Implicitization of surfaces via Geometric Tropicalization

María Angélica Cueto

Institut Mittag-Leffler (Sweden)
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Columbia University (USA)

CIAM-KTH Workshop
Feb 21st 2011

Three references:

Sturmfels, Tevelev, Yu: The Newton polytope of the implicit equation (2007)
Sturmfels, Tevelev: Elimination theory for tropical varieties (2008)
MAC: Tropical Implicitization (Ch. 5) (2010) ↝ arXiv soon!
(and many, many more!)
Implicitization problem: Classical vs. tropical approach

**Input:** Laurent polynomials \(f_1, f_2, \ldots, f_n \in \mathbb{C}[t_1^{\pm 1}, \ldots, t_d^{\pm 1}]\).

**Algebraic Output:** The *prime* ideal \(I\) defining the Zariski closure \(Y\) of the image of the map:

\[
f = (f_1, \ldots, f_n) : \mathbb{T}^d \rightarrow \mathbb{T}^n
\]

The ideal \(I\) consists of all polynomial relations among \(f_1, f_2, \ldots, f_n\).

**Existing methods:** Gröbner bases and resultants.

- **GB:** always applicable, but often too slow.
- **Resultants:** useful when \(n = d + 1\) and \(I\) is *principal*, with limited use.

**Geometric Output:** Invariants of \(Y\), such as dimension, degree, etc.

**Punchline:** We can effectively compute them using tropical geometry.

**TODAY:** Study the case when \(d = 2\) and \(Y\) is a surface.

M.A. Cueto (Inst. Mittag-Leffler)
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**TODAY:** Study the case when \( d = 2 \) and \( Y \) is a *surface*. 
Example: parametric surface in $\mathbb{T}^3$

Input: Three Laurent polynomials in two unknowns:

\[
\begin{align*}
  x &= f_1(s, t) = 1 + s^3 t + s t^2, \\
  y &= f_2(s, t) = 2s t + 3s + 5t, \\
  z &= f_3(s, t) = -t + s^2 + s t^2.
\end{align*}
\]

Output: The **Newton polytope** of the implicit equation $g(x, y, z)$.

The Newton polytope of $g$ is the convex hull in $\mathbb{R}^3$ of all lattice points $(i, j, k)$ such that $x^i y^j z^k$ appears with *nonzero* coefficient in $g(x, y, z)$. 
**STEP 1:** Draw the three Newton polytopes and their Minkowski sum.
STEP 2: From these four polytopes, build the abstract graph $\Delta$

- Add one fat colored node per polytope $\mathcal{P}_i$ ($i = 1, 2, 3$) and draw the triangle joining these three nodes.
- Add one skinny colored node per edge in the polytope $\mathcal{P}_1 + \mathcal{P}_2 + \mathcal{P}_3$ and draw the corresponding polygon joining nodes of adjacent edges in the polytope (in this case, a 9-gon).
- Join all skinny nodes to the corresponding fat node of the same color.
- In our example: 3 fat nodes, 9 skinny nodes and 21 edges.
**STEP 3:** Realize the abstract graph $\Delta$ as a *weighted* graph $\Gamma$ in $S^2$.

$$f\text{-vector}(\Gamma) = (7, 13, 8).$$
**STEP 3:** Realize the abstract graph $\Delta$ as a **weighted** graph $\Gamma$ in $\mathbb{S}^2$.

\[ f\text{-vector}(\Gamma) = (7, 13, 8). \]

- $\Gamma$ is a balanced weighted **planar** graph in $\mathbb{R}^3$. It is the **tropical variety** $\mathcal{T}(g(x, y, z))$, dual to the Newton polytope of $g$.
- We can recover $g(x, y, z)$ from $\Gamma$ using **numerical linear algebra**.
What is Tropical Geometry?

Given a variety $X \subset \mathbb{T}^n$ with defining ideal $I \subset \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, the tropicalization of $X$ equals:

$$\mathcal{T}X = \mathcal{T}I := \{ w \in \mathbb{R}^n \mid \text{in}_w(I) \text{ contains no monomial}\}.$$
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1. It is a rational polyhedral fan in $\mathbb{R}^n \hookrightarrow \mathcal{T}X \cap S^{n-1}$ is a spherical polyhedral complex.

2. If $I$ is prime, then $\mathcal{T}X$ is pure of the same dimension as $X$.

3. Maximal cones have canonical multiplicities attached to them. With these multiplicities, $\mathcal{T}X$ satisfies the balancing condition.

Example (hypersurfaces):
- $\mathcal{T}(g)$ is the union of all codim. 1 cones in the (inner) normal fan of the Newton polytope $NP(g)$.
- Maximal cones in $\mathcal{T}(g)$ are dual to edges in $NP(g)$, and $m_\sigma$ is the lattice length of the associated edge.
- Multiplicities are essential to recover $NP(g)$ from $\mathcal{T}(g)$. 
What is Geometric Tropicalization?

**AIM:** Given $Z \subset \mathbb{T}^N$ a **surface**, compute $\mathcal{T}Z$ from the geometry of $Z$.

**KEY FACT:** $\mathcal{T}Z$ can be characterized in terms of **divisorial valuations**.

**Theorem (Geometric Tropicalization [Hacking - Keel - Tevelev])**

Consider $\mathcal{T}\mathbb{N}$ with coordinate functions $\chi_1, \ldots, \chi_N$, and let $Z \subset \mathcal{T}\mathbb{N}$ be a closed smooth surface. Suppose $Z \supset Z$ is any compactification, whose boundary divisor has $m$ irreducible components $D_1, \ldots, D_m$ which are smooth and with no triple intersections (C.N.C.). Let $\Delta$ be the graph:

$V(\Delta) = \{1, \ldots, m\}$; $(i, j) \in E(\Delta) \iff D_i \cap D_j \neq \emptyset$.

Realize $\Delta$ as a graph $\Gamma \subset \mathbb{R}^N$ by $[D_k] = (\text{val}_{D_k}(\chi_1), \ldots, \text{val}_{D_k}(\chi_N)) \in \mathbb{Z}^N$.

Then, $\mathcal{T}Z$ is the cone over the graph $\Gamma$.

**Theorem (Combinatorial formula for multiplicities [C.])**

$m([D_i], [D_j]) = (D_i \cdot D_j) [Z_{[D_i], [D_j]}]$ sat:

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Consider $\mathbb{T}^N$ with coordinate functions $\chi_1, \ldots, \chi_N$, and let $Z \subset \mathbb{T}^N$ be a **closed smooth surface**. Suppose $\overline{Z} \supset Z$ is any compactification, whose boundary divisor has $m$ irreducible components $D_1, \ldots, D_m$ which are smooth and with no triple intersections (**C.N.C.**). Let $\Delta$ be the graph:

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What is Geometric Tropicalization?

**AIM:** Given $Z \subset \mathbb{T}^N$ a **surface**, compute $TZ$ from the **geometry** of $Z$.

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Consider $\mathbb{T}^N$ with coordinate functions $\chi_1, \ldots, \chi_N$, and let $Z \subset \mathbb{T}^N$ be a **closed smooth surface**. Suppose $\overline{Z} \supset Z$ is any compactification, whose boundary divisor has $m$ irreducible components $D_1, \ldots, D_m$ which are smooth and with no triple intersections (**C.N.C.**). Let $\Delta$ be the graph:

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**Theorem (Combinatorial formula for multiplicities [C.])**

$$m([D_i],[D_j]) = (D_i \cdot D_j) \left[ (\mathbb{Z}\langle[D_i],[D_j]\rangle)^{\text{sat}} : \mathbb{Z}\langle[D_i],[D_j]\rangle \right]$$
QUESTION: How to compute $\mathcal{T}Y$ from a parameterization

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ANSWER: Compactify the domain $X = \mathbb{T}^2 \setminus \bigcup_{i=1}^{n}(f_i = 0)$ and use the map $f$ to translate back to $Y$.

Proposition

Given $f: X \subset \mathbb{T}^2 \to Y \subset \mathbb{T}^n$ generically finite map of degree $\delta$, let $\overline{X}$ be a smooth, CNC compactification with associated intersection complex $\Delta$.

Map each vertex $D_k$ of $\Delta$ in $\mathbb{Z}^n$ to a vertex $\widetilde{D}_k$ of $\Gamma \subset \mathbb{R}^n$, where

$$[\widetilde{D}_k] = \text{val}_{D_k}(\chi \circ f) = f^#([D_k]).$$

Then, $\mathcal{T}Y$ is the cone over the graph $\Gamma \subset \mathbb{R}^n$, with multiplicities

$$m_{([\widetilde{D}_i],[\widetilde{D}_j])} = \frac{1}{\delta} (D_i \cdot D_j) \left[ (\mathbb{Z}\langle [\widetilde{D}_i], [\widetilde{D}_j] \rangle)_{\text{sat}} : \mathbb{Z}\langle [\widetilde{D}_i], [\widetilde{D}_j] \rangle \right].$$
Implicitization of generic surfaces

**SETTING:** Let \( f = (f_1, \ldots, f_n) : \mathbb{T}^2 \to Y \subset \mathbb{T}^n \) of \( \deg(f) = \delta \), where
- each \( f_i \in \mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}] \) is irreducible and has fixed Newton polytope,
- we assume generic coefficients.

**GOAL:** Compute the graph \( \Gamma \) of \( \mathcal{I}Y \) from the Newton polytopes \( \{ P_i \}_{i=1}^n \).

**IDEA:** Compactify \( X \) inside the proj. toric variety \( \mathbb{P}(N) \), where \( N \) is the normal fan of \( \sum_{i=1}^n P_i \).
Generically, \( X \) is smooth and has CNC boundary.
The vertices and edges of the boundary intersection complex \( \Delta \) are
- \( V(\Delta) = \{ E_i : \dim P_i \neq 0, 1 \leq i \leq n \} \cup \{ D_\rho : \rho \in N[1] \} \),
- \( (E_i, D_\rho) \in E(\Delta) \iff \rho, \rho' \) are consecutive rays in \( N \).
- \( (E_i, E_j) \in E(\Delta) \iff (f_i = f_j = 0) \) has a solution in \( \mathbb{T}^2 \).

Then, \( \Gamma \) is the realization of \( \Delta \) via \( [E_i] := e_i, [D_\rho] := (\min_{\alpha \in P_i} \{ \alpha \cdot \eta_\rho \}) \forall \rho \in N[1] \), where \( \eta_\rho \) is the primitive lattice vector generating \( \rho \).
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V(\Delta) = \{ E_i : \dim P_i \neq 0, 1 \leq i \leq n \} \bigcup \{ D_\rho : \rho \in \mathcal{N}[1] \},
\]

- \((D_\rho, D_{\rho'}) \in E(\Delta) \) iff \( \rho, \rho' \) are **consecutive** rays in \( \mathcal{N} \).
- \((E_i, D_\rho) \in E(\Delta) \) iff \( \rho \in \mathcal{N}(P_i) \).
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Theorem (Sturmfels-Tevelev-Yu, C.)

The tropical variety $\mathcal{T}Y$ is the cone over the graph $\Gamma$, with multiplicities

\[ m([D_\rho],[D_{\rho'}]) = \frac{1}{\delta} \frac{\gcd\{2\text{-minors of } ([D_\rho][D_{\rho'}])\}}{|\det(\eta_\rho|\eta_{\rho'})|}, \text{ for } \rho, \rho' \text{ consec. rays in } \mathcal{N}. \]

\[ m(e_i,[D_\rho]) = \frac{1}{\delta} (|\text{face}_\rho \mathcal{P}_i \cap \mathbb{Z}^2| - 1) \gcd\{[D_\rho]_j : j \neq i\}, \text{ for } \rho \in \mathcal{N}_i^{[1]}. \]

\[ m(e_i,e_j) = \frac{1}{\delta} \text{length}((f_i = f_j = 0) \cap \mathbb{T}^2), \text{ if } \dim(\mathcal{P}_i + \mathcal{P}_j) = 2. \]

Under further genericity assumptions,

\[ \text{length}((f_i = f_j = 0) \cap \mathbb{T}^2) = MV(\mathcal{P}_i, \mathcal{P}_j). \]
Example (generic surface)

\[ Y = \begin{cases} 
  x = f_1(s, t) = a_1 + a_2 s + a_3 t, \\
  y = f_2(s, t) = b_1 + b_2 t + b_3 s^2, \\
  z = f_3(s, t) = c_1 + c_2 st, 
\end{cases} \]

\[ a_1, \ldots, c_2 \in \mathbb{C}^* \text{ are generic.} \]
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Implicitization of non-generic surfaces

Non-genericity $\iff$ CNC/smoothness condition is violated, i.e. triple intersections among:

$$E_i = (f_i = 0)'s \text{ only} \quad \text{or} \quad E_i'\text{s and } D_{\rho}'\text{s combined.}$$
Implicitization of *non-generic* surfaces

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**Solution 1:**
1. Embed $X$ in $\mathbb{P}(\mathcal{N})$.
2. Resolve triple intersections and singularities by classical blow-ups, and carry divisorial valuations along the way.
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**Solution 1:**
1. Embed $X$ in $\mathbb{P}(\mathcal{N})$.
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**Solution 2:**
1. Embed $X$ in $\mathbb{P}^2_{(s,t,u)} \rightsquigarrow n + 1$ boundary divisors
   $$E_i = (f_i = 0) \quad (1 \leq i \leq n), \quad E_\infty = (u = 0).$$
2. Resolve triple intersections and singularities by blow-ups $\pi : \tilde{X} \to X$, and read divisorial valuations by *columns*

   $$\left(f \circ \pi\right)^*(\chi_i) = \pi^*(E_i - \deg(f_i)E_\infty) = E_i' - \deg(f_i)E_\infty' - \sum_{j=1}^{r} b_{ij} H_j \quad \forall i.$$  

The graph $\Delta$ is obtained by gluing resolution diagrams and adding pairwise intersections.
Example (non-generic surface)

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  x = f_1(s, t) = s - t, \\
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Affine Charts:

\[ E_1 := (s - t = 0) \]
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\[ H_1 \quad H_2 \quad E_\infty \quad E_1 \quad E_2 \quad E_3 \]

(7, 11, 6)
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\[ E_1 : (u - s^2 = 0) \]
\[ E_2 : (-u^2 + s = 0) \]
\[ E_\infty : (u = 0) \]
Further questions

1. What if we allow coefficients on an *arbitrary* closed non-archimedean valued field, e.g. $\mathbb{C}\{t\}$, $\mathbb{Q}_p$, ...? $\leadsto$ Berkovich spaces!
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2. Special surfaces are tropicalized via resolution of singularities, which is hard to do in practice.
   - Is there an alternative approach? \( \leadsto \) combinatorial resolutions?
   - Can we predict the graph \( \Gamma \) from the topology/geometry of the singularities on the domain \( X \)? \( \leadsto \) Enriques/dual diagrams, clusters of infinitely near points, ...
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3. If $\dim Z > 2$, geometric tropicalization requires the boundary of a compactification $\overline{Z}$ to have simple normal crossings. Can we replace it with combinatorial normal crossings?