m^i(F) = m^i_X(F)\colon H^i(X, F) \to \operatorname{Ext}^{3-i}_X(F, \mathbb{G}_m)^\vee$ the map induced by the Yoneda pairing, where $^\vee$ denotes taking the $\mathbb{Q}/\mathbb{Z}$-dual. Then, it suffices to show that $m^i(F)$ is an isomorphism for all constructible F.

3 Proof of the "geometric" portion: sheaves with torsion prime to p

In this section, we will prove Artin-Verdier duality for the case of sheaves with torsion prime to p.

Let us first recall the following fact. Let S a locally Noetherian scheme with $n \in \mathbb{Z}$ invertible on S and F a sheaf of $\mathbb{Z}/n\mathbb{Z}$-sheaves on $S_{\text{ét}}$. Then, for all $i \geq 0$, we have

$$\operatorname{Ext}^i_{\mathbb{Z}/n\mathbb{Z}}(F, \mu_n) = \operatorname{Ext}^i_S(F, \mathbb{G}_m),$$

where the LHS is computed in the category of $\mathbb{Z}/n\mathbb{Z}$-sheaves on $S_{\text{ét}}$ and the RHS is computed in $\mathsf{Ab}(S_{\text{ét}})$.

To see this, let $I^\bullet$ be an injective resolution of $\mathbb{G}_m$. Then, we compute Ext by the sequence with $\operatorname{Hom}_S(F, -)$ and taking cohomology. Evidently $\operatorname{Hom}_S(F, I) = \operatorname{Hom}_{\mathbb{Z}/n\mathbb{Z}}(F, I[n])$, and noting that $(I[n])^\bullet$ gives an injective resolution of μ_n in the category of $\mathbb{Z}/n\mathbb{Z}$-sheaves on $S_{\text{ét}}$, the result follows.

The key input to the geometric portion of Artin-Verdier duality is provided by Poincaré duality for curves over an algebraically closed field. We have an isomorphism

$$H^i(\overline{X}, \overline{F}) \xrightarrow{\sim} \left(\operatorname{Ext}^{2-i}_{\overline{X}}(\overline{F}, \mathbb{G}_m)\right)^\vee$$

for any constructible $\mathbb{Z}/n\mathbb{Z}$-sheaf F (theorem V.2.1 in [7]).

Again, by the Hochschild-Serre spectral sequences, we have

$$H^i(G, H^j(\overline{X}, \overline{F})) \Rightarrow H^{i+j}(X, F)$$

and similarly

$$H^i(G, \operatorname{Ext}^j_{\overline{X}}(\overline{X}, \mathbb{G}_m)) \Rightarrow \operatorname{Ext}^{i+j}_X(F, \mathbb{G}_m).$$

Because G has cohomological dimension 1, these spectral sequences give short exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & H^1(G, H^{j-1}(\overline{X}, \overline{F})) \arrow[r] \arrow[d, "m'"'] & H^j(X, F) \arrow[r] \arrow[d, "m^i(F)"'] & H^0(G, H^j(\overline{X}, \overline{F})) \arrow[r] \arrow[d, "m''"'] & 0 \\
0 \arrow[r] & H^0(G, \operatorname{Ext}^{3-j}_{\overline{X}}(\overline{F}, \mathbb{G}_m))^\vee \arrow[r] & \operatorname{Ext}^{3-j}_X(F, \mathbb{G}_m)^\vee \arrow[r] & H^1(G, \operatorname{Ext}^{2-j}_{\overline{X}}(\overline{F}, \mathbb{G}_m))^\vee \arrow[r] & 0
\end{tikzcd}$$

The first and third vertical arrows are isomorphisms by Poincaré duality and Pontryagin duality, which tells us that for $r = 0, 1$,

$$H^r(G, M) \times H^{1-r}(G, M^\vee) \to \mathbb{Q}/\mathbb{Z}$$

is a non-degenerate pairing for all finite Galois modules M.

Then, the middle vertical arrow is an isomorphism as well, and the geometric portion of Artin-Verdier duality follows.

4 Proof of the "arithmetic" portion: sheaves with p-torsion

In this section, we will prove the arithmetic portion of Artin-Verdier duality, i.e. the case of sheaves with p-torsion. Hence, F will now denote a constructible p-adic sheaf on X

Let us first being with the case of **punctual sheaves**: $F = i_* P$ for some inclusion of a closed point $i: x \hookrightarrow X$ and P a finite étale sheaf on x. In this case, we can again conclude via Pontryagin duality. To see this, we will identify $H^i(X, i_* P)$ with $H^i(G, P)$, $\mathrm{Ext}^{3-i}_X(i_* P, \mathbb{G}_m)$ with $H^{1-i}(G, \mathrm{Hom}(P, \mathbb{Q}/\mathbb{Z}))$, and $H^3(X, \mathbb{G}_m)$ with $H^1(G, \mathbb{Q}/\mathbb{Z})$.

Recall that $R^i i^! \mathbb{G}_m = \begin{cases} \mathbb{Z} & q = 1 \\ 0 & q \neq 1 \end{cases}$ (this comes from the divisor exact sequence—look at the proof of theorem V.2.1 in [5]). Then, the spectral sequence $\mathrm{Ext}^i_x(P, R^j i^! \mathbb{G}_m) \Rightarrow \mathrm{Ext}^{i+j}_X(i_* P, \mathbb{G}_m)$ gives rise to an isomorphism

$$\mathrm{Ext}^i_X(i_* P, \mathbb{G}_m) \cong \mathrm{Ext}^{i-1}_x(P, \mathbb{Z}) \cong \mathrm{Ext}^{i-1}_G(P, \mathbb{Z}).$$

We also have $H^i(X, i_* P) = H^i(G = \mathrm{Gal}(\overline{\kappa(x)}/\kappa(x)), P)$ (the former being étale cohomology and the latter Galois cohomology).

There is a local-global Ext spectral sequence (for Galois cohomology) that tells us that

$$H^i(G, \mathrm{Ext}^i_{\mathbb{Z}}(P, \mathbb{Q})) \Rightarrow \mathrm{Ext}^{i+j}_G(P, \mathbb{Q}).$$

Since the LHS is zero, we have $\mathrm{Ext}^i_G(P, \mathbb{Q}) = 0$ for all i. Taking the exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$, we see that for all $i \geq 0$,

$$\mathrm{Ext}^{i+1}_G(P, \mathbb{Z}) \cong \mathrm{Ext}^i_G(P, \mathbb{Q}/\mathbb{Z}).$$

The local-global Ext spectral sequence similarly gives us

$$H^i(G, \mathrm{Hom}_{\mathbb{Z}}(P, \mathbb{Z})) \cong \mathrm{Ext}^i_G(P, \mathbb{Q}/\mathbb{Z}).$$

This completes the necessary identifications for the punctual case. We then have the following commutative diagram, which completes our proof of Artin-Verdier duality in the punctual setting via Pontryagin duality:

$$\begin{array}{ccc}
H^i(X, i_*P) \times \operatorname{Ext}_X^{3-i}(i_*P, \mathbb{G}_m) & \longrightarrow & H^3(X, \mathbb{G}_m) \\
\cong \big\downarrow & & \big\downarrow \\
H^i(G, P) \times \operatorname{Ext}_X^{2-i}(P, \mathbb{Z}) & \longrightarrow & H^2(G, \mathbb{Z}) \\
\cong \big\downarrow & & \cong \big\downarrow \\
H^i(G, P) \times H^{1-i}(G, \operatorname{Hom}(P, \mathbb{Q}/\mathbb{Z})) & \longrightarrow & H^1(G, \mathbb{Q}/\mathbb{Z})
\end{array}$$

Next, we show that Artin-Verdier duality only needs verification in a few degrees.

Lemma 6. *Let F' be a constructible p-torsion sheaf on U. Then, $H^i_c(U, F') = 0$ for $i \geq 3$ and $\operatorname{Ext}^i_U(F, \mathbb{G}_m) = 0$ for $i > 3$.*

Proof. By Artin-Schreier theory (theorem 7.2.13 in [4]), the p-cohomological dimension of X is at most $\dim X + 1 = 2$. So the first result is immediate.

For the claim about Ext, note that $\operatorname{Ext}^i_U(\text{punctual sheaf}, \mathbb{G}_m) = 0$ for $i > 3$ by Artin-Verdier duality (for punctual sheaves). If $j' \colon U' \subset U$ is an inclusion of nonempty open subsets with i' the inclusion of the complement, we have an exact sequence $0 \to j'_! j'^* F' \to F \to i'_* i'^* F \to 0$, which by what we just said implies $\operatorname{Ext}^i_U(j'_! j'^* F', \mathbb{G}_m) \cong \operatorname{Ext}^i_U(F', \mathbb{G}_m)$ for $i > 3$. Also, the LHS is isomorphic to $\operatorname{Ext}^i_{U'}(j'^* F, \mathbb{G}_m)$ because j^* preserves injectives. So it suffices to show the claim for locally constant sheaves F'.

By considering a Jordan-Hölder sequence of F', we can reduce to the case $F' = \mathbb{Z}/p\mathbb{Z}$. By considering the exact sequence $0 \to \mathbb{Z} \xrightarrow{\cdot p} \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \to 0$, and noting that $\underline{\operatorname{Ext}}^i_U(\mathbb{Z}, \mathbb{G}_m) = 0$ for $i \geq 1$, we have $\underline{\operatorname{Ext}}^i_U(F', \mathbb{G}_m) = 0$ for $i \geq 2$. The p-cohomological dimension of U is at most 2, so using the local-global Ext spectral sequence gives the desired result. ☺

The rest of the argument proceeds inductively, and we begin by proving the base case. We also verify that that the sizes of the groups in the duality theorem are the same. One should view this as the "meat" of the arithmetic portion of the proof.

Lemma 7.

(i) $|H^i(X, \mathbb{Z}/p\mathbb{Z})| = |\operatorname{Ext}^{3-i}_X(\mathbb{Z}/p\mathbb{Z}, \mathbb{G}_m)|$ for all $i \in \mathbb{Z}$ (and both are finite).

(ii) $m^0(\mathbb{Z}/p\mathbb{Z})$ is an isomorphism.

Proof. We first prove (i). For $i < 0$ and $i > 2$, the result follows from the previous lemma and lemma 4.

For $i = 0$, note that $|H^0(X, \mathbb{Z}/p\mathbb{Z})|$ and $|\operatorname{Ext}^3_X(\mathbb{Z}/p\mathbb{Z}, \mathbb{G}_m)|$ are both equal to p by lemma 4.

For $i = 1$, recall that $H^1(X, \mathbb{Z}/p\mathbb{Z})$ classifies $\mathbb{Z}/p\mathbb{Z}$-torsors on X and is equal to $\operatorname{Hom}_{\text{cts}}(\pi_1(X), \mathbb{Z}/p\mathbb{Z})$. As a consequence of (geometric) class field theory[1], this is the same as $\operatorname{Hom}(\operatorname{Pic}(X), \mathbb{Z}/p\mathbb{Z})$. Then, $|H^1(X, \mathbb{Z}/p\mathbb{Z})|$ and $|\operatorname{Ext}^2_X(\mathbb{Z}/p\mathbb{Z}, \mathbb{G}_m)|$ both have size $|\operatorname{Pic}(X)/p\operatorname{Pic}(X)|$ by lemma 4.

[1]The key point being that $\widehat{\operatorname{Pic}(X)} \cong \pi_1^{\text{ab}}(X)$.

For $i = 2$, note that $\operatorname{Pic}(X) \cong \operatorname{Pic}^0(X) \times \mathbb{Z}$ and $\operatorname{Pic}^0(X)$ is finite. The Artin-Schreier sequence implies that $|H^2(X, \mathbb{Z}/p\mathbb{Z})| = |H^1(X, \mathbb{Z}/p\mathbb{Z})|/p = |\operatorname{Pic}(X)/p\operatorname{Pic}(X)|/p = |\operatorname{Pic}(X)[p]| = |\operatorname{Ext}^1_X(\mathbb{Z}/p\mathbb{Z}, \mathbb{G}_m)|$, as desired.

For (ii), the map $m^0(\mathbb{Z}/p\mathbb{Z})\colon \mathbb{Z} = H^0(X, \mathbb{Z}) \to \operatorname{Ext}^3_X(\mathbb{Z}, \mathbb{G}_m)^\vee \cong (\mathbb{Q}/\mathbb{Z})^\vee \cong \widehat{\mathbb{Z}}$ is given by inclusion. We have the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r, "\cdot p"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & H^0(X, \mathbb{Z}/p\mathbb{Z}) \arrow[r] \arrow[d, "m^0(\mathbb{Z}/p\mathbb{Z})"'] & 0 \\
0 \arrow[r] & \widehat{\mathbb{Z}} \arrow[r, "\cdot p"] & \widehat{\mathbb{Z}} \arrow[r] & \operatorname{Ext}^3_X(\mathbb{Z}/p\mathbb{Z}, \mathbb{G}_m)^\vee &
\end{tikzcd}$$

By an easy diagram chase, it's clear that $m^0(\mathbb{Z}/p\mathbb{Z})$ is injective (note that we have $H^0(X, \mathbb{Z}/p\mathbb{Z}) \to \widehat{\mathbb{Z}}/p\widehat{\mathbb{Z}} \cong \mathbb{Z}/p\mathbb{Z} \hookrightarrow \operatorname{Ext}^3_X(\mathbb{Z}/p\mathbb{Z}, \mathbb{G}_m)^\vee$). Hence, by (i), it must be an isomorphism. ☺

The next lemma roughly tell us how to reduce from constructible sheaves to constant sheaves. The idea is that we can embed a constructible sheaf into a direct sum of pushforwards of constant sheaves (we will see this later). Let $\mathcal{C}_X$ denote either the 1. category of constructible $\mathbb{Z}/p\mathbb{Z}$-sheaves on X or 2. category of constructible p-torsion sheaves on X.

Lemma 8. *Let X and Y be smooth proper curves over k, and suppose $\pi\colon Y \to X$ is finite and dominant. Then, for all i and any constructible sheaf F on Y, we have $m^i_X(\pi_* F)$ is an isomorphism iff $m^i_Y(F)$ is an isomorphism.*

Proof. By Tag 0BCX ([7]), there exists a norm map

$$\pi_* \mathbb{G}_m \to \mathbb{G}_m,$$

which is compatible with the trace morphism:

$$\begin{tikzcd}
H^3(X, \pi_* \mathbb{G}_m) \arrow[r, "\cong"] \arrow[d, "\text{norm}"'] & H^3(Y, \mathbb{G}_m) \arrow[r, "\cong"] & \mathbb{Q}/\mathbb{Z} \arrow[d, "\cong"] \\
H^3(X, \mathbb{G}_m) \arrow[rr, "\cong"'] & & \mathbb{Q}/\mathbb{Z}
\end{tikzcd}$$

By exactness of π_*, we have a morphism $\operatorname{Ext}^i_Y(F, \mathbb{G}_m) \to \operatorname{Ext}^i_X(\pi_* F, \pi_* \mathbb{G}_m)$, which composed with the norm gives the map

$$N\colon \operatorname{Ext}^i_Y(F, \mathbb{G}_m) \to \operatorname{Ext}^i_X(\pi_* F, \mathbb{G}_m).$$

We have a commutative diagram as follows:

$$\begin{tikzcd} H^i(Y,F)\times \mathrm{Ext}^{3-i}_Y(F,\mathbb{G}_m) \arrow[r] \arrow[d, "N"'] & H^3(Y,\mathbb{G}_m)\cong H^3(X,\pi_*\mathbb{G}_m) \arrow[r] \arrow[d, "\mathrm{norm}"'] & \mathbb{Q}/\mathbb{Z} \arrow[d, "\cong"'] \\ H^i(X,\pi_*F)\times \mathrm{Ext}^{3-i}_X(\pi_*F,\mathbb{G}_m) \arrow[r] & H^3(X,\mathbb{G}_m) \arrow[r] & \mathbb{Q}/\mathbb{Z} \end{tikzcd}$$

It follows that if F is punctual, then by Artin-Verdier duality, N is necessarily an isomorphism.

If π is étale, N is automatically an isomorphism[2].

The norm map is also compatible with restriction to open subschemes. If $j: U \subset X$ is an open immersion, $\bar{j}: V \subset Y$ is the base change along π, and $\pi': V \to U$ is the base change of π along j, then we have the following diagram:

$$\begin{tikzcd} \mathrm{Ext}^i_Y(\bar{j}_!F,\mathbb{G}_m) \arrow[r, "N"] \arrow[d, "\cong"'] & \mathrm{Ext}^i_X(\pi_*\bar{j}_!F,\mathbb{G}_m) \arrow[r, "\cong"] & \mathrm{Ext}^i_X(j_!\pi'_*F,\mathbb{G}_m) \arrow[d, "\cong"] \\ \mathrm{Ext}^i_V(F,\mathbb{G}_m) \arrow[rr, "N"] & & \mathrm{Ext}^i_U(\pi'_*F,\mathbb{G}_m) \end{tikzcd}$$

Then, one N being an isomorphism implies the other is.

Now, for a general F, consider the exact sequence $0 \to j_!j^*F \to F \to i_*i^*F \to 0$, where j is chosen so that $\pi|_V$ is étale (Tag 0C1C, [7]). Then, we conclude by the discussion above. ☺

Let us now return to the induction.

Lemma 9. *Suppose there is a positive integer r such that $m^i(F)$ is an isomorphism for all $i < r$ and $F \in \mathcal{C}_X$. Then, $m^r(F)$ is injective.*

Proof. Recall that $H^r(X,-)$ is coeffaceable for constructible sheaves, i.e. for any $F \in \mathcal{C}_X$, we can find $G \in \mathcal{C}_X$ with an injection $f: F \hookrightarrow G$ so that $H^r(X,f) = 0$ (proposition 5.6.13 in [4]). Then, we have an exact sequence $0 \to F \to G \to G/F \to 0$ and by assumption $\ker m^r(F)$ maps injectively into $\ker m^r(G)$. Take any $x \in \ker m^r(F) \subset H^r(X,F)$. It goes to 0 in $H^r(X,F')$, so we necessarily have $x = 0$, as desired. ☺

Let (C_r) denote the following condition: $m^r(C)$ is an isomorphism for all X smooth proper curves over k and for all finite constant sheaves $C \in \mathcal{C}_X$.

Lemma 10. *Suppose there is a nonnegative integer r such that $m^i(F)$ is an isomorphism for all $i < r$, X a smooth proper curve over k, and $F \in \mathcal{C}_X$, and that (C_r) holds.*

Then, for all smooth proper curves X over k and $F \in \mathcal{C}_X$, we have $m^r(F)$ is an isomorphism and $m^{r+1}(F)$ is injective.

[2]There is some discussion about this here: `https://mathoverflow.net/questions/266877/norm-theorem-for-finite-etale-morphisms-between-dedekind-affine-schemes`

Proof. The claim that $m^{r+1}(F)$ is injective is immediate from the preceding lemma (assuming the claim that $m^r(F)$ is an isomorphism).

Recall (proposition 5.8.11 in [4]) that there are finite morphisms $\pi_i: Y_i \to X$ with integral normal schemes Y_i and finite constant sheaves $C_i \in \mathcal{C}_{Y_i}$ such that $F \hookrightarrow \bigoplus_i \pi_{i*}C_i$. We claim that each $m^r(\pi_{i*}C_i)$ is an isomorphism.

If π_i is not dominant, then $\pi_{i*}C_i$ is necessarily a punctual sheaf, and so our proof of Artin-Verdier duality for punctual sheaves tells us that $m^r(\pi_{i*}C_i)$ is an isomorphism.

If π_i is dominant, then we necessarily have that Y_i is a smooth proper curve over $\mathcal{O}(Y_i)$ (a finite field), so condition (C_r) implies that $m_Y^r(C_i)$ is an isomorphism. By lemma 8, it follows that $m^r(\pi_{i*}C_i)$ is also an isomorphism.

Hence, $m^r\left(\bigoplus_i \pi_{i*}C_i\right)$ is an isomorphism. Now, consider the exact sequence $0 \to F \to \bigoplus_i \pi_{i*}C_i \to Q \to 0$, where Q is the cokernel of $F \hookrightarrow \bigoplus_i \pi_{i*}C_i$. Then, we have the following commutative diagram:

$$\begin{tikzcd}
H^{r-1}(X,G) \arrow[r] \arrow[d, "\cong"'] & H^{r-1}(X,Q) \arrow[r] \arrow[d, "\cong"'] & H^r(X,F) \arrow[r] \arrow[d, "m^r(F)"'] & H^r(X,G) \arrow[r] \arrow[d, "\cong"'] & H^r(X,Q) \arrow[d, "m^r(Q)"'] \\
\mathrm{Ext}_X^{2-r}(G,\mathbb{G}_m)^\vee \arrow[r] & \mathrm{Ext}_X^{2-r}(Q,\mathbb{G}_m)^\vee \arrow[r] & \mathrm{Ext}_X^{3-r}(F,\mathbb{G}_m)^\vee \arrow[r] & \mathrm{Ext}_X^{3-r}(G,\mathbb{G}_m)^\vee \arrow[r] & \mathrm{Ext}_X^{3-r}(Q,\mathbb{G}_m)^\vee
\end{tikzcd}$$

By a diagram chase, it's clear that the middle arrow is injective. This holds for any constructible sheaf F, so it holds for G as well. Then, by the five lemma, it follows that the middle arrow is an isomorphism, as desired. ☺

We are almost done with the proof of the Artin-Verdier duality. The remainder is now mostly a matter of piecing together the moving parts in a coherent way.

First, let us consider the case of constructible $\mathbb{Z}/p\mathbb{Z}$-sheaves on X. Since any such constant sheaf is a direct sum of $\mathbb{Z}/p\mathbb{Z}$'s, it follows that (C_r) is equivalent to requiring that $m^r(\mathbb{Z}/p\mathbb{Z})$ is an isomorphism for all smooth proper curves X over k. By lemma 7, we know (C_0) holds. Lemma 7 also ensures that "injective" and "isomorphic" are equivalent (because the sizes of the groups are the same), and so the previous lemma lets us conclude the case of constructible $\mathbb{Z}/p\mathbb{Z}$-sheaves via induction.

In particular, $m^r(\mathbb{Z}/p\mathbb{Z})$ is an isomorphism for every r. By considering the exact sequence $0 \to \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p^2\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \to 0$, it follows that $m^r(\mathbb{Z}/p^2\mathbb{Z})$ is an isomorphism as well for all r. We can similarly show that for any constant p-torsion sheaf that Artin-Verdier duality holds. Then, by the previous lemma, Artin-Verdier duality holds for all constructible sheaves, and the induction is complete.

5 Concluding remarks

At this point, we should add a few words describing the development of Artin-Verdier duality. Deninger, for instance, extended the duality theorem to non-torsion sheaves, namely to a class of sheaves that are called "$\mathbb{Z}$-constructible." Others, including Artin, Mazur, and Milne, have extended some of these results to flat cohomology. There is of course much more to this story, and we recommend taking a look at Milne's textbook ([3]) for a detailed account.

References

[1] Deninger, C. 1984, On Artin-Verdier duality for function fields, Math. Z. 188, 91–100.

[2] Mazur, B. 1973, Notes on étale cohomology of number fields, Ann. Sci. Ecole Norm. Sup. 6, 521–552.

[3] Milne, J. S. 2006, Arithmetic duality theorems, BookSurge, LLC, Charleston, SC.

[4] Fu, L. 2015, Étale cohomology theory, Nankai Tracts in Mathematics 13.

[5] Milne, J. S. 1980, Étale cohomology, Princeton Mathematical Series 33, Princeton University Press, Princeton, N.J.

[6] Serre, J. P. and Ion P. (trans.) 1997, Galois Cohomology, Springer-Verlag, New York.

[7] The Stacks Project Authors 2022, Stacks Project http://stacks.math.columbia.edu.