FINITE HILBERT STABILITY OF CANONICAL CURVES, II. THE EVEN-GENUS CASE

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Abstract. We prove that a generic canonically embedded curve of even genus has semistable m^{th} Hilbert point for all $m \geq 2$. More precisely, we prove that a generic canonically embedded trigonal curve of even genus has semistable m^{th} Hilbert point for all $m \geq 2$. Furthermore, we show that the analogous result fails for bielliptic curves. Namely, the Hilbert points of bielliptic curves are asymptotically semistable but become non-semistable below a definite threshold value depending on g.

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1. Introduction

This paper is a sequel to [AFS11], where we proved that a general smooth curve of odd genus, canonically or bicanonically embedded, has semistable m^{th} Hilbert point for all $m \geq 2$. Here, we prove an analogous result for canonically embedded curves of even genus. Our main result is the following.

Theorem 1.1 (Main Result)**.** *Suppose $C \subset \mathbb{P}\mathrm{H}^0(C, K_C)$ is a general smooth curve of even genus, embedded by the complete linear system $|K_C|$. Then the m^{th} Hilbert point of C is semistable for every $m \geq 2$.*

We refer to our previous paper [AFS11] for an extended discussion of the geometric motivation behind this result and its applications to the Hassett-Keel log minimal model program for $\overline{M}_g$, as well as an informal description of the method of proof. As in [AFS11], this generic stability result is obtained by proving that a

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very special singular curve has semistable Hilbert points. The singular curve we used in [AFS11] was a balanced canonical ribbon of odd genus. The singular curve that we will use here is the so called *balanced double A_{2k+1}-curve* of even genus.

A *double A_{2k+1}-curve* is any curve obtained by gluing three copies of $\mathbb{P}^1$ along two A_{2k+1} singularities (see Figure 1). In every even genus $g = 2k$, double A_{2k+1}-curves have $2k-4$ moduli corresponding to the crimping of the A_{2k+1}-singularities, i.e., deformations that preserve the analytic type of the singularities as well as the normalization of the curve; we refer to [vdW10] for a comprehensive discussion of crimping of curve singularities. We note that the parameter space of crimping for an A_{2k+1}-singularity with automorphism-free branches has dimension k. However, in the case of a double A_{2k+1}-curve, the presence of automorphisms of the pointed $\mathbb{P}^1$'s reduces the dimension of crimping moduli by 4.

Among double A_{2k+1}-curves, there is a unique double A_{2k+1}-curve with a $\mathbb{G}_m$-action, corresponding to the trivial choice of crimping data. We call this curve the *balanced double A_{2k+1}-curve*. Our motivation for considering double A_{2k+1}-curves comes from the Hassett-Keel program for $\overline{M}_{2k}$, where we expect the $2k-4$ dimensional locus of double A_{2k+1}-curves to replace the locus in the boundary divisor $\Delta_k \subset \overline{M}_{2k}$ consisting of curves $C_1 \cup C_2$ such that each C_i is a hyperelliptic curve of genus k. Indeed, this prediction has already been verified in $g = 4$ by the second author who showed that the divisor $\Delta_2 \subset \overline{M}_4$ is contracted to the point corresponding to the unique genus 4 double A_5-curve in the final non-trivial log canonical model of $\overline{M}_4$ [Fed11].

It is not too difficult to see that the balanced double A_{2k+1}-curve is trigonal, i.e., it lies in the closure of the locus of canonically embedded smooth trigonal curves; see Proposition 2.2. From this observation, we obtain a slight strengthening of our Main Result:

Theorem 1.2 (Stability of trigonal curves)**.** *Suppose $C \subset \mathbb{P}\mathrm{H}^0(C, K_C)$ is a general smooth trigonal curve of even genus, embedded by the complete linear system $|K_C|$. Then the m^{th} Hilbert point of C is semistable for every $m \geq 2$.*

This result leads to two related questions: Is it true that all smooth trigonal curves have semistable m^{th} Hilbert points for all $m \geq 2$? Similarly, do other curves with low Clifford index have this property? Surprisingly, the answer to both questions is no. It is not too difficult to see that the 2^{nd} Hilbert point of a trigonal curve with a positive Maroni invariant is non-semistable; see [FJ11] for a quick proof. In the final section of this paper, we will present a heuristic which suggests that a smooth trigonal curve has a semistable m^{th} Hilbert point for $m \geq 3$. We also prove that the m^{th} Hilbert point of a smooth bielliptic curve becomes non-semistable below a certain definite threshold value of m, depending on g. This is complemented by a proof of the semistability of a generic bielliptic curve of odd genus for large values of m.

The outline of this paper is as follows. In Section 2, we prove the basic facts about the balanced double A_{2k+1}-curve necessary to prove semistability by the strategy described in [AFS11]. In Section 3, we construct the monomial bases necessary to prove semistability of the Hilbert points of the balanced double A_{2k+1}-curve. As a result, we obtain a proof of Theorems 1.1 and 1.2; see Corollary 3.2. In Section 4, we discuss finite Hilbert stability of trigonal curves with a positive Maroni invariant and bielliptic curves.

We work over the field of complex numbers $\mathbb{C}$.

2. The balanced double A_{2k+1}-curve

In this section, we give an explicit description of the pluricanonical linear system $\mathrm{H}^0(C, \omega_C^m)$ of the balanced double A_{2k+1}-curve C. In addition, we prove the key fact that $\mathrm{H}^0(C, \omega_C)$ is a multiplicity-free representation of $\mathrm{Aut}(C)$. Following the strategy of [AFS11], this allows us to prove the semistability of the m^{th} Hilbert point of C by writing down monomial bases for $\mathrm{H}^0(C, \omega_C^m)$. In Section 3, we construct the requisite monomial bases and thus prove the semistability of the Hilbert points of C.

Let us begin by giving a precise description of the balanced double A_{2k+1}-curve. Let C_0, C_1, C_2 denote three copies of $\mathbb{P}^1$, and label the uniformizers at 0 (resp., at ∞) by s_0, s_1, s_2 (resp., by t_0, t_1, t_2). Fix an integer $k \geq 2$, and let C be the arithmetic genus $g = 2k$ curve obtained by gluing three $\mathbb{P}^1$'s along two A_{2k+1} singularities with trivial crimping. More precisely, we impose an A_{2k+1} singularity at $(\infty \in C_0) \sim (0 \in C_1)$ by gluing $C_0 \smallsetminus 0$ and $C_1 \smallsetminus \infty$ into an affine singular curve

$$\operatorname{Spec} \mathbb{C}[x, y]/(y^2 - x^{2k+2}) \simeq \operatorname{Spec} \mathbb{C}[(t_0, s_1), (t_0^{k+1}, -s_1^{k+1})]. \tag{2.1}$$

Similarly, we impose an A_{2k+1} singularity at $(\infty \in C_1) \sim (0 \in C_2)$ by gluing $C_1 \smallsetminus 0$ and $C_2 \smallsetminus \infty$ into

$$\operatorname{Spec} \mathbb{C}[x, y]/(y^2 - x^{2k+2}) \simeq \operatorname{Spec} \mathbb{C}[(t_1, s_2), (t_1^{k+1}, -s_2^{k+1})]. \tag{2.2}$$

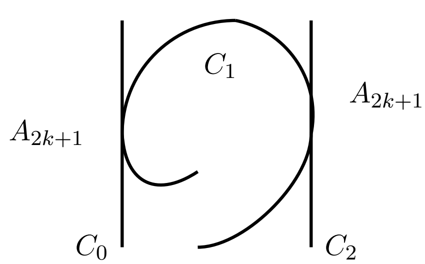

Figure 1. Double A_{2k+1}-curves

The automorphism group of C is given by $\mathrm{Aut}(C) = \mathbb{G}_m \rtimes \mathbb{Z}_2$ where $\mathbb{Z}_2$ acts via $s_i \leftrightarrow t_{2-i}$ and $\mathbb{G}_m = \operatorname{Spec} \mathbb{C}[\lambda, \lambda^{-1}]$ acts via

$$\begin{aligned} \lambda \cdot s_0 &= \lambda s_0, \\ \lambda \cdot s_1 &= \lambda^{-1} s_1, \\ \lambda \cdot s_2 &= \lambda s_2. \end{aligned}$$

Using the description of the dualizing sheaf on a singular curve as in [Ser88, Ch.IV] or [BHPVdV04, Ch.II.6], we can write down a basis of $\mathrm{H}^0(C, \omega_C)$ as follows:

$$\begin{aligned} x_1 &= \left(ds_0, \frac{ds_1}{s_1^2}, 0\right) & y_1 &= \left(0, ds_1, \frac{ds_2}{s_2^2}\right) \\ x_2 &= \left(s_0 ds_0, \frac{ds_1}{s_1^3}, 0\right) & y_2 &= \left(0, s_1 ds_1, \frac{ds_2}{s_2^3}\right) \\ &\vdots & &\vdots \\ x_k &= \left(s_0^{k-1} ds_0, \frac{ds_1}{s_1^{k+1}}, 0\right) & y_k &= \left(0, s_1^{k-1} ds_1, \frac{ds_2}{s_2^{k+1}}\right) \end{aligned} \tag{2.3}$$

It is straightforward to generalize this description to the spaces of pluricanonical differentials.

Lemma 2.1. *For $m \geq 2$, the product map $\operatorname{Sym}^m \mathrm{H}^0(C, \omega_C) \to \mathrm{H}^0(C, \omega_C^m)$ is surjective and a basis of $\mathrm{H}^0(C, \omega_C^m)$ consists of the following $(2m-1)(2k-1)$ differentials:*

$$\begin{aligned} \omega_0 &= \left((ds_0)^m, \frac{(ds_1)^m}{s_1^{2m}}, 0\right) & \eta_0 &= \left(0, (ds_1)^m, \frac{(ds_2)^m}{s_2^{2m}}\right) \\ \omega_1 &= \left(s_0 (ds_0)^m, \frac{(ds_1)^m}{s_1^{2m-1}}, 0\right) & \eta_1 &= \left(0, s_1 (ds_1)^m, \frac{(ds_2)^m}{s_2^{2m+1}}\right) \\ &\vdots & &\vdots \\ \omega_{m(k-1)} &= \left(s_0^{m(k-1)} (ds_0)^m, \frac{(ds_1)^m}{s_1^{m(k+1)}}, 0\right) & \eta_{m(k-1)} &= \left(0, s_1^{m(k-1)} (ds_1)^m, \frac{(ds_2)^m}{s_2^{m(k+1)}}\right) \end{aligned}$$

and

$$\chi_{-k(m-1)+1} = \left(0, s_1^{k(m-1)-m-1}(ds_1)^m, 0\right)$$
$$\vdots$$
$$\chi_i = \left(0, s_1^{-i-m}(ds_1)^m, 0\right)$$
$$\vdots$$
$$\chi_{k(m-1)-1} = \left(0, \frac{(ds_1)^m}{s_1^{(m-1)(k+1)}}, 0\right)$$

Proof. By Riemann-Roch formula, $h^0(C, \omega_C^m) = (2m-1)(2k-1)$. Thus, it suffices to observe that the given $(2m-1)(2k-1)$ differentials all lie in the image of the map $\mathrm{Sym}^m \mathrm{H}^0(C, \omega_C) \to \mathrm{H}^0(C, \omega_C^m)$. Using the basis of $\mathrm{H}^0(C, \omega_C)$ given by (2.3), one easily checks that the differentials $\{\omega_i\}_{i=0}^{m(k-1)}$ are precisely those arising as m-fold products of x_i's, the differentials $\{\eta_i\}_{i=0}^{m(k-1)}$ are those arising as m-fold products of y_i's, and the differentials $\{\chi_i\}_{i=-k(m-1)+1}^{k(m-1)+1}$ are those arising as mixed m-fold products of x_i's and y_i's. □

Next, we show that $|\omega_C|$ is a very ample linear system, so that C admits a canonical embedding, and the corresponding Hilbert points are well defined.

Proposition 2.2. *ω_C is very ample. The complete linear system $|\omega_C|$ embeds C as a curve on a balanced rational normal scroll*

$$\mathbb{P}^1 \times \mathbb{P}^1 \xhookrightarrow{|\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,k-1)|} \mathbb{P}^{g-1}.$$

Moreover, C_0 and C_2 map to $(1,0)$-curves on $\mathbb{P}^1\times\mathbb{P}^1$, and C_1 maps to a $(1,k+1)$-curve. In particular, C is a $(3,k+1)$ curve on $\mathbb{P}^1\times\mathbb{P}^1$ and has a g^1_3 cut out by the $(0,1)$-ruling.

Proof. To see that the canonical embedding of C lies on a balanced rational normal scroll in $\mathbb{P}^{2k-1}$, recall that the scroll can be defined as the determinantal variety (see [Har92, Lecture 9]):

$$\operatorname{rank}\left(\begin{array}{cccc|cccc} x_1 & x_2 & \cdots & x_{k-1} & y_k & y_{k-1} & \cdots & y_2 \\ x_2 & x_3 & \cdots & x_k & y_{k-1} & y_{k-2} & \cdots & y_1 \end{array}\right) \leq 1. \tag{2.4}$$

From our explicit description of the basis of $\mathrm{H}^0(C, \omega_C)$ given by (2.3), one easily sees that the differentials x_i's and y_i's on C satisfy the determinantal description of (2.4). Moreover, we see that $|\omega_C|$ embeds C_0 and C_2 as degree $k-1$ rational normal curves in $\mathbb{P}^{2k-1}$ lying in the class $(1,0)$ on the scroll. Also, we see that $|\omega_C|$ embeds C_1 via the very ample linear system

$$\mathrm{span}\{1, s_1, \dots, s_1^{k-1}, s_1^{k+1}, \dots, s_1^{2k}\} \subset |\mathcal{O}_{\mathbb{P}^1}(2k)|$$

as a curve in the class $(1, k+1)$. It follows that $|\omega_C|$ separates points and tangent vectors on each component of C. We now prove that $|\omega_C|$ separates points of different components and tangent vectors at the A_{2k+1}-singularities. First, observe that C_0 and C_2 span different subspaces. Therefore, being $(1,0)$ curves, they must be distinct and non-intersecting. Second, C_0 and C_1 are the images of two branches of an A_{2k+1}-singularity and so have contact of order at least $k+1$. However, being $(1,0)$ and $(1,k+1)$ curves on the scroll, they have order of contact at most $k+1$. It follows that C_0 and C_1 on S meet in a precisely A_{2k+1}-singularity. We conclude that $|\omega_C|$ is a closed embedding at each A_{2k+1}-singularity.

We can also directly verify that $|\omega_C|$ separates tangent vectors at an A_{2k+1} singularity of C, say the one with uniformizers s_1 and t_0. The local generator of ω_C at this singularity is

$$x_k = \left(-\frac{dt_0}{t_0^{k+1}}, \frac{ds_1}{s_1^{k+1}}, 0\right).$$

We observe that on the open affine chart $\operatorname{Spec}\mathbb{C}[(t_0, s_1), (t_0^{k+1}, -s_1^{k+1})]$ defined in Equation (2.1) we have $y_1 = (0, s_1^{k+1}) \cdot x_k$ and $x_{k-1} = (t_0, s_1) \cdot x_k$. Under the identification $\mathbb{C}[x,y]/(y^2 - x^{2k+2}) = \mathbb{C}[(t_0, s_1), (t_0^{k+1}, -s_1^{k+1})]$, we have $(t_0, s_1) = x$ and $(0, s_1^{k+1}) = (x^{k+1} - y)/2$. We conclude that sections y_1 and x_{k-1} of ω_C span the cotangent space $(x,y)/(x,y)^2$ and thus separate tangent vectors at the singularity $x = y = 0$.

□

Finally, the following elementary observation is the key to analyzing the stability of Hilbert points of C.

Lemma 2.3. $\mathrm{H}^0(C, \omega_C)$ *is a multiplicity-free* $\operatorname{Aut}(C)$*-representation, i.e., no irreducible* $\operatorname{Aut}(C)$*-representation appears more than once in the decomposition of* $\mathrm{H}^0(C, \omega_C)$ *into irreducibles.*

Proof. Consider the basis of $\mathrm{H}^0(C, \omega_C)$ given in (2.3). Then $\mathbb{G}_m \subset \operatorname{Aut}(C)$ acts on x_i with weight i and on y_i with weight $-i$. Thus $\mathrm{H}^0(C, \omega_C)$ decomposes into $g = 2k$ distinct characters of $\mathbb{G}_m$. □

3. Monomial bases and semistability

Since $\mathrm{H}^0(C, \omega_C)$ is a multiplicity-free representation of $\mathbb{G}_m \subset \operatorname{Aut}(C)$ by Lemma 2.3, we can apply the Kempf-Morrison Criterion [AFS11, Proposition 2.3] to prove semistability of C. Namely, to prove that the m^{th} Hilbert point of the canonically embedded balanced double A_{2k+1}-curve C is semistable, it suffices to check that for every one-parameter subgroup $\rho\colon \mathbb{G}_m \to \mathrm{SL}(g)$ acting diagonally on the basis $\{x_1, \dots, x_k, y_1, \dots, y_k\}$ with integer weights $\lambda_1, \dots, \lambda_k, \nu_1, \dots, \nu_k$, there exists a monomial basis for $\mathrm{H}^0(C, \omega_C^m)$ of non-positive ρ-weight. Explicitly, this means that we must exhibit a set $\mathcal{B}$ of $(2m-1)(2k-1)$ degree m monomials in the variables $\{x_i, y_i\}_{i=1}^k$ with the properties that:

(1) $\mathcal{B}$ maps to a basis of $\mathrm{H}^0(C, \omega_C^m)$ via $\mathrm{Sym}^m \mathrm{H}^0(C, \omega_C) \to \mathrm{H}^0(C, \omega_C^m)$.
(2) $\mathcal{B}$ has non-positive ρ-weight, i.e., if $\mathcal{B} = \{m_i\}_{i=1}^{(2m-1)(2k-1)}$, and $m_i = \prod_{j=1}^k x_j^{a_{ij}} y_j^{b_{ij}}$, then
$$\sum_{i=1}^{(2m-1)(2k-1)} \sum_{j=1}^{k} (a_{ij}\lambda_j + b_{ij}\nu_j) \leq 0.$$

Theorem 3.1. *If $C \subset \mathbb{P}\mathrm{H}^0(C, \omega_C)$ is a canonically embedded balanced double A_{2k+1}-curve, then the Hilbert points $[C]_m$ are semistable for all $m \geq 2$.*

As an immediate corollary of this result, we obtain a proof of Theorem 1.2 and hence of Theorem 1.1:

Corollary 3.2 (Theorem 1.2)**.** *A general smooth trigonal curve of genus $g = 2k$ embedded by the complete canonical linear system has a semistable m^{th} Hilbert point for every $m \geq 2$.*

Proof of Corollary. By Proposition 2.2 the canonical embedding of the balanced double A_{2k+1}-curve C lies on a balanced surface scroll in $\mathbb{P}^{2k-1}$ in the divisor class $(3, k+1)$. It follows that C deforms flatly to a smooth curve in the class $(3, k+1)$ on the scroll. Such a curve is a smooth trigonal canonically embedded curve. The semistability of a general deformation of C follows from the openness of semistable locus. □

Proof of Theorem 3.1. Recall from Lemma 2.1 that
$$\mathrm{H}^0(C, \omega_C^m) = \mathrm{span}\{\omega_i\}_{i=0}^{m(k-1)} \oplus \mathrm{span}\{\eta_i\}_{i=0}^{m(k-1)} \oplus \mathrm{span}\{\chi_i\}_{i=-k(m-1)+1}^{k(m-1)-1}.$$
Now, given a one-parameter subgroup ρ as above, we will construct the requisite monomial basis $\mathcal{B}$ as a union
$$\mathcal{B} = \mathcal{B}_\omega \cup \mathcal{B}_\eta \cup \mathcal{B}_\chi,$$
where $\mathcal{B}_\omega, \mathcal{B}_\eta$, and $\mathcal{B}_\chi$ are collections of degree m monomials which map onto the bases of the subspaces spanned by $\{\omega_i\}_{i=0}^{m(k-1)}, \{\eta_i\}_{i=0}^{m(k-1)}$ and $\{\chi_i\}_{i=-k(m-1)+1}^{k(m-1)-1}$, respectively.

To construct $\mathcal{B}_\omega$ and $\mathcal{B}_\eta$, we use Kempf's proof of the stability of Hilbert points of a rational normal curve. More precisely, consider the component C_0 of C with the uniformizer s_0 at $0 \in C_0$. Evidently, $\omega_C|_{C_0} \simeq \mathcal{O}_{\mathbb{P}^1}(k-1)$. The restriction map $\mathrm{H}^0(C, \omega_C) \to \mathrm{H}^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(k-1))$ identifies $\{x_i\}_{i=1}^k$ with a basis of $\mathrm{H}^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(k-1))$ given by $\{1, s_0, \dots, s_0^{k-1}\}$. Under this identification, the subspace $\mathrm{span}\{\omega_i\}_{i=0}^{m(k-1)}$ is identified with $\mathrm{H}^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(m(k-1)))$. Set $\lambda = \sum_{i=1}^k \lambda_i/k$. Given a one-parameter subgroup $\widetilde{\rho}\colon \mathbb{G}_m \to \mathrm{SL}(k)$ acting on $(x_1, \dots, x_k)$ diagonally with weights $(\lambda_1-\lambda,\dots,\lambda_k-\lambda)$, Kempf's result on the semistability of a rational normal curve in $\mathbb{P}^{k-1}$ [Kem78, Corollary 5.3], implies the existence of a monomial basis $\mathcal{B}_\omega$ of $\mathrm{H}^0(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}(m(k-1)))$ with non-positive $\widetilde{\rho}$-weight. Under the above identification, $\mathcal{B}_\omega$ is a monomial basis of $\operatorname{span}\{\omega_i\}_{i=0}^{m(k-1)}$ of ρ-weight at most $m(m(k-1)+1)\lambda$. Similarly, if $\nu=\sum_{i=1}^k \nu_i/k$, we deduce the existence of a monomial basis $\mathcal{B}_\eta$ of $\operatorname{span}\{\eta_i\}_{i=0}^{m(k-1)}$ whose ρ-weight is at most $m(m(k-1)+1)\nu$. Since $\lambda+\nu=0$, it follows that the total ρ-weight of $\mathcal{B}_\omega\cup\mathcal{B}_\eta$ is non-positive.

Thus, to construct a monomial basis $\mathcal{B}$ of non-positive ρ-weight, it remains to construct a monomial basis $\mathcal{B}_\chi$ of non-positive ρ-weight for the subspace

$$\operatorname{span}\{\chi_i\}_{i=-k(m-1)-1}^{k(m-1)-1}\subset \mathrm{H}^0(C,\omega_C^m).$$

In Lemma 3.3, proved below, we show the existence of such a basis. Thus, we obtain the desired monomial basis $\mathcal{B}$ and finish the proof. □

Note that if we define the *weighted degree* by $\deg(x_i)=i$ and $\deg(y_i)=-i$, then a set $\mathcal{B}_\chi$ of $2k(m-1)-1$ degree m monomials in $\{x_1,\dots,x_k,\,y_1,\dots,y_k\}$ maps to a basis of $\operatorname{span}\{\chi\}_{i=k(m-1)-1}^{k(m-1)+1}$ if and only if it satisfies the following two conditions:

(1) Each monomial has both x_i and y_i terms,
(2) Each weighted degree from $(m-1)k-1$ to $-(m-1)k+1$ occurs exactly once.

We call such a set of monomials a *χ-basis*. The following combinatorial lemma completes the proof of Theorem 3.1.

Lemma 3.3. *Suppose $\rho\colon\mathbb{G}_m\to\mathrm{SL}(2k)$ is a one-parameter subgroup which acts on $\{x_1,\dots,x_k,y_1,\dots,y_k\}$ diagonally with integer weights $\lambda_1,\dots,\lambda_k,\nu_1,\dots,\nu_k$ satisfying $\sum_{i=1}^k(\lambda_i+\nu_i)=0$. Then there exists a χ-basis with non-positive ρ-weight.*

Proof of Lemma 3.3 for $m=2$. Take the first χ-basis to be

$$\begin{aligned}\mathcal{B}_1:=\{&x_ky_1,x_{k-1}y_1,x_{k-1}y_2,x_{k-2}y_2,x_{k-2}y_3,\dots\\ &\dots,x_iy_{k-i},x_iy_{k-i-1},\dots,x_2y_{k-1},x_1y_{k-1},x_1y_k\}\end{aligned}$$

In this basis, all variables except x_k and y_k occur twice and x_k, y_k occur once each. Thus

$$w_\rho(\mathcal{B}_1)=2(\lambda_1+\dots+\lambda_{k-1})+2(\nu_1+\dots+\nu_{k-1})+\lambda_k+\nu_k=-(\lambda_k+\nu_k).$$

Take the second χ-basis to be

$$\mathcal{B}_2:=\{x_ky_1,x_ky_2,\dots,x_ky_i,\dots,x_ky_k,x_{k-1}y_k,x_{k-2}y_k,\dots,x_iy_k,\dots,x_1y_k\}.$$

We have

$$w_\rho(\mathcal{B}_2)=(k-1)(\lambda_k+\nu_k).$$

For any one-parameter subgroup ρ, we must have either $\lambda_k+\nu_k \geq 0$ or $\lambda_k+\nu_k \leq 0$. Thus, either $\mathcal{B}_1$ or $\mathcal{B}_2$ gives a χ-basis of non-positive weight. □

Proof of Lemma 3.3 for $m \geq 3$. We will prove the Lemma by exhibiting one collection of χ-bases whose ρ-weights sum to a positive multiple of $\lambda_k + \nu_k$ and a collection of χ-bases whose ρ-weights sum to a negative multiple of $\lambda_k+\nu_k$. Since, for any given one-parameter subgroup ρ, we have either $\lambda_k+\nu_k \geq 0$ or $\lambda_k+\nu_k \leq 0$, it follows at once that one of our χ-bases must have non-positive weight. We begin by writing down χ-bases maximizing the occurrences of x_k and y_k while balancing the occurrences of the other variables. Define T_1 as the set of degree m monomials of the ideal

$$x_k^{m-1}(y_1,\dots,y_{k-1},y_k)+x_k^{m-2}y_k(y_1,\dots,y_{k-1},y_k,x_1,\dots,x_{k-1})+\cdots \\ +x_ky_k^{m-2}(y_1,\dots,y_{k-1},y_k,x_1,\dots,x_{k-1})+y_k^{m-1}(x_1,\dots,x_{k-1}).$$

The ρ-weight of T_1 is

$$\left(k(m-1)+(2k-1)\binom{m-1}{2}\right)(\lambda_k+\nu_k)+(m-1)(\lambda_1+\nu_1+\cdots+\lambda_{k-1}+\nu_{k-1}).$$

Note that T_1 misses only the weighted degrees

$$k(m-3), k(m-5), \dots, -k(m-5), -k(m-3).$$

For each $s=1,\dots,k-1$, define

$$T_2(s) := \{x_k^{m-3}y_k(x_{k-s}x_s), x_k^{m-4}y_k^2(x_{k-s}x_s),\dots,y_k^{m-2}(x_{k-s}x_s)\}$$
$$T_2'(s) := \{y_k^{m-3}x_k(y_{k-s}y_s), y_k^{m-4}x_k^2(y_{k-s}y_s),\dots,x_k^{m-2}(y_{k-s}y_s)\}$$

For each s, the sets $T_1\cup T_2(s)$ and $T_1\cup T_2'(s)$ are χ-bases. Using the relation $\sum_{i=1}^k(\lambda_i+\nu_i)=0$, one sees at once that the sum of the ρ-weights of such bases, as s ranges from 1 to $k-1$, is a *positive multiple* of $(\lambda_k+\nu_k)$.

We now write down bases minimizing the occurrences of x_k and y_k. We handle the case when k is even and odd separately.

Case of even k: If $k = 2\ell$, we define the following set of monomials where the weighted degrees range from $k(m-1)-1$ to m:

$$S_1 := \begin{cases} \left.\begin{matrix} x_k^{m-1}y_1, & x_k^{m-2}x_{k-1}y_1, & \dots, & x_{k-1}^{m-1}y_1, \\ x_{k-1}^{m-1}y_2, & x_{k-1}^{m-2}x_{k-2}y_2, & \dots, & x_{k-2}^{m-1}y_2, \\ & \vdots & & \\ x_{\ell+2}^{m-1}y_{\ell-1} & x_{\ell+2}^{m-2}x_{\ell+1}y_{\ell-1}, & \dots, & x_{\ell+1}^{m-1}y_{\ell-1} \end{matrix}\right\} & \text{m terms in each of the $(\ell-1)$ rows} \\ \left.\begin{matrix} x_{\ell+1}^{m-1}y_\ell, & x_{\ell+1}^{m-2}x_\ell y_\ell, & \dots, & x_{\ell+1}^2 x_\ell^{m-3}y_\ell, \\ x_\ell^{m-1}y_{\ell-1}, & x_\ell^{m-2}x_{\ell-1}y_{\ell-1}, & \dots, & x_\ell^2 x_{\ell-1}^{m-3}y_{\ell-1}, \\ & \vdots & & \\ x_2^{m-1}y_1, & x_2^{m-2}x_1y_1, & \dots, & x_2^2x_1^{m-3}y_1 \end{matrix}\right\} & \text{$(m-2)$ terms in each of the ℓ rows} \end{cases}$$

Let ι be the involution of the set $\{x_i, y_i\}_{i=1}^k$ exchanging x_i and y_i. In the set $S_1 \cup \iota(S_1)$, the variables x_k and y_k occur $(m^2-m) - \binom{m}{2}$ times, $x_{\ell+1}$ and $y_{\ell+1}$ occur $(m^2-m)-1$ times, x_ℓ and y_ℓ occur $(m^2-m)-m$ times, and x_1 and y_1 occur $m^2-m-(\binom{m}{2}-1)$ times while all of the other variables occur m^2-m times. To complete $S_1 \cup \iota(S_1)$ to a χ-basis, we define, for each $s = 1, \dots, k-1$, the following set of monomials where the weighted degrees range from $m-1$ to $1-m$:

$$S_2(s) := \begin{Bmatrix} x_{\ell+1}y_\ell x_1^{m-2} & \\ x_\ell y_\ell (x_s y_s)^i x_1^{m-2i-2}, & \text{for } 0 \le 2i \le m-2, \\ x_\ell y_\ell (x_s y_s)^i y_1^{m-2i-2}, & \text{for } 0 \le 2i < m-2, \\ (x_k y_s y_{k-s})(x_s y_s)^i x_1^{m-2i-3}, & \text{for } 0 \le 2i \le m-3, \\ (x_k y_s y_{k-s})(x_s y_s)^i y_1^{m-2i-3}, & \text{for } 0 \le 2i < m-3, \\ y_{\ell+1}x_\ell y_1^{m-2}. & \end{Bmatrix}$$

For each $s = 1, \dots, k-1$, the sets $S_1 \cup \iota(S_1) \cup S_2(s)$ and $S_1 \cup \iota(S_1) \cup \iota(S_2(s))$ are χ-bases. We compute that in the union

$$\bigcup_{s=1}^{k} (S_1 \cup \iota(S_1) \cup S_2(s)) \cup (S_1 \cup \iota(S_1) \cup \iota(S_2(s)))$$

of $2(k-1)$ χ-bases the variables x_k and y_k each occurs

$$2(k-1)(m^2-m) - (k-1)(m^2-2m+2)$$

times while all of the other variables occur

$$2(k-1)(m^2-m) + (m-2)(m-1)$$

times.

Using the relation $\sum_{i=1}^{k}(\lambda_i+\nu_i)=0$, we conclude that the sum of the ρ-weights of all such χ-bases is a *negative multiple* of $(\lambda_k+\nu_k)$.

Case of odd k: If $k=2\ell+1$ is odd, χ-bases of non-positive ρ-weight can be constructed analogously to the case when k is even. For the reader's convenience, we spell out the details. We define of the following set of monomials where the weighted degrees range from $k(m-1)-1$ to $m-1$:

$$S_1 := \left\{\begin{array}{l} \left.\begin{array}{llll} x_k^{m-1}y_1, & x_k^{m-2}x_{k-1}y_1, & \ldots, & x_{k-1}^{m-1}y_1, \\ & \vdots & & \\ x_{\ell+3}^{m-1}y_{\ell-1} & x_{\ell+3}^{m-2}x_{\ell+2}y_{\ell-1}, & \ldots, & x_{\ell+2}^{m-1}y_{\ell-1} \end{array}\right\} \begin{array}{l} m \text{ terms in} \\ \text{each of the} \\ (\ell-1) \text{ rows} \end{array} \\ \left.\begin{array}{llll} x_{\ell+2}^{m-1}y_\ell, & x_{\ell+2}^{m-2}x_{\ell+1}y_\ell, & \ldots, & x_{\ell+2}^2x_{\ell+1}^{m-3}y_\ell, \\ x_{\ell+1}^{m-2}y_{\ell-1}, & x_{\ell+1}^{m-1}x_\ell y_\ell, & \ldots, & x_{\ell+1}^2x_\ell^{m-3}y_{\ell-1}, \\ & \vdots & & \\ x_3^{m-1}y_1, & x_3^{m-2}x_2y_1, & \ldots, & x_3^2x_2^{m-3}y_1 \end{array}\right\} \begin{array}{l} (m-2) \text{ terms} \\ \text{in each of the} \\ \ell \text{ rows} \end{array} \\ x_{\ell+2}y_\ell x_2^{m-2}, \\ x_{\ell+1}y_\ell x_2^{m-2}, x_{\ell+1}y_\ell x_2^{m-3}x_1, \ldots, x_{\ell+1}y_\ell x_2 x_1^{m-3}, x_{\ell+1}y_\ell x_1^{m-2} \end{array}\right.$$

Let ι be the involution exchanging x_i and y_i. In the set of monomials $S_1\cup\iota(S_1)$, the variables x_k and y_k occur $\binom{m}{2}$ times, $x_{\ell+1}$ and $y_{\ell+1}$ occur $m^2-m-(m-1)$ times, and x_1 and y_1 occur $m^2-m-\binom{m-1}{2}$ times, while all of the other variables occur m^2-m times. Finally, for each $s=1,\ldots,k-1$, we define the following set of monomials where the weighted degrees range from $m-2$ to $2-m$:

$$S_2(s) := \left\{\begin{array}{ll} x_{\ell+1}y_{\ell+1}(x_sy_s)^i x_1^{m-2-2i}, & \text{for } 0\le 2i\le m-2, \\ x_{\ell+1}y_{\ell+1}(x_sy_s)^i y_1^{m-2-2i}, & \text{for } 0\le 2i< m-2, \\ (x_ky_sy_{k-s})(x_sy_s)^i x_1^{m-3-2i}, & \text{for } 0\le 2i\le m-3, \\ (x_ky_sy_{k-s})(x_sy_s)^i y_1^{m-3-2i}, & \text{for } 0\le 2i< m-3 \end{array}\right\}$$

For each $s=1,\ldots,k-1$, the sets $S_1\cup\iota(S_1)\cup S_2(s)$ and $S_1\cup\iota(S_1)\cup\iota(S_2(s))$ are χ-bases. We compute that in the union

$$\bigcup_{s=1}^{k}\left(S_1\cup\iota(S_1)\cup S_2(s)\right)\cup\left(S_1\cup\iota(S_1)\cup\iota(S_2(s))\right)$$

of $2(k-1)$ χ-bases the variables x_k and y_k each occurs

$$2(k-1)\binom{m}{2}+2(k-1)(m-2)$$

times while all of the other variables occur

$$2(k-1)(m^2-m)+(m-2)(m-1)$$

times.

Using the relation $\sum_{i=1}^{k}(\lambda_i+\nu_i)=0$, we conclude that the total ρ-weight of these χ-bases is a *negative multiple* of $(\lambda_k+\nu_k)$ and we're done. ☐

4. Non-semistability results

The generic semistability results of Theorem 1.1 and [AFS11, Theorem 1.2] raise a natural question of whether Hilbert points of smooth canonically embedded curves can at all be non-semistable. An indirect way to see that the answer is affirmative is as follows. Denote by $\overline{H}_{g,1}^{m}$ the closure of the locus of m^{th} Hilbert points of smooth canonical curves. Next, it is proved in [HH08, Section 5] that an application of Grothendieck-Riemann-Roch formula allows to write the polarization on the GIT quotient $\overline{H}_{g,1}^{m}/\!/\,\mathrm{SL}(g)$ as a linear combination

$$(4.1)\quad (m(m-1)(4g+2)-(m-1)(g-1)+1)\lambda-\frac{gm(m-1)}{2}\delta \\ \sim\left[8+\frac{4}{g}-\frac{2(g-1)}{gm}+\frac{2}{gm(m-1)}\right]\lambda-\delta$$

of a tautological divisor λ (the first Chern class of the Hodge bundle) and the boundary divisor δ (at least on the locus parameterizing curves with mild singularities). By generalizing the proof of [CH88, Proposition 4.3], it is not too difficult to see that if $B\to\overline{\mathcal{M}}_g$ is a family of stable curves whose general fiber is canonically embedded and the slope $(\delta\cdot B)/(\lambda\cdot B)$ is greater than $(8+\frac{4}{g})-\frac{2(g-1)}{gm}+\frac{2}{gm(m-1)}$, then every curve in B (with a well-defined m^{th} Hilbert point) must have a non-semistable m^{th} Hilbert point.

Two observations now lead to a candidate for a non-semistable canonically embedded curve. The first is that $(8+\frac{4}{g})-\frac{2(g-1)}{gm}+\frac{2}{gm(m-1)}\leq 8$ for $g\geq 2m+1+1/(m-1)$. The second is that there are families of bielliptic curves of slope 8 (such can be constructed by taking a double cover of a trivial family of elliptic curves). In the following result, we establish that bielliptic curves indeed become non-semistable for small values of m, and show that generic bielliptic curves are semistable for m large enough.

Theorem 4.1. *A smooth bielliptic curve of genus g has non-semistable m^{th} Hilbert point for all $m\leq(g-3)/2$. A general bielliptic curve of odd genus $g=2k+1$ has semistable m^{th} Hilbert point for $m\geq(g-1)/2$.*

Proof. Let C be a bielliptic canonical curve. Then C is a quadric section of a projective cone over an elliptic curve $E\subset\mathbb{P}^{g-2}$ embedded by a complete linear system of degree $g-1$. Choose projective coordinates $[x_0:\ldots:x_{g-1}]$. Suppose that the vertex of the cone has coordinates $[0:0:\ldots:0:1]$. Let ρ be the one-parameter subgroup of $\mathrm{SL}(g)$ acting with weights $(-1,-1,\ldots,-1,g-1)$. There are

$$s_m := h^0(\mathbb{P}^{g-2}, \mathcal{O}_{\mathbb{P}^{g-2}}(m)) - h^0(E, \mathcal{O}_E(m)) = \binom{g-2+m}{m} - m(g-1)$$

degree m hypersurfaces containing E. Thus

$$\dim \mathrm{H}^0(C, \mathcal{I}_C(m)) \cap (x_0, x_2, \ldots, x_{g-2})^m = s_m$$

and so there are at most

$$h^0(C, \mathcal{O}_C(m)) - s_m = m(g-1)$$

elements in $\mathrm{H}^0(C, \mathcal{O}_C(m))$ of ρ-weight $(-m)$. The remaining $(m-1)(g-1)$ elements in $\mathrm{H}^0(C, \mathcal{O}_C(m))$ have ρ-weight at least $g-m$. Thus the ρ-weight of any monomial basis of $\mathrm{H}^0(C, \mathcal{O}_C(m))$ is at least

$$(m-1)(g-1)(g-m) - m(m(g-1)) = (g-1)((g+1)m - 2m^2 - g). \tag{4.2}$$

If $m \leq (g-3)/2$, then (4.2) is positive, and so C has a non-semistable m^{th} Hilbert point.

To prove the generic semistability of bielliptic curves in the range $m \geq (g-1)/2$, we recall [AFS11, Theorem 4.12] which shows that the odd genus g canonically embedded rosary has a semistable m^{th} Hilbert point if and only if $g \leq 2m+1$. It remains to observe that the canonically embedded rosary deforms to a canonically embedded smooth bielliptic curve in the Hilbert scheme of canonically embedded curves. This is accomplished in Lemma 4.2 below. □

Lemma 4.2. *The canonically embedded rosary deforms flatly to a canonically embedded bielliptic curve.*

Proof. Let C be the rosary of genus $g = 2k+1$ introduced by Hassett and Hyeon [HH08, Section 8.1]. We use the notation of [AFS11, Section 3.2].

Consider $\mathbb{P}^{g-2}$ with projective coordinates $[x_0 : \ldots : x_{g-2}]$ and define $E \subset \mathbb{P}^{g-2}$ to be the union of $g-1$ lines $L_i : \{x_{i+1} = \cdots = x_{i+g-3} = 0\}$, for $i = 0, \ldots, g-2$ (we use the convention that $x_{i+g-1} = x_i$). Then E is a nodal curve of arithmetic genus 1. Since $\mathrm{H}^1(C, \mathcal{O}_C(1)) = 0$, we can deform E in a flat family to a smooth elliptic curve by [Kol96, p.83]. Using the basis $(\eta, \omega_0, \ldots, \omega_{g-2})$ of $\mathrm{H}^0(C, \omega_C)$ described in [AFS11, Lemma 3.6], we observe that the rosary C is cut out by the quadric

$$y^2 = x_0x_1 + x_1x_2 + \cdots + x_{g-2}x_0$$

on the projective cone over E in $\mathbb{P}^{g-1}$. Since E deforms to a smooth elliptic curve, it follows that C deforms to a smooth bielliptic curve. □

Remark 4.3 (Trigonal curves of higher Maroni invariant)**.** Theorem 1.2 shows that the general trigonal curve with Maroni invariant 0 has a semistable m^{th} Hilbert point for all $m \geq 2$. In joint work of the second author with Jensen, it is shown that every trigonal curve with Maroni invariant 0 has a semistable 2^{nd} Hilbert point and every trigonal curve with a positive Maroni invariant has a non-semistable 2^{nd} Hilbert point [FJ11]. In view of the asymptotic stability of the canonically embedded curves [Mum77], this result suggests that *every* smooth trigonal curve of Maroni invariant 0 has a semistable m^{th} Hilbert point for every $m \geq 2$. One also expects that for a general smooth trigonal curve of positive Maroni invariant already the third Hilbert point is semistable. Indeed, Equation 4.1 shows that the polarization on $\overline{H}^3_{g,1}$ is a multiple of

$$\left(\frac{22}{3}+\frac{5}{g}\right)\lambda-\delta. \tag{4.3}$$

On the other hand, the maximal possible slope for a family of generically smooth trigonal curves of genus g is $36(g+1)/(5g+1)$ by [SF00]. We note that

$$36(g+1)/(5g+1) \leq \left(\frac{22}{3}+\frac{5}{g}\right)$$

whenever $(g-3)(2g-5) \geq 0$. Thus we expect that the 3^{rd} Hilbert point of a genus $g \geq 4$ canonically embedded trigonal curve is stable.

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