These notes are based heavily on the introduction to Morse Homology in Salamon's "Introduction to Floer Homology" [Sal99] and Akaho's "A Crash Course of Floer Homology for Lagrangian Intersections" [Aka]. McDuff provides a brief sketch of Lagrangian Floer homology in [McD06]. Ono also provides a brief introduction in [Ono06], and goes on to discuss the obstruction to existence.

1 Introduction & Motivation

Lagrangian Floer homology is an infinite-dimensional generalization of Morse homology, first studied by Floer in 1988. Floer was interested in the Arnold Conjecture (now a theorem of Floer, Hofer-Salamon, Ono, Lu-Tian, Fukaya-Ono, and Ruan):

Theorem 1. *(Arnold Conjecture) Let (M, ω) be a compact symplectic manifold and H_t be a smooth time-dependent 1-periodic ($H_t = H_{t+1}$) Hamiltonian function on M with nondegenerate 1-periodic solutions (if ψ_t is the flow of X_{H_t}, nondegenerate just means that a 1-periodic solution x satisfies $\det(\mathbb{1} - d\psi_1(x(0))) \neq 0$). Then the number of 1-periodic solutions is bounded below by the sum of the Betti numbers of M.*

Exercise 1. If $H_t = H$ is independent of time, then H is a Morse function and the Arnold conjecture follows from Morse theory.

In 1988, Floer proved a related result about intersections of Lagrangian submanifolds using what we now call Lagrangian Floer homology:

Theorem 2. *Let (M, ω) be a compact symplectic manifold, L a Lagrangian submanifold, and ϕ_t the flow of a Hamiltonian function H. If $\int_{D^2} u^*(\omega) = 0$ for all $u : D^2 \to M$ with $\partial(u(D^2)) \subset L$ and L is transverse to $\phi_1(L)$, then the number of elements in $\{L \cap \phi_1(L)\}$ is bounded below by the sum of the ($\mathbb{Z}_2$) Betti numbers of L.*

2 Morse Homology

A Morse function on a manifold M is a smooth function $f : M \to \mathbb{R}$ with nondegenerate critical points. A critical point x_0 is nondegenerate if the Hessian - the matrix of second derivatives - is nonsingular at that point. Every critical point of a Morse function can be assigned a Morse index, which is just the number of negative eigenvalues of the Hessian and is denoted by $\mu(x_0)$. Given a metric g on M, you can define the negative gradient flow φ_s to be the flow of $-\nabla f$ with respect to g. Then you can define the stable (unstable) manifold of a critical point x_0, $W^s(x_0)$ ($W^u(x_0)$), to be the points z in M such that $\lim_{s\to\infty} \varphi_s(z) = x_0$ ($\lim_{s\to-\infty} \varphi_s(z) = x_0$). The gradient flow is called Morse-Smale if the stable and unstable manifolds of any two critical points intersect transversely. Then we have

Lemma 1. *$\mathcal{M}(y, x) = W^s(x) \cap W^u(y)$, the space of negative gradient flow lines from y to x, is a smooth submanifold of M of dimension $\mu(y) - \mu(x)$.*

Note that each of these gradient flow lines has an $\mathbb{R}$ action (given by $s \to (s + t)$), so we can quotient out by this action to get the moduli space $\hat{\mathcal{M}}(y, x)$. Therefore, if $\mu(y) - \mu(x) = 1$, the reduced space is a 0-dimensional manifold - a disjoint union of points. If this is compact, we can count the number of points(either $\mathbb{Z}_2$ or with sign).

The space $\hat{\mathcal{M}}(y, x)$ can be compactified by adding "broken" trajectories - that is, gradient flow lines from y to x which pass through critical points $x = x_0, x_1, \ldots x_m = y$. The lemma above and the fact that each moduli space has an $\mathbb{R}$ action (this implies that $\mathcal{M}(y, x) = \varphi$ if

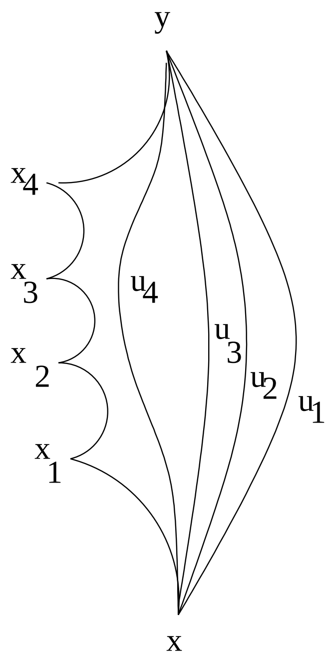

Figure 1: Sequence of gradient trajectories converging to a broken trajectory.

$\mu(y) - \mu(x) \leq 0$) then show that $\mu(x) < \mu(x_1) < \ldots < \mu(y)$. Therefore, if $\mu(y) - \mu(x) = 1$, there are no broken trajectories and $\mathcal{M}(y, x)$ is compact.

Exercise 2. Prove this: namely, show that for any sequence $u^i \in \mathcal{M}(y, x)$, there exists a subsequence (also called u^i), $x = x_0, x_1, \ldots x_m = y$, gradient lines $v_j \in \mathcal{M}(x_j, x_{j+1})$, and sequences $s^i_j \in \mathbb{R}$ such that, $\forall j$, $u^i(t + s^i_j)$ converges to $v_j(t)$ uniformly on compact subsets (see Figure **??**).

We then can construct a homology complex generated by the critical points of f, with boundary operator

$$\partial(y) = \sum_{\substack{x \in \mathrm{Crit}(f) \\ \mu(y)-\mu(x)=1}} \sharp\{u \in \hat{\mathcal{M}}(y, x)\} x$$

$\partial^2 = 0$ exactly because the 1-dimensional part of the reduced space has once-broken trajectories in its boundary, as in Figure **??**. ∂^2 counts once-broken trajectories, but we can also show by a gluing argument that any once-broken trajectory is in the compactification of the 1-dimensional part of the reduced space. Since a 1-manifold has an even number of boundary points (or because the signs of the boundary of the 1-manifold cancel properly), we have $\partial^2 = 0$, and thus we can define the homology of this chain complex by

$$\mathrm{HM}_*(M, f, \mathbb{Z}_2) = \frac{\ker(\partial)}{\mathrm{im}\ (\partial)}$$

Morse theory show that there is a natural isomorphism from this homology to the singular homology.

Exercise 3. Compute the Morse homology of the torus using this Morse-Smale gradient flow in Figure **??**

For more details, see Schwarz's book on Morse homology [Sch93].

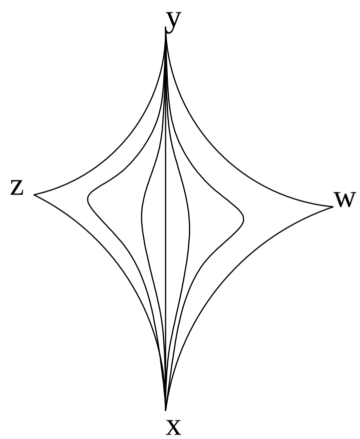

Figure 2: $\partial^2 = 0$.

3 Floer Homology

Floer used the idea of Morse homology to approach the Arnold conjecture. However, numerous problems arise when you do this. The biggest of these is that the natural manifold on which the Lagrangian intersections are critical points of some function is infinite-dimensional. Secondly, the compactification of the moduli space, in general, is not as simple. Finally, one needs to show that the homology complex obtained is independent of the metric chosen and the Hamiltonian.

The first and last of these problems can be managed, but the second one is more difficult. We'll impose a very strong condition on M - namely that $\int_{D^2} u^*(\omega) = 0$ for all $u : D^2 \to M$ with $\partial(u(D^2)) \subset L$ to avoid dealing with this problem.

In fact, none of these are the "big problem" in Lagrangian Floer homology. We would like to define Lagrangian Floer homology for general Lagrangian submanifolds L_0 and L_1 - then you need to worry about transversality, as well as the second problem from above. We'll talk a little about this later, but this is a very, very big problem.

Let (M, ω) be a $2n$ dimensional symplectic manifold. Let L be a Lagrangian submanifold of M, which is to say that L is an n-dimensional submanifold of M such that $TL^\omega = TL$. Finally, let ϕ_t be the flow of a Hamiltonian vector field X_H such that $\phi_1(L)$ intersects L transversally. For any fixed point x_0 of ϕ_t, define

$$\Omega = \{\gamma : [0,1] \to M | \gamma(0) \in L, u(1) \in \phi_1(L), \gamma \text{ is homotopic to } \phi_t(x_0)\}$$

This has universal cover

$$\tilde{\Omega} = \{u : [0,1] \times [0,1] \to M | u(s,0) \in L, u(s,1) \in \phi_1(L), u(0,t) = \phi_t(x_0)\}$$

Now we define a function on this space $\tilde{\Omega}$ which will play the role of the Morse function:

$$F(u) = \int_0^1 \int_0^1 u^*(\omega)$$

In fact, this is a function on Ω, because of our condition that $\int_{D^2} u^*(\omega) = 0$ for all $u : D^2 \to M$ with $\partial(u(D^2)) \subset L$. So we can consider F to be a map from the infinite-dimensional manifold Ω to R. Ω has tangent space

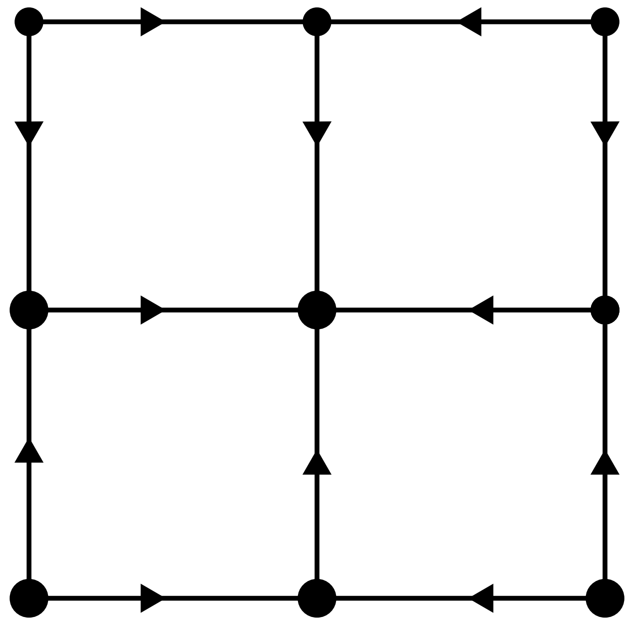

Figure 3: Torus with a Morse-Smale gradient flow.

$$T_\gamma(\Omega) = \{\xi(t) \in \gamma^*TM | \xi(0) \in T_{\gamma(0)}L, \xi(1) \in T_{\gamma(1)}\phi_1(L)$$

Now, since we want to do gradient flows for this function, we need to choose a metric. We actually want to define a metric on Ω, so g can depend on t. Of course, on a symplectic manifold, choosing a metric is equivalent to choosing an almost complex structure compatible with ω. Let J_t be a 1-periodic family of such almost complex structures. Then $g_t(u, v) = \omega(u, J_t(v))$. The metric on Ω will be the obvious one:

$$\langle \xi_1, \xi_2 \rangle = \int_0^1 g_t(\xi_1(t), \xi_2(t))dt$$

Then we see immediately that

$$\begin{aligned} dF_\gamma(\xi) &= \int_0^1 \omega(\frac{d\gamma}{dt}, \xi)dt \\ &= \int_0^1 \omega(\frac{d\gamma}{dt}, J_t(-J_t\xi))dt \\ &= \langle \frac{d\gamma}{dt}, -J_t\xi \rangle \\ &= \langle J_t\frac{d\gamma}{dt}, \xi \rangle \\ \nabla F &= J_t\frac{d\gamma}{dt} \end{aligned}$$

Now we can define, for $x, y \in L \cap \phi_1(L)$ the space of J_t holomorphic strips from x to y with boundary on L and $\phi_1(L)$:

$$\mathcal{M}(x, y) = \{u : \mathbb{R} \to \Omega | \frac{\partial u}{\partial s} = -\nabla F, \lim_{t\to-\infty} u(s, [0, 1]) = x, \lim_{t\to\infty} u(s, [0, 1]) = y\}$$

We can express this $\frac{\partial u}{\partial s} = -\nabla F$ condition as a PDE (with boundary conditions):

$$\bar{\partial} u := \frac{\partial u}{\partial s} + J_t(u(s,t))\frac{\partial u}{\partial t} = 0$$

To prove that this space is finite dimensional, one actually has to look at a linearized version of this operator between appropriate Sobolev spaces. This operator will be a Fredholm operator, which means that it has finite-dimensional kernel and cokernel. Then the infinite-dimensional version of the implicit function theorem tells us that the dimension of the moduli space is equal to the Fredholm index. The actual statement is this:

Theorem 3. *Let $k > \frac{p}{2}$. For each $x, y \in L \cap \phi_1(L)$, there exist smooth Banach manifolds $\mathcal{P}^{p,k}(x,y) \subset \{u \in W^{k,p}_{loc}(\mathbb{R} \times I, M) | u(s,0) \in L, u(s,1) \in \phi_1(L)\}$ such that $\bar{\partial}$ defines a smooth section of a smooth Banach space bundle $\mathcal{L}$ over $\mathcal{P}^{p,k}(x,y)$ with fiber over u given by $W^{p,k}(u^*TM)$ such that $\mathcal{M}(x,y)$ is the zero set of this section. The tangent space consists of all $\xi \in W^{p,k}(u^*TM)$ such that $\xi(s,0) \in TL$ and $\xi(s,1) \in T\phi_1(L)$. The linearizations*

$$E_u := D\bar{\partial}(u) : T_u\mathcal{P}(x,y) \to W^{p,k}(u^*TM)$$

are Fredholm operators for $u \in \mathcal{M}(x,y)$. There exists a dense set of almost complex structures $\mathcal{J}_{reg}(L)$ such that if $J_t \in \mathcal{J}_{reg}(L)$, then E_u is surjective for all $u \in \mathcal{M}(x,y)$.

The index of the Fredholm operator E_u is equal to the Maslov-Viterbo index $\mu_u(x,y)$ (this is also a result of Floer, and uses the spectral flow). The Maslov-Viterbo index comes from a symplectic trivialization to $[0,1] \times [0,1] \times \mathbb{C}^n$ with the standard symplectic form:

$$\Phi : u^*TM \to [0,1] \times [0,1] \times \mathbb{C}^n$$

which is chosen so that it is constant on $0 \times [0,1]$ and $1 \times [0,1]$ and such that $\Phi(T_x\phi_1(L)) = iT_xL$ and $\Phi(T_yL) = iT_y(\phi_1(L))$. Then we can construct a closed loop in the the Lagrangian subspaces Λ_n of $\mathbb{C}^n$ by $\gamma : \partial[0,1]^2 \to \Lambda_n$:

$$\begin{aligned}
\gamma(s,0) &= \Phi(T_{u(s,0)}L) \\
\gamma(1,t) &= e^{i\pi\frac{t}{2}}\Phi(T_yL) \\
\gamma(s,1) &= \Phi(T_{u(s,1)}\phi_1(L)) \\
\gamma(0,t) &= e^{-i\pi\frac{t}{2}}\Phi(T_x\phi_1(L))
\end{aligned}$$

The Maslov-Viterbo index $\mu_u(x,y)$ is then equal to the value of the Maslov class (the positive generator of $\mathrm{H}_1(\Lambda_n, \mathbb{Z})$) on this loop.

We can define the moduli space $\hat{\mathcal{M}}(x,y)$ by quotienting out by the $\mathbb{R}$ action $a \to u(s+a,t)$. Once more, as in the Morse homology case, we can show that the compactification of $\mathcal{M}(p,q)$ is given by "broken" gradient trajectories - that is, holomorphic strips which break into holomorphic strips between intervening intersection points x_i - this is enough to show us that the 0-dimensional part of the reduced moduli space is compact, since any broken trajectory would then have a piece with $\mu(x,y) \leq 0$. Also, as in the Morse homology case, there is a gluing argument which tells us that any once-broken trajectory is in the compactification of the 1-dimensional part of the reduced moduli space. The important part of both of these statements is that they break in a way that respects the Maslov-Viterbo index. Namely,

$$\mu(x,y) = \sum_{0}^{m-1} \mu(x_j, x_{j+1})$$

Thus, we can define a chain complex generated by the intersection points $x_i \in L \cap \phi_1(L)$ with boundary operator given by counting elements of the 0-dimensional part of the moduli space:

$$\partial(x) = \sum_{\mu(x,y)=1} \sharp\hat{\mathcal{M}}(x,y)y$$

Provided we are counting mod 2, this will be a chain complex, and the resulting homology is called the Lagrangian Floer homology. You can show that this homology is independent of the choice of J_t by looking at a larger moduli space where you vary J. You can use a similar method to show that the homologies induced by different ϕ_t are also isomorphic. Finally, one can show that for a sufficiently small H, we can identify $\phi_1(L)$ with the graph of $-df$ in a neighborhood of L (denoted by $N(L)$), where f is a Morse function. Then for an appropriate metric, the gradient flow lines of f will correspond to the holomorphic strips between L and $\phi_1(L)$, and the homology obtained will be exactly the Morse homology. This completes the proof of Floer's result, since we now see that we must have at least as many intersections as the sum of the Betti numbers.

4 Details & Complications

- Discs. Big problems in Lagrangian Floer theory arise when you have nontrivial J-holomorphic discs with boundary on the lagrangian. We avoided that today completely. You can also assume that the lagrangian is monotone, which means that the $\omega(A) = \lambda c_1(A)$ for any class $A \in H_*(M, L)$ which is represented by a nontrivial J-holomorphic disc with boundary on L. In this case, we can show that a generic choice of J_t will allow us to avoid such discs, at least in the parts of the moduli space that we need.

- Transversality. It would be nice to define Lagrangian Floer homology for any pair of lagrangians L_0 and L_1. If these two lagrangians don't intersect transversally, though, we run into problems. Then one needs to consider the effect of perturbing the lagrangians by a hamiltonian symplectomorphism to obtain transversal lagrangians.

- Analysis. That little paragraph about Fredholm operators glosses over a large and complicated tangle of analysis. It's not something that we can go over in an hour, but if you're interested in Floer homologies, it does crop up in most of them at some point. Appendix A in "J-Holomorphic Curves and Symplectic Topology" [MS04] is an excellent reference. Salamon's "Lectures on Floer Homology" in the 1999 Park City notes [Sal99] are also an excellent (though more concise) reference.

- Gluing. Floer's gluing theorem, which allows us to show that $\partial^2 = 0$, is more complicated than it looks. Salamon has a good discussion of this in the Hamiltonian Floer theory case in the Park City notes [Sal99].

- Existence. Lagrangian Floer theory, unlike Hamiltonian Floer theory, does not exist for all pairs of Lagrangians in all M. Fukaya-Oh-Ohta-Ono have shown that there is a specific obstruction to existence of Lagrangian Floer homology.

5 Connection to Heegaard Floer Homology

Heegaard Floer homology is a version of Lagrangian Floer Homology which looks at the complex generated by the intersections of curves in a Heegaard splitting. A Heegaard splitting or Heegaard decomposition is the decomposition of a 3-manifold into two solid handlebodies glued along curves on the surface of the handlebody. These curves are called α_i and β_i and there are g of each, where g is the genus of the handlebodies. In traditional Heegaard Floer homology, one looks at intersections of $\alpha_1 \times \alpha_2 \times \ldots \alpha_g$ and $\beta_1 \times \beta_2 \times \ldots \beta_g$ in the gth symmetric product of Σ_g. These are not actually lagrangians, though (they are in Σ^g, but not in the symmetric product). McDuff discusses Lagrangian Floer homology and its relation to Heegaard Floer in this traditional setting in [McD06].

Lipshitz reformulated Heegaard Floer homology in terms of lagrangians in [Lip06]. He considers $\Sigma \times [0,1] \times \mathbb{R}$. Then one can consider Heegaard Floer homology as generated by g-tuples of $\{x_i \times [0,1]\}$ where $x_i \in \alpha_i \cap \beta_i$. The boundary map counts holomorphic strips asymptotic to such a g-tuple at $-\infty$ and ∞ with boundary on the lagrangian cylinders $\alpha_i \times 1 \times \mathbb{R}$ and $\beta_i \times 0 \times \mathbb{R}$.

References

[Aka] Manabu Akaho, *A crash course of floer homology for lagrangian intersections.*

[Lip06] Robert Lipshitz, *A cylindrical reformulation of Heegaard Floer homology*, Geom. Topol. **10** (2006), 955–1097 (electronic). MR MR2240908 (2007h:57040)

[McD06] Dusa McDuff, *Floer theory and low dimensional topology*, Bull. Amer. Math. Soc. (N.S.) **43** (2006), no. 1, 25–42 (electronic). MR MR2188174 (2006g:57060)

[MS04] Dusa McDuff and Dietmar Salamon, *J-holomorphic curves and symplectic topology*, American Mathematical Society Colloquium Publications, vol. 52, American Mathematical Society, Providence, RI, 2004. MR MR2045629 (2004m:53154)

[Ono06] Kaoru Ono, *Development in symplectic Floer theory*, International Congress of Mathematicians. Vol. II, Eur. Math. Soc., Zürich, 2006, pp. 1061–1082. MR MR2275635 (2007k:53153)

[Sal99] Dietmar Salamon, *Lectures on Floer homology*, Symplectic geometry and topology (Park City, UT, 1997), IAS/Park City Math. Ser., vol. 7, Amer. Math. Soc., Providence, RI, 1999, pp. 143–229. MR MR1702944 (2000g:53100)

[Sch93] Matthias Schwarz, *Morse homology*, Progress in Mathematics, vol. 111, Birkhäuser Verlag, Basel, 1993. MR MR1239174 (95a:58022)