Implicitization of surfaces via Geometric Tropicalization

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Three references:

Sturmfels, Tevelev, Yu: The Newton polytope of the implicit equation (2007)
Sturmfels, Tevelev: Elimination theory for tropical varieties (2008)
(and many, many more!)
Implicitization problem: Classical vs. tropical approach

**Input:** Laurent polynomials $f_1, f_2, \ldots, f_n \in \mathbb{C}[t_1^{\pm 1}, \ldots, t_d^{\pm 1}]$.

**Algebraic Output:** The *prime* ideal $I$ defining the Zariski closure $Y$ of the image of the map:

$$f = (f_1, \ldots, f_n) : \mathbb{T}^d \to \mathbb{T}^n$$

The ideal $I$ consists of all polynomial relations among $f_1, f_2, \ldots, f_n$.

**Existing methods:** Gröbner bases and resultants.
- **GB:** always applicable, but often too slow.
- **Resultants:** useful when $n = d + 1$ and $I$ is principal, with limited use.

**Geometric Output:** Invariants of $Y$, such as dimension, degree, etc.

**Punchline:** We can *effectively* compute them using tropical geometry.

**TODAY:** Study the case when $d = 2$ and $Y$ is a surface.
Example: parametric surface in $\mathbb{T}^3$

Input: Three Laurent polynomials in two unknowns:

\[
\begin{align*}
    x & = f_1(s, t) = 3 + 5s + 7t, \\
    y & = f_2(s, t) = 17 + 13t + 11s^2, \\
    z & = f_3(s, t) = 19 + 47st.
\end{align*}
\]

Output: The Newton polytope of the implicit equation $g(x, y, z)$.

**STRATEGY:** Recover the Newton polytope of $g(x, y, z)$ from the Newton polytopes of the input polynomials $f_1, f_2, f_3$. 
\( Y = \begin{cases} 
  x = f_1(s, t) = 3 + 5s + 7t, \\
  y = f_2(s, t) = 17 + 13t + 11s^2, \\
  z = f_3(s, t) = 19 + 47st. 
\end{cases} \)

\[ \leadsto \text{Newton polytope of } g(x, y, z). \]

\( \Gamma \) is a balanced weighted planar graph in \( \mathbb{R}^3 \). It is the tropical variety \( \mathcal{T}(g(x, y, z)) \), dual to the Newton polytope of \( g \).

- We can recover \( g(x, y, z) \) from \( \Gamma \) using numerical linear algebra.
What is Tropical Geometry?

Given a variety \( X \subset \mathbb{T}^n \) with defining ideal \( I \subset \mathbb{C}[x_1^{\pm1}, \ldots, x_n^{\pm1}] \), the tropicalization of \( X \) equals:

\[
\mathcal{T}X = \mathcal{T}I := \{ w \in \mathbb{R}^n \mid \text{in}_w I \text{ contains no monomial} \}.
\]

1. It is a rational polyhedral fan in \( \mathbb{R}^n \leadsto \mathcal{T}X \cap S^{n-1} \) is a spherical polyhedral complex.

2. If \( I \) is prime, then \( \mathcal{T}X \) is pure of the same dimension as \( X \).

3. Maximal cones have canonical multiplicities attached to them.

Example (hypersurfaces):

- Maximal cones in \( \mathcal{T}(g) \) are dual to edges in the Newton polytope \( \text{NP}(g) \), and \( m_\sigma \) is the lattice length of the associated edge.
- Multiplicities are essential to recover \( \text{NP}(g) \) from \( \mathcal{T}(g) \).
What is Geometric Tropicalization?

**AIM:** Given $Z \subset \mathbb{T}^N$ a surface, compute $\mathcal{T}Z$ from the geometry of $Z$.

**KEY FACT:** $\mathcal{T}Z$ can be characterized in terms of divisorial valuations.
What is Geometric Tropicalization?

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**Theorem (Geometric Tropicalization [Hacking - Keel - Tevelev, C.])**

Consider $\mathbb{T}^N$ with coordinate functions $\chi_1, \ldots, \chi_N$, and let $Z \subset \mathbb{T}^N$ be a closed smooth surface. Suppose $\overline{Z} \supset Z$ is any normal and $\mathbb{Q}$-factorial compactification, whose boundary divisor has $m$ irreducible components $D_1, \ldots, D_m$ with no triple intersections (C.N.C.). Let $\Delta$ be the graph:

$$V(\Delta) = \{1, \ldots, m\} \quad ; \quad (i, j) \in E(\Delta) \iff D_i \cap D_j \neq \emptyset.$$ 

Realize $\Delta$ as a graph $\Gamma \subset \mathbb{R}^N$ by $[D_k] := (\text{val}_{D_k}(\chi_1), \ldots, \text{val}_{D_k}(\chi_N)) \in \mathbb{Z}^N$.

Then, $\mathcal{T}Z$ is the cone over the graph $\Gamma$. 

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Tropical Implicitization of surfaces

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What is Geometric Tropicalization?

**AIM:** Given $Z \subset \mathbb{T}^N$ a surface, compute $T_Z$ from the geometry of $Z$.

**KEY FACT:** $T_Z$ can be characterized in terms of divisorial valuations.

**Theorem (Geometric Tropicalization [Hacking - Keel - Tevelev, C.])**

Consider $\mathbb{T}^N$ with coordinate functions $\chi_1, \ldots, \chi_N$, and let $Z \subset \mathbb{T}^N$ be a closed smooth surface. Suppose $\overline{Z} \supset Z$ is any normal and $\mathbb{Q}$-factorial compactification, whose boundary divisor has $m$ irreducible components $D_1, \ldots, D_m$ with no triple intersections (C.N.C.). Let $\Delta$ be the graph:

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Realize $\Delta$ as a graph $\Gamma \subset \mathbb{R}^N$ by $[D_k] : = (\text{val}_{D_k}(\chi_1), \ldots, \text{val}_{D_k}(\chi_N)) \in \mathbb{Z}^N$.

Then, $T_Z$ is the cone over the graph $\Gamma$.

**Theorem (Combinatorial formula for multiplicities [C.])**

$$m([D_i], [D_j]) = (D_i \cdot D_j) \left[ (\mathbb{Z}[D_i], [D_j])^{\text{sat}} : \mathbb{Z}[D_i], [D_j] \right]$$
QUESTION: How to compute $\mathcal{T}Y$ from a parameterization

$$f = (f_1, \ldots, f_n): \mathbb{T}^2 \to Y \subset \mathbb{T}^n$$
QUESTION: How to compute $\mathcal{T} Y$ from a parameterization

$$f = (f_1, \ldots, f_n): \mathbb{T}^2 \dashrightarrow Y \subset \mathbb{T}^n \ ?$$

ANSWER: Compactify the domain $X = \mathbb{T}^2 \setminus \bigcup_{i=1}^{n} (f_i = 0)$ and use the map $f$ to translate back to $Y$.

Proposition

Given $f: X \subset \mathbb{T}^2 \rightarrow Y \subset \mathbb{T}^n$ generically finite map of degree $\delta$, let $\overline{X}$ be a normal, $\mathbb{Q}$-factorial, CNC compactification with intersection complex $\Delta$. Map each vertex $D_k$ of $\Delta$ in $\mathbb{Z}^n$ to a vertex $[\tilde{D}_k]$ of $\Gamma \subset \mathbb{R}^n$, where

$$[\tilde{D}_k] = \text{val}_{D_k}(\chi \circ f) = f^#([D_k]).$$

Then, $\mathcal{T} Y$ is the cone over the graph $\Gamma \subset \mathbb{R}^n$, with multiplicities

$$m_{([\tilde{D}_i],[\tilde{D}_j])} = \frac{1}{\delta} (D_i \cdot D_j) \left[ (\mathbb{Z}\langle [\tilde{D}_i],[\tilde{D}_j] \rangle)^{sat} : \mathbb{Z}\langle [\tilde{D}_i],[\tilde{D}_j] \rangle \right].$$
Implicitization of generic surfaces

**SETTING:** Let \( f = (f_1, \ldots, f_n) : \mathbb{T}^2 \to Y \subset \mathbb{T}^n \) of \( \deg(f) = \delta \), where we fix the Newton polytope of each \( f_i \) and allow generic coefficients.

**GOAL:** Compute the graph \( \Gamma \) of \( T_Y \) from the Newton polytopes \( \{P_i\}_{i=1}^n \).

**IDEA:** Compactify \( X \) inside the proj. toric variety \( X_{\mathcal{N}} \), where \( \mathcal{N} \) is the common refinement of all \( \mathcal{N}(P_i) \). Generically, \( \overline{X} \) is smooth with CNC.

The vertices and edges of the boundary intersection complex \( \Delta \) are

\[
V(\Delta) = \{ E_i : \text{dim } P_i \neq 0, 1 \leq i \leq n \} \cup \{ D_\rho : \rho \in \mathcal{N}^{[1]} \},
\]

- \( (D_\rho, D_{\rho'}) \in E(\Delta) \) iff \( \rho, \rho' \) are consecutive rays in \( \mathcal{N} \).
- \( (E_i, D_\rho) \in E(\Delta) \) iff \( \rho \in \mathcal{N}(P_i) \).
- \( (E_i, E_j) \in E(\Delta) \) iff \( (f_i = f_j = 0) \) has a solution in \( \mathbb{T}^2 \).

Then, \( \Gamma \) is the realization of \( \Delta \) via

\[
[E_i] := e_i \quad (1 \leq i \leq n), \quad [D_\rho] := \text{trop}(f)(\eta_\rho) \quad \forall \text{ ray } \rho \quad (\eta_\rho \text{ prim. vector}.)
\]

**Theorem [Sturmfels-Tevelev-Yu, C.]:** \( T_Y \) is the weighted cone over \( \Gamma \).
Implicitization of *non-generic* surfaces

*Non-genericity* $\leftrightarrow$ CNC condition is violated.

**Solution 1:**
1. Embed $X$ in $X_\mathcal{N}$.
2. Resolve triple intersections and singularities by classical blow-ups, and carry divisorial valuations along the way.

The graph $\Delta$ is obtained by gluing resolution diagrams and adding pairwise intersections.
Implicitization of *non-generic* surfaces

*Non-genericity* $\leftrightarrow$ CNC condition is violated.

**Solution 1:**
1. Embed $X$ in $X_N$.
2. Resolve triple intersections and singularities by classical blow-ups, and carry divisorial valuations along the way.

**Solution 2:**
1. Embed $X$ in $\mathbb{P}^2_{(s,t,u)} \leadsto n + 1$ boundary divisors
   \[ E_i = (f_i = 0) \quad (1 \leq i \leq n), \quad E_\infty = (u = 0). \]
2. Resolve triple intersections and singularities by blow-ups $\pi: \tilde{X} \rightarrow X$, and read divisorial valuations by columns
   \[ (f \circ \pi)^*(\chi_i) = \pi^*(E_i - \deg(f_i)E_\infty) = E_i' - \deg(f_i)E_\infty' - \sum_{j=1}^{r} b_{ij}H_j \quad \forall i. \]

The graph $\Delta$ is obtained by gluing resolution diagrams and adding pairwise intersections.
Example (non-generic surface)

\[ Y = \begin{cases} 
  x = f_1(s, t) = s - t, \\
  y = f_2(s, t) = t - s^2, \\
  z = f_3(s, t) = -1 + st, 
\end{cases} \]

Affine Charts:

- \( E_1 := (s - t = 0) \)
- \( E_2 := (t - s^2 = 0) \)
- \( E_3 := (1 - st = 0) \)
- \( E_\infty := (u = 0) \)
- \( E_2 := (u - s^2 = 0) \)
- \( E_3 := (-u^2 + s = 0) \)

\( \Gamma \) (7, 11, 6)