# Notes from “Abelian reasons and a variety of examples to care about abelian varieties”

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Notes from the student seminar “Abelian reasons and a variety of examples to care about abelian varieties (Spring 2022)” (http://www.math.columbia.edu/~mhaseliu/AV.html). I’ve followed (**i.e. copied directly**) Bhargav Bhatt’s course notes fairly closely (http://www-personal.umich.edu/~stevmatt/abelian and added some comments—these are mostly for myself but maybe they will be useful to someone else. Let me know if you find any typoes I’ve made!

## Contents

(TODO: add notes from first three talks)

**Theorem 1** (Rigidity). *Let $f\colon X \to S$ be proper flat with $\kappa(s) \cong H^0(X_s, \mathcal{O}_{X_s})$ for all $s \in S$ and $S$ connected. Also, let $g\colon Y \to S$ be separated. If $\pi\colon X \to Y$ is an $S$-map such that the restriction $\pi_s\colon X_s \to Y_s$ is constant (i.e. factors through $s$), then $\pi$ is constant (i.e. factors through $S$).*

**Theorem 2** (Seesaw theorem). *Let $f\colon X \to S$ be proper flat with geometrically integral fibers. Let $L \in \mathrm{Pic}(X)$. Then,*

*(i) $Z := \{s \in S\colon L_s \text{ is trivial}\}$ is a closed subset of $S$.*

*(ii) $L_{Z_{\mathrm{red}}}$ is pulled back from $Z_{\mathrm{red}}$ (the reduced induced structure on $Z$).*

*(iii) $\exists!$ closed subscheme structure on $Z$ such that $L_Z$ is pulled back from $Z$, and $Z$ is universal with this property among all $S$-schemes.*

# 1 02/17: Projectivity of abelian varieties

We will continue discussing the theorem of the cube and see some applications to abelian varieties. In particular, the theorem of the square will motivate the existence of dual abelian varieties. After introducing some important constructions, such as the Mumford bundle and the Picard functor, we will prove that abelian varieties are projective.

**Theorem 3.** *Let $S$ be a connected Noetherian scheme, $X \to S, Y \to S$ be proper and flat with geometrically integral fibers, and $L \in \mathrm{Pic}(X \times_S Y)$. Suppose there are sections $e_X \in X(S)$ and $e_Y \in Y(S)$ such that $L|_{\{e_X\}\times_S Y}$ and $L|_{X\times_S\{e_Y\}}$ are trivial and there is a point $s \in S$ such that $L|_{X_s \times_s Y_s}$ is trivial. Then, $L$ is trivial.*

*Proof.* Let $\pi\colon P = X \times_S Y \to S$ be the standard projection and $Z \subset S$ the universal subscheme from the Seesaw theorem. We are done if we can show $Z = S$. Note that $s \in Z$ so $Z$ is nonempty, so it suffices to show that $Z$ is closed under generalization. Can assume $S = \mathrm{Spec}(R)$ with $(R, \mathfrak{m})$ a DVR and $Z = \mathrm{Spec}(R/I)$. WTS $I = 0$.

If $I \neq 0$, we can find an ideal $J \subset I$ such that $I/J \cong k = R/\mathfrak{m}$; e.g. take $J$ to be $\mathfrak{m}I$ since $\mathfrak{m}I = I$ implies $I = 0$ (by NAK) and $I/\mathfrak{m}I$ is a $k$-vector space injecting into $R/\mathfrak{m} = k$. Now, we have $W = \mathrm{Spec}(R/J) \subset S$ strictly contains $Z$ (scheme-theoretically), i.e. we have a closed immersion $Z \to W$ that is not an isomorphism. Now, using some universal property related to $Z$, we will show that $W$ cannot have this same property, which will let us achieve our contradiction.

In particular, $\mathrm{Pic}(\text{local ring})$ is trivial, so $L_{P_Z}$, which is pulled back from $Z$, is necessarily trivial. If we can show that $L_{P_W}$ is also trivial, we get a contradiction. We have the SES (on $Z$) given by $0 \to k \to R/J \to R/I \to 0$, which can be pulled back to $P$ ($\pi$ is flat) and tensored with $L$ (which is exact) to get $0 \to L_{P_s} \to L_{P_W} \to L_{P_Z} \to 0$. Suppose $L_{P_Z} = \mathcal{O}_{P_Z} \cdot s$. If we can lift $s$ to a section of $L_{P_W}$, say $t$, then the map $\mathcal{O}_{P_W} \to L_{P_W}$ given by multiplication by $t$ descends to an isomorphism $k \to L_{P_W} \otimes_{\mathcal{O}_{P_W}} k$, so NAK implies $L_{P_W} = \mathcal{O}_{P_W} \cdot t$. So if we can show that $\delta\colon H^0(P_Z, L_{P_Z}) \to H^1(P_s, L_{P_s})$ sends $s$ to 0, then we are done. Note that we have maps $H^1(P_s, L_{P_s}) \to H^1(X_s, L_{X_s})$ and $H^1(P_s, L_{P_s}) \to H^1(Y_s, L_{Y_s})$ given by pullback, which are both trivial by assumption that the existence of sections $e_X, e_Y$ that make $L|_{\{e_X\}\times_S Y}$ and $L|_{X\times_S\{e_Y\}}$ trivial. But Kunneth, the fact that the fibers are geometrically integral, and $L|_{X_s \times_s Y_s} = L_{P_s}$ being trivial imply that $H^1(P_s, L_{P_s}) \to H^1(X_s, L_{X_s}) \times H^1(Y_s, L_{Y_s})$ is bijective, so $\delta(s) = 0$, and the result follows. ☺

**Corollary 4** (Theorem of the cube)**.** *Let $k$ be a field, $X$ and $Y$ be proper, geometrically integral schemes of finite type over $k$, and $Z$ be connected of finite type over $k$. If $L \in \mathrm{Pic}(X \times Y \times Z)$ and $x \in X(k), y \in Y(k), z \in Z(k)$ such that $L$ is trivial on $\{x\} \times Y \times Z, X \times \{y\} \times Z, X \times Y \times \{z\}$, then $L$ is trivial.*

*Proof.* Consider $X \times Y \times Z = (X \times Z) \times_Z (Y \times Z) \to Z$ and apply the theorem above. ☺

**Corollary 5.** *Let $A$ be an abelian variety over $k$, and $Z$ be a $k$-scheme with maps $f, g, h \colon Z \to A$. Then, for any $L \in \mathrm{Pic}(A)$, we have $(f+g+h)^* L \cong (f+g)^* L \otimes (g+h)^* L \otimes (f+h)^* L \otimes f^* L^{-1} \otimes g^* L^{-1} \otimes h^* L^{-1}$.*

**Corollary 6** (Theorem of the square)**.** *Let $A$ be an abelian $S$-scheme and $L \in \mathrm{Pic}(A)$, then for any $x, y \in A(S)$, we have $t_x^*(L) \otimes t_y^*(L) \cong t_{x+y}^*(L) \otimes L$, up to line bundles pulled back from $S$.*

*Proof.* Let $c_x$ be the constant map $A \to S \to A$ with $x = S \to A$, and similarly for $c_y$ and $c_e$. Then, apply the previous corollary to $c_x, c_y, c_e$. ☺

**Corollary 7.** *If $A$ is an abelian variety over $k$ and $L \in \mathrm{Pic}(A)$, then $[n]^* L \cong L^{(n^2+n)/2} \otimes [-1]^* L^{(n^2-n)/2}$.*

*Remark* 8. If $A$ is an abelian $S$-scheme, the **Picard functor** of $A$ is the functor $\mathrm{Pic}_{A/S} \colon (\mathsf{Sch}/S)^{\mathsf{op}} \to \mathsf{Ab}$ sending $T/S$ to $\mathrm{Pic}(A_T)/\mathrm{Pic}(T)$. The theorem of the square implies that the map $\phi_L \colon A \to \mathrm{Pic}_{A/S}$ of presheaves sending a $T$-valued point $x \colon T \to A$ to $\phi_L(x) = t_x^*(L_T) \otimes L_T^{-1}$ on $A \times_S T$ is actually a group homomorphism.

**Definition 9.** Let $A/k$ be an abelian variety and $L \in \mathrm{Pic}(A)$. The kernel of $\phi_L$ is denoted $K(L)$ (as a presheaf). The **Mumford bundle** of $L$ is denoted $\Lambda(L)$, and is the line bundle $m^*(L) \otimes \mathrm{pr}_1^*(L) \otimes \mathrm{pr}_2^*(L)$ on $A \times A$.

Applying the Seesaw theorem to $A \times A \to A$, we can also view $K(L)$ as the maximal closed subscheme of $A$ such that $\Lambda(L)|_{K(L) \times A}$ is pulled back from $K(L)$. To see how this is related to the $\phi_L$, consider a $T$-point $x \colon T \to A$ with $T$ a $k$-scheme. Let $t_x$ be defined as the map $T \times A \cong (T \times A) \times_T T \to (T \times A) \times_T (T \times A) \to T \times A$ given by $(\mathrm{Id}, x_T)$ followed by $m$. Note that the composition $\mathrm{pr}_1 \circ t_x = m \circ (x, \mathrm{Id})$, so $(x, \mathrm{Id})^* \Lambda(L) = t_x^*(L_T) \otimes \mathrm{pr}_1^* x^*(L^{-1}) \otimes L_T^{-1}$. We have $\mathrm{pr}_1^* x^*(L^{-1})$ is pulled back from $T$, so it follows a map $x \colon T \to A$ factors through $K(L)$ precisely when $t_x^*(L_T) \otimes L_T^{-1}$ is pulled back from $T$, i.e. $|K(L)|$ is the set of points $x \in A$ such that $t_x^*(L) \cong L$.

**Corollary 10.** *$K(L)$ is a subgroup scheme of $A$.*

*Proof.* Immediate from the theorem of the square. ☺

**Corollary 11.** *If $L$ is ample, then $K(L) \to k$ is finite.*

*Proof.* We first assume $k$ is algebraically closed.

We claim $K(L)^{\circ}_{\mathrm{red}}$ is an abelian subvariety (where $\circ$ denotes the connected component containing the identity). It is a proper, reduced, connected group scheme over $k = \overline{k}$, so its smooth locus is nonempty. Using the group structure, we can push the smooth locus everywhere (more precisely, use the differential criterion from earlier—$f$ is smooth of rel. dim. $d$ if it is locally of finite presentation, flat, has all nonempty fibers of equidimension $d$, and $\Omega_{X/Y}$ is finite locally free of rank $d$). Thus, $B = K(L)^{\circ}_{\mathrm{red}}$ is an abelian subvariety.

Next, let $M = L|_B \in \mathrm{Pic}(B)$. Clearly, we have $\Lambda(M) = \Lambda(L)|_{B\times B}$. We have the section $K(L) \to K(L) \times A$ sending defined by $(\mathrm{Id}, e)$ of $K(L) \times A \to K(L)$. Note that $\Lambda(L)|_{K(L)\times A}$ pulled back along $(\mathrm{Id}, e)$ is $i^* L \otimes i^* L^{-1} \otimes c_e^* L^{-1}$ with $i\colon K(L) \hookrightarrow A$. We are working over a field so the $c_e$ term is trivial, and so $\Lambda(L)|_{K(L)\times A}$ pulled back along $(\mathrm{Id}, e)$ is trivial, and hence $\Lambda(L)$ pulled back along $K(L) \times A \to K(L)$ is trivial. Hence, $\Lambda(L)_{B\times B}$ is trivial, and hence $\Lambda(M)$ is trivial. Pulling back $\Lambda(M)$ along $(\mathrm{Id}, -\mathrm{Id})\colon B \to B \times B$ gives $M \otimes [-1]^* M$ is then trivial.

Note that $K(L)$ is proper, so it has finitely-many connected components, so $K(L)$ is finite iff $B = 0$. For general $k$, we can base change to the algebraic closure to check that $K(L)$ is finite, i.e. we can assume $k$ is algebraically closed for convenience. It suffices to show $\dim B = 0$, since $B$ is an abelian variety. Since $L$ is ample, $M$ is ample and hence $[-1]^* M$ is also ample, so $M \otimes [-1]^* M$ is also ample.

In general, $\mathcal{O}_X$ being ample is true iff $X$ is quasiaffine, which is true iff $X \to \mathrm{Spec}\,\Gamma(\mathcal{O}_X)$ is quasicompact and an open immersion. Since $X \to \mathrm{Spec}\,\Gamma(\mathcal{O}_X) \to k$ is proper, we have $X \to \mathrm{Spec}\,\Gamma(\mathcal{O}_X)$ is proper. Proper + open immersion implies isomorphism, so $X$ is affine. Affine + proper implies zero dimensional, so the result follows. ☺

*Remark* 12. Without further conditions, it's clear that the converse to the previous corollary is false: note $\phi_{L^{-1}}(x) = t_x^*(L_T^{-1}) \otimes L_T = -\phi_L(x)$, so $K(L) = K(L^{-1})$. The key point is to add some effectivity.

The rest of this talk will be devoted to showing that abelian varieties are projective. By descent, it suffices to check projectivity after base change to the algebraic closure, so we will assume that $k = \overline{k}$ in the sequel. Our next proposition tells us that every map out of a simple abelian variety to a $k$-variety is either constant or finite (according to Bhatt's notes, appearing first in http://van-der-geer.nl/~gerard/AV.pdf).

**Proposition 13.** *Let $A/k$ be an abelian variety and $f\colon A \to Y$ a morphism of $k$-varieties. For each $k$-point $a \in A(k)$, set $F_a = (f^{-1}(f(a)))^{\circ}_{\mathrm{red}}$. Then, $F_e$ is an abelian subvariety, and $F_a = a + F_e = t_a(F_e)$.*

*Proof.* Recall the rigidity lemma tells us for $k$-varieties $X, Y, Z$ with $X$ complete that a map $X \times Y \to Z$ that is constant on one fiber above $Y$ (i.e. $X \times \{y\} \to \{z\}$ for some $y \in Y(k), z \in Z(k)$) implies $X \times Y \to Z$ factors through $Y$ (apply our version of the rigidity lemma with $X \times Y \to Z \times Y$ over $Y$).

Consider $\phi \coloneqq f \circ m\colon A \times F_a \to A \times A \to A \to Y$. Note that $\phi$ is constant on the fiber above $e \in A(k)$, so $A \times F_a \to Y$ factors through $A$, say $\overline{\phi}\colon A \to Y$. Suppose $\sigma\colon A \to A \times F_a$ is the section of $A \times F_a \to A$ given by $(\mathrm{Id}, a)$ (the constant map sending everything to $a\colon A \to k \to A$). Now, let

$b \in A(k)$. Then, we have $\overline{\phi}(b) = \phi(b, a) = f(b + a)$, so

$$f(b - a + F_a) = \phi(b - a, F_a) = \overline{\phi}(b - a) = f(b - a + a) = f(b).$$

Taking $a = e$ gives $f(b + F_e) = f(b)$, so $b + F_e \subset F_b$. Taking $b = e$ gives $f(-a + F_a) = f(e)$, so $-a + F_a \subset F_e$. So $F_a = a + F_e$.

Finally, to check $F_e$ is an abelian (sub-)variety, we need to verify that for any $x \in F_e$, we have $x + F_e \subset F_e$. But we already showed that $x + F_e \subset F_x = F_e$, so we're done. ☺

*Remark* 14. The same argument should be fine for arbitrary $k$.

**Lemma 15.** *Let $A$ be an abelian variety over $k$ and $L = \mathcal{O}_A(D)$ for an effective Cartier divisor $D \subset A$. Then, $L^{\otimes 2}$ is globally generated.*

*Proof.* Let $a \in A(k)$. Saying that $L^{\otimes 2}$ is globally generated is saying that the complete linear system $|2D|$ is base-point free, i.e. there exists an effective divisor $E$ so that $a \notin E$ (i.e. the linear system defines a morphism into projective space). Consider $U = A \backslash D - a \subset A$ and $[-1]^* U$, which are both dense open subsets of $A$. Then, $U \cap [-1]^* U$ is also an dense open subset of $A$. If $b$ is a $k$-point of this intersection, note that $b \in U$ implies that $b + a \in A \backslash D$. Also, $b \in [-1]^* U$, so $-b \in U$, so $-b + a \in A \backslash D$. Then, $a \notin b + D$ and $a \notin -b + D$, so $a \notin t_{-b}(D) \cup t_b(D)$. Then, if $E = t_b(D) + t_{-b}(D)$, the theorem of the square implies that $E$ is an effective divisor linearly equivalent to $2D$, as desired. ☺

**Theorem 16.** *If $A$ is an abelian variety over $k$, then $A$ is projective over $k$.*

*Proof.* First, recall that an arbitrary morphism from an affine scheme to a separated scheme is affine. If $U \subset A$ is a nonempty affine open, $D = A \backslash U$, and $y$ a generic point of $Y$, then the map $g \colon \operatorname{Spec} \mathcal{O}_{A,y} \to A$ is affine, so $g^{-1}(U) = \operatorname{Spec} \mathcal{O}_{A,y} \backslash \{y\}$ is affine. Since $A$ is normal, it follows that the dimension of the local ring is 1. Then, $D$ is an effective Cartier divisor. Let $L = \mathcal{O}_A(D)$. We claim that $L$ is ample.

By the previous lemma, we know $L^{\otimes 2}$ is globally generated, i.e. there is a morphism $f \colon A \to \mathbb{P}^m$ such that $f^* \mathcal{O}_{\mathbb{P}^m}(1) = L^{\otimes 2}$. It follows that we can find some hyperplane $H$ in $\mathbb{P}^m$ such that pulling back the section corresponding to $H$ gives the section corresponding to $2D$. For any closed point $x \in \mathbb{P}^m \backslash H$, we have $f^{-1}(x) \subset A \backslash D = U$. Note that there exists an $x$ with $f^{-1}(x)$ nonempty; else, $f$ would be degenerate. Also, note that $A \to \mathbb{P}^m \to k$ and $\mathbb{P}^m \to k$ are proper, so $A \to \mathbb{P}^m$ is proper, so $f^{-1}(x)$ is proper. It is also affine because $U \to \mathbb{P}^m$ is affine (by the first sentence of this proof). Then, $f^{-1}(x)$ is finite. By the previous proposition, it follows that $f$ is quasi-finite. By ZMT, we get that $f$ is finite (since it is also proper), so $L$ is ample, since $L^{\otimes 2}$ is the pullback of the ample line bundle $\mathcal{O}_{\mathbb{P}^m}(1)$ by a finite map (this follows from the Leray spectral sequence, the cohomological criterion of ampleness, and the fact that higher pushforwards of finite morphisms vanish). ☺

Blackboxing some facts about Chern classes, we will prove that abelian varieties of dimension $g$ cannot be embedded into projective $(2g-1)$-space. Then, we will discuss the structure of torsion subgroups of abelian varieties over algebraically closed fields.

We quickly recall some facts about Chern classes. If $H^*$ is a Weil cohomology theory, then to any coherent sheaf $F \in \mathbb{P}^m$, we can associate the $i$th Chern class $c_i(F) \in H^{2i}(\mathbb{P}^m)$. It is compatible with pullback, the total Chern class, defined as the sum of $c_i(F)$ over all $i$ (living in $H^{2*}(\mathbb{P}^m)$) is a map from the $K$-theory of $\mathbb{P}^m$ to the cohomology ring $H^{2*}(\mathbb{P}^m)$, vector bundles of rank $r$ have trivial Chern class above degree $r$, and $h = c_1(\mathcal{O}(1)) \in H^2(\mathbb{P}^m)$ satisfies $(h|_X)^d \neq 0$ for any $X$ a subvariety of $\mathbb{P}^m$.

**Proposition 17.** *If $A/k$ is an abelian variety of dimension $g$, then $A$ cannot be embedded in $\mathbb{P}^{2g-1}$.*

*Proof.* Let $A \hookrightarrow \mathbb{P}^m$ be a closed immersion cut out by $I_A$. We want to show that $m \geq 2g$. There is a conormal exact sequence $0 \to I_A/I_A^2 \to \Omega_{\mathbb{P}^m}|_A \to \Omega_A \to 0$ because $A/k$ is smooth. As a result, we have $I_A/I_A^2$ is locally free of rank $m-g$, and so $c_i(I_A/I_A^2) = 0$ for $i > m-g$. Then, $c_{\text{tot}}(\Omega_{\mathbb{P}^m}|_A) = c_{\text{tot}}(I_A/I_A^2)c_{\text{tot}}(\Omega_A)$. Note that $\Omega_A = \mathcal{O}_A^{\oplus g}$, so it follows that $c_{\text{tot}}(\Omega_A) = c_{\text{tot}}(\mathcal{O}_A)^g = 1$—this follows from compatibility with pullback of Chern classes. So $c_i(\Omega_{\mathbb{P}^m}|_A) = 0$ for $i > m-g$.

The Euler exact sequence tells us that $0 \to \Omega_{\mathbb{P}^m} \to \mathcal{O}(-1)^{m+1} \to \mathcal{O}_{\mathbb{P}^m} \to 0$, so taking Chern classes tells us that $c_{\text{tot}}(\Omega_{\mathbb{P}^m}) = c_{\text{tot}}(\Omega_{\mathbb{P}^m})c_{\text{tot}}(\mathcal{O}_{\mathbb{P}^m}) = c_{\text{tot}}(\mathcal{O}(-1)^{m+1}) = c_{\text{tot}}(\mathcal{O}(-1))^{m+1} = (1-h)^{m+1}$. Then, $c_{\text{tot}}(\Omega_{\mathbb{P}^m}|_A) = (1-h|_A)^{m+1}$. But $c_i(\Omega_{\mathbb{P}^m}|_A) = 0$ for $i > m-g$ and $h^g|_A \neq 0$, so $m-g \geq g$, so $m \geq 2g$, as desired. ☺

We will need some general constructions about degrees of coherent sheaves to understand the torsion subgroups of abelian varieties.

Let $X$ be a projective variety of dimension $g$, $L \in \operatorname{Pic}(X)$, and $F$ a coherent sheaf on $X$. Then, define $P_{F,L}(n)$ to be the polynomial sending $n \mapsto \chi(F \otimes L^{\otimes n})$. The **degree** of $F$ with respect to $L$ is denoted by $d_L(F) \coloneqq g! \cdot (\text{coefficient of } n^g \text{ in } P_{F,L}(n))$. The degree of $L$ is $d_L(\mathcal{O}_X)$. We also define the rank of a coherent sheaf to be the rank of the sheaf at the generic point.

*Remark* 18. It is easy to see that $d_L$ is additive in short exact sequences (using the definition of cohomology and SESs).

Also, for any positive integer $k$, we have $\deg(L^{\otimes k}) = k^g \deg(L)$.

If $L$ is ample, then $d_L(F) \geq 0$ for all $F$, and this inequality is strict if $\dim(F) \coloneqq \dim(\operatorname{Supp}(F)) = g$.

**Lemma 19.** *Let $X$ be a Noetherian integral scheme with $\xi$ the generic point. Suppose $F$ is a coherent sheaf on $X$ and let $r = \operatorname{rk}(F_\xi)$. Then, there is some sheaf of ideals $I \subset \mathcal{O}_X$ such that $I^{\oplus r} \hookrightarrow F$, and is an isomorphism at $\xi$.*

*Proof.* Tag 01YE. The general idea is as follows. We have an isomorphism of $\mathcal{O}_X^r$ and $F$ at the stalk $\xi$, so we can find some nonempty open $U \subset X$ and a morphism $\psi\colon \mathcal{O}_U^r \to F|_U$, which is an iso at $\xi$. But then we have an iso in a neighborhood of $\xi$; we can shrink $U$ and assume it is iso on

$U$. Now, take $I \subset \mathcal{O}_X$ to be the ideal cutting out the complement of $U$, and we get some morphism $(I^n)^{\oplus r} = I^n\mathcal{O}_X^{\oplus r} \to F$ which induces $\psi$ over $U$ (there is some colimit of such morphisms that is precisely $\psi$). Injectivity follows by noting that if $A$ is a domain and $I \subset A$, then $I \hookrightarrow I \otimes_A \mathrm{Frac}(A)$ and checking at the generic point (so that if we have blah to blah to blah with first blah to third blah injective then first blah to second blah is injective). ☺

**Lemma 20.** *We have $d_L(F) = \mathrm{rk}(F) \cdot \deg(L)$ and for any projective variety of dimension $g$ with $f\colon Y \to X$ finite, $\deg(f^*L) = \deg(f)\deg(L)$.*

*Proof.* The previous lemma gives us an exact sequence $0 \to I^r \to F \to Q \to 0$. Applying $d_L$ gives $d_L(F) = d_L(Q) + d_L(I^r) = rd_L(I)$, since $Q$ is supported on a lower-dimensional subvariety. Also, from the exact sequence $0 \to I \to \mathcal{O}_X \to \mathcal{O}_X/I \to 0$, we see that $\deg(L) = d_L(I)$, so the first result follows.

For the second, note that $P_{\mathcal{O}_Y, f^*L}(n) = \chi(f^*L^n) = \chi(f_*f^*L^n) = \chi((f_*\mathcal{O}_Y) \otimes L^n) = P_{f_*\mathcal{O}_Y, L}(n)$, using the fact that finite pushforward is exact (hence commutes with cohomology) and the projection formula. Then, $\deg(f^*L) = d_{f^*L}(\mathcal{O}_Y) = d_L(f_*\mathcal{O}_Y) = \mathrm{rk}(f_*\mathcal{O}_Y)d_L(\mathcal{O}_X) = \mathrm{rk}(f_*\mathcal{O}_Y)\deg(L) = \deg(f)\deg(L)$, as desired. ☺

**Theorem 21.** *If $A/S$ is an abelian scheme with $n \in \mathbb{Z}\backslash\{0\}$, then $[n]\colon A \to A$ is finite flat. If $S = k$, then $\deg([n]) = n^{2g}$ where $g = \dim A$. As a corollary, $A(k = \overline{k})$ is divisible.*

*Proof.* Let us first assume that $S$ is $k$. Then, picking an ample line bundle $L$ on $A$ and setting $M = L \otimes [-1]^*L$ gives a symmetric ample line bundle. Then, we have $[n]^*M = M^{(n^2+n)/2+(n^2-n)/2} = M^{n^2}$. Since $M^{n^2}$ is also ample, we must have $[n]$ is finite (pullback of ample line bundle by $f$ is ample implies $f$ is finite, positive dimensional variety cannot have trivial ample line bundle). Then, globally, $[n]$ is finite because it is proper + quasifinite. The dimensions of fibers are 0, and hence by miracle flatness we get $[n]$ is flat.

Next, to compute the degree of $[n]$, note that $\deg(M^{n^2}) = \deg([n]^*M) = \deg([n]) \cdot \deg(M)$. We also have $\deg(M^{n^2}) = (n^2)^g \deg M$, and since $\deg M > 0$ (since it is ample), we get $\deg([n]) = n^{2g}$. ☺

**Lemma 22.** *If $k$ is perfect and $\mathrm{char}(k) = p$, $\mathrm{Spec}\, R$ is an affine and smooth over $k$, then $\ker(d\colon R \to \Omega_{R/k})$ is precisely $R^p \subset R$.*

*Proof.* Tag 031W says that the result holds if $K/k$ is an extension. The idea is that we can take a min poly $P(T)$ of $a \in K$ in $k(x_1, \dots, x_r)[T]$, which is separable, so that $P(T) = T^d + \sum_{i=1}^d a_i T^{d-i}$, so $0 = dP(a) = \sum_{i=1}^d a^{d-i} da_i \in \Omega_{K/k} = \bigoplus_{i=1}^r K dx_i$, so that $da_i = 0$ and hence $a_i$ is a power of $p$. But then if the coefficients of the min poly are powers of p, so is the root.

Now, we can assume $R$ is a domain (connected components are the same as irreducible componentS), so the claim holds for $\mathrm{Frac}(R)$ first. Then, noting that we have a composition $R \to \Omega_{R/k} \to \Omega_{\mathrm{Frac}(R)/k} = R \subset \mathrm{Frac}(R) \to \Omega_{\mathrm{Frac}(R)/k}$, the required claim holds. ☺

*Remark* 23. The previous lemma implies that if $f: S \to R$ is a map of $k$-algebras such that $\Omega_{S/k} \otimes_S R = f^*\Omega_{S/k} \to \Omega_{R/k}$ is the zero map, then $f$ factors uniquely as $S \to R^p \subset R$ (since if $d(f(s)) = 0$, then $f(s)$ lands in $R^p$).

Globally, this just becomes the fact that if we have a morphism $f: X \to Y$ of $k$-schemes with $X$ smooth, then $f^*\Omega_{Y/k} \to \Omega_{X/k}$ being 0 implies $f$ factors as $X \to X^{(1)} \to Y$ as morphisms of $k$-schemes, where $X \to X^{(1)}$ is the relative Frobenius ($X^{(1)}$ is just $X \times_{\operatorname{Spec} k} \operatorname{Spec} k$ with $\operatorname{Spec} k \to \operatorname{Spec} k$ given by the $p$-th power Frobenius).

**Theorem 24.** *Let $A/k = \overline{k}$ be an abelian variety of dimension $g$ and $n \in \mathbb{Z}\backslash\{0\}$.*

- *(i) If $n$ is invertible on $k$, then $A[n] \cong (\mathbb{Z}/n)^{2g}$.*
- *(ii) If $\operatorname{char}(k) = p > 0$, then there is a unique integer $0 \le i \le g$ (called the $p$-**rank of** $A$) such that for all $m$, we have $A[p^m](k) \cong (\mathbb{Z}/p^m)^i$.*

*Proof.* From earlier, we know that if $n \in k^\times$, then $[n]$ is finite etale. Then, $A[n]$ is finite etale over $k$—general fact about quotients by group schemes; also, $\ker \to A \to k$ is finite by looking at dimensions of fibers. The category of finite etale $k$-schemes is equivalent to the category of finite sets, so for (1), it suffices to show that $G = A[n](k) \cong (\mathbb{Z}/n)^{2g}$ as abelian groups. We already know $G$ is abelian and $\#G = n^{2g} = \deg([n])$ and $n \cdot G = 0$. This also holds for any $m|n$, since $G[m] = A[m](k)$, i.e. $\#G[m] = m^{2g}$ and $m \cdot G[m] = 0$. Then, by the classification of finite abelian groups, we get $G \cong (\mathbb{Z}/n)^{2g}$.

For (2), we induct on $m$. We want to use the previous lemma to factor $[p]$ through the relative Frobenius. To do this, we check that $[p]$ induces the zero map on differentials $[p]^*\Omega_{A/k} \to \Omega_{A/k}$. But we know on tangent spaces that $T_e([p]): T_e(A) \to T_e(A)$ is multiplication by $p$ (as done earlier), and is hence the zero map. This holds for tangent spaces at all points by moving stuff; dualizing implies the maps on stalks of cotangents sheaves are also 0, so $[p]$ indeed induces the zero map on differentials. Then, we can factor $[p]$ as $X \to X^{(1)} \to X$. We call the first map $F$, the relative Frobenius, and the second map $V$, the Verschiebung.

$F$ is a homeomorphism, so there is a bijection between the $p$-torsion $k$-points and the $k$-points $V^{-1}(e)(k)$. Since $\deg([p]) = p^{2g}$ and $\deg(F) = p^g$, we get $\deg(V) = p^g$. Then, $\#V^{-1}(e)(k) \le \deg(V) = p^g$. Since we know that $A[p](k)$ is already a $p$-torsion abelian group, it follows that it must be of the form $(\mathbb{Z}/p)^i$ for some $0 \le i \le g$.

Next, for $m > 1$, we know $[p^m]$ is surjective, so it follows that we have a SES

$$0 \to A[p](k) \to A[p^m](k) \to A[p^{m-1}](k) \to 0.$$

The first term is iso to $(\mathbb{Z}/p)^i$ and the third is iso to $(\mathbb{Z}/p^{m-1})^i$, so the middle has $p^i \cdot p^{(m-1)i} = p^{mi}$ elements.

We also know that $A[p^m](k)/p \cong A[p](k) \cong (\mathbb{Z}/p)^i$, so NAK implies $A[p^m](k)$ is a quotient of $(\mathbb{Z}/p^m)^i$ (take $(\mathbb{Z}/p^m)^i \to (\mathbb{Z}/p)^i \to A[p^m](k)$). By cardinality, it follows that they are the same! ☺

# 3 03/10: The dual abelian variety

We will begin discussing the dual abelian variety following Mumford's approach. Given an abelian variety $A$, the dual is roughly the solution of the moduli problem associating to each $k$-scheme $T$ a family of degree zero line bundles parametrized by $T$. We will see that the representing object (i.e. the dual abelian variety) has a remarkably simple description as the "quotient" $A/K(L)$. To make sense of this, we will appeal (likely without proof) to the theory of quotients of schemes by group actions.

Recall that we have a map $\operatorname{Pic}(A) \to \operatorname{Hom}(A, \operatorname{Pic}_{A/k})$ given by $L \mapsto \phi_L$, where $\phi_L(x) = t_x^*(L_T) \otimes L_T^{-1}$ if $x$ is a $T$-point of $A$. By the theorem of the square, note that this map is a homomorphism. Also, $\phi_L = 0$ iff for every point $x: T \to A$, we have $t_x^*(L_T) \cong L_T$.

**Definition 25.** Define the degree zero line bundles of $A$ to be $\operatorname{Pic}^0(A) \coloneqq \{L \in \operatorname{Pic}(A) : \phi_L = 0\} \subset \operatorname{Pic}(A)$.

First, the following lemma tells us how to produce some easy examples of line bundles living in $\operatorname{Pic}^0(A)$ provided that we already one in $\operatorname{Pic}(A)$.

**Lemma 26.** *For any $L \in \operatorname{Pic}(A)$ and $x \in A(k)$, we have $\phi_L(x) = t_x^*(L) \otimes L^{-1} \in \operatorname{Pic}^0(A)$. In other words, this induced a map $\operatorname{Pic}(A)/\operatorname{Pic}^0(A) \to \operatorname{Hom}(A(k), \operatorname{Pic}^0(A))$.*

*Proof.* Apply the theorem of the square. Let's just check that $t_y^*(\phi_L(x)) = \phi_L(x)$ for all $y \in A(k)$—the argument is pretty much identical for $T$-points. Then, we have $t_y^*(t_x^*(L) \otimes L^{-1}) = t_{x+y}^* L \otimes t_y^* L^{-1} = t_x^* L \otimes t_y^* L \otimes L^{-1} \otimes t_y^* L^{-1} = t_x^* L \otimes L^{-1}$, as desired. ☺

This lemma tells us that we can characterize degree zero line bundles using the Mumford bundle.

**Lemma 27.** *If $L \in \operatorname{Pic}(A)$, then $L \in \operatorname{Pic}^0(A)$ iff $\Lambda(L)$ is trivial.*

*Proof.* By the Seesaw theorem, there is some maximal closed subscheme $K(L) \subset A$ such that $\Lambda(L)|_{K(L)\times A}$ is pulled back from $K(L)$. Recall that $\Lambda(L) = m^*L \otimes \operatorname{pr}_1^*(L^{-1}) \otimes \operatorname{pr}_2^*(L^{-1})$ and that $K(L) = A$ precisely when $\phi_L = 0$, i.e. when $L \in \operatorname{Pic}^0(A)$. Saying that $K(L) = A$ is the same as $\Lambda(L)$ being pulled back from $A$ (the first coordinate). Consider the section $(\operatorname{Id}, e): A \to A \times A$ of $\operatorname{pr}_1$, and note that pulling back $\Lambda(L)$ along this gives $L \otimes L^{-1} \otimes \mathcal{O}_A$ is trivial. ☺

This lemma tells us that degree zero line bundles behave linearly as opposed to general line bundles, which act quadratically.

**Lemma 28.** *If $L \in \operatorname{Pic}^0(A)$ and $x, y \in A(T)$ for some $k$-scheme $T$, then $(x+y)^*L \cong x^*L \otimes y^*L$. For example, $[n]^*L = L^{\otimes n}$.*

*Proof.* Use the previous lemma and pull back the Mumford bundle along the map $(x, y): T \to A \times A$. ☺

**Lemma 29.** *If $L \in \operatorname{Pic}(A)$, then there is some $M \in \operatorname{Pic}^0(A)$ such that $[n]^*L \cong L^{\otimes n^2} \otimes M$.*

*Proof.* From a corollary of the theorem of the cube, we have $[n]^*L = L^{(n^2+n)/2} \otimes [-1]^*L^{(n^2-n)/2} = L^{n^2} \otimes (L \otimes [-1]^*L^{-1})^{(n^2-n)/2}$. So it suffices to show that this latter term is in $\mathrm{Pic}^0(A)$, i.e. because $\mathrm{Pic}^0(A)$ is a group, it suffices to show $L \otimes [-1]^*L^{-1}$ is.

Let $x \in A(k)$. Then, it's easy to see $t_x^*(L \otimes [-1]^*L^{-1}) = t_x^*L \otimes [-1]^*(L \otimes t_{-x}^*(L^{-1})) \otimes [-1]^*L^{-1}$. Since $L \otimes t_{-x}^*(L^{-1}) \in \mathrm{Pic}^0(A)$ by an earlier lemma, it follows that $[-1]^*(L \otimes t_{-x}^*(L^{-1})) = (L \otimes t_{-x}^*(L^{-1}))^{\otimes -1} = L^{-1} \otimes t_{-x}^*L$. Then, $t_x^*(L \otimes [-1]^*L^{-1})$ simplifies to $L \otimes [-1]^*L^{-1}$ (using the theorem of the square), as desired. ☺

**Lemma 30.** *If $L \in \mathrm{Pic}(A)$ has finite order, then $L \in \mathrm{Pic}^0(A)$.*

*Proof.* Suppose $L^{\otimes n} = \mathcal{O}_A$. Then, $n\phi_L = 0$ (via the homomorphism $L \mapsto \phi_L$), so $0 = n\phi_L(x) = \phi_L(nx)$ for all $x \in A(k)$. Since $A(k)$ is divisible, it follows that $\phi_L = 0$, as desired. I'm not sure how to get around the fact that $x \in A(k)$ (I don't think it's divisible for $T$-points) unless we use the silly fact that a morphism of $k$-varieties is determined on its closed points (I think $\mathrm{Pic}(A)$ should be a variety—at least separated and finite type over $k$?). ☺

**Lemma 31.** *Let $S$ be connected and of finite type over $k$. If $L \in \mathrm{Pic}(A \times S)$ and $s, t \in S(k)$, then $L_s \otimes L_t^{-1} \in \mathrm{Pic}^0(A)$.*

*Proof.* By shrinking $S$, we can assume $L|_{\{e\}\times S}$ is trivial. We can also assume $L_s$ is trivial and show that $L_t$ is then in $\mathrm{Pic}^0(A)$ for all $t$. We want to show that $\Lambda(L_t)$ is trivial. Let $\mu: S \times A \times A \to S \times A$ be multiplication on $S \times A$ (sending $(s, a, b)$ to $(s, a+b)$) and consider $M = \Lambda(L) = \mu^*L \otimes \mathrm{pr}_{12}^*(L^{-1}) \otimes \mathrm{pr}_{13}^*(L^{-1})$ on $S \times A \times A$. Note that for all $t$, we have $M|_{\{t\}\times A \times A} \cong \Lambda(L_t)$. Since $M|_{\{s\}\times A\times A}, M|_{S\times\{e\}\times A}, M|_{S\times A\times\{e\}}$ are trivial, the theorem of the cube tells us $M$ is trivial, so $\Lambda(L_t)$ is trivial too. ☺

**Lemma 32.** *If $L \in \mathrm{Pic}^0(A)$ is non-trivial, then $H^i(A, L) = 0$ for all $i \geq 0$.*

*Proof.* For $i = 0$, note that if $H^0$ is not 0, then we can write $L = \mathcal{O}_A(D)$ for some effective Cartier divisor $D$. Note that $[-1]^*L = L^{-1}$ is then also effective, which is impossible.

Now, let $i > 0$ be the smallest $i$ such that $H^i$ is not 0. The identity can be factored as $A \to A \times A \to A$, where the first map is $(\mathrm{Id}, e)$ and the second $m$. The identity on cohomology factors as $H^i(A, L) \to H^i(A \times A, m^*L) \to H^i(A, L)$, so it suffices to show $H^i(A \times A, m^*L) = 0$. By Kunneth, we have $H^i(A \times A, m^*L) = H^i(A \times A, \mathrm{pr}_1^* L \otimes \mathrm{pr}_2^* L) = \bigoplus \text{stuff} \otimes \text{stuff}$, and each term is 0 by minimality and the base case $i = 0$. ☺

**Proposition 33.** *If $L \in \mathrm{Pic}(A)$ is ample, then $\phi_L: A(k) \to \mathrm{Pic}^0(A)$ is surjective and has kernel $K(L)(k)$.*

*Proof.* We already showed the fact about kernels in a previous talk. So we just need to show surjectivity. This relies on a trick that Mumford uses again and again.

Suppose $M$ is not in the image of $\phi_L$ and consider $K = \Lambda(L) \otimes \mathrm{pr}_1^*(M^{-1}) \cong m^*L \otimes \mathrm{pr}_2^* L^{-1} \otimes \mathrm{pr}_1^*(L^{-1} \otimes M^{-1})$. For any $x \in A(k)$, we have

$$K|_{A\times\{x\}} \cong t_x^*(L) \otimes L^{-1} \otimes M^{-1} \text{ and } K|_{\{x\}\times A} \cong t_x^*(L) \otimes L^{-1}.$$

For every $x$, we necessarily have that $K|_{A\times\{x\}}$ is nontrivial (because $M$ is not in the image); it is also a degree zero line bundle because $t_x^*(L) \otimes L^{-1}$ and $M$ are. By the previous lemma, it follows that $K$ has no cohomology when restricted to any fiber in the second coordinate. By the semicontinuity theorem, we have $R^i \mathrm{pr}_{2*} K = 0$ for all $i$. By the Leray spectral sequence, it follows that $H^i(A \times A, K) = 0$ for all $i$.

Also, $K|_{\{x\}\times A}$ is trivial iff $x \in K(L)(k)$, so $R^i \mathrm{pr}_{1*}(K)$ has support contained in $K(L)$, which is finite because $L$ is ample. Any coherent sheaf supported on a finite subscheme has no higher cohomology, so the Leray spectral sequence tells us that $H^i(A \times A, K) \cong \bigoplus_{x \in K(L)} \left(R^i \mathrm{pr}_{1*} K\right)_x$.

Combining the results from the last two paragraphs yields $R^i \mathrm{pr}_{1*} K = 0$ for all $i$. Then, by proposition 4.6 of Bhatt, it follows that $H^i(A, K|_{\{x\}\times A}) = 0$ for all $i$ and $x \in A(k)$. In particular, for $x = e$ and $i = 0$, we get $H^0(A, \mathcal{O}_A) = 0$, which is a contradiction! ☺

We'll now blackbox the theory of quotients by finite group schemes. Given a field $k$, a finite group scheme $G/k$, and $X/k$ finite type, then we say that an action $G \times X \to X$ on $X$ is **free** if $G \times X \to X \times X$ is injective on the level of points. E.g. translation is a free action on an abelian variety.

A quasicoherent sheaf $F$ on $X$ is said to be $G$-**equivariant** if it comes with isomorphisms $\lambda_g : a_g^* F_S \to F_S$ (where $S$ is a $k$-scheme, $g \in G(S)$, $a_g : X_S \to X_S$ is the induced action on $X_S$ by $g$) so that $\lambda_{h\cdot g} = \lambda_g \circ a_g^* \lambda_h$.

**Theorem 34.** *Let $X/k$ be finite type and separated and $G$ a finite group scheme over $k$ acting freely on $X$. Suppose any finite subset of $X$ is contained in an affine open subset of $X$ (e.g. $X$ is quasiprojective—then, we can always a hypersurface that avoids any finite set of points + complement of hypersurface is affine by the Veronese). Then, there is a universal $G$-invariant morphism $\pi : X \to X/G$ such that:*

*(i)* $\pi$ *is finite, flat, surjective, and* $G \times X \cong (G \times X/G) \times_{X/G} X \to X \times_{X/G} X$ *is an isomorphism (i.e. a torsor).*

*(ii)* $\deg(\pi) = \mathrm{rk}(G) \coloneqq \dim_k(\Gamma(G))$.

*(iii) If* $f$ *is* $X \times G \to X \to X/G$*, then* $\mathcal{O}_{X/G} \to \pi_* \mathcal{O}_X \rightrightarrows f_* \mathcal{O}_{X\times G}$*, with the second maps given by the action of* $G$ *and* $\mathrm{pr}_2$*, is an equalizer.*

*(iv) If* $k$ *is algebraically closed, then* $X(k)/G(k) \to (X/G)(k)$ *is an iso and* $|X|/|G| \to |X/G|$ *is a homeo.*

*(v)* $\pi^*$ *induces an equivalence of categories (via descent)*

$$\mathsf{QCoh}(X/G) \cong \{G\text{-equivariant quasicoherent sheaves on } X\}.$$

*(vi) There is a norm map* $\mathrm{Nm} : \mathrm{Pic}(X) \to \mathrm{Pic}(X/G)$ *such that the composition with* $\pi^*$ *is the multiplication-by-*$\mathrm{rk}(G)$ *map* $L \to L^{\otimes \mathrm{rk}(G)}$ *with* $L \in \mathrm{Pic}(X/G)$.

Let us now describe the moduli problem associated to the dual abelian variety. The main goal now is the following theorem:

**Theorem 35.** *Let $A/k$ be an abelian variety. Let $C_A$ be the category of triples $(S, L, \iota)$, where $S$ is a $k$-scheme, $L \in \operatorname{Pic}(A \times S)$ such that $L|_{\{s\}\times A} \in \operatorname{Pic}^0(A)$ for all $s \in A(k)$, and $\iota\colon L|_{S\times\{e\}} \cong \mathcal{O}_S$ is a choice of isomorphism ("rigidification").*

*Then, $C_A$ has a final object called the* ***dual abelian variety*** *(denoted $(A^t, \mathcal{P}, \iota_{\text{univ}})$), i.e. for any $k$-scheme $S$, $k$-morphisms $S \to A^t$ are in bijection with triples $(S, L, \iota)$ given by pulling $(A^t, \mathcal{P}, \iota_{\text{univ}})$ back along $S \to A^t$.*

*Remark* 36. The purpose of adding a rigidification is so that we don't have to worry about automorphisms of objects—otherwise, we would need to use the language of stacks.

The general strategy will to pick some ample line bundle $L$ on $A$, define $A^t$ to be $A/K(L)$, and construct $\mathcal{P}$ by descending the Mumford bundle from $A \times A$ to $A^t \times A = (A \times A)/(K(L) \times 0)$. Then, we will show that this satisfies the universal property as described in the theorem.

To make this work precisely, we will need the following lemma along with the theory of quotients.

**Lemma 37.** *$\Lambda(L)$ is $K(L)$-equivariant: a $K(L)$-equivariant structure on $\Lambda(L)$ is uniquely determined after specifying a rigidification $L|_{\{e\}} \cong k$.*

*Proof.* Let $x \in K(L)(T)$, where $T$ is some $k$-scheme. Then, there exists some line bundle $M_0$ pulled back from $T$ so that $t_x^*(L_T) \cong L_T \otimes M_0$. Then, we have

$$\begin{aligned}
t_{(x,e)}^*(\Lambda(L)_T) &= t_{(x,e)}^*(\Lambda(L_T)) \\
&= t_{(x,e)}^*\left(m_T^*(L_T) \otimes \operatorname{pr}_1^* L_T^{-1} \otimes \operatorname{pr}_2^{-1}(L_T^{-1})\right) \\
&= m_T^* t_x^*(L_T) \otimes \operatorname{pr}_1^* t_x^* L_T^{-1} \otimes \operatorname{pr}_2^* L_T^{-1} \\
&\cong m_T^* L_T \otimes m_T^* M_0 \otimes \operatorname{pr}_1^* L_T^{-1} \otimes \operatorname{pr}_1^* M_0^{-1} \otimes \operatorname{pr}_2 L_T^{-1} \\
&= m_T^* L_T \otimes \operatorname{pr}_1^* L_T^{-1} \otimes \operatorname{pr}_2 L_T^{-1} \\
&= \Lambda(L_T),
\end{aligned}$$

since $m_T^* M_0 = \operatorname{pr}_1^* M_0$ (since $\operatorname{pr}_T \circ m_T = \operatorname{pr}_T \circ \operatorname{pr}_1$—they are both map of $T$-schemes, so all maps to $T$ agree).

To ensure that these isomorphisms are canonical (so that we get compatibility with the $K(L)$ action on $A$), we use the rigidification. It suffices to define this isomorphism after pulling back along the inclusion $\iota\colon A_T \to A_T \times_T A_T$ (given by $(\mathrm{Id}, e)$), since any two isomorphisms, either on $A_T$ or on $A_T \times_T A_T$, differ by global units $H^0(A_T, \mathcal{O}_{A_T}^\times) = H^0((A \times A)_T, \mathcal{O}_{(A\times A)_T}^\times) = H^0(T, \mathcal{O}_T^\times)$ by Kunneth, so that the restriction $H^0((A \times A)_T, \mathcal{O}_{(A\times A)_T}^\times) \to H^0(A_T, \mathcal{O}_{A_T}^\times)$ is an iso.

Now, note that $t_{(x,e)} \circ \iota = \iota \circ t_x\colon A_T \to (A \times A)_T$, from which it follows that $\iota^* t_{(x,e)}^*(\Lambda(L_T)) = t_x^* \iota^*(\Lambda(L_T))$. Also, $\iota^*(\Lambda(L_T)) = \iota^*(m_T^*(L_T) \otimes \operatorname{pr}_1^* L_T^{-1} \otimes \operatorname{pr}_2^{-1}(L_T^{-1})) = L_T \otimes L_T^{-1} \otimes (L^{-1}|_{\{e\}} \times A_T) = L^{-1}|_{\{e\}} \times A_T$. Then, we just need to fix an iso between $t_x^*(L^{-1}|_{\{e\}} \times A_T)$ and $L^{-1}|_{\{e\}} \times A_T$, which is given by simply having $t_x$ only on the second factor and be constant on the first fact (we fix a rigidification $L|_{\{e\}} \cong k$). ☺

This allows us to descend the Mumford bundle $\Lambda(L)$ to $\mathcal{P}$. To define $\iota_{\text{univ}}$, note that we need to write down an iso $\mathcal{P}_{A^t \times \{e\}} \cong \mathcal{O}_{A^t}$. To do this, it suffices to write an isomorphism $\Lambda(L)|_{A \times \{e\}} \cong \mathcal{O}_A$ that is compatible with the $K(L)$-action.

The LHS is given by $L \otimes L^{-1} \otimes (L^{-1}|_{\{e\}} \times A) \cong L^{-1}|_{\{e\}} \times A$. Hence, after choosing an iso $L|_{\{e\}} \cong k$ as in the previous lemma, we get $\Lambda(L)|_{A \times \{e\}} \cong \mathcal{O}_A$ that is compatible with the $K(L)$ action. So this gives us the triple $(A^t, \mathcal{P}, \iota_{\text{univ}})$.

Next time, we will check the universal property and complete the construction of the dual abelian variety.

# 4 03/24: Constructing the dual abelian variety and Fourier-Mukai transforms

We will construct the dual abelian variety following Mumford's approach, concluding our discussion from last time. Along the way, we will compute the cohomology of the Poincare bundle, as well as the cohomology of the structure sheaf of an abelian variety. This will naturally lead us to Fourier-Mukai transforms, and we will state the famous Fourier-Mukai equivalence $D(A) \cong D(A^t)$ for an abelian variety $A$ and its dual $A^t$.

Last time, we constructed $(A^t, \mathcal{P}, \iota_{\mathrm{univ}})$. It remains to show that for any object $(S, F, \iota)$ in our category of triples parametrizing degree zero line bundles, we have a unique map $S \to A^t$ such that $F$ and $\iota$ arise from $\mathcal{P}, \iota_{\mathrm{univ}}$ under pullback.

Consider $M = \mathrm{pr}_{13}^*(F^{-1}) \otimes \mathrm{pr}_{23}^*(\mathcal{P})$ living in $S \times A^t \times A$, and let $\Gamma_S$ be the maximal closed subscheme of $S \times A^t$, obtained from the Seesaw theorem, such that $M|_{\Gamma_S \times A}$ arises from pullback from $\Gamma_S$. Then, it suffices to show that $\Gamma_S \subset S \times A^t \to S$ is an isomorphism; the unique map $S \to A^t$ is given by $S \cong \Gamma_S \subset S \times A^t \to A^t$

Indeed, for any morphism $\phi: S \to A^t$, let $\Gamma_\phi: S \to S \times A^t$ denote the graph. We then have $(\phi \times \mathrm{Id})^*\mathcal{P} \cong F$ iff $\Gamma_\phi$ factors through $\Gamma_S$. But $\Gamma_S$ is already the graph of $S \cong \Gamma_S \subset S \times A^t \to A^t$, so uniqueness follows.

We'll divide up the proof into several steps:

(i) **Some easy reductions:** By descent and noting that $\Gamma_S$ is compatible with base change, we can assume $k$ is algebraically closed. Moreover, for varieties (or just finitely presented schemes over a base ring), if the induced morphism on Specs of local rings is an iso, it follows that the original morphism is an iso (because we can uniquely extend to neighborhoods on both sides). Since completions of Noetherian local rings are faithfully flat, we can pass to the completions. Moreover (okay, this is probably overkill), by the Cohen structure theorem, a complete Noetherian local ring containing its residue field is just a power series in the field mod some ideal, which is an inverse limit of Artinian local rings. Hence, we may assume $S$ is an Artinian local ring with residue field $k$. Then, $S$ is just a single point, say $\{s\}$, and write $S = \mathrm{Spec}(B)$.
We can also assume that $F|_{\{s\}\times A} \cong \mathcal{O}_A$ by replacing $M$ with $M \otimes \mathrm{pr}_3^*(\mathcal{P}^{-1}|_{\{b\}\times A})$ (this doesn't change $\Gamma_S$), where $b \in A^t(k)$ such that $F|_{\{s\}\times A} \cong \mathcal{P}|_{\{b\}\times A}$ (such $b$ exists because $A^t(k) \to \mathrm{Pic}^0(A)$ is surjective and we have $F|_{\{s\}\times A} \in \mathrm{Pic}^0(A)$ by definition).

(ii) **Cohomology of $M$ is free using $\mathrm{pr}_{13}$:** By an earlier claim (from last lecture), we know that for a line bundle $L$ on $A\times S$, we have $L|_{\{s\}} \in \mathrm{Pic}^0(A)$ implies $L|_{\{t\}} \in \mathrm{Pic}^0(A)$ for $s, t \in A(k)$. Since $M|_{\{s\}\times A^t \times \{e\}}$ is a degree zero line bundle, it follows that $M|_{\{s\}\times A^t\times\{a\}} \in \mathrm{Pic}^0(A^t)$ for any $a \in A(k)$. Now, $M|_{\{s\}\times A^t\times\{a\}}$ is trivial for only finitely many $a$: Let $\pi: A \to A^t$ be the quotient, and note $\pi^*(M|_{\{s\}\times A^t\times\{a\}}) \cong t_a^*(L)\otimes L^{-1}$, which is trivial iff $a \in K(L)$, which is finite because $L$ is ample. Hence, $\bigoplus_{i\in\mathbb{Z}} R^i\,\mathrm{pr}_{13,*}(M)$ has finite, discrete support (by a result from last time—all cohomology groups of a non-trivial degree zero line bundle are 0). Then, arguing as we did before, the Leray spectral sequence gives us an iso $H^i(S \times A^t \times A, M) \cong H^0(S\times A, R^i\,\mathrm{pr}_{13,*} M)$. By shrinking $S\times A$ so that $F$ is trivial (but still contains the support), it follows that we can assume $F$ is trivial. Then, we get

$$\begin{aligned} H^i(S \times A^t \times A, M) &\cong H^0(S \times A, R^i \operatorname{pr}_{13,*} M) \\ &\cong H^0(S \times A, R^i \operatorname{pr}_{13,*}(\operatorname{pr}_{13}^* F^{-1} \otimes \operatorname{pr}_{23}^* \mathcal{P})) \\ &\cong H^0(S \times A, F^{-1} \otimes R^i \operatorname{pr}_{13,*}(\operatorname{pr}_{23}^* \mathcal{P})) \text{ (projection formula)} \\ &\cong H^0(S \times A, R^i \operatorname{pr}_{13,*}(\operatorname{pr}_{23}^* \mathcal{P})) \\ &\cong H^i(S \times A^t \times A, \operatorname{pr}_{23}^* \mathcal{P}) \text{ (again by the Leray SS)} \\ &\cong B \otimes_k H^i(A^t \times A, \mathcal{P}) \text{ (by flat base change),} \end{aligned}$$

so $H^i(S \times A^t \times A, M)$ is a free $B$-module.

(iii) **Vanishing of cohomology of $M$ using $\operatorname{pr}_{12}$:** Now, consider $M|_{\{s\}\times\{a\}\times A}$ with $a \in A^t(k)$. If it is nontrivial, then we again have from last time (since it lives in $\operatorname{Pic}^0(A)$) that all cohomology groups $R^i \operatorname{pr}_{12,*}(M)$ are trivial. Note that $M|_{\{s\}\times\{a\}\times A} = \mathcal{P}|_{\{a\}\times A}$ is trivial precisely when $a = e$, so the support of $\bigoplus R^i \operatorname{pr}_{12,*}(M) \subset \{s\} \times \{e\}$. Let $R = B \otimes_k \mathcal{O}_{A^t,e}$. Then, since $R^i \operatorname{pr}_{12,*}(M)$ is supported at $(s,e)$, we can view it as an Artinian $R$-module (more precisely, it is the skyscraper corresponding to this Artinian $R$-module supported at $(s,e)$); note that it is also an Artinian $\mathcal{O}_{A^t,e}$-module. There is a perfect complex $K^\bullet = (K^0 \to \cdots \to K^g)$ that universally computes the cohomology of $R^i \operatorname{pr}_{12,*}(M)$.
In general, Mumford claims that if $\mathcal{O}$ is an Artinian local ring of dimension $g$ and $K^\bullet = (K^0 \to \cdots \to K^g)$ is a complex of perfect $\mathcal{O}$-modules such that $H^i(K^\bullet)$ is an Artinian $\mathcal{O}$-module. Then, $H^{<g}(K^\bullet) = 0$. To see this, let us induct on $g$. The case $g = 0$ is trivial. Pick $x \in \mathfrak{m} - \mathfrak{m}^2$; then, $\mathcal{O}/x$ is also regular and of dimension $g-1$ (do this exercise!). Let $\overline{K}^\bullet = K^\bullet \otimes_{\mathcal{O}} \mathcal{O}/x$. Now, consider the SES $0 \to K^\bullet \to K^\bullet \to \overline{K}^\bullet \to 0$ given by multiplication by $x$ in the first map. We get the LES $H^p(K^\bullet) \to H^p(K^\bullet) \to H^p(\overline{K}^\bullet) \to H^{p+1}(K^\bullet) \to H^{p+1}(K^\bullet)$, which shows that $H^i(\overline{K}^\bullet)$ are Artinian. Then, by induction, these vanish for $i < g-1$, so $H^{p+1}(K^\bullet) \to H^{p+1}(K^\bullet)$ (multiplication by $x$) is injective for $p < g-1$. Since each one is injective, some power of $x$ will eventually kill $H^{p+1}(K^\bullet)$, so it follows that $H^{p+1}(K^\bullet) = 0$ for $p < g-1$, as desired.
So $R^i \operatorname{pr}_{12,*}(M) = 0$ for $i < g$. Again, the Leray SS tells us that $H^i(S \times A^t \times A, M) \cong H^0(S \times A^t, R^i \operatorname{pr}_{12,*}(M))$ (since each $R^i \operatorname{pr}_{12,*}(M)$ has finite support), so $H^i(S \times A^t \times A, M) = 0$ for $i < g$ and is some free $B$-module $N$ in degree $g$.

(iv) **Scheme structure on $\Gamma_S$:** We will only sketch this. The idea is roughly the same as how we proved the Seesaw theorem: Take the dual of the perfect complex $K^\bullet$, and let $Q$ be the cokernel of $K^{1,\vee} \to K^{0,\vee}$. Then, repeating the same argument shows that $\operatorname{Hom}_R(Q,k)$ is 1-dimensional over $k$ so that NAK implies $Q$ is cyclic, i.e. of the form $R/I$. Then, using the universal property of $\Gamma_S$ lets us verify that $\Gamma_S = \operatorname{Spec}(R/I) \subset \operatorname{Spec} R$.

(v) **$\Gamma_S \to S$ is an iso:** We want to show that $B \to R \to R/I$ is an iso. First, $H^i(K^{\bullet,\vee})$ is Artinian because it is true for $K^\bullet$. Again, we have that $0 \to K^{\bullet,\vee} \to Q \to 0$ is a resolution. Since $I$ kills $Q$, we also have that $I$ kills $H^i(\operatorname{Hom}_R(K^{\bullet,\vee}, R))$ (indeed, for any $R$-linear functor applied to $K^{\bullet,\vee}$), so $I \cdot H^i(K^\bullet) = 0$. Now, $N$ is non-zero (why?) and $I \cdot N = 0$, so $I \cap B = \{0\}$ and $B \to R/I$ is injective.
For surjectivity, by NAK, it suffices to verify the claim after modding out by $\mathfrak{m}_B$. So we can assume $S$ is $k$. Then, $\operatorname{Spec} R/I \to A^t$ is the maximal closed subscheme such that $\mathcal{P}$ is

pulled back from $\operatorname{Spec} R/I$. Then, $\pi^{-1}(\operatorname{Spec}(R/I))$ is the maximal closed subscheme such that $(\pi, \mathrm{Id})^*\mathcal{P} = \Lambda(L)$ is pulled back from $\operatorname{Spec} R/I$. So $K(L) = \pi^{-1}(\operatorname{Spec}(R/I))$. So it follows that $R/I$ is just $k$, and the result follows.

**Corollary 38.** *If $A$ is an abelian variety of dimension $g$, then $H^i(A^t \times A, \mathcal{P}) = 0$ for $i < g$ and $k$ for $i = g$. This implies that $R^i \operatorname{pr}_{1,*}(\mathcal{P}) = 0$ for $i < g$ and $k(e)$ for $i = g$.*

*Proof.* The $i < g$ case is clear from the proof above by setting $S = \operatorname{Spec} k$. For $i = g$, write $Q = R/I$ as in the proof, and recall $Q = H^g(K^{\bullet,\vee})$ so that $K^{\bullet,\vee}$ is a resolution of $k$. It is a fact that any two resolutions over a regular local ring are homotopy equivalent, so we can use the Koszul resolution instead (I won't write it down here). It is also self-dual. Then, the result follows. ☺

**Corollary 39.** *If $A$ is an abelian variety of dimension $g$, then $H^i(A, \mathcal{O}_A)$ has rank $\binom{g}{i}$.*

*Proof.* Like in the previous corollary, we can use the Koszul complex, which universally computes cohomology. Now, note that since $R^i \operatorname{pr}_{1,*}(\mathcal{P})$ is just a skyscraper (0 for $i \neq g$ and $k(e)$ for $i = g$), it follows that on $e \times A$, we have $R^i \operatorname{pr}_{1,*}(\mathcal{P})|_{\{e\}\times A}$ can be identified with $H^i(A, \mathcal{O}_A)$. Then, the corresponding complex is

$$0 \to \Lambda^g k^g \to \Lambda^{g-1} k^g \to \cdots \to k^g \to k \to 0,$$

and all of the maps are 0 because we are taking the fiber (i.e. tensoring by $k$). So the result follows. ☺

We now move on to duality. Given a morphism of abelian varieties $f\colon A \to B$, we can construct a morphism $f^t\colon B^t \to A^t$. To see this, consider $g = (\mathrm{Id}, f)\colon B^t \times A \to B^t \times B$ and $g^*\mathcal{P}_B$. To construct $f^t$, we use the universal property of $A^t$, i.e. we need to check that $g^*\mathcal{P}_B|_{\{s\}\times A} \in \operatorname{Pic}^0(A)$ for all $s \in B^t$ and $g^*\mathcal{P}_B|_{B^t \times \{e\}} \cong \mathcal{O}_{B^t}$. The latter is just $\mathcal{P}_B|_{B^t\times\{e\}}$ because homomorphisms send $e$ to $e$, and this already has a trivialization. For the former, note that it is iso to $f^*(\mathcal{P}_B|_{\{s\}\times B})$ and that $\mathcal{P}_B|_{\{e\}\times B}$ is trivial (and hence $g^*\mathcal{P}_B|_{\{s\}\times A}$ is in $\operatorname{Pic}^0(A)$). Then, by a lemma from last time, it follows that $g^*\mathcal{P}_B|_{\{s\}\times A}$ is in $\operatorname{Pic}^0(A)$ for all $s \in B^t(k)$.

As a result, we have a unique $f^t\colon B^t \to A^t$ such that $(f^t, \mathrm{Id})^*\mathcal{P}_A \cong g^*\mathcal{P}_B = (\mathrm{Id}, f)^*\mathcal{P}_B$.

**Proposition 40.** *If $f\colon A \to B$ is an isogeny (i.e. is surjective and has finite kernel), then so is $f^t\colon B^t \to A^t$.*

*Proof.* We will use the following lemma, which characterizes isogenies. Since $f$ is an isogeny, we have $\dim(B^t) = \dim(B) = \dim(A) = \dim(A^t)$ (because $\pi\colon A \to A^t, B \to B^t$ is finite and surjective). So by the following lemma, it suffices to show that $f^t$ is surjective.

To see this, let $L$ be an ample line bundle of $B$, so that $\phi_L\colon B \to B^t$ is a polarization. Since $A \to B$ is finite, pulling back $L$ to $A$ is still ample. We get the following diagram (from the universal property):

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\phi_{L|_A}"] & B \arrow[d, "\phi_L"] \\ A^t & B^t \arrow[l, "f^t"] \end{tikzcd}$$

Since $\phi_{L|_A}: A \to A^t$ is surjective it follows that $f^t: B^t \to A^t$ is surjective too. ☺

**Theorem 41.** *There is a canonical biduality $A \to (A^t)^t$.*

*Proof.* First, let us construct such a map. Consider $\mathcal{P}_A \in \mathrm{Pic}(A \times A^t)$. Note that $\mathcal{P}_A|_{\{x\}\times A^t} \in \mathrm{Pic}^0(A^t)$ for all $x \in A(k)$ because it holds for $x = e$ (since $\mathcal{P}_A|_{\{e\}\times A^t} \cong \mathcal{O}_{A^t}$). Next, to construct a rigidification $\mathcal{P}_A|_{A\times\{e\}} \cong \mathcal{O}_A$, consider the homomorphism $A(k) \to \mathrm{Pic}^0(A)$ given by sending $x \mapsto \Lambda(L)|_{A\times\{x\}}$, which descends to an isomorphism (as shown before) $A^t(k) \to \mathrm{Pic}^0(A)$ sending $x \mapsto \mathcal{P}_A|_{A\times\{x\}}$.

So we have a morphism $\alpha: A \to (A^t)^t$ (using the universal property of $(A^t)^t$) such that $(\alpha, \mathrm{Id})^*(\mathcal{P}_{A^t}) \cong \mathcal{P}_A$. Let $\pi = (\alpha, \mathrm{Id})$. Then, we claim that $\pi$ is an isomorphism (which implies $\alpha$ is).

First, $\pi$ is flat since the explicit description of $\alpha$ is something like $S \cong \Gamma_S \to S \times A^t \to S$ (from the proof), which has fibers of the same dimension; we can then use miracle flatness.

Next, $\pi$ is finite because $A \to (A^t)^t \to A^t$ (the latter given by the dual of a polarization $\phi_L: A \to A^t$) is the same as $\phi_L: A \to A^t$ using the universal property; $\phi_L$ is an isogeny and its dual is also an isogeny, so both are finite.

Finally, $\pi$ is of degree 1 by using the fact that for a proper variety $X$, $G$ finite acting freely on $X$, $\pi: X \to X/G$, and $F \in \mathsf{Coh}(X/G)$, then $\chi(X/G, F)\deg(\pi) = \chi(X, \pi^* F)$ (c.f. Mumford). Letting $\pi = (\alpha, \mathrm{Id})$ and $F = \mathcal{P}_{A^t}$, we have $\deg(\pi)(-1)^g = \chi(A \times A^t, \pi^*\mathcal{P}_{A^t}) = \chi(A \times A^t, \mathcal{P}_A) = (-1)^g$.

So $\pi$ is an iso, as desired. ☺

**Lemma 42.** *Let $f: A \to B$ be a homomorphism of abelian varieties. TFAE (and $f$ is then called an* ***isogeny****):*

*(i)* $f$ *is surjective and* $\dim A = \dim B$.

*(ii)* $\ker f$ *is a finite group scheme and* $\dim A = \dim B$.

*(iii)* $f$ *is finite, flat, and surjective.*

*Proof.* Relatively straightforward, c.f. EvdGM (Prop 5.2). Also Tag 047T (subgroup is closed immersion, so $\ker f \to A \to k$ is proper—only need to check quasi-finiteness by ZMT). ☺

*Remark* 43. There is an alternative approach using Cartier duality. If $f: A \to B$ is an isogeny with kernel $K$, then $f^t$ is an isogeny with kernel $K^t$, where $K^t$ is the Cartier dual of $K$, i.e. the finite group scheme given by $\underline{\mathrm{Hom}}(-, \mathbb{G}_m)$, which on points sends $T \mapsto \mathrm{Hom}_T(G_T, \mathbb{G}_{m,T})$.

The dual $A^t$ can also be described as $\underline{\mathrm{Ext}}^1(A, \mathbb{G}_m)$, i.e. rigidified line bundles on $A$ are in bijection with extensions by $\mathbb{G}_m$. The idea is roughly as follows (the only reference I can find for this is https://www.raymondvanbommel.nl/talks/duality_av.pdf and allegedly Serre's textbook on algebraic groups and class fields, which unfortunately doesn't use schemes). Let $L \in \mathrm{Pic}^0(A)$ and $L|_e \cong k$. Then, the claim is that this corresponds to some extension $0 \to \mathbb{G}_m \to E \to A \to 0$. Let $E$ be $\underline{\mathrm{Isom}}(\mathcal{O}_A, L)$ over $A$, which happens to be represented by a scheme. Pick two points $(a, \alpha), (b, \beta) \in E(T)$, where $T$ is some $S$-scheme; we have an iso $\alpha: \mathcal{O}_T \to a^*L$ (similarly for $\beta$ and $b$). By the theorem of the square, we have $(a+b)^*L \cong a^*L \otimes b^*L \otimes e_T^*L^{-1}$. Using the rigidification, we have $e_T^*L \cong \mathcal{O}_T$, so we get a canonical iso $(a+b)^*L \cong \mathcal{O}_T$ via $\alpha$ and $\beta$. This exhibits a way of adding $(a,\alpha)$ and $(b,\beta)$. The map $\mathbb{G}_{m,S} \to E$ is given by sending $x \in \mathbb{G}_{m,S}(T)$ to $(e_T, \cdot x: \mathcal{O}_T \to e_T^*L \cong \mathcal{O}_T)$ and $E \to A$ is given by forgetting the isomorphism.

Using this, consider the sequence $0 \to K \to A \to B \to 0$. Hitting it with $\underline{\mathrm{Hom}}(-, \mathbb{G}_m)$ (i.e. Cartier duality) gives the sequence $\underline{\mathrm{Hom}}(A, \mathbb{G}_m) \to \underline{\mathrm{Hom}}(K, \mathbb{G}_m) \to \underline{\mathrm{Ext}}^1(B, \mathbb{G}_m) \to \underline{\mathrm{Ext}}^1(A, \mathbb{G}_m) \to \underline{\mathrm{Ext}}^1(K, \mathbb{G}_m)$. The first and last terms are 0 because $A$ is proper and $\mathbb{G}_m$ is affine (by the rigidity lemma) and because $K$ is 0-dimensional. The induced map $\underline{\mathrm{Ext}}^1(B, \mathbb{G}_m) \to \underline{\mathrm{Ext}}^1(A, \mathbb{G}_m)$ corresponds to $f^t: B^t \to A^t$ (why is this true?) from which it follows that $K^t = \underline{\mathrm{Hom}}(K, \mathbb{G}_m)$.

We now move on to Fourier-Mukai transforms, which will allow us to go between the derived category of an abelian variety and the derived category of its dual using the integral transform of the Poincare bundle.

**Definition 44.** Let $X$ be a scheme over $k$.

(i) $D(X)$ is the full subcategory of the derived category of $\mathcal{O}_X$-modules $K$ such that $H^i(K) \in \mathsf{QCoh}(X)$ for all $i$. Note: it is not always true that $D(X) = D(\mathsf{QCoh}(X))$ (but it is if $X$ has affine diagonal)—this isn't *useful* because $\mathsf{QCoh}(X)$ has ugly injective resolutions.

(ii) $D^b(X)$ is the full subcategory of $D(X)$ that is bounded, i.e. cohomology vanishes eventually in both directions.

(iii) $D^b_{\mathrm{coh}}(X)$ is the full subcategory of $D(X)$ such that $H^i(K) \in \mathsf{Coh}(X)$ for all $i$ and is also bounded.

**Definition 45.** Let $X, Y$ be schemes over $k$ and $K \in D(X \times Y)$. Then, define

$$\phi_K: D(X) \to D(Y), N \mapsto R\,\mathrm{pr}_{2,*}(L\,\mathrm{pr}_1^*(N) \otimes K)$$

and

$$\psi_K: D(Y) \to D(X), M \mapsto R\,\mathrm{pr}_{1,*}(L\,\mathrm{pr}_2^*(N) \otimes K).$$

$\phi_K$ and $\psi_K$ are called **integral transforms** and $K$ is called the **kernel** (of $\phi_K$ or $\psi_K$).

*Remark* 46. Many functors between derived categories are secretly integral transforms! Orlov proved that every fully faithful functor arises as an integral transform.

**Example 47.**

(i) Let $\Delta: X \to X \times_k X$ be the graph. Set $K = R\Delta_*\mathcal{O}_X$. Then, we have $\phi_K = \mathrm{Id}$:

$$\begin{aligned} N \mapsto R\,\mathrm{pr}_{2,*}(L\,\mathrm{pr}_1^* N \otimes R\Delta_*\mathcal{O}_X) &\cong R\,\mathrm{pr}_{2,*}(R\Delta_*(L\Delta^* L\,\mathrm{pr}_1^* N \otimes \mathcal{O}_X)) \text{ (projection formula)} \\ &= R\,\mathrm{pr}_{2,*}(R\Delta_* N) \\ &= N. \end{aligned}$$

(ii) Let $f\colon X \to Y$ be a morphism of $k$-schemes and $\Gamma\colon X \to X \times_k Y$ be the graph with $i\colon \Gamma \hookrightarrow X \times_k Y$ the inclusion. Set $K = Ri_* \mathcal{O}_\Gamma$. Then, we have $\phi_K = Rf_*$ and $\psi_K = Lf^*$:

$$\begin{aligned} N \mapsto R\,\mathrm{pr}_2(L\,\mathrm{pr}_1^* N \otimes Ri_* \mathcal{O}_\Gamma) &\cong R\,\mathrm{pr}_{2,*}(Ri_*(Li^* L\,\mathrm{pr}_1^* N \otimes \mathcal{O}_\Gamma)) \text{ (projection formula)} \\ &= R\,\mathrm{pr}_{2,*}(Ri_* N) \\ &= Rf_* N. \end{aligned}$$

We can finally state the famous Fourier-Mukai equivalence.

**Theorem 48** (Mukai)**.** *Let $A$ be an abelian variety. Then, $\phi_{\mathcal{P}_A}\colon D(A) \to D(A^t)$ is an equivalence of triangulated categories.*

## 5 03/31: The Fourier-Mukai equivalence

We will restate the Fourier-Mukai equivalence. Then, we will prove it.

For today, let us fix an abelian variety $A/k$ of dimension $g$ and set $\phi_A$ to be $\phi_{\mathcal{P}_A}$.

**Proposition 49.** *Let $X, Y, Z$ be $k$-schemes. Let $K \in D(X \times Y)$ and $L \in D(Y \times Z)$. Define the* ***convolution***

$$K * L = \mathrm{pr}_{13,*}(\mathrm{pr}_{12}^* K \otimes \mathrm{pr}_{23}^* L) \in D(X \times Z).$$

*Then,*

$$\phi_L \circ \phi_K = \phi_{K*L}.$$

*Proof.* The argument is essentially formal, using with the input of flat base change and the projection formula applied to the diagram below.

$$\begin{tikzcd}
&&& X \times Y \times Z \arrow[dll, "p_{XY}"'] \arrow[drr, "p_{YZ}"] \arrow[ddlll, bend right, "p_X"'] \arrow[ddrrr, bend left, "p_Z"] &&& \\
& X \times Y \arrow[dl, "\mathrm{pr}_X"] \arrow[drr, "\mathrm{pr}_Y"'] &&&& Y \times Z \arrow[dll, "\mathrm{pr}_Y"] \arrow[dr, "\mathrm{pr}_Z"'] & \\
X &&& Y &&& Z
\end{tikzcd}$$

☺

Next, we have a lemma generalizing the fact that $m^* L \cong \mathrm{pr}_1^* L \otimes \mathrm{pr}_2^* L$ for $L \in \mathrm{Pic}^0(A)$.

**Lemma 50.** *Let $\mu\colon A \times A^t \times A \to A \times A^t$ be the map sending $(a, b, c) \mapsto (m(a, c), b)$ on points. Then,*

$$\mu^*(\mathcal{P}_A^{-1}) \otimes \mathrm{pr}_{12}^*(\mathcal{P}_A) \otimes \mathrm{pr}_{23}^*(\mathcal{P}_A)$$

*is trivial.*

*Proof.* Use the theorem of the cube to check that the fibers above $e$ on each coordinate is trivial. ☺

**Theorem 51** (Mukai). *$\phi_{\mathcal{P}_A}\colon D(A) \to D(A^t)$ is an equivalence of triangulated categories. Moreover,*

$$\phi_{\mathcal{P}_{A^t}} \circ \phi_{\mathcal{P}_A} \cong [-1]^*[-g].$$

*Proof.* First, write $\phi_{\mathcal{P}_{A^t}} \circ \phi_{\mathcal{P}_A} = \phi_{\mathcal{P}_A * \mathcal{P}_{A^t}}$. Next, if $\Gamma \subset A \times A$ is the graph of $[-1]$, it follows (from the example at the end of last time) that it remains to show $\mathcal{P}_A * \mathcal{P}_{A^t} = \mathcal{O}_\Gamma[-g]$.

Write

$$\mathcal{P}_A * \mathcal{P}_{A^t} = \mathrm{pr}_{13,*}(\mathrm{pr}_{12}^* \mathcal{P}_A \otimes \mathrm{pr}_{23}^* \mathcal{P}_{A^t}) = \mathrm{pr}_{13,*}(\mu^* \mathcal{P}_A).$$

Now, consider the following diagram:

$$\begin{tikzcd}
A \times A^t \times A \arrow[r, "\mu"] \arrow[d, "\mathrm{pr}_{13}"'] & A \times A^t \arrow[d, "\mathrm{pr}_1"] \\
A \times A \arrow[r, "m"'] & A
\end{tikzcd}$$

By flat base change, we then have $\mathcal{P}_A * \mathcal{P}_{A^t} \cong m^* \mathrm{pr}_{1,*} \mathcal{P}_A \cong m^* k(e)[-g]$, with the latter from our computation of the cohomology of the Poincare bundle. We also have:

$$\begin{tikzcd}
\Gamma \arrow[r, hook, "i"] \arrow[d, "m"'] & A \times A \arrow[d, "m"] \\
\{e\} \arrow[r, hook, "i"] & A
\end{tikzcd}$$

Again, by flat base change, we have $m^* k(e)[-g] \cong \mathcal{O}_\Gamma[-g]$, as desired. ☺

*Remark* 52. One can check that $D^b_{\mathrm{coh}}(A) \cong D^b_{\mathrm{coh}}(A^t)$ under this equivalence (by some general properties—check Huybrechts).

**Proposition 53.** *Let $x \in A^t(k)$ and $M_x = t_x^* L \otimes L^{-1} \in \mathrm{Pic}^0(A)$ be degree zero line bundle associated to $x$. Then, for any $F \in D(A)$, we have*

$$\phi_A(F \otimes M_x) \cong t_x^* \phi_A(F).$$

*Similarly, we have*

$$\phi_A(t_x^* F) \cong N_{-x} \otimes \phi_A(F),$$

*where $x \in A(k)$.*

*Proof.* Let us express the LHS as $\mathrm{pr}_{2,*}(\mathrm{pr}_1^*(F) \otimes \mathrm{pr}_1^*(M_x) \otimes \mathcal{P}_A)$ and the RHS as

$$\begin{aligned}
t_x^* \phi_A(F) &\cong t_x^* \mathrm{pr}_{2,*}(\mathrm{pr}_1^* F \otimes \mathcal{P}_A) \\
&\cong \mathrm{pr}_{2,*}(t_{(e,x)}^* \mathrm{pr}_1^* F \otimes t_{(e,x)}^* \mathcal{P}_A) \\
&\cong \mathrm{pr}_{2,*}(\mathrm{pr}_1^* F \otimes t_{(e,x)}^* \mathcal{P}_A).
\end{aligned}$$

Then, it suffices to show that

$$\mathrm{pr}_1^*(M_x) \otimes \mathcal{P}_A \cong t_{(e,x)}^* \mathcal{P}_A.$$

Then, we conclude by using the Seesaw theorem. First, we can check that on any fiber $A \times \{y\}$, both sides restrict to the same thing. This means that the LHS tensored by the inverse of the RHS is pulled back from $A^t$. So it suffices to check that on the fiber $\{e\} \times A^t$, both sides restrict to the same thing, which is indeed true. ☺

**Corollary 54.** *Let $F = \mathcal{O}_A$. Then, we have*

$$\begin{aligned}\phi_A(M_x) &\cong t_x^* \phi_A \mathcal{O}_A \\ &\cong t_x^* \operatorname{pr}_{2,*} \mathcal{P}_A \\ &\cong t_x^* k(e)[-g] \\ &\cong k(-x)[-g],\end{aligned}$$

*since we have the diagram*

$$\begin{tikzcd} \{-x\} \arrow[r, hook] \arrow[d] & A \arrow[d, "t_x"] \\ \{e\} \arrow[r, hook] & A \end{tikzcd}$$

**Proposition 55.** *Let $M, N \in D(A)$. Then,*

$$\phi_A(M) \otimes \phi_A(N) \cong \phi_A(M * N).$$

*Proof.* The proof is essentially formal. Consider the following diagram:

$$\begin{tikzcd} & A \times A \times A^t \arrow[rr, "\mu=(m,\mathrm{Id})"] \arrow[d, "\operatorname{pr}_{12}"'] \arrow[ddl, bend right, "\operatorname{pr}_1"'] \arrow[ddr, bend left, "\operatorname{pr}_2"] \arrow[rrr, bend left, "\operatorname{pr}_3"] & & A \times A^t \arrow[r, "\operatorname{pr}_2"] \arrow[d, "\operatorname{pr}_1"] & A^t \\ & A \times A \arrow[rr, "m"'] \arrow[dl, "\operatorname{pr}_1"] \arrow[dr, "\operatorname{pr}_2"] & & A & \\ A & & A & & \end{tikzcd}$$

Using flat base change, the projection formula, and our lemma from earlier (the one where we said "$\mu^*(\mathcal{P}_A^{-1}) \otimes \operatorname{pr}_{12}^*(\mathcal{P}_A) \otimes \operatorname{pr}_{23}^*(\mathcal{P}_A)$"), one can verify that

$$\phi_A(M * N) \cong \operatorname{pr}_{3,*}((\operatorname{pr}_{13}^* \mathcal{P}_A \otimes \operatorname{pr}_1^* M) \otimes (\operatorname{pr}_{23}^* \mathcal{P}_A \otimes \operatorname{pr}_2^* N).$$

Since $\operatorname{pr}_3$ is the fiber product of $\operatorname{pr}_2$ with itself, we can then use Kunneth (use the derived version on the Stacks project!), it follows that

$$\begin{aligned}\phi_A(M * N) &\cong \operatorname{pr}_{2,*}(\mathcal{P}_A \otimes \operatorname{pr}_1^* M) \otimes \operatorname{pr}_{2,*}(\mathcal{P}_A \otimes \operatorname{pr}_2^* N) \\ &\cong \phi_A(M) \otimes \phi_A(N),\end{aligned}$$

as desired. ☺

**Lemma 56.** *Let $x \in A^t(k)$ and $M_x = t_x^* L \otimes L^{-1}$ be the corresponding degree zero line bundle. Then, if $F \in D(A)$, we have*

$$R\Gamma(A, F \otimes M_x) \cong \phi_A(F)|_{\{x\}} \in D(k)$$

*and $G \in D(A^t)$, then*

$$R\Gamma(A, \phi_{A^t}(G) \otimes M_x) \cong G[-g]|_{\{-x\}} \in D(k).$$

*Proof.* The second claim follows from the first by plugging in $F = \phi_{A^t}(G)$. For the first claim, consider the following:

$$\begin{tikzcd} A \times \{x\} \arrow[r, hook, "i"] \arrow[d, "\mathrm{pr}_2"'] & A \times A^t \arrow[d, "\mathrm{pr}_2"] \\ \{x\} \arrow[r, hook, "i"] & A^t \end{tikzcd}$$

By flat base change, we get

$$\begin{aligned} \phi_A(F)|_{\{x\}} &\cong \mathrm{pr}_{2,*}(\mathrm{pr}_1^* F \otimes \mathcal{P}_A)|_{\{x\}} \\ &\cong R\Gamma(A, (\mathrm{pr}_1^* F \otimes \mathcal{P}_A)|_{A \times \{x\}}) \\ &\cong R\Gamma(A, F \otimes M_x), \end{aligned}$$

as desired. ☺

**Corollary 57.** *Taking $F = \mathcal{O}_A$ and $x = e$ gives $R\Gamma(A, \mathcal{O}_A) \cong k(e)[-g]|_{\{x\}}$ (both sides are actually exterior algebras!).*

**Corollary 58.** *Taking $G \in \mathrm{Pic}(A^t)$ and $x = e$ gives $R\Gamma(A, \phi_{A^t}(G)) \cong G[-g]|_{\{e\}}$. The RHS is non-canonically isomorphic to $k(e)[-g]$, so taking Euler characteristics gives $\chi(A, \phi_{A^t}(G)) = (-1)^g$.*

**Proposition 59.** *If $A \to B$ is a homomorphism of abelian varieties, then $\phi_B \circ f_* \cong (f^t)^* \circ \phi_A$.*

*Proof.* Note that $(\mathrm{Id}, f^t)^* \mathcal{P}_A \cong (f, \mathrm{Id})^* \mathcal{P}_B$ from our discussion of the dual abelian variety. For any $F \in D(A)$, note that

$$\begin{aligned} (f^t)^* \circ \phi_A(F) &\cong (f^t)^*(\mathrm{pr}_{2,*}(\mathrm{pr}_1^* F \otimes \mathcal{P}_A)) \\ &\cong \mathrm{pr}_{2,*}((\mathrm{Id}, f^t)^* \mathrm{pr}_1^* F \otimes (\mathrm{Id}, f^t)^* \mathcal{P}_A)) \\ &\cong \mathrm{pr}_{2,*}(\mathrm{pr}_1^* F \otimes (f, \mathrm{Id})^* \mathcal{P}_B) \\ &\cong \mathrm{pr}_{2,*}(f, \mathrm{Id})_*(\mathrm{pr}_1^* F \otimes (f, \mathrm{Id})^* \mathcal{P}_B) \\ &\cong \mathrm{pr}_{2,*}((f, \mathrm{Id})_* \mathrm{pr}_1^* F \otimes \mathcal{P}_B) \\ &\cong \mathrm{pr}_{2,*}(\mathrm{pr}_1^* f_*(F) \otimes \mathcal{P}_B) \\ &\cong \phi_B \circ f_*(F), \end{aligned}$$

using flat base change twice and the projection formula. ☺

We can generalize degree zero line bundles by considering homogeneous vector bundles in the derived category.

**Definition 60.** $E \in D(A)$ is **homogeneous** if for all $x \in A(k)$, we have an isomorphism $t_x^* E \cong E$.

**Theorem 61.** *The functor $\phi_A[g] \colon D(A) \to D(A^t)$ restricts to an equivalence of categories*

$$\{\textit{homogeneous vector bundles on } A\} \longleftrightarrow \{\textit{coherent sheaves on } A^t \textit{ with finite support}\}.$$

*Under this equivalence, the rank of a homogeneous vector bundle $E$ is the same as the length of the coherent sheaf $\phi_A(E)[g]$.*

The proof of this theorem requires the following lemma, which we will only sketch.

**Lemma 62.** *Suppose $G \in D^b_{\mathrm{coh}}(A)$ satisfies $G \otimes L \cong G$ for all $L \in \operatorname{Pic}^0(A)$. Then, $G$ has finite support.*

*Proof.* We can assume $G$ is a coherent sheaf placed in degree 0. If the support is not finite, then we can find a curve in the support, say $C$. The normalization, say $\tilde{C}$, comes with finite map $f\colon \tilde{C} \to C$. Consider $\overline{G} \coloneqq f^*G/\text{torsion}$, which is a vector bundle on $\tilde{C}$. By assumption, $f^*G \otimes f^*L \cong f^*G$ for degree zero $L$, so modding out by torsion gives $\overline{G} \otimes f^*L \cong \overline{G}$. Taking determinants gives that $f^*L^{\otimes \operatorname{rk} \overline{G}}$ is trivial. This gives us a map $\pi = ([\operatorname{rk} \overline{G}] \circ f, \mathrm{Id})$, so that $\pi^*\mathcal{P}_A$ is trivial along the fibers of $\mathrm{pr}_2$. The Seesaw theorem then implies that it is pulled back from $A^t$. This then implies that the map associated to $\mathcal{P}_{A^t}$, i.e. $\tilde{C} \to (A^t)^t$ is constant with image equal to the point corresponding to the line bundle on $A^t$. This map $\tilde{C} \to (A^t)^t \cong A$ turns out to be the same as $\tilde{C} \to A = \tilde{C} \to C \subset A$, so we get a contradiction. ☺

*Proof of theorem.* First, recall that for any extension $k(x) \to N \to k(x)$ in $D(A^t)$, we can write this as $\phi_A(M_{-x})[g] \to \phi_A(M) \to \phi_A(M_{-x})[g]$ for some $M \in D(A)$ (and $M_x$ defined earlier). Then, by duality, we have an exact triangle $M_{-x}[g] \to M \to M_{-x}[g]$. Then, $M = H^{-g}(M)[g]$.

Suppose now that $N$ is a coherent sheaf on $A^t$ with finite support. We will show that we can obtain a homogeneous vector bundle on $A$. Write $N = \phi_A(M)$. Since we can write $N$ as a sequence of extensions by $k(x)$, we can use the first paragraph to conclude that $M$ is a concentrated in a single degree (degree $-g$) and expressed as a sequence of extensions by $M_{-x}$. Then, $M[-g]$ is a vector bundle, and it is homogeneous because $\phi_A(t_x^*(M)) \cong N_{-x} \otimes \phi_A(M) \cong N_{-a} \otimes N \cong N \cong \phi_A(M)$ (using that $N$ has finite support). So applying $\phi_A^{-1}[-g]$ sends a coherent sheaf on $A^t$ with finite support to a homogeneous vector bundle on $A$.

In the other direction, suppose we have a homogeneous vector bundle $E$ on $A$. Then, $\phi_A(E)$ is invariant under tensoring by $L$ for any $L \in \operatorname{Pic}^0(A^t)$. By our lemma, it follows that $\phi_A(E)$ has finite support. If it has at least two non-zero cohomology sheaves (lives in more than one degree), then $E$ would necessarily as well, which is a contradiction! So we are done. ☺

# 6 04/07: Cohomology of ample line bundles

We will discuss which line bundles $L$ give rise to finite $K(L)$. We will then compute the cohomology of such line bundles using the machinery of the Fourier-Mukai transform that we have developed so far. This will be used later to understand (semi-)stable vector bundles on elliptic curves.

**Definition 63.** $L \in \operatorname{Pic}(A)$ is said to be **non-degenerate** if $K(L)$ is finite.

**Lemma 64.** *If $L \in \operatorname{Pic}(A)$, then $\phi_L^*(\phi_A(L)) = R\Gamma(A, L) \otimes L^{-1}$. In particular, each cohomology sheaf $\mathcal{H}^i(\phi_L^*(\phi_A(L)))$ is a vector bundle (over $A$) and $\mathcal{H}^i(\phi_A(L))$ is a vector bundle on $A^t$.*

*Proof.* Since $\phi_L$ is faithfully flat, it suffices to show the first claim (it's clear that the first thing is a vector bundle because we are just pulling back things from $D(k)$). Consider the following diagram:

$$\begin{tikzcd} A \arrow[d, "="'] & A \times A \arrow[l, "p_1"'] \arrow[r, "p_2"] \arrow[d, "\alpha"'] & A \arrow[d, "\phi_L"] \\ A & A \times A^t \arrow[l, "\mathrm{pr}_1"'] \arrow[r, "\mathrm{pr}_2"] & A^t \end{tikzcd}$$

Using flat base change and the projection formula, we get

$$\begin{aligned} \phi_L^* \phi_A(L) &= \phi_L^*(\mathrm{pr}_{2,*}(\mathrm{pr}_1^* L \otimes \mathcal{P}_A)) \\ &\cong p_{2,*}\alpha^*(\mathrm{pr}_1^* L \otimes \mathcal{P}_A) \\ &\cong p_{2,*}(p_1^* L \otimes \Lambda(L)) \\ &\cong p_{2,*}(p_1^* L \otimes m^* L \otimes p_1^* L^{-1} \otimes p_2 L^{-1}) \\ &\cong p_{2,*} m^* L \otimes L^{-1}. \end{aligned}$$

Now, consider this diagram:

$$\begin{tikzcd} A \times A \arrow[r, "m"] \arrow[d, "p_2"'] & A \arrow[d] \\ A \arrow[r] & k \end{tikzcd}$$

Applying flat base change to this gives

$$\phi_L^* \phi_A(L) \cong f^* f_* L \otimes L^{-1} \cong R\Gamma(A, L) \otimes L^{-1}.$$

☺

**Lemma 65.** *If $L \in \operatorname{Pic}(A)$ is non-degenerate, then $\chi(A, L)^2 = \operatorname{rk}(K(L))$ (recall $\operatorname{rk}(K(L)) = \dim_k(\mathcal{O}(A^t))$).*

*Proof.* From an earlier lemma about Euler characteristics, recall that

$$\chi(A, \phi_L^* \phi_A(L)) = \operatorname{rk}(K(L)) \chi(A^t, \phi_A(L))$$

(since $\mathrm{rk}(K(L)) = \deg(\phi_L)$). We computed the Euler characteristic appearing on the RHS last time, so we have

$$\chi(A, \phi_L^* \phi_A(L)) = \mathrm{rk}(K(L))(-1)^g.$$

Let us compute the LHS in a different way. From the previous lemma, we know that $\chi(A, \phi_L^*\phi_A(L)) = \chi(A, R\Gamma(A, L) \otimes L^{-1})$. The projection formula tells us that $R\Gamma(A, R\Gamma(A, L) \otimes L^{-1}) = R\Gamma(A, L) \otimes R\Gamma(A, L^{-1})$, so we have

$$\chi(A, \phi_L^* \phi_A(L)) = \chi(A, L)\chi(A, L^{-1}).$$

Also, by differential properties of abelian varieties, we know that the canonical bundle is trivial, so Serre duality tells us that

$$\chi(A, \phi_L^* \phi_A(L)) = \chi(A, L)^2(-1)^g.$$

So

$$\mathrm{rk}(K(L)) = \chi(A, L)^2.$$

☺

**Lemma 66.** *If $L \in \mathrm{Pic}(A)$ is non-degenerate, then $L *_A [-1]^* L^{-1}$ is finitely-supported.*

*Proof.* It suffices to show that if $i \colon K(L) \hookrightarrow A$ is the inclusion for any line bundle $L \in \mathrm{Pic}(A)$, then $L *_A [-1]^* L^{-1} \cong i_*(L|_{K(L)})[-g]$.

Consider the diagram

$$\begin{tikzcd} A \times A \arrow[r, "\eta"] \arrow[dr, "m"'] & A \times A \arrow[d, "p_1"] \\ & k \end{tikzcd}$$

where $\eta$ sends $(x, y)$ to $(m(x, y), -y)$ (so that $\eta^2 = \mathrm{Id}$). Then, $\eta_* = \eta^*$, and we have

$$\begin{aligned} m_*(p_1^* L \otimes p_2^*[-1]^* L^{-1}) &= p_{1,*}(\eta^* p_1^* L \otimes \eta^* p_2^* [-1]^* L^{-1}) \\ &\cong p_{1,*}(m^* L \otimes p_2 L^{-1}) \\ &\cong p_{1,*}(\Lambda(L) \otimes p_1^* L) \\ &\cong p_{1,*}(\Lambda(L)) \otimes L \end{aligned}$$

by the projection formula. Now, consider the diagram

$$\begin{tikzcd} A \times A \arrow[r, "\alpha = (\phi_L, \mathrm{Id})"] \arrow[d, "p_1"'] & A^t \times A \arrow[d, "\mathrm{pr}_1"] \\ A \arrow[r, "\phi_L"'] & A^t \end{tikzcd}$$

By definition, we have $\alpha^* \mathcal{P}_A = \Lambda(L)$, so applying flat base change twice (also to a diagram with left vertical $K(L) \to \{e\}$ given by $\phi_L$) and our earlier result on cohomology of the Poincare bundle gives

$$\begin{aligned}\mathrm{RHS} &\cong p_{1,*}(\alpha^*\mathcal{P}_A)\otimes L\\ &\cong \phi_L^*(k(e)[-g])\otimes L\\ &\cong i_*\mathcal{O}_{K(L)}[-g]\otimes L\\ &\cong i_*(L|_{K(L)})[-g],\end{aligned}$$

as desired. ☺

**Theorem 67.** *Let $L \in \operatorname{Pic}(A)$ be non-degenerate.*

*(i) There is a vector bundle $E$ on $A^t$ such that $\phi_A(L) = E[-i(L)]$ for some integer $0 \le i(L) \le g$. As a corollary, we have $R\Gamma(A, L) \cong E[-i(L)]|_{\{e\}}$ by a fact from the Fourier-Mukai transform from last time, and hence $H^i(A, L) = E|_{\{e\}}$ for $i = i(L)$ and 0 otherwise. $i(L)$ is called the **index**.*

*(ii) $\left(\dim H^{i(L)}(A, L)\right)^2 = \operatorname{rk}(K(L)) = \deg(\phi_L)$.*

*Proof.* It suffices to show the existence of a vector bundle $E$ with $0 \le i(L) \le g$ and $\phi_A(L) = E[-i(L)]$. Since $\phi_L$ is faithfully flat, it suffices to show

$$\phi_L^*\phi_A(L) = E'[-i(L)]$$

for some vector bundle $E'$.

By a previous lemma the LHS is $R\Gamma(A, L)\otimes L^{-1}$, so it remains to show that $R\Gamma(A, L)$ in a single degree, which is between 0 and $i(L)$.

By the previous lemma and using the fact that Fourier-Mukai exchanges convolutions for tensor products, we have

$$\phi_A(L)\otimes\phi_A([-1]^*L^{-1}) \cong \phi_A(i_*(L|_{K(L)})[-g]).$$

By the big theorem from last time, we know the RHS is a homogeneous vector bundle, and is hence of the form $E''[-g]$ for some homogeneous vector bundle $E''$ on $A$. Hitting both sides with $\phi_L^*$ gives

$$(R\Gamma(A, L)\otimes L^{-1})\otimes(R\Gamma(A, [-1]^*L^{-1})\otimes[-1]^*L) \cong E'''[-g]$$

with $E'''$ some vector bundle on $A$. Rearranging gives

$$R\Gamma(A, L)\otimes R\Gamma(A, [-1]^*L^{-1}) \cong E''''[-g]$$

for some $E''''$ a vector bundle on $A$. The result follows. ☺

**Corollary 68.** *If $L \in \operatorname{Pic}(A)$ is very ample, then $H^i(A, L) = 0$ for $i > 0$ (because $H^0$ is non-zero).*

We'll show soon that we can replace "very ample" with just "ample." The key point is that $i(L) = i(L^{\otimes n})$.

**Definition 69.** Say $L, L' \in \mathrm{Pic}(A)$ are **algebraically equivalent** if there is a smooth variety $V$ and a line bundle $\mathcal{L} \in \mathrm{Pic}(V \times A)$ such that $\mathcal{L}|_{\{c_0\}\times A} = L$ and $\mathcal{L}|_{\{c_1\}\times A} = L'$ for some closed points $c_0, c_1 \in V$.

**Example 70.** Every $L \in \mathrm{Pic}^0(A)$ is algebraically equivalent to the trivial line bundle. This follows from the fact that $L$ corresponds to some $x \in A^t(k)$ so that $\mathcal{P}_A|_{\{x\}\times A} \cong L$, and also $\mathcal{P}_A|_{\{e\}\times A}$ is trivial.

**Lemma 71.** *If $L, L' \in \mathrm{Pic}(A)$ are algebraically equivalent and non-degenerate, then they have the same index.*

*Proof.* By assumption, there exists $\mathcal{L}$ and $c_0, c_1 \in V$ such that $\mathcal{L}|_{\{c_0\}\times A} = L$ and $\mathcal{L}|_{\{c_1\}\times A} = L'$. Let $M$ be the line bundle $\mathcal{L} \otimes \mathrm{pr}_2^* L^{-1}$. Notice that $M|_{\{c_0\}\times A}$ is trivial, and so $M|_{\{c\}\times A} \in \mathrm{Pic}^0(A)$ for all $c$. So we get a map $g\colon V \to A^t$ such that $(g, \mathrm{Id})^*\mathcal{P}_A = M$.

Since $\phi_L$ is surjective, it follows that $L^{-1} \otimes L'$ can be written as $\phi_L(x)$ for some $x \in A$, i.e. $t_x^*L \otimes L^{-1} \cong L^{-1} \otimes L'$, i.e. $t_x^*L \cong L'$. But this means that $H^*(A, L) \cong H^*(A, L')$ under the automorphism $t_x$, so the result follows. ☺

**Lemma 72.** *Let $f\colon A \to B$ be an isogeny and $L \in \mathrm{Pic}(B)$ be non-degenerate. Then, $i(L) = i(f^*L)$.*

*Proof.* First, we need to verify that the RHS even makes sense—why is $f^*L$ non-degenerate? To see this, it suffices to show that $\phi_{f^*L} = f^t \circ \phi_L \circ f$, since each term in the composition is an isogeny.

Let us use the universal property of the dual, i.e. using the fact that there is a unique map $A \to A^t$ such that pulling back $\mathcal{P}_A$ under this map gives rise to a prescribed line bundle. In this case, let the line bundle be $\Lambda(f^*L) \cong \phi_{f^*L}^*\mathcal{P}_A$. Consider the following diagram (we want to show that the top horizontal arrow makes the diagram commute):

$$\begin{tikzcd}
A \times A \arrow[r, "\phi_{f^*L}\times \mathrm{Id}"] \arrow[d, "f\times\mathrm{Id}"'] & A^t \times A \\
B \times A \arrow[r, "\phi_L \times \mathrm{Id}"'] & B^t \times A \arrow[u, "f^t\times\mathrm{Id}"'] \arrow[d, "\mathrm{Id}\times f"] \\
& B^t \times B
\end{tikzcd}$$

We want to show that $(f \times \mathrm{Id})^*(\phi_L \times \mathrm{Id})^*(f^t \times \mathrm{Id})^*\mathcal{P}_A \cong \Lambda(f^*L)$ (more precisely we should also check the rigidifications too, but we'll just ignore this). Now, by definition, we have the LHS is isomorphic to

$$\begin{aligned}
(f \times \mathrm{Id})^*(\phi_L \times \mathrm{Id})^*(\mathrm{Id} \times f)^*\mathcal{P}_B &\cong (\phi_L \circ f, f)^*\mathcal{P}_B \\
&\cong (f, f)^*\Lambda(L) \\
&\cong \Lambda(f^*L),
\end{aligned}$$

as desired.

Let us now compute the index. Recall from last time that $f^t_*(\phi_B(L)) = \phi_A(f^*L)$, and $f^t$ is an isogeny, we have $f^t_*(\phi_B(L)) = f^t_*(E[-i(L)]) = E'[-i(L)]$ for some vector bundle $E'$ on $A$. So we are done. ☺

**Lemma 73.** *For sufficiently large $n > 0$ and a non-degenerate line bundle $L \in \mathrm{Pic}(A)$, we have $i(L) = i(L^{\otimes n})$.*

*Proof.* We can actually just take $n = m^2$. Since $[m]$ is an isogeny, we have $i(L) = i([m]^*L)$ by the previous lemma. Earlier in our discussion about degree zero line bundles, we showed there exists a degree zero line bundle $N$ such that $[m]^*L \cong L^{m^2} \otimes N$. Since $N$ is algebraically equivalent to the trivial line bundle (see the example above), it follows that $[m]^*L$ and $L^{m^2}$ are also algebraically equivalent. Then, $i(L) = i([m]^*L) = i(L^{m^2})$. ☺

*Remark* 74. The previous lemma holds for any positive integer $n > 0$. The general idea (due to Zarhin) is to observe the fact (!) that $n$ can be expressed as a sum of four squares: $n = a^2+b^2+c^2+d^2$. Then, there is a morphism $f\colon A^4 \to A^4$ defined by a matrix $M$ corresponding to multiplication by the quarternion $a + bi + cj + dk$. Note that $i(L^{\boxtimes 4}) = 4i(L)$ (I think by Kunneth). Also, one can check that $f^*L^{\boxtimes 4}$ is algebraically equivalent to $(L^n)^{\boxtimes 4}$ (the reason being that $M^t \cdot M = n\,\mathrm{Id}_4$). So $4i(L) = i(L^{\boxtimes 4}) = i(f^*L^{\boxtimes 4}) = 4i(L^n)$, as desired.

**Corollary 75.** *If $L$ is ample, then $i(L) = 0$ and $(\dim H^0(A, L))^2 = \deg(\phi_L)$.*

*Proof.* Simply choose $m$ large enough and apply the previous lemma. ☺

# 7 04/14: Atiyah's theorem on vector bundles on elliptic curves

Using the Fourier-Mukai equivalence, we will explain a simplification of Atiyah's original paper ("Vector bundles over an elliptic curve"). To build up to the proof, we will also review the notions of (semi-)stability of a vector bundle and the Harder-Narasimhan filtration. In particular, Atiyah's theorem describes an equivalence between the category of semistable vector bundles on an elliptic curve (of given slope) with the category of torsion coherent sheaves.

First, let $C$ be a smooth projective, geometrically connected curve over a field $k$. If $F$ is a coherent sheaf on $C$, then we can define two important numerics as follows:

**Definition 76.** The **rank** $r(F) \geq 0$ is defined as the dimension of the vector space $F_\eta$, where $\eta$ is the generic point of $C$. The **degree** $d(F) \in \mathbb{Z}$ is defined by decomposing $F$ as a direct sum of a vector bundle $E$ and a torsion sheaf $T$ and defining $d(F) = d(E) + d(T)$, where $d(E)$ is the degree of the determinant of $E$ and $d(T)$ is the length of $T$.

Note that $r$ and $d$ are additive in short exact sequences, giving rise to a function $(r, d)\colon K_0(C) \to \mathbb{Z}^{\oplus 2}$, which we can descend to the bounded coherent derived category.

The **slope** $\mu(F)$ is defined to be $d(F)/r(F) \in \mathbb{Q} \cup \{\infty\}$ (this is not additive clearly). If $F$ is moreover a vector bundle, we say that $F$ is **semistable** (resp. **stable**) if for any non-trivial quotient bundle $Q$ of $F$ we have $\mu(Q) \geq \mu(F)$ (resp. >).

*Remark* 77. This requires a bit of justification. Some helpful notes: `https://ocw.mit.edu/courses/18-725-algebraic-geometry-fall-2015/ec341c7a2524e5dba7c3e939f322613a_MIT18_725F15_notes.pdf`

**Lemma 78.** *Let $0 \to E_1 \to E \to E_3 \to 0$ be a SES of coherent sheaves on $C$. Then,*

$$\min\{\mu(E_1), \mu(E_2)\} \leq \mu(E) \leq \max\{\mu(E_1), \mu(E_2)\}.$$

*Proof.* Let $d_i = d(E_i)$ and $r_i = r(E_i)$. Note that $d_1/r_1$ and $d_2/r_2$ are upper/lower bounds on $(d_1 + d_2)/(r_1 + r_2)$, so the result follows. ☺

**Lemma 79.** *Let $E$ be a vector bundle on $C$. The, $E$ is semistable (resp. stable) iff for any non-trivial subsheaf $F \subset E$, we have $\mu(F) \leq \mu(E)$ (resp. <).*

*Proof.* Suppose $E$ is semistable and $F$ a non-trivial subsheaf of $E$. Then, let $\overline{F}$ be the saturation of $E$ (take the preimage of the torsion subsheaf of $E/F$ under the projection $E \to E/F$, this is defined so that $E/\overline{F}$ gives a vector bundle). First, note that $\mu(F) \leq \mu(\overline{F})$ using the following exact sequence

$$0 \to F \to \overline{F} \to \overline{F}/F \to 0.$$

Since $\overline{F}$ and $F$ have the same rank, the previous lemma tells us that $\mu(\overline{F}) \geq \min\{\mu(F), \infty\} = \mu(F)$.

To see that $\mu(\overline{F}) \leq \mu(E)$, consider the exact sequence

$$0 \to \overline{F} \to E \to E/\overline{F} \to 0,$$

so $\mu(E) \geq \min\{\mu(\overline{F}), \mu(E/\overline{F})\}$ by the previous lemma. $E$ is semistable so $\mu(E) \leq \mu(E/\overline{F})$, so $\mu(\overline{F}) \leq \mu(E)$.

Conversely, if $\mu(E) \geq \mu(F)$ for any non-trivial subsheaf, let $E \to Q$ be any quotient subbundle and consider the sequence $0 \to \ker \to E \to Q \to 0$. Then, by the lemma, we have $\mu(E) \leq \mu(Q)$ because $\ker$ is a subsheaf of $E$.

A similar argument works for $E$ stable. ☺

**Lemma 80.** *Let $E, F$ be semistable vector bundles on $C$ such that $\mu(E) > \mu(F)$. Then, $\mathrm{Hom}(E, F) = 0$.*

*Proof.* Suppose we have a non-zero map $E \to F$. Then, $\mathrm{im}$ is a subsheaf of $F$, so $\mu(F) \geq \mu(\mathrm{im})$. Also, $E$ surjects onto $\mathrm{im}$, so $\mu(E) \leq \mu(\mathrm{im})$. This is a contradiction. ☺

**Lemma 81.** *Let $0 \to E_1 \to E \to E_2 \to 0$ be a SES of vector bundles on $C$ with the same slope $\mu$. Then, $E$ is semistable iff $E_1$ and $E_2$ are.*

*Proof.* Suppose $E$ is semistable. Then, for any nontrivial subsheaf $F \subset E_1 \subset E$, we have $\mu(F) \leq \mu(E) = \mu = \mu(E_1)$, so $E_1$ is semistable. For any nontrivial quotient bundle $E \to E_2 \to Q$, the same argument works.

Now, suppose $E_1$ and $E_2$ are both semistable. Let $F$ be a nontrivial subsheaf of $E$. It suffices to show that $\mu(F) \leq \mu(E)$.

Consider the SES

$$0 \to F \cap E_1 \to F \to F/(F \cap E_1) \to 0.$$

We have $\min\{\mu(F \cap E_1), \mu(F/(F \cap E_1)\} \leq \mu(F) \leq \max\{\mu(F \cap E_1), \mu(F/F \cap E_1)\}$.

By semistability, assuming $F \cap E_1$ and $F/(F \cap E_1)$ are non-zero, we have $\mu(F \cap E_1) \leq \mu(E_1) = \mu$ and $\mu(F/(F \cap E_1) \leq \mu(E_2) = \mu$, so $\mu(F) \leq \max\{\mu, \mu\} = \mu(E)$. The case where either of them is zero is easy. ☺

**Example 82.** Every line bundle is stable. More generally, extensions of line bundles of the same slope are semistable by the lemma.

*Remark* 83. If $C = \mathbb{P}^1$, then every vector bundle of slope $a$ is of the form $\bigoplus_{i=1}^n \mathcal{O}_C(a)$ (using the Grothendieck-Birkhoff theorem). In fact, there is an equivalence of categories between $k$-vector spaces and semistable vector bundles over $\mathbb{P}^1$ of slope $a$ given by $V \mapsto V \otimes \mathcal{O}(a)$.

**Theorem 84** (Harder-Narashimhan filtration)**.** *Let $E$ be a vector bundle on $C$. Then, there is a unique filtration $0 = E_0 \subset \cdots \subset E_\ell = E$ such that $Q_i = E_i/E_{i-1}$ is semistable and $\mu(Q_{i-1}) > \mu(Q_i)$.*

*Proof.* Let us first establish uniqueness by induction on the rank $r$ of $E$. Suppose we have two Harder-Narasimhan filtrations

$$0 = E_0 \subset \cdots \subset E_m = E$$

$$0 = E'_0 \subset \cdots \subset E'_n = E,$$

with successive quotients having slopes $\lambda_1 > \cdots > \lambda_m$ and $\lambda_1' > \lambda_2' > \cdots > \lambda_n'$. It suffices to show that $E_1 = E_1'$, since then we can quotient everything by $E_1 = E_1'$ and then apply the induction hypothesis to $E/E_1 = E/E_1'$.

Suppose $\lambda_1 \neq \lambda_1'$, WLOG $\lambda_1 > \lambda_1'$. Then, $\lambda_1 > \lambda_i'$ for all $i$. Then, there are no non-zero maps from $E_1$ to $E_i'/E_{i-1}'$. Since $E$ admits a finite filtration such that its quotients are given by $E_i'/E_{i-1}'$, it follows that there are no non-zero maps from $E_1$ to $E$, which is a contradiction (because $E_1 \subset E$ is a non-zero map!).

So $\lambda_1 = \lambda_1'$. Again, $\lambda_1 > \lambda_i'$ for all $i > 1$. By the same argument, it follows that there are no non-zero maps from $E_1$ to $E/E_1'$, i.e. the composite $E_1 \to E \to E/E_1'$ is zero, i.e. $E_1 \subset E_1'$. By symmetry, $E_1' \subset E_1$, so we get $E_1 = E_1'$. This completes the uniqueness part of the proof.

For existence, choose an inclusion $E \subset L^{\oplus n}$ for a sufficiently ample line bundle $L$ on $C$ and some $n$. We can do this because a sufficiently high twist $E^\vee(a)$ has finitely many global sections, giving a surjection $\bigoplus_{i=1}^n \mathcal{O}_X \to E^\vee(a)$, so that $\bigoplus_{i=1}^n \mathcal{O}_X(-a) \to E^\vee$ is a surjection, then take duals. Since $L$ is a line bundle, it it semistable, so $L^{\oplus n}$ is also semistable. So for any $F \subset E$ a subsheaf, we have $\mu(F) \leq \mu(L^{\oplus n})$.

The slopes are rational numbers and the rank of $F$ is at most the rank of $E$, so the denominators are necessarily bounded below. Then, this bounded set of rational numbers will attain a maximum, i.e. there exists a subsheaf of maximal slope. Of course, then any such subsheaf is also necessarily semistable.

Let $F$ and $G$ be two subsheaves of this maximal slope $\mu$. Then, $\mu(F+G) = \mu$ because $F+G$ is a quotient of $F \oplus G$ (note $\mu(F \oplus G) = \mu$ as well and is hence semistable). So there is a maximal subsheaf of slope $\mu$, which we denote by $E_1$.

Moreover, $E_1$ is saturated. Else, taking the exact sequence $0 \to E_1 \to \overline{E_1} \to \overline{E_1}/E_1 \to 0$ would imply $\overline{E_1}$ has larger slope than that of $E_1$, which contradicts maximality. Then, $E/E_1$ is a vector bundle.

We can then induct by taking a filtration for $E/E_1$ and then take preimages. It remains to show that the slopes are indeed strictly decreasing. Again, by induction, it suffices to check $\mu(E_1) > \mu(E_2/E_1)$. Else, consider the SES

$$0 \to E_1 \to E_2 \to E_2/E_1 \to 0.$$

Then, $\mu(E_2) \geq \mu(E_1) = \mu$, so $\mu(E_2) = \mu(E_1)$ and hence $E_2 = E_1$ by maximality, which is a contradiction (I guess we can just remove $E_2$ from the filtration and move on to $E_3$, etc.). ☺

**Corollary 85.** *Suppose $C$ has genus 1 (e.g. an elliptic curve). Then, the Harder-Narasimhan filtration of any vector bundle $E$ is split. In particular, any indecomposable vector bundle is semistable.*

*Proof.* Let $E_0 \subset \cdots \subset E_\ell = E$ be the Harder-Narasimhan filtration of $E$. Let the quotients be $Q_i = E_i/E_{i-1}$. Let us induct on the length of the filtration. The base case is obviously trivial. By induction, we can assume all but the last term is split, i.e. $E_{\ell-1} = \bigoplus_{i=1}^{\ell-1} E_i/E_{i-1}$, and it remains to split the SES

$$0 \to E_{\ell-1} = \bigoplus_{i=1}^{\ell-1} E_i/E_{i-1} \to E_\ell \to E_\ell/E_{\ell-1} \to 0.$$

Consider $\mathrm{Ext}^1(E_\ell/E_{\ell-1}, E_{\ell-1}) = \bigoplus_{i=1}^{\ell} \mathrm{Ext}^1(E_\ell/E_{\ell-1}, E_i/E_{i-1})$. By Serre duality, we have

$$\begin{aligned}\mathrm{Hom}(E_i/E_{i-1}, E_\ell/E_{\ell-1}) &\cong H^{1-1}(C, \underline{\mathrm{Hom}}(E_i/E_{i-1}, E_\ell/E_{\ell-1})) \\ &\cong \mathrm{Ext}^1(\underline{\mathrm{Hom}}(E_i/E_{i-1}, E_\ell/E_{\ell-1}), \omega_C)^\vee \\ &\cong \mathrm{Ext}^1((E_i/E_{i-1})^\vee \otimes E_\ell/E_{\ell-1}, \mathcal{O}_C)^\vee \\ &\cong \mathrm{Ext}^1(E_\ell/E_{\ell-1}, E_i/E_{i-1})^\vee.\end{aligned}$$

Since $E_i/E_{i-1}$ has slope larger than that of $E_\ell/E_{\ell-1}$, it follows the LHS is 0. So the result follows. ☺

We now finally arrive at the statement of Atiyah's theorem on vector bundles on elliptic curves. Let $E$ be an elliptic curve over $k$ and fix a principal polarization (i.e. an isomorphism $\phi_L: E \to E^t$) given by $L = O(e)$ (which we note is ample). From last time, we know $\phi_L$ has degree equal to the square root of $\dim H^0(E, L) = 1$, so $\phi_L$ is indeed an isomorphism. We use $\phi_L$ in the sequel to identify $E$ with its dual.

**Theorem 86** (Atiyah)**.** *Let $\mu \in \mathbb{Q}$. Then, there is an equivalence of categories between semistable vector bundles on $E$ of slope $\mu$ and torsion coherent sheaves on $E$.*

*If $F$ is a semistable vector bundle on $E$, then the length of $F$, viewed as a torsion coherent sheaf, is given by the gcd of the rank of $F$ and the degree of $F$.*

**Corollary 87.** *The category of torsion coherent sheaves on $E$ is independent of $E$, and there are semistable vector bundles of any slope.*

*Remark* 88. We include the zero vector bundle in the category of semistable vector bundles (of any given slope). I suppose this makes the above corollary a little stupid.

**Lemma 89.** *Let $F \in D^b_{\mathrm{coh}}(E)$. Then, $d(\phi_E(F)) = -r(F)$ and $r(\phi_E(F)) = d(F)$. Hence, $\mu(\phi_E(F)) = -1/\mu(F)$.*

*Proof.* By Riemann-Roch, we have $\chi(E, F) = d(F)$ and $\chi(\{e\}, F|_{\{e\}}) = r(F)$. Also, by Fourier-Mukai, we know $R\Gamma(E, \phi_E(F)) = F[-1]|_{\{e\}}$ and $R\Gamma(E, F) \cong \phi_E(F)|_{\{e\}}$, so taking Euler characteristics gives $-r(F) = d(\phi_E(F))$ and $r(\phi_E(F)|_{\{e\}}) = d(F)$, as desired. ☺

**Lemma 90.** *If $F$ is a vector bundle on $E$ with $\mu(F) < 0$, then $\phi_E(F[1])$ is a semistable vector bundle with slope $-1/\mu(F)$.*

*Proof.* We can assume $F$ is indecomposable. Then, $\phi_E(F[1])$ is also indecomposable. For any degree zero line bundle $L$, we have $\mathrm{Hom}(L, F) = 0$ by the assumption on slope. The former is also $H^0(E, F \otimes L^{-1})$, so it follows that $\phi_E(F)|_{\{x_L\}} \cong R\Gamma(E, F \otimes L)$ is concentrated in degree 1 for any $L \in \mathrm{Pic}^0(E)$ (where $x_L$ is the point in $E^t$ corresponding to $L$). Then, we see that $\phi_E(F)[1]$ is free at every closed point, and it is moreover coherent, so it is actually free in a neighborhood of every closed point, so it is in fact a vector bundle. We showed earlier that an indecomposable vector bundle is semistable.

Finally, the slope computation follows from the previous lemma. ☺

**Lemma 91.** *Let $\mu \in \mathbb{Q}$. Then, there is an equivalence of categories between vector bundles of slope 0 and vector bundles of slope $\mu$, which is obtained by some sequence of applying the operations $F \mapsto \phi_E(F)[1]$, $F \mapsto F \otimes O(e)$, and their inverses.*

*Proof.* If $\mu = 0$, we are done already. If $\mu < 0$, we can go to $-\mu^{-1}$ via the first operation. We can also go from $\mu$ to $\mu + 1$ via the second operation. Note that $\mathrm{SL}_2(\mathbb{Z})$ acts transitively on $\mathbb{P}^1(\mathbb{Q})$ and has generators $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, which correspond to the two operations. The result follows. ☺

*Proof of Atiyah's theorem.* By the previous lemma, it suffices to show that the category of semisimple vector bundles of slope 0 is equivalent to the category of torsion coherent sheaves. More precisely, we'll show that if $F$ is a semistable vector bundle of slope 0, then $\phi_E(F)[1]$ is a torsion coherent sheaf on $E$.

Suppose there is a non-zero map $L \to F$ with $L \in \mathrm{Pic}^0(E)$. Then, $L$ is saturated (else, $\overline{L}$ would have strictly larger slope, so the we would have a non-zero map $L \to \overline{L} \to F$, which contradicts semistability). So $F/L$ is a vector bundle of degree 0, and hence slope 0, and hence also semistable by a previous lemma (all three have the same slope). Since $\phi_E(L)[1]$ is a skyscraper, it suffices to show the claim for $F/L$.

So we can assume that there are no non-zero maps $L \to F$, i.e. $H^0(F \otimes L) = 0$ for all $L \in \mathrm{Pic}^0(E)$. Then, by the same argument as in an earlier lemma, it follows that $\phi_E(F)[1]$ is a vector bundle. But its rank is equal to the degree of $F$ (by a previous lemma), which is 0. So $\phi_E(F)[1] = 0$, so $F$ is 0.

Conversely, note that any torsion coherent sheaf is generated by extensions of skyscrapers. The inverse functor can also just be taken to be $[-1]^* \phi_E(-)$.

Finally, note that the length of $\phi_E(F)[1]$ is $\chi(E, \phi_E(F)[1]) = r(F) = \gcd(r(F), d(F) = 0)$. Now, observe that $\gcd(r(F), d(F))$ is preserved under equivalence of the categories of semisimple vector bundles of differing slope, since $\gcd(r(F), d(F)) = \gcd(-d(F), r(F)) = \gcd(r(F), d(F) + r(F))$. ☺

*Remark* 92. Let $k$ be algebraically closed. Then, for an indecomposable vector bundle $F$ of degree $d$ and rank $r$, TFAE: 1. $F$ is stable, 2. $F$ is simple (meaning that that $\mathrm{Hom}(F, F) = k$), 3. $\gcd(d, r) = 1$, and 4. $F$ is simple as an object of the category of semistable vector bundles of slope $\mu = d/r$.

The reason is that stable vector bundles have endomorphism rings that are division rings (a variant of Schur's lemma, which follows almost immediately from definition), which are isomorphic to $k$ when $k = \overline{k}$. Then, the point is that stable bundles (resp. skyscrapers) are simple objects of the category of semistable bundles of slope $\mu$ (resp. torsion coherent sheaves).

# 8 04/21: Symmetric powers of curves and Jacobians

We will first describe a few important constructions—the Picard scheme, the Albanese, symmetric powers of curves, and the Jacobian—and see how they are related to each other. Then, we will say a bit about Hacon's work on generic vanishing and Beilinson-Polishchuk's work on the Torelli theorem via Fourier-Mukai transforms.

Let us assume that $k$ is of characteristic 0 and algebraically closed and that $X$ is a smooth proper scheme/$k$ with $x \in X(k)$. Recall that the Picard functor $\underline{\mathrm{Pic}}(X)\colon \mathsf{Sch}/k \to \mathsf{Ab}$ is given on the level of points by sending $T$ to $(L \in \mathrm{Pic}(X \times T), \iota\colon L|_{\{x\}\times T} \cong \mathcal{O}_T)/\sim$. Note that $\underline{\mathrm{Pic}}(X)(k) = \mathrm{Pic}(X)$.

**Theorem 93.** *The Picard functor is representable by a locally finitely-presented group scheme/$k$ and the identity component $\underline{\mathrm{Pic}}^0(X)$ is an abelian variety/$k$.*

*Remark* 94. The assumption on the characteristic of $k$ is only used in the smoothness of $\underline{\mathrm{Pic}}^0(X)$.

**Definition 95.** The **Poincare bundle** $\mathcal{P}_X$ is the line bundle over $X \times \underline{\mathrm{Pic}}(X)$ corresponding to the identity map $\underline{\mathrm{Pic}}(X)$ to itself. It also comes with a universal trivialization $\mathcal{P}_X|_{\{x\}\times \mathrm{Pic}(X)} \cong \mathcal{O}_{\underline{\mathrm{Pic}}(X)}$.

**Proposition 96.** *For a smooth proper $k$-scheme $X$, the abelian variety $\underline{\mathrm{Pic}}^0(X)$ has dimension* $\dim H^1(X, \mathcal{O}_X)$.

*Proof.* We'll show that $T_e\underline{\mathrm{Pic}}^0(X) \cong H^1(X, \mathcal{O}_X)$. Recall that the former can be written as $\ker(\underline{\mathrm{Pic}}(X)(k[\varepsilon]) \to \underline{\mathrm{Pic}}(X)(k)) = \ker(\mathrm{Pic}(X \times_k k[\varepsilon]) \to \mathrm{Pic}(X))$. Consider the exponential exact sequence

$$1 \to 1 + \varepsilon\mathcal{O}_{X\times_k k[\varepsilon]} \to \mathcal{O}^\times_{X\times_k k[\varepsilon]} \to \mathcal{O}_X^\times \to 1.$$

Note that the first term is isomorphic to $\mathcal{O}_X$ via sending (on local sections) $1 + \varepsilon s \mapsto s$. Then, the LES gives

$$\to H^0(X, \mathcal{O}^\times_{X\times_k k[\varepsilon]}) \to H^0(X, \mathcal{O}_X^\times) \to H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}^\times_{X\times_k k[\varepsilon]}) \to H^1(X, \mathcal{O}_X^\times) \to,$$

which is the same as (because $X$ is smooth and proper)

$$\to H^0(X, \mathcal{O}^\times_{X\times_k k[\varepsilon]}) \to k^\times \to H^1(X, \mathcal{O}_X) \to \mathrm{Pic}(X \times_k k[\varepsilon]) \to \mathrm{Pic}(X) \to .$$

The first map is then necessarily surjective, so we get $H^1(X, \mathcal{O}_X)$ is $\ker(\mathrm{Pic}(X \times_k k[\varepsilon]) \to \mathrm{Pic}(X))$, as desired. ☺

We now introduce the Albanese scheme, which is the universal abelian variety that receives a map from $X$. There is actually a more explicit description, which we show is equivalent. Note that the Poincare bundle induces a map $X \to \underline{\mathrm{Pic}}^0(X)^t$ (note that this is constructed essentially the same way as the double dual map), which essentially maps a point $y \in X(k)$ to $\mathcal{P}_X|_{\{y\}\times\underline{\mathrm{Pic}}^0(X)} \in \mathrm{Pic}^0(\underline{\mathrm{Pic}}^0(X))$.

**Definition 97.** The **Albanese scheme** of $X$ is $\mathrm{Alb}(X) \coloneqq \underline{\mathrm{Pic}}^0(X)^t$ with natural map $X \to \underline{\mathrm{Pic}}^0(X)^t$ the **Albanese map**.

**Proposition 98.** *The Albanese map $a$ as above is indeed the universal map $f\colon X \to B$ (with $B$ an abelian variety) such that $f(x) = e$.*

*Proof.* We have $a(x) = e$ ($\mathcal{O}_{\underline{\mathrm{Pic}}(X)}$) via the universal trivialization. Given a map $X \to B$, we get a pullback map $f^*\colon \underline{\mathrm{Pic}}(B) \to \underline{\mathrm{Pic}}(X)$, which induces a pullback map $f^*\colon \underline{\mathrm{Pic}}^0(B) \to \underline{\mathrm{Pic}}^0(X)$. Taking duals gives $\mathrm{Alb}(X) \to B$. By using the universal property of the dual, it is easy to check that $X \to \mathrm{Alb}(X) \to B$ is the same as the original map $f$ (this is quite similar to the proof that $A \to (A^t)^t \to A^t$ is the same as $A \to A^t$ given by some polarization). ☺

Next, let us define symmetric powers of a smooth projective curve $C$. Fix an integer $r \geq 0$ and denote by $C^r$ the product of $r$ copies of $C$. It has a natural $S_r$-action, and so using the theory of quotients by finite groups, we can define the symmetric power of a curve.

**Theorem 99.** *Let $\mathrm{Sym}^r(C) \coloneqq C^r/S_r$. This is a smooth projective variety/$k$ and its $k$-points can be identified with the effective divisors of degree $r$, denoted by $\mathrm{Div}_{\mathrm{eff}}(C)_r$.*

*Proof.* The idea is that we take $S_r$-invariants on the level of affines, then glue them together. Then, $\mathrm{Sym}^r(C)$ exists as a $k$-scheme and is normal because taking invariants is integrally closed. Also, $\mathrm{Sym}^r(C)$ is projective (probably some easy argument, but this follows e.g. from GIT).

Consider $x \in C(k)$ and the worse case $y = (x, x, \dots, x) \in C^r(k)$ with image $z \in \mathrm{Sym}^r(C)$. Since $C$ is smooth, we have that $\widehat{\mathcal{O}}_{C,x} \cong k\llbracket t \rrbracket$. Then, we have $\widehat{\mathcal{O}}_{C^r,x} \cong k\llbracket t_1, \dots, t_r \rrbracket$ so that $\widehat{\mathcal{O}}_{\mathrm{Sym}^r(C),z} \cong k\llbracket t_1, \dots, t_r \rrbracket^{S_r}$, which turns out to be a power series ring in the elementary symmetric functions in $r$ variables, i.e. $\mathrm{Sym}^r(C)$ is still smooth at $z$ (we end up getting something like $k\llbracket t_1, \dots, t_{r'} \rrbracket^{S_{r'}} \otimes k\llbracket t_1, \dots, t_{r''} \rrbracket^{S_{r''}} \otimes \cdots$ for a general point, and something similar works).

Finally, note that we have $\mathrm{Sym}^r(C)(k) = C^r(k)/S_r$, and hence we have a map $C^r(k) \to \mathrm{Div}_{\mathrm{eff}}(C)_r$ sending $(x_1, \dots, x_r) \mapsto \sum [x_i]$, which clearly descends to $\mathrm{Sym}^r(C)(k)$. ☺

*Remark* 100. It might be surprsing that this is smooth—this is special to the case of a curve (c.f. https://www.jmilne.org/math/xnotes/JVs.pdf).

*Remark* 101. We can say something a bit stronger. In general, for any $k$-scheme $T$, the $T$-points of $\mathrm{Sym}^r(C)$ are given by relative effective Cartier divisors on $C \times_k T$ of degree $r$. These are effective Cartier divisors $Z$ of degree $r$ on $C \times_k T$ such that the induced map $Z \to T$ is finite flat and of degree $d$. This is equivalent (c.f. Milne again) to each fiber of $Z \to T$ being a degree $r$ divisor on the corresponding fiber in $C \times_k T \to T$.

The next goal is to relate the symmetric powers with the Picard scheme. Consider the map $C^r \to \underline{\mathrm{Pic}}(C)$ given on $k$-points by sending $(x_1, \dots, x_r) \mapsto \mathcal{O}_C([x_1] + \cdots + [x_r])$. Since this map is $S_r$-invariant, it descends to a map $\mathrm{Sym}^r(C) \to \underline{\mathrm{Pic}}(C)$, which we denote by $\sigma_r$.

Also, consider the map $\deg\colon \underline{\mathrm{Pic}}(C) \to \mathbb{Z}$, where we view $\mathbb{Z}$ as a discrete scheme comprising $\mathbb{Z}$-many copies of $k$. Let $\underline{\mathrm{Pic}}(C)_r \coloneqq \deg^{-1}(r)$, which is a connected component of $\underline{\mathrm{Pic}}(C)$. Since $\sigma_r$ lands in $\underline{\mathrm{Pic}}(C)_r$, we re-define $\sigma_r$ to be a map landing in $\underline{\mathrm{Pic}}(C)_r$.

Next, we define the Jacobian of $C$.

**Definition 102.** The **Jacobian** of $C$ is $\underline{\mathrm{Pic}}^0(C)$.

**Theorem 103.** *Let* $r \geq 0$.

*(i) Each fiber of* $\sigma_r$ *is a projective space (possibly empty).*

*(ii) For* $r > g - 1$, $\sigma_r$ *is surjective.*

*(iii) For* $r > 2g - 2$, $\sigma_r$ *is a projective bundle.*

*(iv) For* $r \leq g$, $\sigma_r$ *is birational onto its image* $W^r \coloneqq \sigma_r(\mathrm{Sym}^r(C))$.

*(v)* $\underline{\mathrm{Pic}}(C)_0$ *is connected and* $\mathrm{Jac}(C) = \underline{\mathrm{Pic}}(C)_0$.

*Proof.* For (i), note that for $L \in \underline{\mathrm{Pic}}(C)_r$, we have $\sigma_r^{-1}(L) = \mathbb{P}(H^0(C, L))$ since it comprises all effective divisors $D$ on $C$ such that $\mathcal{O}_C(D) \cong L$ (i.e. the complete linear system of $L$).

For (ii), it suffices to show $H^0(C, L) \neq 0$. By Riemann-Roch, note that $\dim H^0(C, L) \geq \chi(C, L) = r + 1 - g > 0$, as desired.

For (iii), we want to construct a vector bundle $\mathcal{E}$ on $\underline{\mathrm{Pic}}(C)_r$ such that $\mathrm{Sym}^r(C) \cong \mathbb{P}(\mathcal{E})$ over $\underline{\mathrm{Pic}}(C)_r$. As a candidate, let us take $\mathrm{pr}_{2,*}\mathcal{P}_C$, where $\mathcal{P}_C$ is the Poincare bundle on $C \times \underline{\mathrm{Pic}}(C)_r$ (i.e. $\mathcal{P}_C|_{C\times\{L\}} \cong L$).

By cohomology and base change (c.f. Hartshorne or Milne or what we did earlier), it's clear that $\mathcal{E}$ is a vector bundle because $H^1(C, L) = H^0(C, K_C \otimes L^{-1})^\vee = 0$. Note that the fiber above $L$ in $\mathbb{P}(\mathcal{E}) \to \underline{\mathrm{Pic}}(C)_r$ is given by $\mathbb{P}(H^0(C, L))$, since the stalk of $\mathcal{E}$ at $L$ (by flat base change) is $\mathrm{pr}_{2,*}(\mathcal{P}|_{C\times\{L\}}) = \mathrm{pr}_{2,*}(L) = H^0(C, L)$.

We can construct a map $\mathrm{Sym}^r(C) \to \mathbb{P}(\mathcal{E})$ by descending one from $C^r \to \mathbb{P}(\mathcal{E})$ (locally, this will look like $\sigma_r^{-1}(U) \to U \times_k \mathbb{P}(H^0(C, L))$ sending a collection of points to the corresponding effective divisor in the complete linear system). On fibers we have an isomorphism.

For (iv), first observe that $W^r$ is irreducible (image of an irreducible). We want to show $\mathrm{Sym}^r(C) \to W^r$ is birational, and to do so it suffices to find a point in the target whose fiber is scheme-theoretically a single point (the fiber is a projective space so the fiber is reduced).

In other words, we want to find a degree $r$ line bundle $L$ on $C$ such that $\dim H^0(C, L) = 1$. This follows from the general fact that for $0 \leq r \leq g$, there is a non-empty open $U \subset C^r$ such that $\dim H^0(C, \mathcal{O}_C(\sum[x_i])) = 1$ for any $(x_1, \ldots, x_r) \in U$ (c.f. Milne JVs, Lemma 5.2).

Finally, for (v), showing that $\underline{\mathrm{Pic}}(C)_0$ is connected is the same as showing that $\underline{\mathrm{Pic}}(C)_r$ is connected for some big $r$ by just using the group structure. We know that for $r$ sufficiently large that $\sigma_r$ is surjective, and since the image of a connected is still connected, the result follows. ☺

Now, we'd like to relate the Albanese and the Jacobian—it turns out they are dual to each other!

First, let us define the infinite symmetric product of $C$. Note that for $r, s \geq 0$, we have a clear map $C^r \times C^s \to C^{r+s}$ given by concatenation. Moreover, $S_r \times S_s$ acts on the left and there is a natural map $S_r \times S_s \to S_{r+s}$ so that we get a map $\mathrm{Sym}^r(C) \times \mathrm{Sym}^s(C) \to \mathrm{Sym}^{r+s}(C)$. This basically just takes a degree $r$ divisor, a degree $s$ divisor, and adds them together to get a degree $r + s$ divisor.

**Definition 104.** The **infinite symmetric product** of $C$ is $\mathrm{Sym}(C) \coloneqq \bigsqcup_{r\geq 0} \mathrm{Sym}^r(C)$. It has a commutative monoid structure via the maps above.

Note there is an obvious map $C \cong \mathrm{Sym}^1(C) \subset \mathrm{Sym}(C)$.

**Lemma 105.** *This obvious map is universal among all maps from $C$ to a commutative monoid scheme.*

*Proof.* Let $f: C \to M$ be any map from $C$ into a commutative monoid scheme $M$ and consider $C^r \to M^r \to M$, where the second map is given by summing. This is clearly $S_r$-equivariant, so it descends to $f_r: \mathrm{Sym}^r(C) \to M$. Combining these together gives a map $F: \mathrm{Sym}(C) \to M$, and it's easy to see that $C \to \mathrm{Sym}(C) \to M$ is the same as $f$. ☺

Let $\sigma$ be the map bundling up all the $\sigma_r$'s from earlier:

$$\sigma \coloneqq \mathrm{Sym}(C) = \bigsqcup_{r\geq 0} \mathrm{Sym}^r(C) \to \mathrm{Pic}(C) = \bigsqcup_{r\in\mathbb{Z}} \underline{\mathrm{Pic}}(C)_r.$$

**Lemma 106.** *The map $C \mapsto \underline{\mathrm{Pic}}(C)$ sending $x \mapsto \mathcal{O}_C([x])$ is the universal map from $C$ to a commutative group scheme (that is locally finitely presented and with the identity component being an abelian variety—this isn't strictly necessary).*

*Proof.* Again, let $f: C \to G$ be any such map. Then, the previous lemma implies that there exists some $F: \mathrm{Sym}(C) \to G$ (since $G$, in particular, is a commutative monoid scheme). Restricting to degree greater than $2g-2$ gives

$$F_{>2g-2}: \bigsqcup_{r>2g-2} \mathrm{Sym}^r(C) \to G$$

and similarly

$$\bigsqcup_{r>2g-2} \sigma_r: \bigsqcup_{r>2g-2} \mathrm{Sym}^r(C) \to \bigsqcup_{r>2g-2} \underline{\mathrm{Pic}}(C)_r.$$

Using the next remark, we get a factorization $\bigsqcup_{r>2g-2} \mathrm{Sym}^r(C) \to \bigsqcup_{r>2g-2} \underline{\mathrm{Pic}}(C)_r \to G$ (same as $\bigsqcup_{r>2g-2} \mathrm{Sym}^r(C) \to G$), as desired.

Using the group law we can extend this to all $r$. ☺

*Remark* 107. Suppose $f: X \to Y$ is a proper surjective morphism of normal varieties with fibers covered by rational curves and that $g: X \to A$ is any morphism to an abelian variety. Then, $g$ factors through $f$. If $U$ is an affine open of $A$, then $g^{-1}(U)$ is an open subset of $X$, and using that $g$ is constant on the fibers of $f$, we have $f^{-1}(V) = g^{-1}(U)$ for some $V \subset Y$. Note that $f(X \backslash f^{-1}(V)) = Y \backslash V$ by surjectivity, and since $f$ is closed, it follows that $V$ is open. Let $V' \subset V$ be some open affine. Then, $f^{-1}(V') \subset g^{-1}(U) \to U$ with $U$ affine, so we have a factorization $f^{-1}(V') \to \operatorname{Spec}\Gamma\left(\left(f|_{f^{-1}(V')}\right)_* \mathcal{O}_X\right) \to U$. By Stein factorization (c.f. Tag 0AY8), we know $\left(f|_{f^{-1}(V')}\right)_* \mathcal{O}_X = \mathcal{O}_{V'}$, so we have $f^{-1}(V') \to V' \to U$. Patching these together gives our factorization of $f$ through $g$.

**Corollary 108.** *Fix a point $P \in C(k)$. Let $\phi: C \to \mathrm{Jac}(C)$ be the* ***Abel-Jacobi map****, which sends $x \mapsto \mathcal{O}_C([x] - [P])$.*

*Then, $\phi$ is also the Albanese map.*

*Proof.* Let $f: C \to A$ be any map to an abelian variety $A$ with $f(P) = e$. The goal is to show that $f$ factors over $\phi$.

First, by the previous lemma, we know that $f$ factors through $\underline{\mathrm{Pic}}(C)$ uniquely so that we have a map $F: \underline{\mathrm{Pic}}(C) \to A$ (which is a map of commutative monoids, as before). More precisely, we have $F(\mathcal{O}_C([x])) = f(x)$. Let $F_0$ be the map $\underline{\mathrm{Pic}}^0(C) \to \underline{\mathrm{Pic}}(C) \to A$.

Now, consider the following:

$$\begin{aligned} f(x) &= F(\mathcal{O}_C([x])) \\ &= F(\mathcal{O}_C([x] - [P]) \otimes \mathcal{O}_C([P])) \\ &= F_0(\mathcal{O}_c([x] - [P])) + F(\mathcal{O}_C([P])) \\ &= F_0(\phi(x)) + e \\ &= (F_0 \circ \phi)(x). \end{aligned}$$

So $f$ as factors as $F_0 \circ \phi$, and it's easy to check that the factorization is also unique. ☺

As a result (after fixing some $P$), we have $\mathrm{Jac}(C) \cong \mathrm{Alb}(C) = \mathrm{Jac}(C)^t$.

**Theorem 109** (Torelli)**.** *$\mathrm{Jac}(C)$ is equipped with a principal polarization, and this pair determines the curve $C$ up to isomorphism.*

Recall from before that for an ample line bundle $L$, we have $\phi_{A^t}(L)$ is a vector bundle; denote it by $E(L)$. Also, let $D_X(-)$ be the Verdier dual functor $\underline{\mathrm{Hom}}(-, \omega_X[\dim X])$ with $X$ proper smooth over $k$ of dimension $\dim X$.

**Theorem 110** (Hacon)**.** *Let $A$ be an abelian variety/k. If $F \in D^b_{\mathrm{coh}}(A)$, then TFAE:*

*(i) For $L \in \mathrm{Pic}(A^t)$ sufficiently ample, we have $H^i(A, F \otimes E(L)^\vee) = 0$ for $i \neq 0$.*

*(ii) For $L \in \mathrm{Pic}(A^t)$ sufficiently ample, we have $H^i(A^t, \phi_A(D_A(F)) \otimes L) = 0$ for $i \neq 0$.*

*(iii) $\phi_A(D_A(F))$ is concentrated in degree zero.*

Such $F$ are called Hacon complexes (and is a Hacon sheaf is in degree zero), and they play a key role in a proof of Green-Lazarsfeld generic vanishing:

**Theorem 111** (Green-Lazarsfeld)**.** *For $L \in \underline{\mathrm{Pic}}^0(X)$ "general" and $i < \dim(a(X))$ (recall $a$ is the Albanese map), then $H^i(X, L) = 0$ and $H^{\dim X - i}(X, \omega_X \otimes L) = 0$.*

More precisely, we can say something precise about the codimensions of the loci of where the cohomology doesn't vanish.

# 9 04/28: Weil conjectures for abelian varieties