FINITE HILBERT STABILITY OF (BI)CANONICAL CURVES

JAROD ALPER, MAKSYM FEDORCHUK, AND DAVID ISHII SMYTH*

Abstract. We prove that a generic smooth curve of odd genus, canonically or bicanonically embedded, has a semistable m^{th} Hilbert point for all $m \geq 2$. This is accomplished by proving finite Hilbert semistability of special singular curves with $\mathbb{G}_m$-action, a nonreduced canonically embedded curve, called the balanced ribbon, and a bicanonically embedded tacnodal curve, called the rosary. Finally, we give examples of canonically embedded curves whose m^{th} Hilbert points have the property that they are non-semistable for low m, but become semistable past a definite threshold value of m.

Contents

1. Introduction

Geometric Invariant Theory (GIT) was developed by Mumford in order to construct quotients in algebraic geometry, and in particular to construct moduli spaces. To use GIT to construct a moduli space one must typically prove that a certain class of embedded varieties has stable or semistable Hilbert points. The prototypical example of a stability result is Gieseker and Mumford's asymptotic stability theorem for pluricanonically embedded curves [Mum77, Gie82, Gie83]:

Theorem 1.1 (Asymptotic Stability). *Suppose $C \subset \mathbb{P}\mathrm{H}^0(C, K_C^n)$ is a smooth curve, embedded by the complete linear system $|K_C^n|$ for some $n \geq 1$. Then the m^{th} Hilbert point of C is stable for all $m \gg 0$.*

Gieseker and Mumford's arguments are non-effective, and there is no known bound on how large m must be in order to obtain the conclusion of the theorem. In light of this theorem, it is natural to ask: for which finite values of m do pluricanonically embedded smooth curves have stable or semistable Hilbert points? This has been a basic open problem

*The third author was partially supported by NSF grant DMS-0901095 during the preparation of this work.

in GIT since the pioneering work of Gieseker and Mumford, but has gained renewed interest from recent work of Hassett and Hyeon on the log minimal model program for $\overline{M}_g$. Indeed, Hassett and Hyeon observed that a stability result for finite Hilbert points of canonically and bicanonically embedded smooth curves would enable one to use GIT to construct a sequence of new projective birational models of M_g that would constitute steps of the log minimal model program for $\overline{M}_g$ [HH08]. In this paper, we prove the requisite stability result, at least in the case of odd genus.

Theorem 1.2 (Main Result)**.** *Suppose $C \subset \mathbb{P}\mathrm{H}^0(C, K_C^n)$ is a general smooth curve of odd genus, embedded by the complete linear system $|K_C^n|$, where $n = 1, 2$. Then the m^{th} Hilbert point of C is semistable for every $m \geq 2$.*

The main result is proved in Corollaries 4.2 and 4.11. This is, to our knowledge, the first example of a result in which the semistability of *all* Hilbert points of a given variety is established by a uniform method. In the case of canonically and bicanonically embedded curves of odd genus, we recover a weak form of the asymptotic stability theorem by a much simpler proof. Furthermore, as a sidelight to our main result, we give the first example of an embedded curve for which stability of its m^{th} Hilbert point changes from non-semistable to semistable as m increases (Section 4.3). We will explain our method of proof in the next section. First, however, let us conclude this introduction by describing a fascinating application of the result, anticipated in the work of Hassett and Hyeon [HH08], and by considering prospects for future generalizations.

Fix $g \geq 2$, $n \geq 1$, $m \geq 2$, and set $r = (2n-1)(g-1) - 1$ if $n \geq 2$, and $r = g - 1$ if $n = 1$. To an n-canonically embedded smooth genus g curve C we associate its m^{th} Hilbert point $[C]_m \in \mathbb{P}W_m$; these are defined in more detail in Section 2 below. We denote by $\overline{H}_{g,n}^{m}$ the closure in $\mathbb{P}W_m$ of the locus of m^{th} Hilbert points of n-canonically embedded smooth curves of genus g. Then the $\mathrm{SL}(r+1)$-action on $\overline{H}_{g,n}^{m}$ admits a natural linearization $\mathcal{O}(1)$, which defines an open locus $(\overline{H}_{g,n}^{m})^{ss} \subset \overline{H}_{g,n}^{m}$ of semistable points. *Assuming that $(\overline{H}_{g,n}^{m})^{ss}$ is non-empty*, one obtains a GIT quotient

$$(\overline{H}_{g,n}^{m})^{ss} /\!\!/ \mathrm{SL}(r+1) := \mathrm{Proj} \bigoplus_{k \geq 0} \mathrm{H}^0\big(\overline{H}_{g,n}^{m}, \mathcal{O}(k)\big)^{\mathrm{SL}(r+1)},$$

as a projective variety associated to the algebra of $\mathrm{SL}(r+1)$-invariant functions in the homogenous coordinate ring of $\overline{H}_{g,n}^{m}$.

When $m \gg 0$, the critical assumption $(\overline{H}_{g,n}^{m})^{ss} \neq \varnothing$ is satisfied by Theorem 1.1, and the corresponding quotients have been analyzed using GIT [Gie82, Gie83, Sch91, HH09, HH08, HL10, HM10]. The results of this analysis can be summarized as follows:

$$(\overline{H}_{g,n}^{m})^{ss} /\!\!/ \mathrm{SL}(r+1) \simeq \begin{cases} \overline{M}_g & \text{if } n \geq 5, m \gg 0, \\ \overline{M}_g^{ps} & \text{if } n = 3, 4, m \gg 0, \\ \overline{M}_g^{hs} & \text{if } n = 2, m \gg 0. \end{cases}$$

Here, $\overline{M}_g^{ps}$ is the moduli space of pseudostable curves, in which elliptic tails have been replaced by cusps, and $\overline{M}_g^{hs}$ is the moduli space of h-semistable curves, in which elliptic bridges have been replaced by tacnodes. Furthermore, the birational transformations $\overline{M}_g \to \overline{M}_g^{ps} \dashrightarrow \overline{M}_g^{hs}$ constitute the first two steps of the log minimal model program, namely the first divisorial contraction and the first flip [HH09, HH08].

The key point is that the next stage of the log minimal model program cannot be constructed using an asymptotic stability result. Indeed, an examination of the formula for the divisor class of the polarization on the GIT quotient $(\overline{H}_{g,n}^{m})^{ss} /\!\!/ \mathrm{SL}(r{+}1)$ suggests that the next model occurring in the log minimal model program should be $(\overline{H}_{g,2}^{6})^{ss} /\!\!/ \mathrm{SL}(3g-3)$. Thus, in marked contrast to the cases $n \geq 3$, where finite Hilbert linearizations are not expected to yield new birational models of $\overline{M}_g$, it is widely anticipated that in the cases $n = 1, 2$, there will exist several values of m at which the corresponding GIT quotients are expected to undergo nontrivial birational modifications. In fact, for $n = 1$ we expect the number of threshold values of m to grow with g, while for $n = 2$ the only interesting values are $m \leq 6$, irrespectively of g. (For a detailed analysis of the expected threshold values of m see [FS10] and [AFS10].) Until now, the main obstacle to verifying these expectations has been proving $(\overline{H}_{g,n}^{m})^{ss} \neq \varnothing$ for explicit, finite values of m. Theorem 1.2 removes this obstacle, and thus opens the door to analyzing a whole menagerie of new GIT quotients $(\overline{H}_{g,n}^{m})^{ss} /\!\!/ \mathrm{SL}(r+1)$.

Our main result raises several questions about possible generalizations. First, can the result be extended to cover the cases of even genus and of n-canonically embedded curves for $n \geq 3$? As we shall see in the next section, the essential question is for which n and g does there exist an asymptotically stable curve C with the property that $\mathrm{H}^0(C, nK_C)$ is a multiplicity-free representation of $\mathrm{Aut}(C)$. Given such a curve C, our method should extend to prove stability of finite Hilbert points for the given values of n and g. Unfortunately, we do not have a complete classification of the values of n and g for which such curves exist. Our methods, however, can be extended at least to prove generic semistability of finite Hilbert points for canonical curves of even genus. This will the subject of a sequel paper. The second question raised by this work is whether one can prove stability of finite Hilbert points of smooth curves rather than simply semistability? Evidently, our method is itself insufficient to address this problem, but it is important to observe that the problem can now be approached indirectly. It is, in general, much easier to show that a given Hilbert point is not semistable than to show that it is stable. Thus, once semistability is known, a natural strategy for proving stability of the m^{th} Hilbert point of a general smooth curve is simply to show that the m^{th} Hilbert point of any nontrivial isotrivial degeneration of a general smooth curve is not semistable. We plan to carry out this analysis in future work.

Acknowledgements. We learned about the problem of GIT stability of finite Hilbert points many years ago from Brendan Hassett's talks on the log minimal model program for $\overline{M}_g$. Over the past several years we learned about many aspects of GIT from conversations with Ian Morrison and David Hyeon, as well as through their many papers on the topic. In addition, we gained a great deal from conversations with Aise Johan de Jong, Anand Deopurkar, David Jensen, and David Swinarski.

2. GIT background

The proof of our main result is surprisingly simple. In the canonical (resp., bicanonical) case, we exhibit a singular curve C such that the action of $\mathrm{Aut}(C)$ on $V = \mathrm{H}^0(C, \omega_C)$ (resp., $V = \mathrm{H}^0(C, \omega_C^2)$) is multiplicity-free, i.e., no representation occurs more than once in the decomposition of V into irreducible $\mathrm{Aut}(C)$-representations. As Ian Morrison observed some thirty years ago, under this hypothesis, powerful results of Kempf imply that the m^{th} Hilbert point of C is semistable if and only if it is semistable with respect to one-parameter subgroups of $\mathrm{SL}(V)$ which act diagonally on a *fixed basis* of V. Verifying stability with respect to the resulting fixed torus of $\mathrm{SL}(V)$ is a discrete combinatorial problem which we solve explicitly for every $m \geq 2$. We thus prove the semistability of all Hilbert points of C and deduce the semistability of a general smooth curve by openness of the semistable locus. In Section 3 we will give a precise description of the (rather exotic) curves C appearing in our argument. In this section, we recall the relevant definitions from GIT and explain the general framework for proving semistability of Hilbert points due to Mumford, as well as the aforementioned refinements of Kempf.

Let us begin by recalling the definition of the m^{th} Hilbert point of an embedded scheme. If $X \subset \mathbb{P}V$ is a closed subscheme satisfying $h^1(X, \mathcal{I}_X(m)) = 0$, set

$$W_m := \left(\bigwedge^{h^0(X,\mathcal{O}_X(m))} \mathrm{H}^0(\mathbb{P}V, \mathcal{O}(m)) \right)^\vee .$$

The m^{th} *Hilbert point of* $X \subset \mathbb{P}V$ is a point $[X]_m \in \mathbb{P}W_m$, defined as follows. First, consider the surjection

$$\mathrm{H}^0(\mathbb{P}V, \mathcal{O}(m)) \to \mathrm{H}^0(X, \mathcal{O}_X(m)) \to 0.$$

Taking the $h^0(X, \mathcal{O}_X(m))$-fold wedge product and dualizing, we obtain the m^{th} Hilbert point:

$$[X]_m := \left[\bigwedge^{h^0(X,\mathcal{O}_X(m))} \mathrm{H}^0(\mathbb{P}V, \mathcal{O}(m)) \to \bigwedge^{h^0(X,\mathcal{O}_X(m))} \mathrm{H}^0(X, \mathcal{O}_X(m)) \to 0 \right]^\vee \in \mathbb{P}(W_m).$$

Recall that if W is any linear representation of $\mathrm{SL}(V)$, a point $x \in \mathbb{P}(W)$ is *semistable* if the origin of W is not contained in the closure of the orbit of a lift $\widetilde{x} \in W$, where $\widetilde{x}$ is any lift of x. Thus, to show that a Hilbert point $[X]_m \in \mathbb{P}(W_m)$ is semistable, we must prove that $0 \in W_m$ is not in the closure of $\mathrm{SL}(V) \cdot \widetilde{[X]}_m$, where $\widetilde{[X]}_m$ is any lift of $[X]_m$. An obvious necessary condition is that for any *one-parameter subgroup* $\rho\colon \operatorname{Spec} \mathbb{C}[t, t^{-1}] \to \mathrm{SL}(V)$, we have $\lim_{t\to 0} \rho(t) \cdot \widetilde{[X]}_m \neq 0$. A foundational theorem of Mumford asserts that this necessary condition is sufficient.

Proposition 2.1 (Hilbert-Mumford Numerical Criterion)**.** *Let $X \subset \mathbb{P}V$ be as above. The Hilbert point $[X]_m$ is semistable if and only if $\lim_{t\to 0} \rho(t) \cdot \widetilde{[X]}_m \neq 0$ for every one-parameter subgroup $\rho\colon \operatorname{Spec} \mathbb{C}[t, t^{-1}] \to \mathrm{SL}(V)$.*

Given a one-parameter subgroup $\rho\colon \operatorname{Spec} \mathbb{C}[t, t^{-1}] \to \mathrm{SL}(V)$, we may reformulate the condition $\lim_{t\to 0} \rho(t) \cdot \widetilde{[X]}_m \neq 0$ as follows. First, we may choose a basis $\{x_i\}_{i=0}^r$ of V which diagonalizes the action of ρ. Then $\rho(t) \cdot x_i = t^{\lambda_i} x_i$ for some integers λ_i satisfying $\sum_{i=0}^r \lambda_i = 0$. We call $\{x_i\}_{i=0}^r$ a *weighted basis*. If we set $N_m := h^0(X, \mathcal{O}_X(m))$, a basis for $W_m = \bigwedge^{N_m} \mathrm{H}^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(m))$ consists of N_m-tuples $e_1 \wedge \cdots \wedge e_{N_m}$ of distinct monomials of degree m in the variables $\{x_i\}_{i=0}^r$. Now the condition that $\lim_{t\to 0} \rho(t) \cdot \widetilde{[X]}_m \neq 0$ is equivalent to the existence of one such coordinate which is non-vanishing on $[X]_m$ and on which ρ acts with non-positive weight. The condition that a coordinate $e_1 \wedge \ldots \wedge e_{N_m}$ is non-zero on $[X]_m$ is precisely the condition that $\{e_i\}_{i=1}^{N_m}$ form a *monomial basis* for $\mathrm{H}^0(X, \mathcal{O}_X(m))$, i.e., that they map onto a basis of $\mathrm{H}^0(X, \mathcal{O}_X(m))$ via $\mathrm{H}^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(m)) \to \mathrm{H}^0(X, \mathcal{O}_X(m))$. Furthermore, if $e_i = \prod_{j=0}^r x_j^{a_{ij}}$, then ρ acts on $e_1 \wedge \ldots \wedge e_{N_m}$ with weight $\sum_{i=1}^{N_m} \sum_{j=0}^r a_{ij}\lambda_j$. Thus, we call $\sum_{i=1}^{N_m} \sum_{j=0}^r a_{ij}\lambda_j$ the *weight* of the monomial basis $\{e_i\}_{i=1}^{N_m}$. With this terminology, we have the following criterion.

Proposition 2.2 (Numerical Criterion for Hilbert points)**.** $[X]_m$ *is semistable if and only if for every weighted basis of* V*, there exists a degree* m *monomial basis for* $\mathrm{H}^0(X, \mathcal{O}_X(m))$ *of non-positive weight.*

The Hilbert-Mumford criterion reduces the problem of proving semistability of $[X]_m$ to a concrete algebro-combinatorial problem concerning the defining equations of $X \subset \mathbb{P}V$. However, this problem is not discretely computable since it requires checking *all* one-parameter subgroups of $\mathrm{SL}(V)$. A theorem of Kempf allows us, under certain hypotheses on $\mathrm{Aut}(X)$, to check only those one-parameter subgroups of $\mathrm{SL}(V)$ which act diagonally on a fixed basis. This reduces the problem to one which is discretely computable.

In order to state the next proposition, let us establish a bit more terminology. Given an embedding $X \subset \mathbb{P}V$ by a complete linear system, there is a natural action of $\mathrm{Aut}(X)$ on $V = \mathrm{H}^0(X, \mathcal{O}_X(1))$. Given a linearly reductive subgroup $G \subset \mathrm{Aut}(X)$, we say that V is a *multiplicity-free* G-representation (or simply *multiplicity-free* if G is understood) if it contains no irreducible G-representation more than once in its decomposition into irreducible G-representations. We say that a basis of V, say $\{x_i\}_{i=0}^r$, is *compatible with the irreducible decomposition of* V if each irreducible G-representation in V is spanned by a subset of the x_i's. We may now state the reformulation of Kempf's results that we will use. We keep the assumption that X is embedded by a complete linear system $|\mathcal{O}_X(1)|$ and that $h^1(X, \mathcal{I}_X(m)) = 0$.

Proposition 2.3 (Kempf-Morrison Criterion)**.** *Suppose* $G \subset \mathrm{Aut}(X)$ *is a linearly reductive subgroup, and that* $V = \mathrm{H}^0(X, \mathcal{O}_X(1))$ *is a multiplicity-free representation of* G*. Let* $\{x_i\}_{i=0}^r$ *be a basis of* V *which is compatible with the irreducible decomposition of* V*. Then* $[X]_m$ *is semistable if and only if for every one-parameter subgroup* $\rho\colon \operatorname{Spec}\mathbb{C}[t, t^{-1}] \to \mathrm{SL}(V)$ *acting diagonally on* $\{x_i\}_{i=0}^r$*, we have* $\lim_{t\to 0} \rho(t) \cdot \widetilde{[X]_m} \neq 0$*. Equivalently, for every weighted basis* $\{x_i\}_{i=0}^r$ *of* V*, there exists a monomial basis of* $\mathrm{H}^0(X, \mathcal{O}_X(m))$ *of non-positive weight.*

Proof. If $[X_m]$ is not semistable, then [Kem78, Theorem 3.4 and Corollary 3.5] implies that there is a one-parameter subgroup $\rho_*\colon \operatorname{Spec}\mathbb{C}[t, t^{-1}] \to \mathrm{SL}(V)$ with $\lim_{t\to 0} \rho_*(t) \cdot \widetilde{[X]_m} = 0$ such that the parabolic subgroup $P \subseteq \mathrm{SL}(V)$ associated to the ρ_*-weight filtration

$$0 = U_0 \subseteq U_1 \subseteq \cdots \subseteq U_{k-1} \subseteq U_k = V$$

contains $\mathrm{Aut}(X)$. Let $V = \bigoplus_j V_j$ be the decomposition into irreducible G-representations. Since V is multiplicity-free, each U_i can be written as a direct sum of some of the V_j's. The maximal torus $T \subset \mathrm{SL}(V)$ associated to the basis $\{x_i\}_{i=0}^r$ fixes each V_j and thus the filtration. Therefore, $T \subset P$. By [Kem78, Theorem 3.4 (c)(4)], there exists a one-parameter subgroup $\rho\colon \operatorname{Spec}\mathbb{C}[t, t^{-1}] \to T$ such that $\lim_{t\to 0} \rho(t) \cdot \widetilde{[X]_m} = 0$. □

For the sake of concreteness, let us reiterate the Kempf-Morrison criterion in the case of a canonically (resp., bicanonically) embedded curve $C \subset \mathbb{P}^r$. In order to prove that $[C]_m$ is semistable, we must first check that $V = \mathrm{H}^0(C, K_C)$ (resp., $V = \mathrm{H}^0(C, K_C^2)$) is a multiplicity-free representation of some linearly reductive $G \subset \mathrm{Aut}(C)$. Second, we fix a basis $\{x_i\}_{i=0}^r$ of V compatible with the irreducible decomposition of V. Now any one-parameter subgroup ρ acting diagonally on $\{x_i\}_{i=0}^r$ is given by an integer weight vector $(\lambda_0, \dots, \lambda_r)$ satisfying $\sum_{i=0}^r \lambda_i = 0$. To show that $[C]_m$ is semistable with respect to ρ, we must find a set of $N_m = (2m-1)(g-1)$ (resp., $N_m = (4m-1)(g-1)$) monomials, say $\{e_i\}_{i=1}^{N_m}$, where $e_i = \prod_{j=1}^r x_j^{a_{ij}}$, such that

- $\{e_i\}_{i=1}^{N_m}$ form a monomial basis of $\mathrm{H}^0(C, \mathcal{O}_C(m))$.
- The total weight of this basis, namely $\sum_{i=1}^{N_m} \sum_{j=0}^{r} a_{ij}\lambda_j$, is non-positive.

Note that the total weight of any monomial basis is a linear function of $(\lambda_0, \dots, \lambda_r)$. Therefore, each monomial basis determines a half-space of weight vectors for which the $[C]_m$ is ρ-semistable, namely the half-space $\sum_{i=1}^{N_m} \sum_{j=0}^{r} a_{ij}\lambda_j \leq 0$. As soon as one produces sufficiently many monomial bases such that the union of these half-spaces contains all weight vectors $(\lambda_0, \dots, \lambda_r)$, the proof of semistability for $[C]_m$ is completed.

The idea of applying these results of Kempf to the semistability of finite Hilbert points of curves is due to Morrison and Swinarski [MS11]. In their paper, they consider the so-called hyperelliptic Wiman curve C defined by $y^2 = x^{2g+1} - 1$ and embedded bicanonically. They check that the automorphism group, which is cyclic of order $4g+2$, acts on $\mathrm{H}^0(C, K_C^2)$ with $3g-3$ distinct characters. They fix a basis $\mathrm{H}^0(C, K_C^2) = \{x_0, \dots, x_r\}$ compatible with the decomposition of $\mathrm{H}^0(C, K_C^2)$ into characters, and then, for low values of g and m, use a computer to enumerate monomial bases of $\mathrm{H}^0(C, \mathcal{O}_C(m))$ until the associated half-spaces cover the $(\lambda_0, \dots, \lambda_r)$-plane. As we shall see in the next section, however, a judicious choice of a *singular* curve C satisfying the hypotheses of Proposition 2.3 allows one to take advantage of a much richer symmetry group and to write down *by hand* sufficiently many monomial bases for all values of m.

3. Ribbons and Rosaries

As discussed in the previous section, the key to our proof is to find a singular Gorenstein curve C such that $\mathrm{H}^0(C, \omega_C)$ (resp., $\mathrm{H}^0(C, \omega_C^2)$) is a multiplicity-free representation of $\mathrm{Aut}(C)$ in the canonical case (resp., bicanonical case). In this section, we describe the curves we will use. In the canonical case, we will use a certain ribbon with $\mathbb{G}_m$-action, the so-called *balanced ribbon*. In the bicanonical case, we will use the so-called *rosary*, i.e., a cycle of $\mathbb{P}^1$'s attached by tacnodes, introduced by Hassett and Hyeon in their classification of asymptotically stable bicanonical curves [HH08].

A word of motivation as to where on earth these curves come from may be useful. That some class of canonically embedded ribbons should be GIT-semistable is intuitively plausible, since ribbons arise as the scheme-theoretic limit of a family of canonically embedded smooth curves degenerating abstractly to a hyperelliptic curve. The fact that the balanced ribbon of odd genus is the only ribbon with $\mathbb{G}_m$-action that has the potential to be asymptotically Hilbert semistable was proved in [AFS10, Theorem 7.2]. Hence, it was natural to attempt to prove that this curve is, in fact, semistable. In the bicanonical case, we made use of the classification of asymptotically semistable curves in [HH08]. We simply looked through the curves on their list for one with a large enough symmetry group to satisfy the hypotheses of Proposition 2.3. The rosary was the first curve we checked, and it worked!

3.1. **Canonical case: The balanced ribbon with $\mathbb{G}_m$-action.** In this section we will construct, for every odd genus, a special non-reduced curve C of arithmetic genus g whose canonical embedding satisfies the hypotheses of Proposition 2.3. Given a positive odd integer $g = 2k+1$, set $U := \operatorname{Spec} \mathbb{C}[u, \epsilon]/(\epsilon^2)$, $V := \operatorname{Spec} \mathbb{C}[v, \eta]/(\eta^2)$, and identify $U - \{0\}$ and $V - \{0\}$ via the isomorphism

$$u \mapsto v^{-1} + v^{-k-2}\eta,$$
$$\epsilon \mapsto v^{-g-1}\eta.$$

The resulting scheme C is evidently a complete, locally planar curve of arithmetic genus g; see [BE95] for more details on such curves. Note that C admits $\mathbb{G}_m$-action by the formulas

$$\begin{aligned} t \cdot u &= tu, \\ t \cdot v &= t^{-1}v, \\ t \cdot \epsilon &= t^{k+1}\epsilon, \\ t \cdot \eta &= t^{-k-1}\eta. \end{aligned}$$

Since C is locally planar, it is Gorenstein and its dualizing sheaf ω_C is a line bundle. Using adjunction, we may identify global sections of ω_C with regular functions $f(u,\epsilon)$ on U. To be precise, the global sections of ω_C consist of all differentials of the form

$$f(u,\epsilon)\frac{du \wedge d\epsilon}{\epsilon^2}$$

which transform to differentials $h(v,\eta)\frac{dv\wedge d\eta}{\eta^2}$ with $h(v,\eta)$ regular on V. One easily writes down a basis of g functions satisfying this condition to obtain the following lemma.

Lemma 3.1. *A basis for* $\mathrm{H}^0(C,\omega_C)$ *is given by differentials* $f(u,\epsilon)\frac{du\wedge d\epsilon}{\epsilon^2}$ *where* $f(u,\epsilon)$ *runs over the following list of* g *functions:*

$$\begin{aligned} x_0 &:= 1, & y_{k+1} &:= u^{k+1} + \epsilon, \\ x_1 &:= u, & y_{k+2} &:= u^{k+2} + 2u\epsilon, \\ &\ \vdots & &\ \vdots \\ x_{k-1} &:= u^{k-1}, & y_{2k} &:= u^{2k} + ku^{k-1}\epsilon. \\ x_k &:= u^k, \end{aligned}$$

Lemma 3.2. ω_C *is very ample.*

Proof. Using the basis of $\mathrm{H}^0(C,\omega_C)$ from Lemma 3.1, we see that $|\omega_C|$ separates points of $C_{\text{red}} \simeq \mathbb{P}^1$ and defines a closed embedding when restricted to U and V. Thus ω_C is very ample. □

Proposition 3.3. $\mathrm{H}^0(C,\omega_C)$ *is a multiplicity-free representation of* $\mathbb{G}_m \subset \mathrm{Aut}(C)$.

Proof. The basis of $\mathrm{H}^0(C,\omega_C)$ from Lemma 3.1 diagonalizes the action of $\mathbb{G}_m$ with distinct weights; namely, $(-k,\dots,-1,0,1,\dots,k)$. □

In order to apply Proposition 2.3, we will need an effective way of determining when a set of monomials in the g variables $\{x_0,\dots,x_k,y_{k+1},\dots,y_{2k}\}$ forms a monomial basis for $\mathrm{H}^0(C,\omega_C^m)$. To do this, observe that the global sections of ω_C^m are easily identified with regular functions on U via $f(u,\epsilon) \mapsto f(u,\epsilon)\dfrac{(du\wedge d\epsilon)^m}{\epsilon^{2m}}$. The following proposition determines a basis for $\mathrm{H}^0(C,\omega_C^m)$ under this identification.

Proposition 3.4. *For* $m \geq 2$*, the product map* $\mathrm{Sym}^m \mathrm{H}^0(C,\omega_C) \to \mathrm{H}^0(C,\omega_C^m)$ *is surjective. A basis for* $\mathrm{H}^0(C,\omega_C^m)$ *is given by differentials* $f(u,\epsilon)\frac{(du\wedge d\epsilon)^m}{\epsilon^{2m}}$ *where* $f(u,\epsilon)$ *runs over the following* $(2m-1)(g-1)$ *functions on* U*:*

$$\{u^i\}_{i=0}^{2mk-(k+1)}, \quad \{u^i + (i-k)u^{i-k-1}\epsilon\}_{i=k+1}^{2mk}.$$

Proof. We will show that the image of the product map $\phi\colon \operatorname{Sym}^m \mathrm{H}^0(C,\omega_C) \to \mathrm{H}^0(C,\omega_C^m)$ contains the given functions. Since $h^0(C,\omega_C^m) = (2m-1)(g-1)$ by Riemann-Roch, and because the given functions are linearly independent, this will prove the proposition.

Observe that the expansion in u and ϵ of the monomial $x_{i_1}\dots x_{i_\ell}y_{i_{\ell+1}}\dots y_{i_m}$ is simply $u^a + (a-b)\epsilon u^{a-k-1}$, where

$$a = i_1 + \cdots + i_m,$$
$$b = i_1 + \cdots + i_\ell + k(m-\ell).$$

In u-degrees $i = 0,\dots,k$, we obtain u^i as the monomials $x_0^m, x_0^{m-1}x_1,\dots,x_0^m x_k$. For u-degrees $i = (2m-1)k,\dots,2mk$, we obtain functions $u^i + (i-k)u^{i-k-1}\epsilon$ as the monomials $y_{2k}^{m-1}y_k, y_{2k}^{m-1}y_{k+1},\dots,y_{2k}^m$. For the intermediate u-degrees, simply note that since the dimension of the space of functions $\{cu^i + du^{i-k-1}\epsilon \,:\, c,d\in\mathbb{C}\}$ is two, we simply need to exhibit two linearly independent functions of this form as degree m monomials in $\{x_0,\dots,y_{2k}\}$. Using the above product formula, this is an easy exercise which we leave to the reader. □

This result gives a very simple way of checking whether a set $\mathcal{B}$ of degree m monomials in $\{x_0,\dots,y_{2k}\}$ projects to a basis for $\mathrm{H}^0(C,\mathcal{O}_C(m))$. If we simply view the monomials in $\mathcal{B}$ as polynomials in $\mathbb{C}[u,\epsilon]/(\epsilon^2)$ via the identification preceding Lemma 3.1, then $\mathcal{B}$ is a monomial basis for $\mathrm{H}^0(C,\mathcal{O}_C(m))$ if and only if

(1) $\mathcal{B}$ contains one polynomial of u-degree $0,\dots,k$,
(2) $\mathcal{B}$ contains two linearly independent polynomials of u-degree $k+1,\dots,(2m-1)k-1$,
(3) $\mathcal{B}$ contains one polynomial of u-degree $2mk-k,\dots,2mk$.

We can rephrase this as follows.

Lemma 3.5. *A set of degree m monomials*

$$\{x_{i_1}x_{i_2}\dots,x_{i_\ell}y_{i_{\ell+1}},\dots,y_{i_{m-1}}y_{i_m}\}_{(i_1,\dots,i_m)\in S}$$

forms a monomial basis of $\mathrm{H}^0(C,\mathcal{O}_C(m))$ if and only if the following two conditions hold:

(1) *For $1 \le n \le k$ and $(2m-1)k \le n \le 2mk$, there is exactly one index vector $(i_1,\dots,i_m)\in S$ with $i_1+\ldots+i_m = n$.*
(2) *For $k < n < (2m-1)k$, there are exactly two index vectors $(i_1,\dots,i_m)\in S$ satisfying $i_1+\cdots+i_m = n$. Furthermore, for these two index vectors, the associated integers $i_{\ell+1}+\cdots+i_m - k(m-\ell)$ are distinct.*

Proof. Immediate from the preceding observations and the product formula

$$x_{i_1}\dots x_{i_\ell}y_{i_{\ell+1}}\dots y_{i_m} = u^a + (a-b)\epsilon u^{a-k-1},$$

where $a = i_1+\cdots+i_m$ and b=$i_1+\cdots+i_\ell+k(m-\ell)$. □

3.2. **Bicanonical case: The rosary with $\mathbb{G}_m$-action.** In this section we will construct, in every odd genus, a singular curve C whose bicanonical embedding satisfies the hypotheses of Proposition 2.3. For any odd integer g, we define C to be the curve obtained by gluing a ring of $(g-1)$ $\mathbb{P}^1$'s along $(g-1)$ tacnodes. In the terminology of Hassett and Hyeon [HH08, Section 8.1], C is a *rosary*. We label uniformizers at 0 (resp., ∞) of each $\mathbb{P}^1$ by $s_0,\dots,s_{g-2}$ (resp., $t_0,\dots,t_{g-2}$); in particular $s_i = 1/t_i$. The tacnodes are specified by the identifications $t_i \mapsto s_{i+1}$ (with the convention that $s_{i+g-1} = s_i$). Note that $\mathbb{G}_m \rtimes D_{g-1} \subset \operatorname{Aut}(C)$, where the dihedral group D_{g-1} permutes the components and $\mathbb{G}_m$ acts by $s_i \mapsto t^{(-1)^i}s_i$. We should remark that in the case of even genus, one may still define the curve C, but C does not admit a $\mathbb{G}_m$-action and does not satisfy the hypotheses of Proposition 2.3. Thus, in what follows, we always assume g odd.

Lemma 3.6. *(a) A basis for* $\mathrm{H}^0(C, \omega_C)$ *is given by the following differentials:*

$$\begin{aligned} \omega_0 &= \left(ds_0, \frac{ds_1}{s_1^2}, 0, \dots, 0 \right), \\ \omega_1 &= \left(0, ds_1, \frac{ds_2}{s_2^2}, 0 \dots, 0 \right), \\ &\vdots \\ \omega_{g-2} &= \left(\frac{ds_0}{s_0^2}, 0, \dots, 0, ds_{g-2} \right), \\ \eta &= \left(\frac{ds_0}{s_0}, \frac{ds_1}{s_1}, \dots, \frac{ds_{g-2}}{s_{g-2}} \right). \end{aligned}$$

(b) A basis for $\mathrm{H}^0(C, \omega_C^2)$ *is given by the following differentials:*

$$\begin{aligned} x_0 &= \left((ds_0)^2, \frac{(ds_1)^2}{s_1^4}, 0, \dots \right) & y_0 &= \left(\frac{(ds_0)^2}{s_0}, \frac{(ds_1)^2}{s_1^3}, 0, \dots \right) & z_0 &= \left(\frac{(ds_0)^2}{s_0^2}, 0, \dots \right) \\ x_1 &= \left(0, (ds_1)^2, \frac{(ds_2)^2}{s_2^4}, 0, \dots \right) & y_1 &= \left(0, \frac{(ds_1)^2}{s_1}, \frac{(ds_2)^2}{s_2^3}, 0, \dots \right) & z_1 &= \left(0, \frac{(ds_1)^2}{s_1^2}, 0, \dots \right) \\ &\vdots & &\vdots & &\vdots \\ x_{g-2} &= \left(\frac{(ds_0)^2}{s_0^4}, 0, \dots, 0, (ds_{g-2})^2 \right) & y_{g-2} &= \left(\frac{(ds_0)^2}{s_0^3}, 0, \dots, 0, \frac{(ds_{2k-1})^2}{s_{g-2}} \right) & z_{g-2} &= \left(\dots, 0, \frac{(ds_{g-2})^2}{s_{g-2}^2} \right) \end{aligned}$$

Proof. We use duality on singular curves as developed in [Ser88, Ch.IV] or [BHPVdV04, Ch.II.6]. It is straightforward to verify that each differential from (a) is a Rosenlicht differential and hence is an element of $\mathrm{H}^0(C, \omega_C)$. Since these g elements are independent, the claim follows. The proof of (b) is analogous: Each of the $(3g-3)$ linearly independent differentials from (b) is locally a square of a Rosenlicht differential and hence is an element of $\mathrm{H}^0(C, \omega_C^2)$. □

Lemma 3.7. ω_C *is very ample for odd* $g \geq 5$ *and* ω_C^2 *is very ample for odd* $g \geq 3$.

Proof. To see that C is canonically embedded for $g \geq 5$, observe that $|\omega_C|$ embeds each $\mathbb{P}^1$ as a conic in $\mathbb{P}^{g-1}$, and that the planes spanned by these conics are distinct. Namely, the i^{th} conic (uniformized by s_i) lies in the plane $w_{i+1} = w_{i+2} = \cdots = w_{i+g-3} = 0$ (with the convention that $w_{j+g-1} = w_j$). A straightforward computation shows that ω_C^2 is also very ample for $g = 3$. We finish by noting that C is hyperelliptic in genus 3 and thus is not canonically embedded. □

The $\mathbb{G}_m$-action on $\mathrm{H}^0(C, \omega_C)$ is given by

$$\begin{aligned} t \cdot \omega_i &= t^{(-1)^i} \omega_i, \\ t \cdot \eta &= \eta. \end{aligned}$$

The $\mathbb{G}_m$-action on $\mathrm{H}^0(C, \omega_C^2)$ is given by $x_i \mapsto (t^2)^{(-1)^i} x_i$, $y_i \mapsto t^{(-1)^i} y_i$, $z_i \mapsto z_i$. We define the weight of a monomial to be its $\mathbb{G}_m$-weight.

Proposition 3.8. *Both* $\mathrm{H}^0(C,\omega_C)$ *and* $\mathrm{H}^0(C,\omega_C^2)$ *are multiplicity-free representations of* $\mathbb{G}_m \rtimes \mathbb{Z}_{g-1} \subset \mathrm{Aut}(C)$.

Proof. The action of $\mathbb{Z}_{g-1} \subset D_{g-1}$ on the subspace $\langle \omega_i \rangle_{i=0}^{g-2}$ (resp., $\langle x_i \rangle_{i=0}^{g-2}$, $\langle y_i \rangle_{i=0}^{g-2}$, $\langle z_i \rangle_{i=0}^{g-2}$) corresponds to the regular representation of $\mathbb{Z}_{g-1}$ and is thus multiplicity-free. Since the weight of ω_i is ± 1 and of η is 0 (resp., the weight of x_i is ± 2, of y_i is ± 1, and of z_i is 0), it follows that $\mathrm{H}^0(C,\omega_C)$ (resp., $\mathrm{H}^0(C,\omega_C^2)$) is a multiplicity-free representation of $\mathbb{G}_m \rtimes \mathbb{Z}_{g-1}$. □

The following lemmas are elementary and so we omit the proofs.

Lemma 3.9. *The multiplication map* $\mathrm{Sym}^m \mathrm{H}^0(C,\omega_C) \to \mathrm{H}^0(C,\omega_C^m)$ *is surjective. A set* $\mathcal{B}$ *of degree* m *monomials in* $\omega_0,\dots,\omega_{g-2},\eta$ *forms a monomial basis for* $\mathrm{H}^0(C,\omega_C^m)$ *if and only if the following conditions are satisfied:*

1. $\mathcal{B}$ *contains the* $(g-1)$ *monomials* $\{\omega_i^m\}_{i=0}^{g-2}$ *of weight* $\pm m$,
2. $\mathcal{B}$ *contains the* $(g-1)$ *monomials* $\{\omega_i^{m-1}\eta\}_{i=0}^{g-2}$ *of weight* $\pm(m-1)$,
3. $\mathcal{B}$ *contains* $(g-1)$ *linearly independent monomials of each weight* $2-m \le j \le m-2$.

The reader may wish to check, as an example, that $\{\omega_i^j \eta^{m-j}\}_{i=0}^{g-2}$ and $\{\omega_i^{j+1}\omega_{i-1}\eta^{m-j-2}\}_{i=0}^{g-2}$ give $2g-2$ linearly independent monomials, with $(g-1)$ monomials of weights j and $-j$ each. Thus, taking the union of all these monomials, together with $\{\omega_i^m\}_{i=0}^{g-2}$ and $\{\omega_i^{m-1}\eta\}_{i=0}^{g-2}$ gives a monomial basis of $\mathrm{H}^0(C,\omega_C^m)$.

Lemma 3.10. *The multiplication map* $\mathrm{Sym}^m \mathrm{H}^0(C,\omega_C^2) \to \mathrm{H}^0(C,\omega_C^{2m})$ *is surjective. A set* $\mathcal{B}$ *of degree* m *monomials in* $\{x_i\}_{i=0}^{g-2}$, $\{y_i\}_{i=0}^{g-2}$, $\{z_i\}_{i=0}^{g-2}$ *forms a monomial basis for* $\mathrm{H}^0(C,\omega_C^{2m})$ *if and only if the following conditions are satisfied:*

1. $\mathcal{B}$ *contains the* $(g-1)$ *monomials* $\{x_i^m\}_{i=0}^{g-2}$ *of weight* $\pm 2m$,
2. $\mathcal{B}$ *contains the* $(g-1)$ *monomials* $\{x_i^{m-1}y\}_{i=0}^{g-2}$ *of weight* $\pm(2m-1)$,
3. $\mathcal{B}$ *contains* $(g-1)$ *linearly independent monomials of each weight* $2-2m \le j \le 2m-2$.

4. Monomial bases and stability

4.1. **Canonically embedded ribbon.** Let C denote the balanced ribbon as defined in Section 3.1. In this section, we prove the first part of our Main Result.

Theorem 4.1. *If* $C \subset \mathbb{P}\mathrm{H}^0(C,\omega_C)$ *is a canonically embedded balanced ribbon, then the Hilbert points* $[C]_m$ *are semistable for all* $m \ge 2$.

Corollary 4.2. *Suppose* $C \subset \mathbb{P}\mathrm{H}^0(C,K_C)$ *is a canonically embedded general smooth curve of odd genus. Then the* m^{th} *Hilbert point of* C *is semistable for every* $m \ge 2$.

Proof of Corollary. Quite generally, the locus of semistable points $(\overline{H}_{g,n}^m)^{ss} \subset \overline{H}_{g,n}^m$ is open [MFK94]. Since $\overline{H}_{g,n}^m$ is an irreducible variety whose generic point is the m^{th} Hilbert point of a smooth genus g curve, it remains to find a single semistable point in $\overline{H}_{g,n}^m$. The balanced ribbon C deforms to a smooth canonical curve by [Fon93] and so by Proposition 3.4 we have $[C]_m \in \overline{H}_{g,n}^m$. Applying Theorem 4.1 finishes the proof. □

We have already seen that there is a distinguished basis

$$\mathrm{H}^0(C,\omega_C) = \{x_0,\dots,x_k,y_{k+1},\dots,y_{2k}\}$$

on which $\mathbb{G}_m \subset \mathrm{Aut}(C)$ acts with distinguished weights (Proposition 3.3). According to Proposition 2.3 and the discussion immediately following it, to prove Theorem 4.1 it suffices to show the following. For each degree m, and each one-paramater subgroup $\rho\colon \mathbb{G}_m \to \mathrm{SL}(g)$ acting diagonally on the basis $\{x_0, \dots, x_k, y_{k+1}, \dots, y_{2k}\}$ with weights $(\lambda_0, \dots, \lambda_{2k})$, we must write down a monomial basis of $\mathrm{H}^0(C, \mathcal{O}_C(m)) = \mathrm{H}^0(C, \omega_C^m)$ with non-positive weight. For ease of exposition, we will treat the cases $m = 2$ and $m \geq 3$ separately.

4.1.1. *Monomial bases of* $\mathrm{H}^0(C, \omega_C^2)$. First, we define two monomial bases, $\mathcal{B}^+$ and $\mathcal{B}^-$, of $H^0(C, \omega_C^2)$ as follows. We define $\mathcal{B}^+$ to be the set of quadratic monomials divisible by one of x_0, x_k, or y_{2k}. More precisely,

$$\mathcal{B}^+ := \left\{ \begin{matrix} x_0^2, x_0x_1, \dots, x_0x_k, x_0y_{k+1}, \dots, x_0y_{2k}, \\ x_kx_1, x_kx_2, \dots, x_k^2, x_ky_{k+1}, \dots, x_ky_{2k}, \\ y_{2k}x_1, y_{2k}x_2, \dots, y_{2k}x_{k-1}, y_{2k}y_{k+1}, \dots, y_{2k}^2 \end{matrix} \right\}.$$

We define $\mathcal{B}^-$ as follows:

$$\mathcal{B}^- := \left\{ \begin{matrix} x_0^2, x_1^2, \dots, y_{2k}^2, \\ x_0x_1, x_1x_2, \dots, y_{2k-1}y_{2k}, \\ x_0y_{k+1}, x_1y_{k+2}, \dots, x_{k-1}y_{2k}, \\ x_1y_{k+1}, x_2y_{k+2}, \dots, x_{k-1}y_{2k-1} \end{matrix} \right\}.$$

Lemma 4.3. *$\mathcal{B}^+$ and $\mathcal{B}^-$ are monomial bases of* $\mathrm{H}^0(C, \omega_C^2)$. *For any one-parameter subgroup ρ acting on $(x_0, \dots, y_{2k})$ diagonally with weights $(\lambda_0, \dots, \lambda_{2k})$ the ρ-weights of $\mathcal{B}^+$ and $\mathcal{B}^-$ are:*

$$w_\rho(\mathcal{B}^+) = (g-2)(\lambda_0 + \lambda_k + \lambda_{2k}),$$
$$w_\rho(\mathcal{B}^-) = -2(\lambda_0 + \lambda_k + \lambda_{2k}).$$

Proof. Using Lemma 3.5, one easily checks that $\mathcal{B}^+$ and $\mathcal{B}^-$ are monomial bases. The weight computation is an easy exercise using the relation $\lambda_0 + \dots + \lambda_{2k} = 0$. □

Corollary 4.4. *The* 2^{nd} *Hilbert point of C is semistable.*

Proof. Given any weight vector $(\lambda_0, \dots, \lambda_{2k})$, we must have either $\lambda_0 + \lambda_k + \lambda_{2k} \leq 0$ or $\lambda_0 + \lambda_k + \lambda_{2k} \geq 0$. Thus either $\mathcal{B}^+$ or $\mathcal{B}^-$ has non-positive weight with respect to ρ. □

4.1.2. *Monomial bases of* $\mathrm{H}^0(C, \omega_C^m)$ *for* $m \geq 3$. Finding monomial bases in higher degrees is slightly more cumbersome than in the case $m = 2$. First, we will need three monomial bases in every degree $m \geq 3$. Second, the precise form of one of these bases depends on the residue of $g = 2k + 1$ modulo 4. Nevertheless, the proof is conceptually no different than in the case $m = 2$. We begin by defining two higher-degree analogues of the basis $\mathcal{B}^+$ from Section 4.1.1.

Definition 4.5. We define $\mathcal{B}_1^+$ to be the set of degree m monomial generators of the ideal

$$(x_0, x_k)^{m-1} \cdot (x_0, \dots, x_{k-1}, y_{k+1}, \dots, y_{2k}) + (x_k, y_{2k})^{m-1} \cdot (x_0, \dots, x_{k-1}, y_{k+1}, \dots, y_{2k}) + x_k^m.$$

We define $\mathcal{B}_2^+$ to be the set of degree m monomial generators of the ideal

$$(x_0, y_{2k})^{m-1} \cdot (x_1, \dots, x_{k-1}, y_{k+1}, \dots, y_{2k-1}) + x_k \cdot (x_0, y_{2k})^{m-2} \cdot (x_1, \dots, x_{k-1}, y_{k+1}, \dots, y_{2k-1})$$
$$+ (x_0, y_{2k})^m + x_k(x_0, y_{2k})^{m-1} + x_k^2(x_0, y_{2k})^{m-2} + x_k^3(x_0, y_{2k})^{m-3}$$

Lemma 4.6. *$\mathcal{B}_1^+$ and $\mathcal{B}_2^+$ are monomial bases for $\mathrm{H}(C,\omega_C^m)$. For any one-parameter subgroup ρ acting on $(x_0,\dots,y_{2k})$ diagonally with weights $(\lambda_0,\dots,\lambda_{2k})$ the ρ-weights of $\mathcal{B}^+$ and $\mathcal{B}^-$ are:*

$$w_\rho(\mathcal{B}_1^+) = \big((m-1)^2(g-1)-(2m-3)\big)\,\lambda_k + \left(\frac{m(m-1)}{2}(g-1)-1\right)(\lambda_0+\lambda_{2k}),$$
$$w_\rho(\mathcal{B}_2^+) = ((m-1)(g-1)+(2m-5))\,\lambda_k + ((g-1)(m-1)^2-(2m-3))(\lambda_0+\lambda_{2k}).$$

Proof. Using Lemma 3.5, it is easy to see that $\mathcal{B}_1^+$ and $\mathcal{B}_2^+$ are monomial bases. Next, note that in $\mathcal{B}_1^+$ variables $x_1,\dots,x_{k-1},y_{k+1},\dots,y_{2k-1}$ each appear $2m-1$ times, variables x_0 and y_{2k} each appear $2m-2+(g-1)\binom{m}{2}$ times, and x_k appears $(g-1)(m-1)^2+2$ times. Recalling that $\sum_{i=0}^{2k}\lambda_i=0$, we deduce that the ρ-weight of $\mathcal{B}_1^+$ is

$$\left((g-1)\binom{m}{2}-1\right)(\lambda_0+\lambda_{2k}) + \big((g-1)(m-1)^2-(2m-3)\big)\,\lambda_k.$$

The ρ-weight of $\mathcal{B}_2^+$ is computed analogously by observing that in $\mathcal{B}_2^+$ variables $x_1,\dots,x_{k-1}$ and $y_{k+1},\dots,y_{2k-1}$ each appear $2m-1$ times, variables x_0,y_{2k} each appear $(m-1)^2(g-1)+2$ times, and x_k appears $(g-1)(m-1)+(4m-6)$ times. □

Next, we construct higher-degree analogues of the basis $\mathcal{B}^-$ from Section 4.1.1. Suppose first that $k=2\ell$ so that $g=4\ell+1$. We introduce the following sets of monomials:

$$S_0 := \{x_0^m, x_0^{m-1}x_1,\dots,x_0x_1^{m-1},x_1^m,x_1^{m-1}x_2,\dots,y_{2k-1}y_{2k}^{m-1},y_{2k}^m\}$$

$$S_1^{\mathrm{even}} := \left\{\begin{array}{lllll} x_0^{m-1}y_{k+1}, & x_0^{m-2}x_1y_{k+1}, & \dots, & x_0x_1^{m-2}y_{k+1}, & x_1^{m-1}y_{k+1},\\ x_1^{m-1}y_{k+2}, & x_1^{m-2}x_2y_{k+2}, & \dots, & x_1x_2^{m-2}y_{k+2}, & x_2^{m-1}y_{k+2},\\ & \vdots & & & \\ x_{\ell-2}^{m-1}y_{3\ell-1}, & x_{\ell-2}^{m-2}x_{\ell-1}y_{3\ell-1}, & \dots, & x_{\ell-2}x_{\ell-1}^{m-2}y_{3\ell-1}, & x_{\ell-1}^{m-1}y_{3\ell-1},\\ x_{\ell-1}^{m-1}y_{3\ell}, & x_{\ell-1}^{m-2}x_\ell y_{3\ell}, & \dots, & x_{\ell-1}x_\ell^{m-2}y_{3\ell} & \end{array}\right\}$$

$$S_2^{\mathrm{even}} := \left\{\begin{array}{llll} x_\ell^{m-2}x_{\ell+1}y_{3\ell-1}, & x_\ell^{m-3}x_{\ell+1}^2y_{3\ell-1}, & \dots, & x_\ell x_{\ell+1}^{m-2}y_{3\ell-1},\\ x_{\ell+1}^{m-2}x_{\ell+2}y_{3\ell-1}, & x_{\ell+1}^{m-3}x_{\ell+2}^2y_{3\ell-2}, & \dots, & x_{\ell+1}x_{\ell+2}^{m-2}y_{3\ell-2},\\ & \vdots & & \\ x_{k-2}^{m-2}x_{k-1}y_{k+1}, & x_{k-2}^{m-3}x_{k-1}^2y_{k+1}, & \dots, & x_{k-2}x_{k-1}^{m-2}y_{k+1} \end{array}\right\}$$

$$S_3^{\mathrm{even}} := \begin{cases} \left\{\begin{array}{l} x_{k-1}^{m-2}(x_0y_{2k}), x_{k-1}^{m-3}x_k(x_0y_{2k}),\dots,x_{k-1}^2x_k^{m-4}(x_0y_{2k}),\\ x_{k-1}x_\ell^{(m-1)/2}y_{3\ell}^{(m-1)/2)} \end{array}\right\} & \text{if } m \text{ is odd}\\ \left\{\begin{array}{l} x_{k-1}^{m-2}(x_0y_{2k}), x_{k-1}^{m-3}x_k(x_0y_{2k}),\dots,x_{k-1}^2x_k^{m-4}(x_0y_{2k}),\\ x_{k-1}x_kx_\ell^{(m-2)/2}y_{3\ell}^{(m-2)/2)} \end{array}\right\} & \text{if } m \text{ is even} \end{cases}$$

$$S_4^{\mathrm{even}} := \begin{cases} \{x_0y_{2k}x_k^{m-2}\} & \text{if } m \text{ is odd,}\\ \{x_\ell y_{3\ell}x_k^{m-2}\} & \text{if } m \text{ is even} \end{cases}$$

Suppose now that $k = 2\ell + 1$ so that $g = 4\ell + 3$. We then redefine S_1, S_2, S_3, S_4 as follows:

$$S_1^{\text{odd}} := \begin{Bmatrix} x_0^{m-1}y_{k+1}, & x_0^{m-2}x_1y_{k+1}, & \dots, & x_0x_1^{m-2}y_{k+1}, & x_1^{m-1}y_{k+1}, \\ x_1^{m-1}y_{k+2}, & x_1^{m-2}x_2y_{k+2}, & \dots, & x_1x_2^{m-2}y_{k+2}, & x_2^{m-1}y_{k+2}, \\ & \vdots & & & \\ x_{\ell-2}^{m-1}y_{3\ell}, & x_{\ell-2}^{m-2}x_{\ell-1}y_{3\ell}, & \dots, & x_{\ell-2}x_{\ell-1}^{m-2}y_{3\ell}, & x_{\ell-1}^{m-1}y_{3\ell}, \\ x_{\ell-1}^{m-1}y_{3\ell+1}, & x_{\ell-1}^{m-2}x_\ell y_{3\ell+1}, & \dots, & x_{\ell-1}x_\ell^{m-2}y_{3\ell+1} & \end{Bmatrix}$$

$$S_2^{\text{odd}} := \begin{Bmatrix} x_\ell^{m-2}x_{\ell+1}y_{3\ell}, & x_\ell^{m-3}x_{\ell+1}^2y_{3\ell}, & \dots, & x_\ell x_{\ell+1}^{m-2}y_{3\ell}, \\ x_{\ell+1}^{m-2}x_{\ell+2}y_{3\ell-1}, & x_{\ell+1}^{m-3}x_{\ell+2}^2y_{3\ell-1}, & \dots, & x_{\ell+1}x_{\ell+2}^{m-2}y_{3\ell-1}, \\ & \vdots & & \\ x_{k-3}^{m-2}x_{k-2}y_{k+1}, & x_{k-3}^{m-3}x_{k-2}^2y_{k+1}, & \dots, & x_{k-3}x_{k-2}^{m-2}y_{k+1}, \\ x_{k-2}^{m-2}x_\ell y_{3\ell+1}, & x_{k-2}^{m-3}x_{k-1}x_\ell y_{3\ell+1}, & \dots, & x_{k-2}x_{k-1}^{m-3}x_\ell y_{3\ell+1}, x_{k-1}^{m-2}x_\ell y_{3\ell+1} \end{Bmatrix}$$

$$S_3^{\text{odd}} := \begin{cases} \left\{x_{k-1}^{m-2}x_0y_{2k}, \dots, x_{k-1}^2x_k^{m-4}x_0y_{2k}, x_{k-1}x_\ell^{(m-1)/2}y_{3\ell+2}^{(m-1)/2}\right\} & \text{if } m \text{ is odd} \\ \left\{x_{k-1}^{m-2}x_0y_{2k}, \dots, x_{k-1}^2x_k^{m-4}x_0y_{2k}, x_{k-1}x_kx_\ell^{(m-2)/2}y_{3\ell+2}^{(m-2)/2}\right\} & \text{if } m \text{ is even} \end{cases}$$

$$S_4^{\text{odd}} := \begin{cases} \left\{x_k^{m-2}x_0y_{2k}\right\} & \text{if } m \text{ is odd} \\ \left\{x_k^{m-2}x_\ell y_{3\ell+2}\right\} & \text{if } m \text{ is even} \end{cases}$$

Definition 4.7. Let ι be the reflection on the set of monomials defined by $\iota(x_i) = y_{2k-i}$ and $\iota(y_i) = x_{2k-i}$. Set $* :=$ parity of k. We define a set $\mathcal{B}^-$ of degree m monomials by

$$\mathcal{B}^- := S_0 \cup S_1^* \cup S_2^* \cup S_3^* \cup \iota(S_1^*) \cup \iota(S_2^*) \cup \iota(S_3^*) \cup S_4^*.$$

Lemma 4.8. *For $m \geq 3$, $\mathcal{B}^-$ is a monomial basis of $\mathrm{H}^0(C, \mathcal{O}_C(m)) = \mathrm{H}^0(C, \omega_C^m)$. For any one-parameter subgroup ρ acting on $(x_0, \dots, y_{2k})$ diagonally with weights $(\lambda_0, \dots, \lambda_{2k})$ the ρ-weight of $\mathcal{B}^-$ is:*

$$w_\rho(\mathcal{B}^-) = \begin{cases} -(m^2 - 3m + 5)(\lambda_0 + \lambda_{2k}) - (5m - 10)\lambda_k & \textit{if } m \textit{ is odd,} \\ -(m^2 - 3m + 6)(\lambda_0 + \lambda_{2k}) - (5m - 12)\lambda_k & \textit{if } m \textit{ is even.} \end{cases}$$

Proof. Although the precise definition of $\mathcal{B}^-$ depends on the parity of k, our proof of the lemma does not. Thus we suppress the parity of k in what follows. To prove that $\mathcal{B}^-$ is a monomial basis, we make use of the identification of $\mathrm{H}^0(C, \omega_C^m)$ with functions in $\mathbb{C}[u, \epsilon]/(\epsilon^2)$ made in Section 3.1. First, note that $\mathcal{B}^-$ is invariant under ι. Since ι maps a monomial of u-degree i to a monomial of u-degree $2mk - i$, it suffices in view of Lemma 3.5 to show that $\mathcal{B}^-$ contains one polynomial of u-degree $i = 0, \dots, k$ and two linearly independent polynomials of u-degree $i = k + 1, \dots, mk$. Indeed, S_0 contains exactly one polynomial of u-degree $i = 0, \dots, mk$. Moreover, each of these is a pure polynomial in u. It remains to note that $S_1 \cup S_2 \cup S_3 \cup S_4$ contains exactly one polynomial of u-degree $i = k+1, \dots, mk$ and that each such polynomial has an ϵ term. This finishes the proof that $\mathcal{B}^-$ is a monomial basis.

To compute the ρ-weight of $\mathcal{B}^-$, we observe that in $S_1 \cup S_2 \cup S_3 \cup \iota(S_1) \cup \iota(S_2) \cup \iota(S_3) \cup S_4$ each of the variables $x_1, \dots, x_{k-1}, y_{k+1}, \dots, y_{2k}$ occurs exactly $m(m - 1)$ times; each of the variables x_0 and y_{2k} occurs

$$\begin{cases} (m^2+3m-10)/2 & \text{(if } m \text{ is odd)} \\ (m^2+3m-12)/2 & \text{(if } m \text{ is even)} \end{cases} \quad \text{times;}$$

and x_k occurs

$$\begin{cases} m^2-6m+10 & \text{(if } m \text{ is odd)} \\ m^2-6m+12 & \text{(if } m \text{ is even)} \end{cases} \quad \text{times.}$$

It follows that the total ρ-weight of $S_1 \cup S_2 \cup S_3 \cup \iota(S_1) \cup \iota(S_2) \cup \iota(S_3) \cup S_4$ is

$$-\frac{(m^2-5m+10)}{2}(\lambda_0+\lambda_{2k}) - (5m-10)\lambda_k \quad \text{if } m \text{ is odd,}$$
$$-\frac{(m^2-5m+12)}{2}(\lambda_0+\lambda_{2k}) - (5m-12)\lambda_k \quad \text{if } m \text{ is even.}$$

Finally, $w_\rho(S_0) = -m(m-1)(\lambda_0+\lambda_{2k})/2$. The claim follows. ☐

Proposition 4.9. *Let ρ be a one-parameter subgroup acting on the basis $(x_0,\dots,y_{2k})$ diagonally. Then the ρ-weight of $\mathcal{B}^-$ is the negative of an effective linear combination of the ρ-weights of $\mathcal{B}_1^+$ and $\mathcal{B}_2^+$.*

Proof. This follows from the formulae for the ρ-weights of bases $\mathcal{B}_1^+$, $\mathcal{B}_2^+$, and $\mathcal{B}^-$ given in Lemmas 4.6 and 4.8. In the case of odd m, the claim holds because the inequalities

$$\frac{m(m-1)(g-1)-2}{2(m-1)^2(g-1)-2(2m-3)} \le \frac{(m^2-3m+5)}{5m-10} \le \frac{(g-1)(m-1)^2-(2m-3)}{(m-1)(g-1)+(2m-5)},$$

are satisfied for all $g, m \ge 3$. In the case of even m, we require the same inequalities save that the middle term is replaced by $\frac{(m^2-3m+6)}{5m-12}$. ☐

Proof of Theorem 4.1. The case of $m=2$ was handled in Corollary 4.4. If $m \ge 3$, given any one-parameter subgroup ρ acting on the basis $(x_0,\dots,y_{2k})$ diagonally, it follows from Proposition 4.9 that one of the monomial bases $\mathcal{B}_1^+, \mathcal{B}_2^+$, or $\mathcal{B}^-$ has non-positive weight with respect to ρ. ☐

4.2. **Bicanonically embedded rosary.** We continue our study of the rosary C defined in Section 3.2. In this section, we prove the second part of our Main Result.

Theorem 4.10. *If $C \subset \mathbb{P}\mathrm{H}^0(C,\omega_C^2)$ is a bicanonically embedded rosary, then the Hilbert points $[C]_m$ are semistable for all $m \ge 2$.*

Corollary 4.11. *Suppose $C \subset \mathbb{P}\mathrm{H}^0(C,K_C^2)$ is a bicanonically embedded general smooth curve of odd genus. Then the m^{th} Hilbert point of C is semistable for every $m \ge 2$.* ☐

Proof of Theorem 4.10. We need to show that for any one-parameter subgroup $\rho\colon \mathbb{G}_m \to \mathrm{SL}(3g-3)$ acting on the basis

$$(x_0,x_1,\dots,x_{g-2},y_0,y_1,\dots,y_{g-2},z_0,z_1,\dots,z_{g-2})$$

of $\mathrm{H}^0(C,\mathcal{O}_C(1)) = \mathrm{H}^0(C,\omega_C^2)$ diagonally, there is a monomial basis of $\mathrm{H}^0(C,\mathcal{O}_C(m)) = \mathrm{H}^0(C,\omega_C^{2m})$ of non-positive ρ-weight.

We now define several monomial bases of $\mathrm{H}^0(C,\omega_C^{2m})$. To begin, set

$$S_0 := \left\{ x_i^m, \quad x_i^{m-1}y_i, \qquad \text{where } i=0,\dots,g-2 \right\},$$

$$S_1 := \left\{ \begin{matrix} x_i^j z_i^{m-j}, & x_i^j z_{i+1}^{m-j}, & \text{where } i=0,\dots,g-2, \text{ and } j=1,\dots,m-1 \\ x_i^j y_i z_i^{m-j-1}, & x_i^j y_i z_{i+1}^{m-j-1}, & \text{where } i=0,\dots,g-2, \text{ and } j=0,\dots,m-2 \end{matrix} \right\},$$

$$S_2 := \begin{cases} \{(y_{i-1}y_i)^\ell z_i, \quad \text{where } i=0,\dots,g-2\} & \text{if } m=2\ell+1 \text{ is odd,} \\ \{(y_{i-1}y_i)^\ell z_i^2, \quad \text{where } i=0,\dots,g-2\} & \text{if } m=2\ell+2 \text{ is even.} \end{cases}$$

$$S_2' := \begin{cases} \{(y_{i-1}y_i)^\ell z_i, \quad \text{where } i=0,\dots,g-2\} & \text{if } m=2\ell+1 \text{ is odd,} \\ \{(y_{i-1}y_i)^{\ell+1}, \quad \text{where } i=0,\dots,g-2\} & \text{if } m=2\ell+2 \text{ is even.} \end{cases}$$

We define the following sets of monomials:

$$\mathcal{B}_1^+ := S_0 \cup S_1 \cup S_2,$$
$$\mathcal{B}_2^+ := S_0 \cup S_1 \cup S_2'.$$

(When $g=3$, we take $\mathcal{B}_1^+ = \mathcal{B}_2^+ = S_0 \cup S_1 \cup S_2 \cup S_2'$ because then both S_2 and S_2' contain only one element.) Using Lemma 3.10, it is easy to see that $\mathcal{B}_1^+$ and $\mathcal{B}_2^+$ are monomial bases of $\mathrm{H}^0(C, \omega_C^{2m})$. If we let $X^\lambda, Y^\lambda, Z^\lambda$ denote the sum of the ρ-weights of the x_i's, y_i's, z_i's, respectively, then one easily checks that the average of the ρ-weights of $\mathcal{B}_1^+$ and $\mathcal{B}_2^+$ is

$$(2m^2-2m+1)X^\lambda + (3m-2)Y^\lambda + (2m^2-2m+1)Z^\lambda. \tag{1}$$

To define an alternate pair of monomial bases, we set

$$T_1 := \left\{ \begin{matrix} x_i^{j-m} y_i^{2m-j}, & x_i^{j-m+1} y_i^{2m-j-2} z_i, & \text{where } i=0,\dots,g-2, \text{ and } j=m,\dots,2m-2 \\ y_i^j z_i^{m-j}, & y_i^j z_{i+1}^{m-j}, & \text{where } i=0,\dots,g-2, \text{ and } j=2,\dots,m-1 \\ y_i z_i^{m-1}, & y_i z_{i+1}^{m-1}, & \text{where } i=0,\dots,g-2. \end{matrix} \right\}$$

and define:

$$\mathcal{B}_1^- := S_0 \cup T_1 \cup S_2,$$
$$\mathcal{B}_2^- := S_0 \cup T_1 \cup S_2'.$$

One easily checks that $\mathcal{B}_1^-$ and $\mathcal{B}_2^-$ are monomial bases of $\mathrm{H}^0(C, \omega_C^{2m})$ and that the average of their ρ-weights is

$$m^2 X^\lambda + (2m^2-m)Y^\lambda + m^2 Z^\lambda. \tag{2}$$

To complete the proof of Theorem 4.10, simply note that since $X^\lambda + Y^\lambda + Z^\lambda = 0$ for any one-parameter subgroup ρ, either Equation (1) or Equation (2) is non-positive. □

4.3. **Canonically embedded rosary.** Let C denote the rosary defined in Section 3.2. In this section, we analyze finite Hilbert stability of the *canonical embedding* of C. We find that C is the first known example of a curve in arbitrary (odd) genus where stability of its Hilbert points depends on m: $[C]_m$ is semistable for large m but becomes non-semistable for small m. More precisely, we have the following result.

Theorem 4.12. *For odd $g \geq 5$, the genus g canonically embedded rosary $C \subset \mathbb{P}\mathrm{H}^0(C, \omega_C)$ has a semistable Hilbert point $[C]_m$ if and only if $g \leq 2m+1$.*

Proof. We first show that $[C]_m$ is semistable for $g \leq 2m+1$. This is accomplished using the same technique as in the previous sections, namely by using Lemma 3.9 to find non-positive monomial bases. Let $\rho\colon \mathbb{G}_m \to \mathrm{SL}(g)$ be a one-parameter subgroup acting on the basis $(\omega_0, \dots, \omega_{g-2}, \eta)$ diagonally with weights $(\lambda_0, \dots, \lambda_{g-2}, \lambda_{g-1})$. Set $W := \sum_{i=0}^{g-2} \lambda_i$.

First, we find a basis whose ρ-weight minimizes the coefficient of λ_{g-1}: We define a basis $\mathcal{B}^+$ to be the following set of monomials:

$$\mathcal{B}^+ := \left\{ \begin{array}{ll} \omega_i^m,\ \omega_i^{m-1}\eta, & \text{for } i=0,\dots,g-2, \\ \omega_i^{m-j}\omega_{i-1}^j,\ \omega_i^j\omega_{i-1}^{m-j}, & \text{for } i=0,\dots,g-2,\ 1\le m-2j\le m-2, \\ \omega_i^{m-j-1}\omega_{i-1}^j\eta,\ \omega_i^j\omega_{i-1}^{m-j-1}\eta, & \text{for } i=0,\dots,g-2,\ 2\le m-2j\le m-2, \\ \begin{cases} \omega_i^\ell\omega_{i-1}^\ell & \text{if } m=2\ell \\ \omega_i^{\ell-1}\omega_{i-1}^{\ell-1}\eta & \text{if } m=2\ell-1 \end{cases} & \text{for } i=0,\dots,g-2 \end{array} \right\}$$

The ρ-weight of $\mathcal{B}^+$ is

$$(2m^2-2m+1)W+(m-1)(g-1)\lambda_g=(2m^2-2m+1-(m-1)(g-1))W.$$

We now find a basis which maximizes the coefficient of λ_{g-1}. Namely, we set

$$\mathcal{B}^- := \left\{ \begin{array}{lll} \omega_i^m, & \omega_i^{m-1}\eta, & \text{for } i=0,\dots,g-2, \\ \omega_i^j\eta^{m-j}, & \omega_i\omega_{i-1}^{j+1}\eta^{m-j-2}, & \text{for } i=0,\dots,g-2,\ j=1,\dots,m-2, \\ \omega_i\omega_{i-1}\eta^{m-2}, & & \text{for } i=0,\dots,g-2. \end{array} \right\}$$

Then the ρ-weight of the basis $\mathcal{B}^-$ is

$$(m^2+m-1)W+(m-1)^2(g-1)\lambda_g=(m^2+m-1-(m-1)^2(g-1))W.$$

If $(g,m)\neq(5,2)$ and $g\le 2m+1$, then either $\mathcal{B}^+$ or $\mathcal{B}^-$ has non-positive weight with respect to ρ. If $(g,m)=(5,2)$, then it is easy to find three explicit monomial bases one of whose weights is guaranteed to be non-positive. This finishes the proof of semistability.

Conversely, suppose $g\ge 2m+2$. Consider the one-parameter subgroup ρ acting with weight (-1) on ω_i's and weight $g-1$ on η. If $\mathcal{B}$ is a monomial basis of $\mathrm{H}^0(C,\mathcal{O}_C(m))=\mathrm{H}^0(C,\omega_C^m)$, then for each odd ℓ each monomial of weight $\pm(m-\ell)$ with respect to $\mathbb{G}_m\subset\mathrm{Aut}(C)$ necessarily has an η term (see Lemma 3.9). It follows that the variable η of weight $(g-1)$ occurs at least $(m-1)(g-1)$ times among monomials of $\mathcal{B}$. The remaining at most $m(2m-1)(g-1)-(m-1)(g-1)$ variables occuring in $\mathcal{B}$ all have weight (-1). It follows that the total ρ-weight of $\mathcal{B}$ is at least

$$\begin{aligned}&(g-1)(m-1)(g-1)+((2m-1)(g-1)m-(m-1)(g-1))(-1)\\ &\qquad=(g-1)((m-1)(g-1)-(2m^2-2m+1))\ge(g-1)(3m-2)>0.\end{aligned}$$

Thus ρ destabilizes C. $\square$

References

[AFS10] Jarod Alper, Maksym Fedorchuk, and David Smyth. Singularities with $\mathbb{G}_m$-action and the log minimal model program for $\overline{M}_g$, 2010. Submitted for publication. Available at `arXiv:1010.3751 [math.AG]`.

[BE95] Dave Bayer and David Eisenbud. Ribbons and their canonical embeddings. *Trans. Amer. Math. Soc.*, 347(3):719–756, 1995.

[BHPVdV04] Wolf P. Barth, Klaus Hulek, Chris A. M. Peters, and Antonius Van de Ven. *Compact complex surfaces*, volume 4 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics*. Springer-Verlag, Berlin, second edition, 2004.

[Fon93] Lung-Ying Fong. Rational ribbons and deformation of hyperelliptic curves. *J. Algebraic Geom.*, 2(2):295–307, 1993.

[FS10] Maksym Fedorchuk and David Smyth. Alternate compactifications of moduli spaces of curves, 2010. To appear in the *Handbook of Moduli*, edited by G. Farkas and I. Morrison. `arXiv:1012.0329v2 [math.AG]`.

[Gie82] D. Gieseker. *Lectures on moduli of curves*, volume 69 of *Tata Institute of Fundamental Research Lectures on Mathematics and Physics*. Published for the Tata Institute of Fundamental Research, Bombay, 1982.

[Gie83] David Gieseker. Geometric invariant theory and applications to moduli problems. In *Invariant theory. Proceedings of the 1st 1982 Session of the Centro Internazionale Matematico Estivo (CIME), Montecatini, June 10–18, 1982*, volume 996 of *Lecture Notes in Mathematics*, pages v+159. Springer-Verlag, Berlin, 1983.

[HH08] Brendan Hassett and Donghoon Hyeon. Log minimal model program for the moduli space of curves: the first flip. `arXiv:0806.3444 [math.AG]`, 2008.

[HH09] Brendan Hassett and Donghoon Hyeon. Log canonical models for the moduli space of curves: the first divisorial contraction. *Trans. Amer. Math. Soc.*, 361(8):4471–4489, 2009.

[HL10] Donghoon Hyeon and Yongnman Lee. Log minimal model program for the moduli space of stable curves of genus three. *Math. Res. Lett.*, 17(4):625–636, 2010.

[HM10] Donghoon Hyeon and Ian Morrison. Stability of tails and 4-canonical models. *Math. Res. Lett.*, 17(4):721–729, 2010.

[Kem78] George R. Kempf. Instability in invariant theory. *Ann. of Math. (2)*, 108(2):299–316, 1978.

[MFK94] D. Mumford, J. Fogarty, and F. Kirwan. *Geometric invariant theory*, volume 34 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)]*. Springer-Verlag, Berlin, third edition, 1994.

[MS11] I. Morrison and D. Swinarski. Groebner techniques for low degree Hilbert stability. *Experimental Mathematics*, 20(1):34–56, 2011.

[Mum77] David Mumford. Stability of projective varieties. *Enseignement Math. (2)*, 23(1-2):39–110, 1977.

[Sch91] David Schubert. A new compactification of the moduli space of curves. *Compositio Math.*, 78(3):297–313, 1991.

[Ser88] Jean-Pierre Serre. *Algebraic groups and class fields*, volume 117 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1988. Translated from the French.

(Alper) Departamento de Matemáticas, Universidad de los Andes, Cra 1 No. 18A-10, Edificio H, Bogotá, 111711, Colombia
E-mail address: `jarod@uniandes.edu.co`

(Fedorchuk) Department of Mathematics, Columbia University, 2990 Broadway, New York, NY 10027
E-mail address: `mfedorch@math.columbia.edu`

(Smyth) Department of Mathematics, Harvard University, 1 Oxford Street, Cambridge, MA 01238
E-mail address: `dsmyth@math.harvard.edu`