1. Introduction and prior research

My research integrates themes from combinatorics, algebraic geometry and non-Archimedean geometry, and is centered in the field known as tropical geometry. I specialize in answering geometric questions using combinatorial tools. More specifically, I have focused on tropical implicitization methods and the interplay between non-Archimedean spaces (in the sense of Berkovich) and tropicalization of classical varieties. In the coming years, I plan to further tropical geometry, and explore connections to related geometric and combinatorial questions, summarized in four topics:

- Tropical approach to classical projective varieties (Schubert varieties, secant varieties, Grassmannians);
- Tropicalization techniques under the absence of canonical coordinate systems (Schubert varieties, moduli spaces);
- The use of linear transformations to simplify the combinatorics of tropical objects (repairing tropical plane curves, Bruhat-Tits buildings);
- Explicit computation of tropical varieties and geometric invariants of their algebraic counterparts (restrictive Boltzmann machines, tropical implicitization).

Tropical geometry is a polyhedral version of classical algebraic geometry: algebraic varieties are replaced by weighted, balanced polyhedral complexes, in order to answer open questions or to derive simpler proofs of classical results. These objects preserve just enough data about the original varieties to remain meaningful, while discarding much of their complexity.

Tropical geometry began as a framework to link amoebas [Mik03, Mik06, EKL06], logarithmic limit sets [Ber71], and (real) algebraic geometry [Stu02, MS15]. It synthesized and boosted the pioneering work of Bieri-Groves [BG84] and Viro’s “patchworking” techniques to construct real algebraic varieties by “cutting and pasting” [Vir08, IV96]. The past few years have brought on truly explosive development, establishing deep connections with enumerative algebraic geometry, symplectic and analytic geometry, number theory, mathematical biology, statistical physics, random matrix theory, and mathematical physics.

While I have worked on several combinatorial questions, resulting in two articles [CD07, CCD+13] inspired by the theory of A-discriminants due to Gelfand-Kapranov-Zelevinsky [GKZ08], and four other publications [CM11, CCT11, Cue12b, CCM+14] focused on applications of tropical and polyhedral geometry to phylogenetics and algebraic statistics, the following three themes summarize the breadth and essence of my research interests:

1.1. Computational algebraic geometry. In my doctoral dissertation, I ventured in two directions. The first, developed in [CMS10, CTY10], used computational algebraic geometry to study highly structured projective varieties in tensor spaces, known as restricted Boltzmann machines \( \mathcal{F}(n,k) \) in algebraic statistics and machine learning. In particular, I proved the following theorem:

**Theorem 1.** The variety \( \mathcal{F}(n,k) \) has the expected dimension in all relevant cases. The hypersurface \( \mathcal{F}(2,4) \) in \( \mathbb{P}^{15} \) has degree 110 and its Newton polytope has 17 214 912 vertices and 70 646 facets.
The second one, explored in [CL13, Cue12a], concerned the tropical approach to classical implicitization problems, such as secant varieties of monomial curves and surfaces in 3-space. I did so by enriching the theory of geometric tropicalization of Hacking-Keel-Tevelev [HKT09] with a combinatorial formula for computing multiplicities on the tropical cycle of a smooth variety (see Theorem 2).

These two themes manifest themselves in my work with Tobis and Yu to find the degree and defining equation of $F(2,4)$ [CTY10]. The sheer size of this computational challenge forced us to derive, and efficiently implement, new techniques to go from a tropical hypersurface $Trop(f)$ to the Newton polytope of $f$, building on [DFS07]. Months of strenuous computations proved Theorem 1.

I followed up this special case by studying the general variety $F(n,k)$ from the semialgebraic perspective, together with Morton and Sturmfels [CMS10]. We computed its dimension (Theorem 1) and explored the connection between the tropicalization of this variety and its associated inference functions. The tropical point of view allowed us to organize the geometric information of the space of inference functions and to analyze them with the tools of polyhedral geometry. This novel approach yielded a new interpretation for the space of explanations as a subfan of the secondary fan of the $n$-cube, which connected very nicely with combinatorial and geometric work of Develin and Draisma on tropical secant varieties of toric varieties [Dev06, Dra08].

Tropical implicitization was pioneered by the work of Sturmfels, Tevelev and Yu [STY07]. Their methods are well suited for generic varieties, and are built on the theory of geometric tropicalization [HKT09] and the construction of tropical compactifications as in [Tev07]. However, real life is seldom generic, so it is crucial to attack the non-generic versions of these problems. Most of the known theoretical results concerning this setting rely on the existence of resolution of singularities. These resolutions are by no means explicit, explaining the lack of examples in this theory. One of my main contributions in [Cue12a] is the development of new, explicit techniques for tropical implicitization of non-generic surfaces, including Theorem 2 an essential tool to go from tropical cycles back to the algebraic world.

As a test case, jointly with Lin, I examined tropical surfaces and Chow polytopes associated with secants of monomial projective curves with arbitrary exponents (e.g. the curve $(1 : t^{30} : t^{45} : t^{55} : t^{78})$ in [CL13, Example 7.2]). We reinterpreted classical formulas for computing their degrees [Con98, Ran06], and provided a way of recovering their Chow polytopes from their tropicalizations. I believe that this approach will advance the study of higher secants of monomial curves.

1.2. Geometric tropicalization. As exhibited by Theorem 1, it is often desirable to compute tropicalizations of algebraic varieties without prior knowledge of their defining ideals. The crux of geometric tropicalization is to read off the tropicalization of a smooth closed subvariety of a complex torus $\mathbb{T}^n$ directly from the combinatorics of its boundary in a suitable compactification. To do so, this boundary is required to have simple normal crossings (snc), that is, to behave locally like an arrangement of coordinate hyperplanes. More precisely, let $X \subset \mathbb{T}^n$ be the subvariety and pick a normal and $\mathbb{Q}$-factorial compactification $\overline{X}$ where its divisorial boundary $\partial \overline{X} = \overline{X} \setminus X$ has snc. The combinatorial information of the tropical variety $T_X$ is encoded in an abstract simplicial complex, called the boundary complex $\Delta(\partial \overline{X})$, which resembles the one in [Pay13]. After assigning coordinates to the vertices of this complex by means of divisorial valuations, and extending linearly on cells, we get the support of a complex in the real span of the cocharacter lattice of $\mathbb{T}^n$. Geometric tropicalization [HKT09] says precisely that the cone over this set is the support of the tropical fan $Trop(X)$ and, in particular, that this set does not depend on our choice of $\overline{X}$.

As one might expect, the major difficulty in applying these methods to compute tropical fans in concrete examples lies in the restrictive (and infeasible in practice) assumptions on the compactification $\overline{X}$. After studying in detail the surface case, Sturmfels and Tevelev conjectured that the right condition to impose on $\overline{X}$ is not geometric but combinatorial, requiring the boundary components to intersect in the expected codimension [ST08]. This property ensures that $\Delta(\partial \overline{X})$ is simplicial.
and has the right dimension, namely, one less than the dimension of $X$. In [Cue12a], I proved the *combinatorial normal crossing* (cnc) conjecture in all dimensions, summarized as follows:

**Theorem 2.** There is an explicit combinatorial formula for computing multiplicities on the tropical cycle of any smooth closed subvariety of a complex torus using a compactification with cnc boundary.

This result has been applied in the recent study of the classical moduli space of rational weighted stable curves by Cavalieri, Hampe, Markwig and Ranganathan [CHMR14]. The latter tackles the tropical analog of these Hassett spaces by combining geometric tropicalization and Thuillier skeleta of analytic spaces [Thu07]. These skeleta are induced by toroidal embeddings and agree with the boundary complexes of Payne [Pay13]. The weighted boundary complexes $\Delta(\partial X)$ used in the proof of Theorem 2 are shadows of these skeleta. I am currently discussing with M. Ulirsch, a graduate student at Brown University, about a general framework for combining these two complexes to study tropical moduli spaces, as in the curve case [ACP14, CHMR14]. Part of our work requires to extend the theory of geometric tropicalization to the toroidal setting.

### 1.3. Berkovich skeleta, tropical varieties and faithful tropicalizations.

Kapranov et al. [EKL06] characterized tropical varieties as images of valuations maps. More precisely, given a complete non-Archimedean valued field $K$ with value group $\text{val}(K^*) = \mathbb{R}$, and a subvariety $X \subset \mathbb{T}_K$, the tropicalization of $X$ is the image of $X(K)$ under the negative valuation. Our standard example is the completion of the Puiseux series field $\mathbb{C}\{\{t\}\} = \bigcup n \in \mathbb{N} \mathbb{C}(\{t^{1/n}\})$, where the valuation is the order.

The pioneering work of Gubler [Gub07a, Gub07b], Payne [Pay09], Baker-Payne-Rabinoff [BPR11, BPR13] and Gubler-Rabinoff-Werner [GRW14] has carried the previous characterization to a deeper level, establishing a tight connection between tropicalizations of subvarieties of tori (and toric varieties) and their analytifications in the sense of Berkovich. Analytic spaces in this context are mostly Berkovich analytic spaces, where, roughly speaking, points can locally be described by certain seminorms. Spaces of seminorms or valuations have been present in tropical geometry from its very beginnings by the work of Bieri and Groves [BG84]. Recently, Manon has used such spaces to investigate representations of reductive groups [Man10, Man11].

The tropicalization map defined on the analytification of a closed subvariety $X$ of a toric variety extends the coordinatewise valuation map on the $K$-points of $X$. It is continuous and surjective, and it strongly depends on the given embedding. Work of Payne shows that the Berkovich space $X^{\text{an}}$ is homeomorphic to the projective limit of all tropicalizations of $X$ [Pay09]. We view $X^{\text{an}}$ as a topological object incorporating all choice of embeddings and induced tropicalizations of $X$.

The analytification of any $K$-scheme $X$ with semistable degeneration contains a polyhedral set called its *Berkovich skeleton*, which is a strong deformation retract of the whole space $X^{\text{an}}$ [Ber99, Ber04]. The skeleton depends on the degeneration. In the curve case, they are constructed from semistable models and they form a poset under refinement. Berkovich analytic curves can also be endowed with a polyhedral structure locally modeled on an $\mathbb{R}$-tree. The complement of its set of leaves carries a canonical metric. The tropicalization map on $X^{\text{an}}$ factors through the retraction to suitable extended skeleta (defined with respect to a set of punctures of the embedded curve) [BPR11, BPR13]. By the Poincaré-Lelong formula the resulting map is piecewise linear.

The study in [BPR11] was inspired by earlier work of Katz-Markwig-Markwig [KMM08, KMM09] on smooth plane elliptic cubics whose tropicalization contains a cycle. Indeed, the (canonical) length appearing in the minimal Berkovich skeleton is the valuation of the corresponding $j$-invariant and gives an upper bound for the (lattice) length of the cycle in the tropical elliptic cubic. This bound is attained for generic coefficients when fixing the Newton subdivision of the cubic. The challenge then becomes to detect and repair bad embeddings effectively. In [CST13], Chan and Stumfels describe a procedure to put any plane elliptic cubic with multiplicative reduction into honeycomb form, by resolving a univariate sextic equation. The output curve has the expected cycle length.
My joint work with Markwig \cite{CM14} gives an algorithm for re-embedding plane elliptic cubics and produce a tropicalization with a cycle of the right length while preserving its combinatorial type. We do so by degenerating the Newton polygon and analyzing the initial terms of the discriminant:

**Theorem 3.** A tropical plane elliptic cubic curve whose cycle does not reflect the $j$-invariant can be recursively repaired in dimension 4 by linear tropical modifications of the ambient space.

Every finite subgraph $\Gamma$ of the complement of the set of leaves in any analytic curve $X^{\text{an}}$ maps isometrically to a suitable tropicalization of $X$ \cite{BPR11}, where the metric on this tropicalization is given locally by lattice lengths. We say that this tropicalization represents $\Gamma$ *faithfully*. When all tropical multiplicities equal one, any compact connected subset of a tropicalization is the isometric image of a subgraph of $X^{\text{an}}$ \cite{BPR11}. Theorem 3 gives an effective procedure to construct a faithful tropicalization on the minimal skeleton of a plane elliptic cubic curve.

The results from \cite{BPR11} mentioned above can be seen as a comparison between two polyhedral approximations of analytic curves. The first one is given by all tropicalizations (which approximate the analytic space by Payne’s theorem \cite{Pay09} cited above), whereas the second one comes from Berkovich skeletons associated to semistable models. An interesting challenge, studied in \cite{GRW14}, is to look for a comparison of these two different polyhedral approximations for higher dimensional varieties. Two major difficulties arise in this case. First, there is no polyhedral description of the full Berkovich space generalizing the $\mathbb{R}$-tree description for curves. And second, there are no semistable models available in general. Also, in higher dimensions, skeletons are endowed with piecewise linear structures and not with canonical metrics. Canonical structures are available in the Calabi-Yau case by work of Mustata and Nicaise \cite{MN12}.

My joint work with M. H"abich and A. Werner \cite{CHW14} exhibits the first example of a high-dimensional variety that admits a continuous section to its tropicalization. Its image is a candidate canonical polyhedron (it is a piecewise linear rational skeleton in the sense of Ducros \cite{Duc12}):

**Theorem 4.** There exists a continuous section $\sigma$: $\text{Trop}(\text{Gr}(2,n)) \to \text{Gr}(2,n)^{\text{an}}$ to the tropicalization map $\text{trop}: \text{Gr}(2,n)^{\text{an}} \to \text{Trop}(\text{Gr}(2,n))$ induced by the P"ucker embedding. The tropical Grassmannian $\text{Trop}(\text{Gr}(2,n))$ is homeomorphic to a piecewise linear closed subset of $\text{Gr}(2,n)^{\text{an}}$.

The proof of this result is constructive. It relies on the combinatorial characterization by Speyer-Sturmfels \cite{SS04} of the real points of $\text{Trop}(\text{Gr}(2,n))$ as the space of phylogenetic trees on $n$ leaves. Skeleton norms allow us to to build $\sigma$ via local coordinates on each cone. The combinatorial challenge at state was to choose these local coordinates in a compatible and continuous way.

The existence of the continuous section $\sigma$ is also justified by the properties of the initial (toric) degenerations of the defining Pl"ucker ideal of $\text{Gr}(2,n)$. Indeed, our work \cite{CHW14} shows that they are all irreducible and generically reduced, so all points in $\text{Trop}(\text{Gr}(2,n))$ have multiplicity 1. This condition allows us to pick a canonical point in the fiber of the tropicalization map over any point $\omega$: this is precisely the assignment $\sigma(\omega)$. It is not hard to verify that with this definition, the map $\sigma$ is continuous on each torus orbit induced by the natural action of $\mathbb{T}^n/\mathbb{T}$ (embedded diagonally). This is precisely the approach followed by Gubler, Rabinoff and Werner \cite{GRW14} when relating skeletons and tropicalizations of subvarieties of tori in the presence of Shilov boundaries of size one. However, continuity across different torus orbits remains a mystery in the general setting. The combinatorics of $\text{Trop}(\text{Gr}(2,n))$ becomes crucial to address the continuity of $\sigma$ and prove Theorem 4.

2. Research Objectives, Methods, and Significance

These are four specific topics and long-term goals I will pursue in the coming years:

- **(2.1):** Mustafin degenerations, tropical geometry and Schubert calculus.
- **(2.2):** Monodromy pairing on Mumford curves and their tropical counterparts.
- **(2.3):** Tropical geometry and singularity theory.
- **(2.4):** Tropicalizations of cubic del Pezzo and K3 surfaces.
The first three arise naturally from my previous work. The first one is motivated by [CHSW11] and by discussions with E. Katz [Kat09]. The fourth one is inspired by recent work on tropicalizations of del Pezzo surfaces [RSS14b], Dolgachev’s study of mirror symmetry of polarized K3 surfaces [Dol96] and the Gross-Siebert program on mirror symmetry and tropical geometry [GS06, Gro10].

In the following sections, I survey previous results, define concrete goals, and consider possible approaches to the projects listed above.

2.1. Mustafin degenerations, tropical geometry and Schubert calculus. In the groundbreaking article [SS04], Speyer and Sturmfels showed how to tropicalize the Grassmannian $G(d,n)$ of $d$-planes in $K^n$ using the classical Plücker embedding. This map sends a $d \times n$-matrix of rank $d$, representing a $d$-plane, to the list of its $d \times d$-minors in $P(n^d) - 1$. The ideal of algebraic relations among the maximal minors of a $d \times n$-matrix of indeterminates $(x_{ij})_{i,j}$ defines $G(d,n)$, and is known as the Plücker ideal. The tropical Grassmannian is the tropicalization of this ideal.

The polyhedral structure of the real points of $\text{Trop}(\text{Gr}(2,n))$ and $\text{Trop}(\text{Gr}(3,6))$ is well understood [SS04]. It is a subtle question to extend this structure to the boundary components induced by the matroid stratification, as in [GGMS87]. In my work in progress [Cue14], I address this issue. In particular, since every rank two matroid becomes uniform after removing loops and parallel elements, the compact tropical Grassmannian of lines in $\mathbb{P}^n - 1$ can be characterized as follows:

**Theorem 5** ([Cue14]). The compact tropical Grassmannian $\text{Trop}(\text{Gr}(2,n))$ is the generalized space of phylogenetic trees. The cells are associated to trees with at most $n$ leaves. The leaves are labeled by pairwise disjoint subsets of $\{1,\ldots,n\}$. Inclusion of cells is given by contraction of edges, merging of labels in neighboring leaves, deletion of elements in labels and deletion of leaves with empty labels.

Figure 1 shows the structure of $\text{Trop}(\text{Gr}(2,3))$ and representative boundary cells of all dimensions in $\text{Trop}(\text{Gr}(2,4))$. The remaining ones are obtained by permutations of $\{1,2,3,4\}$.

A more basic example is the tropicalization of $\mathbb{P}^{n-1}$, known as the tropical projective space $\mathbb{T}\mathbb{P}^{n-1} := ((\mathbb{R} \cup \{\infty\})^n \setminus \{(-\infty,\ldots,-\infty)\})/\mathbb{R}(1,\ldots,1)$. The open set $\mathbb{T}\mathbb{G}^{n-1} := \mathbb{R}^n/\mathbb{R}(1,\ldots,1)$ is known as the tropical projective torus. This torus has a natural interpretation in Bruhat-Tits theory. When $K$ is the field of Puiseux series, the torus is homeomorphic to the apartment in the Bruhat-Tits building $\mathcal{B}(\text{SL}(n,K))$ given by the maximal split torus of diagonal endomorphisms in $\text{SL}(n,K)$. Apartments are compactified using fans associated to highest weight vectors and the compactification $\mathbb{P}\mathbb{T}^{n-1}$ of $\mathbb{T}\mathbb{G}^{n-1}$ can be obtained in this way [Wer07].

In the spirit of Payne’s result [Pay09], we can study the projective limit over all $K$-linear re-embeddings of a fixed projective variety over $K$. For $\mathbb{P}^n$ this limit is the compactified building with respect to the identity representation, and yields proper subsets for subvarieties. Theorem 5 should have an interpretation in this framework.

Projective spaces, and Grassmannians in general, have a decomposition into affine spaces known as the Schubert decomposition. For example, $\mathbb{P}^{n-1}$ is decomposed as $K^{n-1} \cup K^{n-2} \cup \ldots \cup K \cup K^0$.\[\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{The complex $\text{Trop}(\text{Gr}(2,3))$ and all relevant boundary cells in $\text{Trop}(\text{Gr}(2,4))$.}
\end{figure}
This construction has no known tropical analog. As a first stepping stone, I will study the following problem using tropical degenerations:

**Problem 6.** Characterize the Schubert decomposition of the tropical Grassmannian and compare it to the tropicalization of the classical Schubert decomposition.

After solving Problem 6, I will investigate tropical analogs of Schubert decompositions of flag manifolds in general. In the classical setting, this is done by fixing a flag after a choice of basis (e.g. a fixed Plücker embedding of the Grassmannian), and showing that the resulting Schubert cells are independent of this basis. However, unlike classical algebraic geometry, tropical geometry is extremely sensitive to choices of coordinate systems. Thus, the first task we need to perform when studying Schubert cell decompositions of flag manifolds is the following:

**Problem 7.** Define the tropicalization of the flag manifold $G/P$ of a reductive group $G$.

We recover the tropical Grassmannian when $G = SL(n, K)$ and $P$ is a maximal parabolic subgroup. Schubert cells are affine spaces indexed by partitions $\lambda \subset (n-d)^d$ whose Young diagram fits inside the $d \times (n-d)$ rectangle. They can be described as closures of Borel group orbits. Unfortunately, group actions do not behave well under tropicalization, since associativity is not preserved [Wer11]. There are two ways to bypass this difficulty, namely by degenerations and analytic geometry. The latter comes naturally since a flag variety (like every variety) gives rise to a Berkovich space.

From the point of view of degenerations, tropical Schubert cells should have a combinatorial structure rich enough to encode all degenerations of a group orbit, and become independent of the chosen Plücker embedding. The Bruhat-Tits building $\mathfrak{B}(G)$, closely connected to tropical convexity, should fulfill these requirements [BT72, BT84]. The current definition of tropicalization only considers torus actions induced by a fixed embedding. The building $\mathfrak{B}(G)$ will allow us to introduce a homogeneous version of tropicalization adapted to Problem 7 incorporating the action of all $K$-points of $G$. In particular, Mustafin degenerations of projective space induced by point configurations in $\mathfrak{B}(SL(n, K))$ should have an analog for tropical Grassmannians [CHSW11, Häb14].

Once the Schubert decomposition of the tropical Grassmannian is established, I will explore the notion of tropical Schubert calculus. Strangely enough, this classical branch of intersection theory on flag manifolds has resisted a translation into tropical geometry. The techniques developed in [CHSW11] and tropical intersection theory [AR10, OP13] will play a key role in this new subject.

### 2.2. Monodromy pairing on Mumford curves and their tropical counterparts.

Non-Archimedean uniformization of abelian varieties, such as Jacobians, is closely related to the monodromy pairing defined by Grothendieck [Gro72]: this pairing is the valuation of the analogue of the Riemann form in the non-Archimedean setting [Pap13].

We fix an algebraically closed, complete non-Archimedean field $K$ of characteristic 0. Tropical compactifications [Tex07] allow us to interpret this monodromy pairing by means of Jacobians [HK12]. Let $J$ be the Jacobian of the tropical compactification of a smooth non-rational curve $X$ over $K$. The connected component of the identity in the special fiber of the Néron model of $J$ over the valuation ring of $K$ is an extension of an abelian variety by a torus. The character lattice $\Lambda$ of this torus is naturally isomorphic to $H_1(\Gamma, Z)$, where $\Gamma$ is the minimal skeleton of $X^{an}$. The monodromy pairing $\Lambda \times \Lambda \to Z$ on $J$ is the length pairing on $\Gamma$ [HK12]. Work of Mikhalkin and Zharkov on tropical Jacobians [MZ08] stimulates a tropical exploration of this pairing.

The case of elliptic curves with multiplicative reduction was treated in [KMM08, BPR11, HK12]. By using the conductor-discriminant formula for the $j$-invariant of a plane elliptic cubic one can define a generic (expected) valuation of the $j$-invariant [KMM08]. This induces a piecewise linear function on the space of cubics with fixed Newton subdivision. Its negative gives the length of the cycle in the tropical cubic curve, thus encoding the monodromy action generically. On the analytic side, the length pairing on the minimal skeleton is the negative valuation of the $j$-invariant [BPR11].
Figure 2. Minimal Berkovich skeleta for Mumford curves of genus 2.

In the seminal paper [Mum72], Mumford extended Tate's uniformization of elliptic curves to (Mumford) curves of higher genus with degenerate reduction through the Schottky uniformization [GvdP80]. Their tropical counterparts become balanced graphs with the maximal number of cycles. This property makes them a suitable playground to explore the monodromy pairing.

**Problem 8. Study the monodromy (length) pairing on Mumford curves from the tropical perspective.**

In the $p$-adic case, the length pairing on the non-archimedean side is related to the valuation of the lattice of the $p$-adic Schottky group and the period matrices [MD73]. Recent numerical algorithms exploit the Schottky presentation to construct their skeleta [RM14]. The task becomes more difficult in the absence of such presentation and is related to Whittaker's conjecture on uniformization of hyperelliptic curves [GGD04, Whi]. Tropical geometry will reveal the valuation of these periods.

The next interesting case is that of Mumford curves of genus two, which I am currently investigating jointly with Markwig and Morrison, a graduate student at UC Berkeley [CMM14]. As depicted in Figure 2 the minimal Berkovich skeleta and the corresponding abstract tropical curves in $M_{2}^{\text{trop}}$ have three combinatorial types (the dumbbell, the figure 8 and the theta graphs) [Cha12]. For each type, a point in the moduli space is determined by three edge lengths. This moduli space is one of the objects of study in [RSS14a] and can be analyze through the lens of matroid theory.

In the algebraic setting, the isomorphism classes of Mumford curves of genus 2 are determined by 3 rational functions, call the (absolute) Igusa invariants [Igu60]. They can be computed from a standard parameterization and they are completely determined by an ordering of six Weierstrass ramification points on $\mathbb{P}^1$. More concretely, assume our hyperelliptic curve is the zero locus of $y^2 = u_0 \prod_{i=1}^{6}(x - \alpha_i)$. The quantities $\omega_i = -\text{val}(\alpha_i)$ for all $i = 1, \ldots, 6$ constitute our tropical parameters. After relabeling, the two maximal cells in $M_{2}^{\text{trop}}$ correspond to the orders $\omega_1 < \omega_2 < \omega_3 < \omega_4 < \omega_5 < \omega_6$ (the dumbbell graph) and $\omega_1 < \omega_2 < \omega_3 < \omega_4 < \omega_5 < \omega_6$ (the theta graph). The Igusa invariants are rational functions on all pairwise differences of the six roots $\alpha_i$ [GL12]. In [CMM14] we show that these invariants are not well suited for tropicalization:

**Proposition 9.** The tropicalization of the Igusa invariants are piecewise linear functions but they do not give a complete set of invariants in $M_{2}^{\text{trop}}$. More precisely, $\text{Trop}_{\bullet\circ\bullet}(I_1) = L_1 + L_0 + L_2$ for all $i = 1, 2, 3$, whereas $\text{Trop}_{\bullet\circ\bullet}(I_1) = L_1 + 12L_0 + L_2$, and $\text{Trop}_{\bullet\circ\bullet}(I_2) = \text{Trop}_{\bullet\circ\bullet}(I_3) = L_1 + 8L_0 + L_2$.

We can already observed a similar phenomenon in the ring of symmetric polynomials: the elementary symmetric functions will not yield a complete set of tropical invariants. Indeed, the valuation of an elementary symmetric function only captures the root with highest valuation. On the contrary, the power sums enable us to recover the valuation of all roots. A similar observation will allow us to find the correct coordinates inducing a complete set of tropical Igusa invariants.

**2.3. Tropical geometry and resolutions of singularities.** As was mentioned in Section 1.1, tropical implicitization of non-generic surfaces in 3-space relies in the resolution of singularities of an arrangement of complex curves in $\mathbb{P}^2$. Its cornerstone is the dual (intersection) complex of the resolved arrangement, build by gluing resolution diagrams [Cue12a]. Each step of the resolution translates to a (baricentric) subdivision of the dual complex. We aim to solve the following:

**Problem 10. Find a tropical algorithm to resolve singularities for arrangements of plane curves.**

The resolution diagrams above can be interpreted as phylogenetic trees. The original plane curves in the arrangement are the leaves of this tree, whereas the exceptional divisors are the internal
nodes. When placing the leaves at infinity, these graphs become tropical trees. We propose a method to construct such trees and solve Problem 10. Fix an intersection point $p$ of at least three of the input curves and perform a base change from $\mathbb{C}$ to the field of Puiseux series. Since blow-ups of points on curves locally correspond to monomial changes of variables, we can pick a value $u$ for a slope and replace the bivariate equation $f_k$ defining a curve $D_k$ containing the point $p$ by the equation $g_k(t,q) = f_k(t,q^u t)$, for a fixed parameter $q$. We repeat this process for all curves containing the same point. We can then interpret each polynomial $g_k$ as a univariate polynomial in $q$ with coefficients in $\mathbb{C}[[t]]$ and view the surface parameterized by $f$ as a curve in $\mathbb{C}[[t]]^n$. By tropicalizing the image of $g : \mathbb{C}[[t]] \to \mathbb{C}[[t]]^n$ we obtain the tropical tree associated with $p$.

Examples provided in [Cue12a] witness the inherit difficulties of Problem 10 the valuations of the exceptional divisors heavily depend on the topology of these plane curves. The standard approach to obtain such valuations was introduced in work of Enriques and Chisini [EC85] and further developed with the notions of Enriques and dual diagrams [Wal04]. In [PP11], P. Popescu-Pampu introduced a new object combining both Enriques and dual graphs, under the name of kite. This kite has a natural interpretation in the valuative tree of Favre and Jonsson [FJ04] and it provides the best framework to study arrangements of plane curves. Combining this approach with the tools of combinatorial resolutions derived from Max Noether’s fundamental theorem [CAMB10] we expect to obtain a classification of tropical surfaces in three-space.

In recent years, tropical geometry has provided new tools to address questions in singularity theory, including the celebrated proofs of Loojenga’s conjecture on the equivalence between the smoothability of a cusp singularity and the minimal resolution of the dual cusp induced by the anticanonical divisor of some smooth rational surface [GHK11, Eng14]. To place this interplay into firm footing, P. Popescu-Pampu and D. Stepanov have developed the local analog of tropicalization, adapted to germs of singularities and their deformations. This notion depends on a toroidal embeddings and a space of local valuations. In joint work in progress with these two authors, I exploit local tropicalization to address a conjecture of Neumann and Wahl [NW05] regarding the Milnor fiber of isolated complete intersection surface singularities with integer homology sphere links.

2.4. Tropicalization of cubic del Pezzo and K3 surfaces. Since the beginning of the field, a recurring goal has been to emulate tropical analogues of classical results in algebraic geometry. The well-know statement “any smooth surface of degree 3 in $\mathbb{P}^3$ contains exactly 27 lines” has resisted a full translation into the tropical world. Work of Vigeland [Vig07, Vig10] partially addresses this question, providing examples of cubic surfaces with infinitely many lines and giving a classification of tropical lines on general smooth tropical cubics in $\mathbb{P}^3$. The complete picture remains a mistery.

Recent work on tropical moduli spaces [RSSS13, RSS14a, RSS14b] has provided the tools to put Vigeland’s combinatorial results into a global framework. Fix $K$ to be a complete non-Archimedean algebraically closed field of characteristic 0 and with non-trivial valuation. Smooth cubic surfaces in $\mathbb{P}^3$ are del Pezzo’s, and can be obtained by blowing up $\mathbb{P}^2$ at six points in general position. Using Cremona transformations, we can assume these 6 points lie on a twisted cubic curve, so we identify them with six points $d_1, \ldots, d_6$ in $K$. We obtain a moduli space by varying these six parameters.

**Problem 11.** Interprethe classification of tropical lines on general smooth tropical cubics in $\mathbb{P}^3$ in terms of a subdivision of the parameter space of $t$-uples $\{d_1, \ldots, d_6\}$.

The tools to solve this problem have been developed in [RSS14b] and are based on the construction of universal Cox rings of del Pezzo surfaces. These Cox rings are finitely generated and are naturally graded by the monoid of effective divisor classes on the surface [BP04]. Furthermore, they admit a presentation as a quotient of a polynomial ring on the classes of (-1)-curves by a quadratic ideal [TVAV09, SX10]. The universal family of cubic surfaces is determined by the Göpel functions described in [DOS88]. It is the open part of the Göpel variety $\mathcal{G} \subset \mathbb{P}^{134}$ constructed in [RSSS13, §5] by means of the representation theory of $W(E_7)$. The universal Cox ring incorporates the parameters $d_1, \ldots, d_6$ into the picture, and it is computed in [RSS14b, Proposition 2.2].
As explained in [RSS14b, Section 5], the anticanonical bundle yields an embedding of each cubic surface \( X \subset \mathbb{P}^3 \subset \mathbb{P}^4 \) and it can be viewed inside \( \text{Cox}(X) \). The defining ideal \( I_X \) is generated by 41 linear forms and a cubic equation, with coefficients in \( \mathbb{Z}[d_1, \ldots, d_6] \). These can be computed from the equations of the universal Cox ring. A combinatorial analysis of the coefficients involved in the generators of \( I_X \) will solve Problem 11. I am developing this approach jointly with A. Deopurkar.

Cox rings of \( K3 \) surfaces of algebraically closed fields of characteristic 0 have been recently studied in [AHL10]. In particular, for generic \( K3 \) surfaces with a non-symplectic involution and Picard number between 2 and 5, the polynomials defining their Cox rings are explicitly calculated in [AHL10, Theorem 3]. The techniques for tropicalizing del Pezzo surfaces in terms of Cox rings developed in [RSS14b] will allow us to define and explore the notion of tropical \( K3 \) surfaces.

### References


Research Statement

Maria Angelica Cueto


