TABULATION OF THE LEGENDRIAN KNOTS BY AMBIENT ISOTOPY

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ABSTRACT. This paper discusses in detail some of the major results known about Legendrian knots, with a primary focus on obtaining a tabulation of the knots. The Legendrian knots of up to 9 crossings are tabulated, and the reasons for the organization chosen are discussed. Further plans for research and extension of this table are discussed.

1. Introduction

A Legendrian Knot is an embedding of a simple closed curve into $\mathbb{R}^3$ which is everywhere tangent to the standard contact structure.

Definition 1. *A* **contact structure** *is a nonintegrable field of planes in* $\mathbb{R}^3$*. The* **standard contact structure** *is the field of planes which corresponds to the kernel of the differential 1-form* $\lambda = dz - ydx$*.*

Since the Legendrian knots are tangent to this contact structure, $y = \frac{dz}{dx}$. This additional condition on the knot is the reason Legendrian knots are studied separately from normal topological knots. Of course, each Legendrian knot is a topological knot in the normal sense, as it is still an embedding of S^1 into $\mathbb{R}^3$. The Legendrian knots are therefore an interesting subclass of the topological knots.

In this sense, Legendrian knots have been studied in a similar manner to the standard knots. Research has been done on connected sums of Legendrian knots, mirrors of Legendrian knots, isotopy of Legendrian knots, and, of course, knot invariants. This paper will focus on the two simplest invariants, the Bennequin and Maslov numbers.

One of the most significant advances in early knot theory was the tabulation of topological knots and their mirrors by Tait and Little in the late nineteenth century. The early knot theorists took advantage of these tables to look for patterns in knots and to provide examples. As knot theory grew as a field, these tables were increased - first by adding more complex knots and later by tabulating invariants of these knots. Topological knots up to 16 crossings have been tabulated by Hoste and Thistlewaite in [HT], along with their polynomials and other invariants. This tremendous resource provides a reference for knot theorists, but no similar reference exists for those working with Legendrian knots.

This paper produces a preliminary such table of the Legendrian knots up to 9 crossings. Due to the time constraints of this project, there is still much work to be done on this table. Plans for such future work are detailed in the last part of the paper.

2. Projections of Legendrian Knots

Although topological knots are generally projected on to whatever plane makes the projection simplest, the rigid structure of the Legendrian knots means that projection onto xz plane generally makes the diagram simple and easy to understand. These projections are called fronts, and in fact are not permissible projections for non-Legendrian knots. One of the requirements for a standard knot projection is that it can have no singular points other than those that represent the crossings of the knot. However, a front projection of a Legendrian knot must necessarily have singular points, in the form of cusps. Because the knot must be tangent to the contact structure, the value of y is always $\frac{dz}{dx}$, the slope of the line in the xz plane. Therefore, if the knot had a vertical tangency, this slope would be undefined, and thus the value of y would be undefined. Therefore, where a normal knot would have a vertical tangency, the front projection has a cusp as the curve turns through the y direction.

Another significant difference between a standard projection and the front projection is that there is no need to denote overpasses and underpasses, as they follow directly from the definition of the knot. Namely, at any crossing, the line with lesser slope has a lesser y value, and thus passes over (in front of, in a sense) the other line.

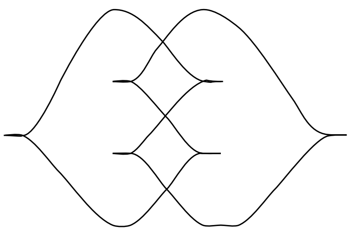

Figure 1. A front projection of the Legendrian trefoil knot 3_1

The knot shown in Figure 1 is ambient isotopic to the trefoil. Since ambient isotopy (the traditional isotopy used by knot theorists) involves creating a homotopy in the space of knots, one would expect a similar equivalence relation to exist for Legendrian knots. This relation exists, and is called Legendrian isotopy. There exist a set of Legendrian Reidemeister moves, with the same property as the traditional Reidemeister moves - if a knot can be continuously deformed into another using only these moves and deformations which do not affect crossings, then they are Legendrian isotopic. The Legendrian Reidemeister moves are presented in Figure 2.

Since the Legendrian knots are a subset of the standard sets, two Legendrian isotopic knots are necessarily ambient isotopic. However, the converse is not true in general. A counterexample, in the form of two Legendrian non-isotopic knots with the same isotopy class (specifically 5_2[1]) was given by Chekanov in [Che]. Although the proof that they are not Legendrian isotopic is well beyond the scope of this paper, the two fronts are presented in Figure 3.

[1]All references to ambient isotopy classes of knots refer to the tabulation of Rolfsen in [Rol]

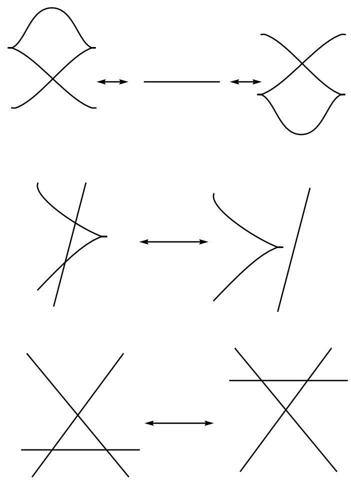

FIGURE 2. The Legendrian Reidemeister moves.

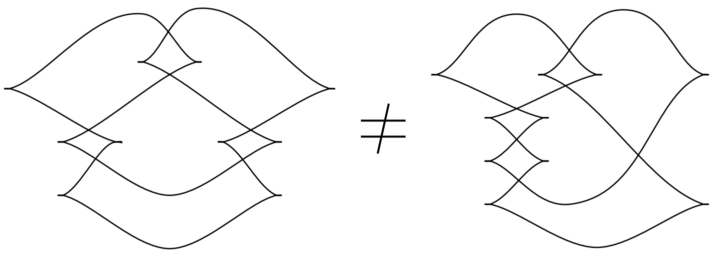

FIGURE 3. Two ambient isotopic knots which are not Legendrian isotopic.

3. LEGENDRIAN KNOT INVARIANTS

The classical invariants of Legendrian isotopy are the Bennequin number and the Maslov Number. The Bennequin number is defined as the number of positive crossings minus the number of negative crossings minus half the number of cusps, where positive crossings and negative crossings are defined in the same way as for the writhe of a knot. As with the writhe, the orientation assigned to the knot is arbitrary - assigning the opposite orientation will not change the Bennequin number. The Bennequin number is denoted by β, and its formula is given in Figure 4.

Theorem 3.1. *The Bennequin number β is a Legendrian isotopy invariant.*

Proof. We approach this proof by examining the three Legendrian Reidemeister moves.

If the first move is performed, either two cusps and a positive crossing are lost, producing a net change of zero, or two cusps and a positive crossing are gained,

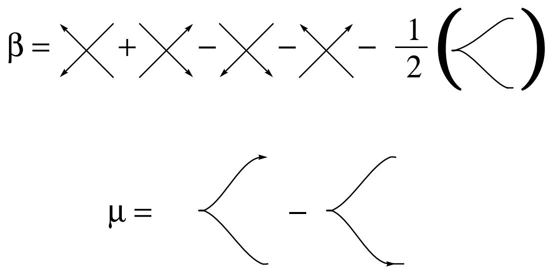

FIGURE 4. Formulae for the Bennequin and Maslov numbers, where the crossing or cusp drawn denotes the number of crossings or cusps of that type in the front projection.

producing a net change of zero. Thus, the first Legendrian Reidemeister move does not change the Bennequin number.

If the second move is performed, a positive crossing and a negative crossing are lost, producing a net change of zero. This also fails to alter the Bennequin number.

If the third move is performed, a positive crossing and a negative crossing are lost, and a positive and negative crossing are gained, also producing a net change of zero. ■

The second classical invariant of the Legendrian knots is the Maslov number. The Maslov number is defined by assigning an orientation (reversing the orientation changes the sign of the Maslov number) and subtracting the number of cusps on which the orientation is downward (in the z direction) from the number of cusps on which the orientation is upward. The formula for the Maslov number, denoted by μ, is also given in Figure 4.

Theorem 3.2. *The Maslov number μ is an invariant under Legendrian isotopy.*

Proof. Since the second and third Legendrian Reidemeister moves do not alter the number of upward or downward cusps, all we need to check is invariance under the first move. Regardless of orientation, the first Legendrian Reidemeister move either removes or adds one upward and one downward cusp. Neither the addition or the removal changes the Maslov number. Therefore the Maslov number is an invariant of Legendrian isotopy. ■

Although there exist other invariants for the Legendrian knots, such as the Chekanov differential graded algebras, they are difficult to define and use.

4. Known Results

This section will serve as a summary of known results, with specific focus on the upper bounds of β and μ for knots of a specific ambient isotopy type. As these results frequently call for proofs from higher level algebraic topology and contact geometry, they will be stated without proof with references to works where they are proven.

Since the Bennequin and Maslov numbers are invariants of Legendrian isotopy, and not ambient isotopy, it is clear that neither invariant can be stated as a constant for any particular ambient isotopy type. Specifically, by adding cusps in what is called a stabilization [Ng3], as shown in Figure 5, both β and μ are changed without altering the ambient isotopy type of the knot (the stabilization is equivalent to either a continuous deformation of the knot or the first Reidemeister move on a loop in the xy plane).

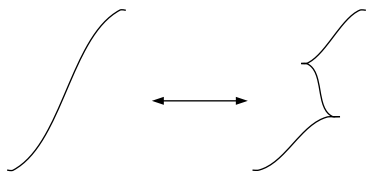

FIGURE 5. A stabilization of the Legendrian knot reduces Bennequin number by 1 and changes the Maslov number by ± 2

However, there do exist bounds for the Bennequin number. Since the stabilization shows us that the Bennequin number can be decreased without bound without changing the ambient isotopy, there can be no lower bound. However, several upper bounds do exist, some for all knots and some for specific knots. These upper bounds are detailed below.

Theorem 4.1. *If $e_y(\gamma)$ denotes the minimal degree of the HOMFLY polynomial for the Legendrian knot γ, then*

$$\beta(\gamma) + \mid \mu(\gamma) \mid < e_y(\gamma)$$

for all Legendrian knots ambient isotopic to γ.

While this theorem, stated in [FT], does give us an inequality, since it depends on both values, it can only give us a bound on one if the other is fixed. Another theorem, gives us a bound on the Bennequin number as a result of Theorem 4.1.

Theorem 4.2. *The Bennequin number for any Legendrian knot is less than the minimal degree in a in the normalized Kauffman polynomial of the knot.*

Since the Kauffman polynomial is ambient isotopy invariant, this relates the Legendrian invariants to the ambient isotopy invariants. A slightly less tight bound, using the minimal degree in a of the normalized Kauffman polynomial modulo 2, was proven in [FT], while this result was proven in [Tab].

For a $(p, -q)$ torus knot $p, q > 0$, Etnyre and Honda note in [EH] that this provides a bound, computed by Epstein in [Eps], of

$$\beta(\gamma) \leq \begin{cases} \text{-pq} & \text{if } q \text{ is even} \\ \text{-pq + p - q} & \text{if } q \text{ is odd} \end{cases}$$

In the same paper, though, Etnyre and Honda show that, while this bound is sharp if q is even, it is not sharp in the cases where q is odd. In fact, they show that the bound for q even applies to all negative torus knots.

Theorem 4.3. *If γ is a $(p, -q)$ torus knot with p,q positive, then*

$$\beta(\gamma) \leq -pq$$

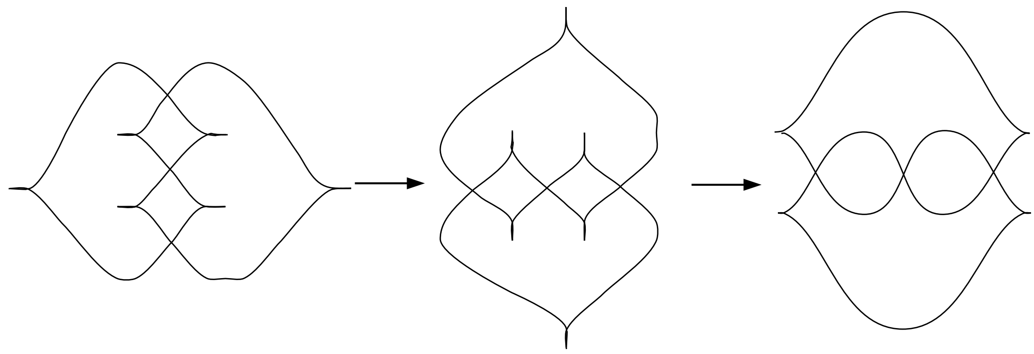

FIGURE 6. Finding the mirror image of the Legendrian trefoil knot 3_1

This shows that the bound given in 4.2 is not sharp for the negative torus knots (The first knot for which this inequality applies is 8_19, the $(4,-3)$ torus knot, included in Appendix A). In fact, Ferrand shows in [Fer] that both this bound and the bound in 4.1 are not sharp.

5. Mirrors of Legendrian Knots

Another question which arises in the tabulation of Legendrian knots is the question of mirror images. A Legendrian knot can be thought of as having two mirrors. One is commonly called the Legendrian mirror.

Definition 2. *The* **Legendrian mirror** *is the image of the knot under the diffeomorphism* $\{x, y, z\} \to \{x, -y, -z\}$ [Ng1].

Since the Legendrian mirror is just a rotation of the knot in space, it has the same ambient isotopy type as the knot itself. Although the Legendrian mirror has some interesting properties, as we are only concerned with a tabulation of ambient isotopy types, we will ignore it for now.

The other mirror of a Legendrian knot is the Legendrian knot of the standard mirror, which we will call the ambient mirror. Since the mirror image in ambient isotopy can be found by switching the overpass and underpass of each crossing of the knot, we can find the ambient mirror of a given Legendrian knot fairly easily.

Since crossings of Legendrian knots are determined by the slopes of the lines at each crossing, the mirror image of a Legendrian knot will interchange these slopes. This can be accomplished by rotating the diagram of the front by $\frac{\pi}{2}$ and interchanging caps and cusps, as shown in Figure 6.

6. Tabulations of Knots

As mentioned in the introduction, the first men to tabulate knots were Tait and Little. Tait cataloged the alternating knots, while Little first created tables of the nonalternating knots. Both tables organized knots by crossing number, because the knots with small numbers of crossings are simple compared to those with large numbers. The larger the number of crossings, the more possible knots with that crossing number there are.

However, beyond the initial number of crossings, the knots were classified in many different ways. In general, further classification depended on the notation used by the individual, and the order in which the diagram was discovered.

In creating a tabulation of the Legendrian Knots, I felt that the diagrams contained within the table should be those with minimum crossing number and maximal Bennequin number. Note that this also ensures that the number of cusps is minimal, as the sum of the crossings is invariant for all minimal crossing number diagrams. However, I also wished that the ambient mirror diagram, as produced in Figure 6, should have maximal Bennequin number. While I was able to do this for all knots of ambient isotopy class up to 9 crossings, I have yet to find a proof that this can actually be done for all ambient isotopy types. Therefore, I offer it as a conjecture.

Conjecture 1. *There exists a Legendrian knot front for every ambient isotopy class such that the knot has maximal Bennequin number and the ambient mirror front produced as in Figure 6 also has maximal Bennequin number for its ambient isotopy class*

My initial impulse was to organize the knots by maximal Bennequin number after the initial organization by crossing number. However, after looking at the tables of maximal Bennequin numbers in [Ng2], I realized that there was no justification for such an organization.

The main reason against organization by maximal bennequin number is convention. Ng organizes his table (which provides only the maximal Bennequin numbers $\overline{\beta}$, and no diagrams) using the numbering of [Rol] because Rolfsen's text has become a standard reference for knot tables. Using an already known system facilitates communication between knot theorists, and thus makes the table more useful.

7. Areas for Future Work

Due to the limited time frame that this project was produced under, as well as the difficult nature of the underlying material, I spent much more of my time this semester uncovering the basic material needed to begin this project than I wished. That being said, there is still tremendous space to expand this project to a much larger document. Future avenues of research that I plan to pursue include:

- Expanding the table given in Appendix A to knots of 10 crossings. This was one of my initial goals for the project this semester, but was left off due to the large number of 10 crossing knots and the amount of time that finding maximal Bennequin number diagrams of the 9 crossing knots took.
- Attempting to prove Conjecture 1.
- Seeking a way of computing Chekanov's Poincaré polynomial ([Che]) that is practical enough to incorporate into the table.

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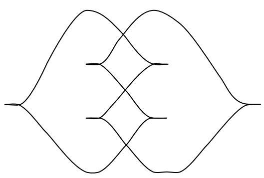
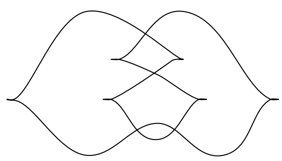
Appendix A. A Tabulation of Legendrian Knots

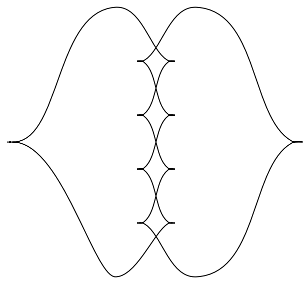
The knot diagrams in this appendix are numbered as in [Rol], and the identification of mirrors is from the same. The Bennequin number for the diagram, $\overline{\beta}$, is guaranteed to be maximal by Theorems 4.2 and 4.3 with the single exception of the mirror of 9_{42}. Although no better diagram exists than the given one, with maximal Bennequin number -5, the bound in Theorem 4.2 only guarantees that $\overline{\beta} \leq -3$. This table shows that the bound in 4.2 is sharp for all knots of 9 crossings or less, With the exception of 9_{42} and 8_19. The computed values of the maximal Bennequin number for these knots verifies the table in [Ng2] with one exception - the maximal Bennequin number for the knot and its mirror image are switched for 9_15. A similar table in [Tan] which covers knots up to 8 crossings is also verified.



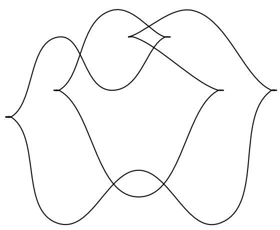
$3_1, \overline{\beta} = -6$



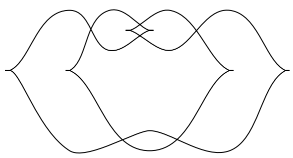
$4_1, \overline{\beta} = -3$



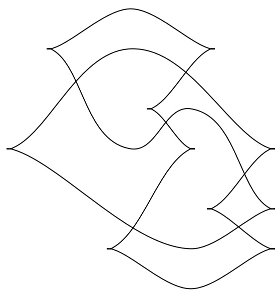
$5_1, \overline{\beta} = -10$



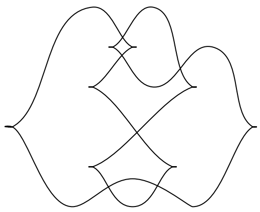
$5_2, \overline{\beta} = -8$



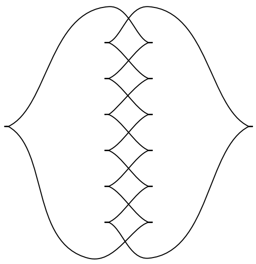
$6_1, \overline{\beta} = -5$



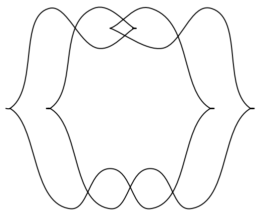
$6_2, \overline{\beta} = -7$



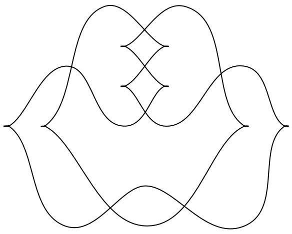
$6_3, \overline{\beta} = -4$



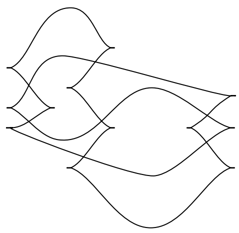
$7_1, \overline{\beta} = -14$



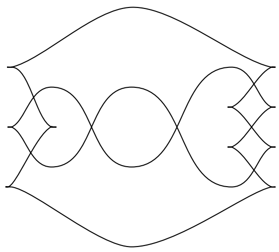
$7_2, \overline{\beta} = -10$



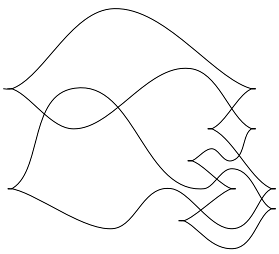
$7_3, \overline{\beta} = 3$



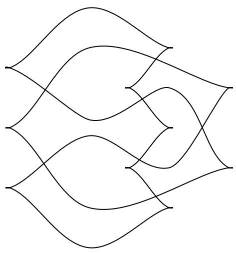
$7_4, \overline{\beta} = 1$



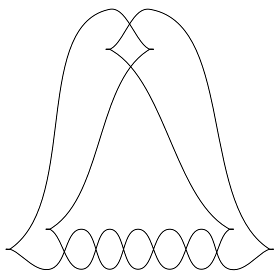
$7_5, \overline{\beta} = -12$



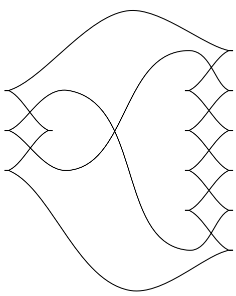
$7_6, \overline{\beta} = -8$



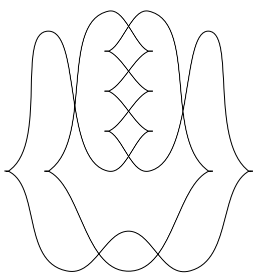
$7_7, \overline{\beta} = -4$



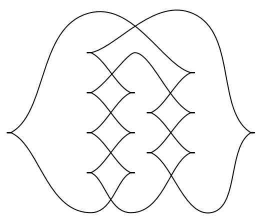
$8_1, \overline{\beta} = -7$



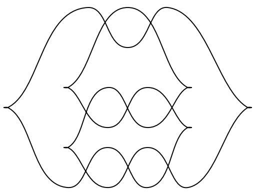
$8_2, \overline{\beta} = -11$



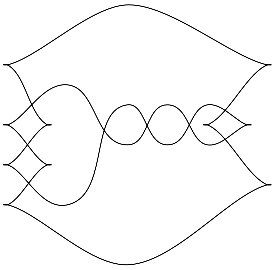
$8_3, \overline{\beta} = -5$



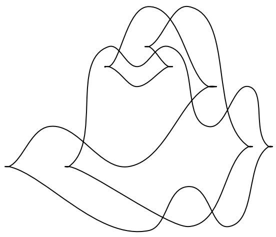
$8_4, \overline{\beta} = -7$



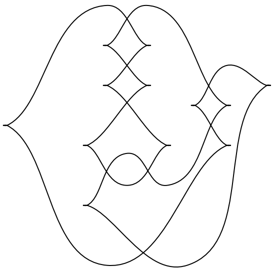
$8_5, \overline{\beta} = 1$



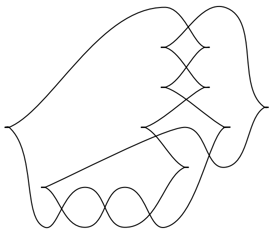
$8_6, \overline{\beta} = -9$



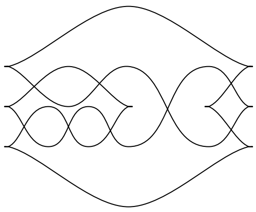
$8_7, \overline{\beta} = -2$



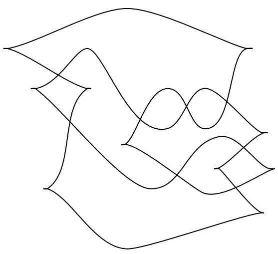
$8_8, \overline{\beta} = -4$



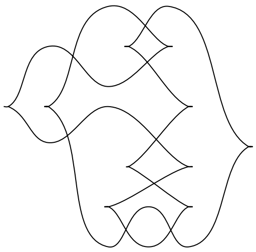
$8_9, \overline{\beta} = -5$



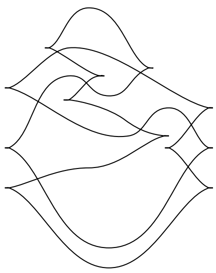
$8_{10}, \overline{\beta} = -2$



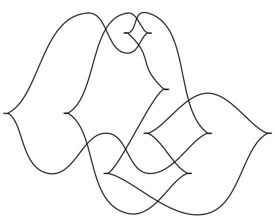
$8_{11}, \overline{\beta} = -9$



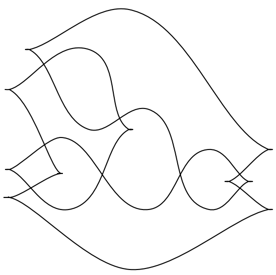
$8_{12}, \overline{\beta} = -5$



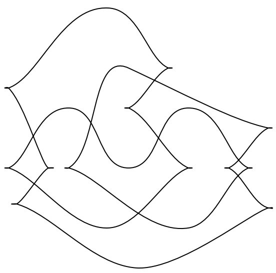
$8_{13}, \overline{\beta} = -4$



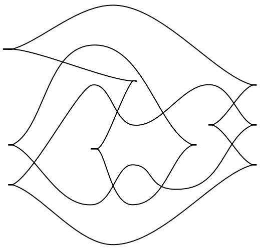
$8_{14}, \overline{\beta} = -9$



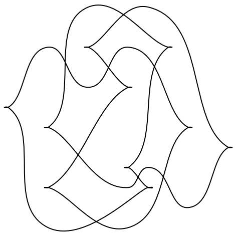
$8_{15}, \overline{\beta} = -13$



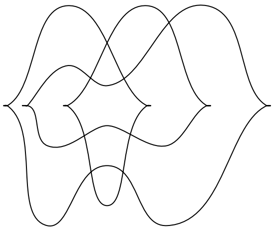
$8_{16}, \overline{\beta} = -8$



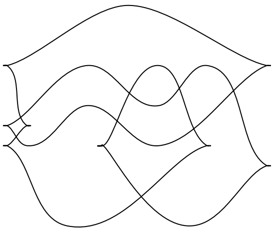
$8_{17}, \overline{\beta} = -5$



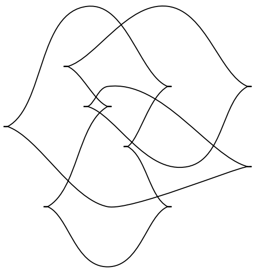
$8_{18}, \overline{\beta} = -5$



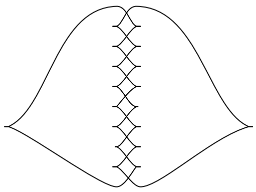
$8_{19}, \overline{\beta} = 5$



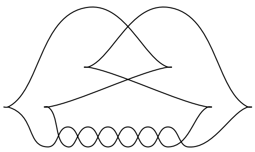
$8_{20}, \overline{\beta} = -6$



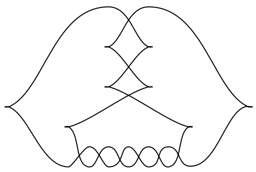
$8_{21}, \overline{\beta} = -9$



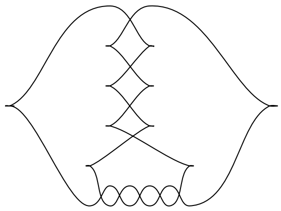
$9_{1}, \overline{\beta} = -18$



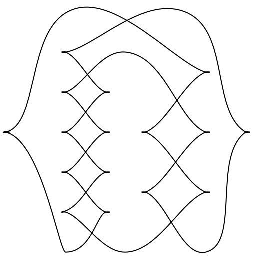
$9_{2}, \overline{\beta} = -12$



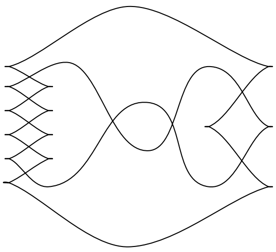
$9_{3}, \overline{\beta} = 5$



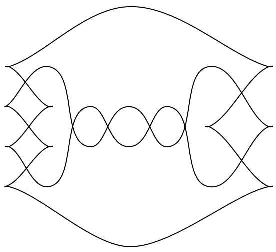
$9_{4}, \overline{\beta} = -14$



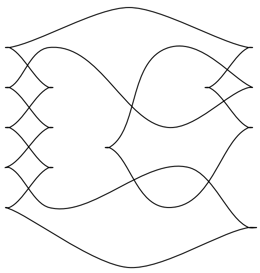
$9_{5}, \overline{\beta} = 1$



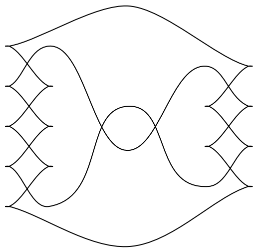
$9_{6}, \overline{\beta} = -16$



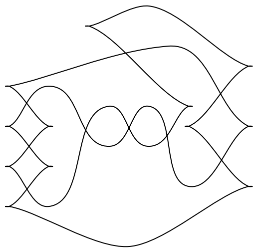
$9_{7}, \overline{\beta} = -14$



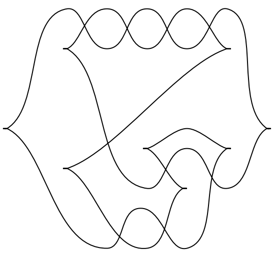
$9_8, \overline{\beta} = -8$



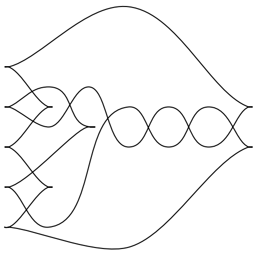
$9_9, \overline{\beta} = -16$



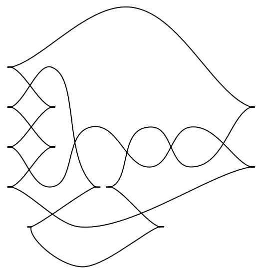
$9_{10}, \overline{\beta} = 3$



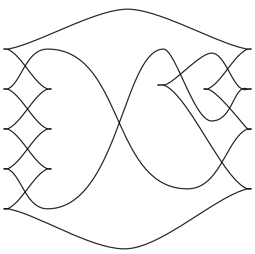
$9_{11}, \overline{\beta} = 1$



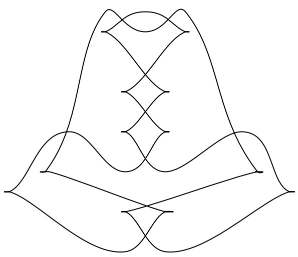
$9_{12}, \overline{\beta} = -10$



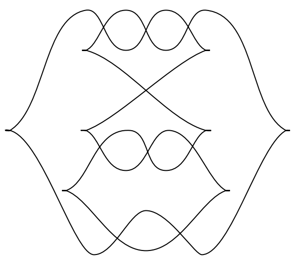
$9_{13}, \overline{\beta} = 3$



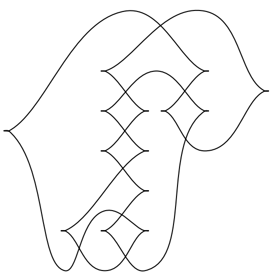
$9_{14}, \overline{\beta} = -4$



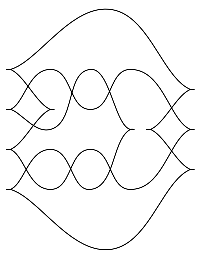
$9_{15}, \overline{\beta} = -10$



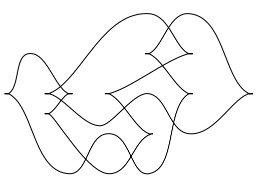
$9_{16}, \overline{\beta} = 5$



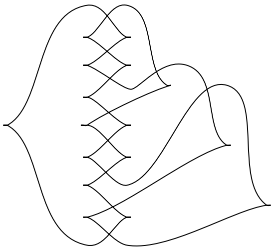
$9_{17}, \overline{\beta} = -8$



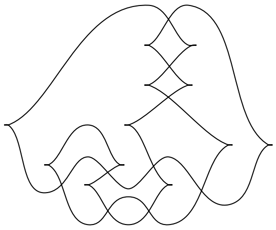
$9_{18}, \overline{\beta} = -14$



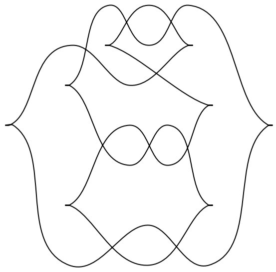
$9_{19}, \overline{\beta} = -6$



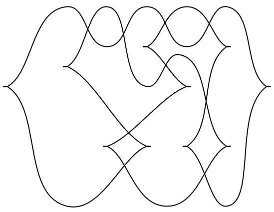
$9_{20}, \overline{\beta} = -12$



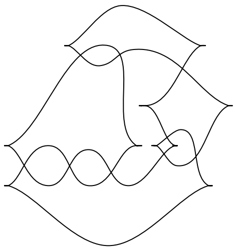
$9_{21}, \overline{\beta} = -1$



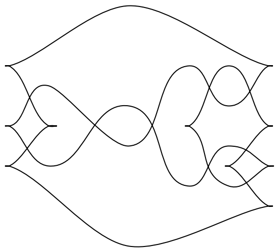
$9_{22}, \overline{\beta} = -3$



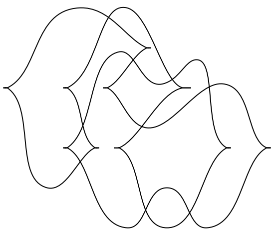
$9_{23}, \overline{\beta} = -14$



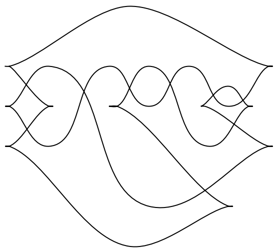
$9_{24}, \overline{\beta} = -6$



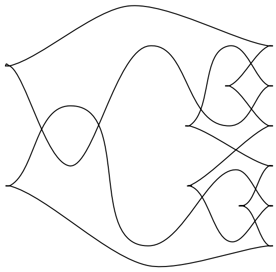
$9_{25}, \overline{\beta} = -10$



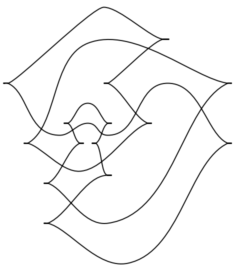
$9_{26}, \overline{\beta} = -2$



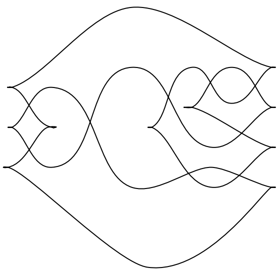
$9_{27}, \overline{\beta} = -6$



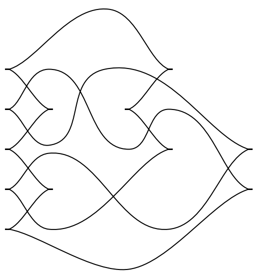
$9_{28}, \overline{\beta} = -9$



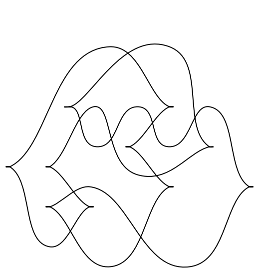
$9_{29}, \overline{\beta} = -8$



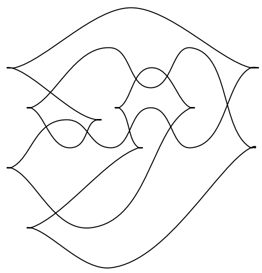
$9_{30}, \overline{\beta} = -6$



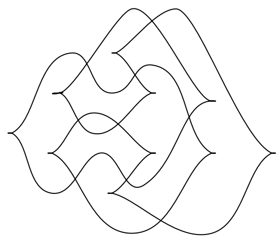
$9_{31}, \overline{\beta} = -9$



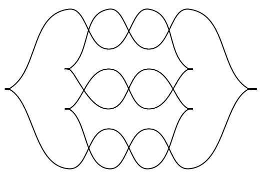
$9_{32}, \overline{\beta} = -2$



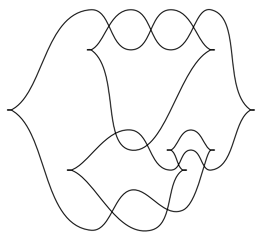
$9_{33}, \overline{\beta} = -6$



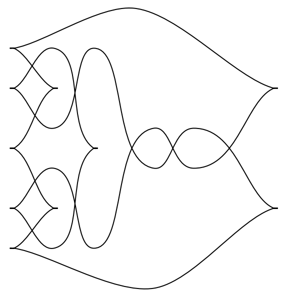
$9_{34}, \overline{\beta} = -6$



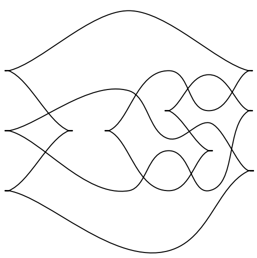
$9_{35}, \overline{\beta} = -12$



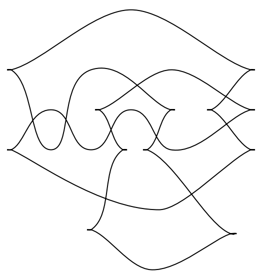
$9_{36}, \overline{\beta} = 1$



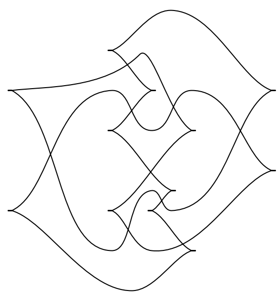
$9_{37}, \overline{\beta} = -6$



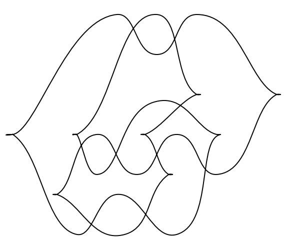
$9_{38}, \overline{\beta} = -14$



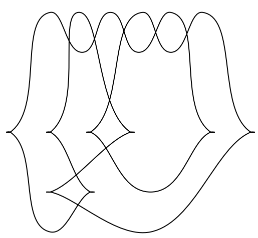
$9_{39}, \overline{\beta} = -1$



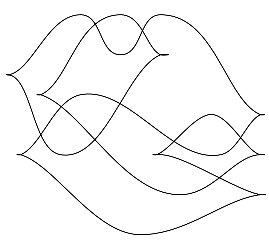
$9_{40}, \overline{\beta} = -9$



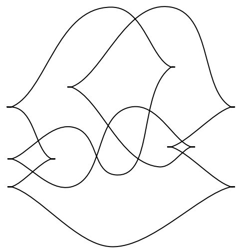
$9_{41}, \overline{\beta} = -7$



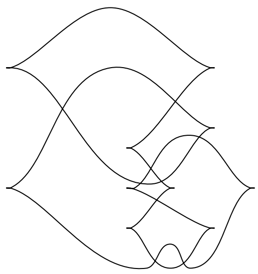
$9_{42}, \overline{\beta} = -3$



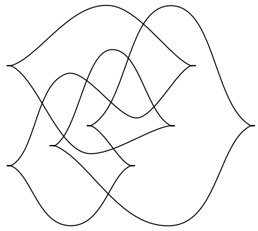
$9_{43}, \overline{\beta} = 1$



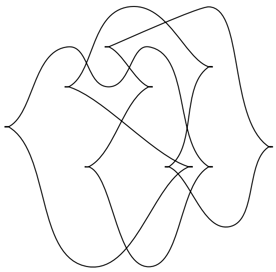
$9_{44}, \overline{\beta} = 6$



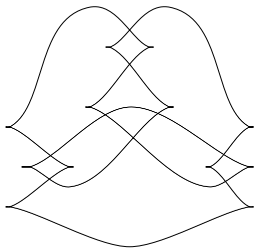
$9_{45}, \overline{\beta} = -10$



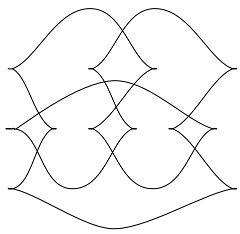
$9_{46}, \overline{\beta} = -7$



$9_{47}, \overline{\beta} = -2$



$9_{48}, \overline{\beta} = -1$



$9_{49}, \overline{\beta} = 3$