A 3-factor Valuation Model for Mortgage-Backed Securities (MBS)

TAKEAKI KARIYA
Research Center for Financial Engineering
KIER, Kyoto University
Sakyo-ku, Kyoto 606-8501 Japan
kariya@kier.kyoto-u.ac.jp

FUMIAKI USHIYAMA
Department of Social Informatics
Kyoto University
Sakyo-ku, Kyoto 606-8501, Japan
ushiyama@kuis.kyoto-u.ac.jp

and

STANLEY R. PLISKA*
Department of Finance
University of Illinois at Chicago
601 S. Morgan Street, Chicago, IL 60607-7124 USA
srpliska@uic.edu

Abstract

In this paper, we generalize the one-factor MBS-pricing model proposed by Kariya and Kobayashi(2000) to a 3-factor model, and describe prepayment behaviors due to refinance and sale of house by incentive response functions. Our valuation of an MBS is made based on a discrete-time no-arbitrage theory to make an association between prepayment behaviors and cash flow patterns therein, and the structure and rationality of our model is demonstrated by valuing an MBS via Monte Carlo simulation and comparing the results.

1 Introduction

Via a no-arbitrage pricing theory in a discrete time setting Kariya and Kobayashi(2000) (abbreviated as KK(2000) or simply KK below) formulated a framework for pricing an MBS(Mortgage-Backed Security) and proposed a one-factor valuation model that has a capacity to describe the burnout-effect of prepayment. The framework directly embeds the heterogeneity of prepayment behaviors into the valuation of an MBS. A special feature of the framework is the treatment of the prepayment options given to loan

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borrowers(mortgagors) in valuing an MBS. In the literature, as represented in Stanton (1995), when a theoretical valuation is attempted, the values of the options are regarded as a gross or lump-sum value and exogenously subtracted from the value of a bond-part. However, an MBS is typically a coupon bond and hence a series of the cash flows from the bond is required to be valued in association with possible occurrences of prepayments in a time horizon. In fact, the distribution of prepayments over a time horizon (heterogeneity of prepayments in time) significantly affects the pattern of cash flows and hence the value of an MBS. The KK framework contains an association between the patterns of cash flows and the heterogeneous prepayment behaviors along with a time horizon and economic uncertainties. But in proposing a specific model, they only treated prepayment due to refinance and expressed the heterogeneity of prepayment behaviors in terms of the differences of incentive thresholds for changes of mortgage interest rates. In doing so, they assumed that the mortgage rate is a linear function of a short-term interest rate that discounts a cash flow of an MBS to a present value. Hence we call this model a one-factor model.

In this paper, we extend the KK model in the following ways;

(1) the distinction of short-term rate as discount factor and mortgage rate as an incentive factor for refinance and

(2) the introduction of prepayment factor due to sale of houses.

It should be emphasized that the short-term rate and mortgage rate are highly correlated but these play different roles in valuation of an MBS. In fact, a big decrease in mortgage rate, which typically involves a decrease in short-term rate, will in general tend to lower the value of an MBS due to refinance, while a decrease in short-term rate tends to increase the value of an MBS through reducing discount rates for distant future cash flows, provided the term to maturity is long. Therefore it is important to distinguish two interest rates and let them play the different roles separately in each position. In other words, the value of an MBS depends on changes of term structure of interest rates. In KK(2000), this effect was treated only by the one-factor of a short-term rate in a Monte Carlo evaluation and the limitation was pointed out in their paper.

The second point (2) is clearly important, especially in valuing US MBS's, because a significant increase in equity value naturally causes to a certain extent sale of houses leading to prepayments. But the sale of houses is also caused by noneconomic or demographic reasons such as death of an owner or a spouse, change of job, etc. In our modelling, this effect is not included in valuation, though the framework enables us to include it so long as the generation process is given. One may use a Poisson process to describe the occurrences of sale of houses due to noneconomic reasons.

In this paper, the heterogeneity of prepayment behaviors is treated as that of incentive thresholds for changes of mortgage rates and changes of prices of houses in loan borrowers. Of course, the differences of the thresholds include the differences of prepayment costs and wealth levels. Our analytical approach to the treatment of this heterogeneity of prepayments is closer to that in a credit risk analysis rather than that in option pricing analysis and in our model, a loan borrower is formulated to prepay only if a change in either his house price or mortgage rate goes over his corresponding threshold for equity or refinance. The heterogeneity is nothing but the difference of these thresholds and the distribution of the thresholds in a loan pool is assumed to be a bivariate normal distribution. Once one of the two variables, i.e., mortgage rate and house price, which are modeled by stochastic processes, hits a corresponding threshold in a 2-dimensional region, a prepayment occurs and the cash flow pattern changes, affecting the value of an MBS. This is our valuation structure and thus it is symbolically expressed as

$$(\{r_n\},\ \{R_n\},\ \{P_n\},\ N(\boldsymbol{\mu}, \boldsymbol{\Sigma}))\,, \tag{1.1}$$

where $\{r_n\}$ and $\{R_n\}$ are respectively short-term rate and mortgage rate processes, $\{P_n\}$ a house price process and $N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ is a bivariate normal distribution with mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$ for the distribution of thresholds in a loan pool. The bivariate normal distribution $N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ describes the heterogeneity of thresholds for prepayments and provides the boundaries that the two incentive factors $\{R_n\}$ and $\{P_n\}$ may hit, while the short-term rate process $\{r_n\}$ acts as a discount factor. Thus, the three-factor structure (1.1) generates prepayments in a loan pool and hence a pattern of cash flows from an MBS. Therefore a value of a given MBS is evaluated through the structure as a forward looking value via the no-arbitrage theory.

There is a large body of literature on the US MBS's, both theoretical and empirical. Among others, Schwartz and Torous(1989) empirically models prepayment or defaults as a function of some explanatory variables. An important issue in a theoretical treatment of prepayment in valuation of an MBS is how an option theory is applied in describing heterogeneous prepayment behaviors. Examples of option-based prepayment models include Dunn and McConnell(1981a,b), Timmis(1985), Dunn and Spatt(1986) and Johnston and Van Drunen(1988). Though cost and lag are introduced as frictional factors in some of these articles, homogeneous prepayment behaviors are basically treated. Stanton(1995) proposed a comprehensive prepayment model to associate heterogeneous behaviors with prepayment cost. While these models have attractive features, and do a reasonable job of explaining actual prepayments, they assume interest rates are the only source of risk. In Kau Keenan, Muller and Epperson(1992,1995), Kau and Keenan(1995), and Deng, Quigley and Van Order(2000) default factor is added to the interest factor in their option based models, though they recognize the importance of the role of house price as a determinant of mortgage termination. Except the last paper, they treat a homogeneous prepayment behavior.

Recently Downing, Stanton and Wallace(2001) develop an option based model that handles both prepayment and default and allow for a direct impact of house prices on mortgage termination. They find that allowing house prices to affect prepayment directly allows the model to describe observed termination behavior significantly.

A common feature of these approaches is that they treat option based models in a continuous time setting and no association has not been made between the cash flow patterns of an MBS and the time distribution of occurrences of prepayments. In other words, options given to mortgagors are separated from the cash flow pattern that changes according to specific occurrences of prepayments in time series and are valued separately from the pattern of changing cash flows. In a discrete time setting, KK(2000) proposed an analytical framework for directly embedding prepayment and default into the cash flow pattern in valuation of an MBS and value an MBS via simulation under an interest model. But they do not distinguish the role of short-term interest rate as a discount factor and the role of mortgage rate as an incentive factor. However, as has been discussed already, lower short-term interest rates discount less for future cash flows and tend to increase value of an MBS, while lower mortgage rates increase incentives for refinance and prepayments, and hence decrease value of the MBS. Nakamura(2001) extends the KK(2000) model to a continuous version and gets a semi-analytic valuation formula for an MBS.

In this paper, we first distinguish these roles by modelling the two interest processes. Secondly we directly treat mortgage termination due to changes of house price. We model the house price process by a discrete time diffusion model with an exponentially smoothing drift model. Hence the house price model is non-Markovian, which is realistic and is allowed because of our discrete time no-arbitrage approach, requiring that relative prices are Martingale under a risk-neutral measure. It is noted that in a continuous time approach a stochastic process is required to be Markovian to develop a no-arbitrage argument.

2 Cash Flows from an MBS with Prepayment

In this section, we describe the cash flow from an MBS with prepayment, where defaults are protected by a guaranty institution. We only consider an MBS based on fixed rate loans with equal monthly payment. Let R_n be the mortgage rate at n, C the coupon of the MBS and S the servicing rate including guaranty. All these rates are annual rates. Also let N be the maturity month, m the current month for valuing the MBS for the remaining period when the prepayment history up to m is given, and n a future month $(0 \leq m \leq n \leq N)$. Also let MB_n be the remaining balance at n when no prepayment occurs. Then, as is well known, the constant monthly payment is

$$MP = MB_0 \times \frac{R_0/12(1+R_0/12)^N}{(1+R_0/12)^N - 1}, \tag{2.1}$$

the initially scheduled interest payment is

$$I_n = MB_{n-1} \times \frac{R_0}{12}, \tag{2.2}$$

and the remaining balance at n with no prepayment is

$$MB_n = MB_0 \times \frac{(1+R_0/12)^N - (1+R_0/12)^n}{(1+R_0/12)^N - 1} \quad (n = 1, \ldots, N). \tag{2.3}$$

Further let $\overline{MB}_n$ be the actual balance at n when prepayments occur and $\overline{I}_n$ the unscheduled interest paid at n under prepayment. To relate actual cash flows with prepayment structure, we assume that there are K loan borrowers in the pool and the loan sizes are equal, where K is only a latent variable to describe proportions of prepayments in terms of the number of borrowers who may prepay in a pool. It is noted that this assumption enables us to switch the concept of prepayment ratio in balances into the concept of prepayment ratio in number of borrowers. It is also assumed that there is no partial prepayment. Let

$$L_n = \text{the number of borrowers who prepay up to } n.$$

Then actual balance at n is expressed in terms of L_n and K as

$$\overline{MB}_n = MB_n(K - L_n) \times \frac{MB_n}{K} = MB_n\left(1 - \frac{L_n}{K}\right), \tag{2.4}$$

and the actual interest paid to investors at n is

$$\begin{aligned} \overline{I}_n &= \overline{MB}_{n-1} \times \frac{R_0}{12} = MB_{n-1}\left(1 - \frac{L_{n-1}}{K}\right)\frac{R_0}{12} \\ &= I_n \times \left(1 - \frac{L_{n-1}}{K}\right). \end{aligned} \tag{2.5}$$

Using these definitions, the total cash flow at n from the MBS is the change of the actual balance from $n-1$ to n and the actual interest at n

with the servicing fee deducted;

(2.6)

$$\begin{aligned}
\overline{CF}_n &= (\overline{MB}_{n-1} - \overline{MB}_n) + \frac{C}{C+S}\overline{I}_n \\
&= MB_{n-1}\left(1 - \frac{L_{n-1}}{K}\right) - MB_n\left(1 - \frac{L_n}{K}\right) + \frac{C}{C+S}I_n\left(1 - \frac{L_{n-1}}{K}\right) \\
&= a_n\left(1 - \frac{L_n}{K}\right) + b_n\left(1 - \frac{L_{n-1}}{K}\right),
\end{aligned}$$

where

$$\begin{aligned}
a_n &= -MB_n \\
b_n &= MB_{n-1} + \frac{C}{C+S}I_n.
\end{aligned}$$

Note that a_n and b_n are known at 0 and hence the unknown variables in (2.6) are only the prepayment proportions L_n/K and L_{n-1}/K in the pool, which are randomly generated by mortgage rate process $\{R_n\}$ and house price process $\{P_n\}$. It should be noted again that the prepayment proportions are switched from the concept of the remaining balances to that of the remaining borrowers. Thus we can relate the actual cash flows to prepayment activities of borrowers in the pool.

Now as in KK(2000), by a general no-arbitrage pricing theory in discrete time frame work, the no-arbitrage value at m of the MBS with maturity N is given by

$$V(m,N) = \sum_{n=m+1}^{N} CF(m,n), \tag{2.7}$$

where

$$\begin{aligned}
CF(m,n) &= E_m^*[\Delta(m,n)CF_n] \\
&= E_m^*\left[\Delta(m,n)\left\{a_n\left(1 - \frac{L_n}{K}\right) + b_n\left(1 - \frac{L_{n-1}}{K}\right)\right\}\right]
\end{aligned} \tag{2.8}$$

with

$$\Delta(m,n) = \exp\left(-\sum_{j=m}^{n-1} r_j h\right) \quad (h = 1/12). \tag{2.9}$$

Here $\{r_j\}$ is an short-term interest rate process and $E_m^*[\cdot]$ is the conditional expectation at m under a risk neutral measure for $\{r_j\}$, $\{R_j\}$ and $\{P_j\}$. Since the model is incomplete, we take an actual measure as such a risk neutral measure, which still guarantees no-arbitrage valuation. Note that $\Delta(m,n)$ randomly discounts a cash flow at n to a value at m.

3 Three-factor model

In KK(2000), it is assumed that a borrower in a pool prepays at n for gains only when the spread of the initial mortgage rate R_0 and the current rate R_n widens more than or equal to his incentive threshold for the first time. Thus the condition of the prepay for the k-th borrower is formulated as

$$u_n^{(1)} = R_0 - R_n \geq d_k^{(1)}. \tag{3.1}$$

And then, they assume that mortgage rate R_n and spot rate r_n are linearly related as $R_n = \alpha(s) + \beta(s) r_n$ to make the process $\{r_n\}$ describe both prepayment due to refinance via (3.1) and the discount factor in (2.9).

In our extended model, we directly use mortgage rates $\{R_n\}$ to describe prepayment behaviors for refinance in (3.1). The threshold $d_k^{(1)}$ in (3.1) in general depends on n and some other state variables such as business condition, but we assume that it is constant over time in this paper. While, a short-term spot rate process $\{r_n\}$ plays as a discount factor for cash flows in (2.8) with (2.9). Naturally $\{R_n\}$ and $\{r_n\}$ need to be specified as processes with high correlation.

On the other hand, we also assume that prepayment behaviors due to sale of houses are described in terms of economic incentives as in the case of refinance in (3.1). More specifically, let P_n be the price level at n. Here it is assumed that it is a common price level for the houses in the pool, and that the k-th borrower sells his mortgaged house if the difference of the current log-price and the initial log-price exceeds or equals his threshold for equity for the first time;

$$u_n^{(2)} = \log P_n - \log P_0 \geq d_k^{(2)}, \tag{3.2}$$

where $d_k^{(2)}$ is also assumed to be constant over time. In general, the house prices and mortgage rates are highly correlated. In this set-up, it is formally expressed that the k-th borrower prepays at $\tau = \min\{\tau_1, \tau_2\}$, where $\tau_i = \min\{n : u_n^{(i)} \geq d_k^{(i)}\} (i = 1, 2)$.

Now let us specify the models for the three factors; $\{r_n\}$, $\{R_n\}$ and $\{P_n\}$. We assume that the monthly spot rate process $\{r_n\}$ and the mortgage rate process $\{R_n\}$ follow Vasicek model;

$$\Delta r_n = \theta_0^{(0)}(\theta_1^{(0)} - r_{n-1})h + \theta_2^{(0)}\sqrt{h}\varepsilon_n^{(0)}, \tag{3.3}$$

$$\Delta R_n = \theta_0^{(1)}(\theta_1^{(1)} - R_{n-1})h + \theta_2^{(1)}\sqrt{h}\varepsilon_n^{(1)}, \tag{3.4}$$

where $\Delta r_n = r_n - r_{n-1}$ and $\Delta R_n = R_n - R_{n-1}$, and the house price process $\{P_n\}$ follows the model;

$$P_n = P_{n-1}\exp(\mu_{n-1}h + \sigma\sqrt{h}\varepsilon_n^{(2)}), \tag{3.5}$$

$$\mu_{n-1} = \phi\mu_{n-2} + (1 - \phi)\log\left(\frac{P_{n-1}}{P_{n-2}}\right). \tag{3.6}$$

where the volatility σ is assumed to be constant and $0 \leq \phi \leq 1$. The drift process driven by past value of process in (3.6) is called an exponentially smoothing model. The parameter $1-\phi$ is the proportion of a recent change in price brought into a change in the drift. The greater $1-\phi$ is, the more volatile the drift is, though it depends on the volatility σ. Here innovations $\boldsymbol{\varepsilon} = (\varepsilon_n^{(0)}, \varepsilon_n^{(1)}, \varepsilon_n^{(2)})$ are assumed to be iid(independently and identically distributed) as 3-dimensional normal with mean $\mathbf{0}$ and covariance matrix $\mathbf{\Lambda}$ with

$$\mathbf{\Lambda} = \begin{pmatrix} 1 & \rho_{01} & \rho_{02} \\ \rho_{10} & 1 & \rho_{12} \\ \rho_{20} & \rho_{21} & 1 \end{pmatrix}.$$

Next, we formulate incentive responsive behaviors by specifying the distribution of thresholds in the pool. The distribution of the thresholds $\{(d_k^{(1)}, d_k^{(2)}) : k = 1, \ldots, K\}$ determines the heterogeneous prepayment behaviors and hence a cash flow pattern of an MBS along realizations of $\{(R_n, P_n) : n = 1, \ldots, N\}$. Therefore the threshold distribution plays an important role in valuation of an MBS. On this distribution, we make

ASSUMPTION. Let the proportion of thresholds lying in a rectangle be denoted by

(3.7)

$$p_{lm} = \frac{1}{K} \times \{\# \text{ of } (d_k^{(1)}, d_k^{(2)}) \text{ in } [(l-1)\eta^{(1)}, l\eta^{(1)}) \times [(m-1)\eta^{(2)}, m\eta^{(2)})\}.$$

Then, p_{lm} is approximated by the area of 2-dimensional normal distribution $N(\boldsymbol{\mu}, \mathbf{\Sigma})$ over $[(l-1)\eta^{(1)}, l\eta^{(1)}) \times [(m-1)\eta^{(2)}, m\eta^{(2)})$ $(l = 1, \ldots, q^{(1)},\ m = 1, \ldots, q^{(2)})$, where

$$\boldsymbol{\mu} = \begin{pmatrix} \mu^{(1)} \\ \mu^{(2)} \end{pmatrix} \text{and} \mathbf{\Sigma} = \begin{pmatrix} (\sigma^{(1)})^2 & \sigma^{(1)}\sigma^{(2)}\delta \\ \sigma^{(1)}\sigma^{(2)}\delta & (\sigma^{(2)})^2 \end{pmatrix}. \tag{3.8}$$

Under this assumption, the K borrowers are allocated into $q^{(1)} \times q^{(2)}$ groups according to their thresholds and the prepayment rate at n in terms of the borrowers is approximated by the corresponding normal volume determined by the maxima of incentive variables $u_n^{(i)}$ $(i = 1, 2)$;

$$\frac{L_n}{K} = \sum_{k=1}^{K} \frac{L_{k,n}}{K} \approx \sum_{l=1}^{q^{(1)}} \sum_{m=1}^{q^{(2)}} g_{lm}(v_n^{(1)}, v_n^{(2)}) p_{lm}, \tag{3.9}$$

where

$$L_{k,n} = \begin{cases} 1 & \text{if the } k\text{-th borrower prepaid up to } n \\ 0 & \text{otherwise,} \end{cases} \tag{3.10}$$

$$g_{lm}(v_n^{(1)}, v_n^{(2)}) = \begin{cases} 1 & \text{if } l\eta^{(1)} \leq v_n^{(1)} \text{ or } m\eta^{(2)} \leq v_n^{(2)} \\ 0 & \text{otherwise} \end{cases} \tag{3.11}$$

with

$$v_n^{(i)} = \max_{j \leq n}\{u_j^{(i)}\} \ (i = 1, 2). \tag{3.12}$$

Let $A(n)$ be the right side of (3.9);

$$A(n) \equiv \sum_{l=1}^{q^{(1)}} \sum_{m=1}^{q^{(2)}} g_{lm}(v_n^{(1)}, v_n^{(2)}) p_{lm}. \tag{3.13}$$

By its definition, $g_{lm}(v_n^{(1)}, v_n^{(2)}) = 1$ if and only if

$$l \leq \frac{v_n^{(1)}}{\eta^{(1)}} \text{ or } m \leq \frac{v_n^{(2)}}{\eta^{(2)}}. \tag{3.14}$$

Since l and m are integers, (3.14) means

$$l \leq \left[\frac{v_n^{(1)}}{\eta^{(1)}}\right] \text{ or } m \leq \left[\frac{v_n^{(2)}}{\eta^{(2)}}\right], \tag{3.15}$$

where $[a]$ denotes the largest integer less than or equal to a. Using the assumption and this fact and distinguishing cases in terms of the maxima $v_n(i)$'s, $A(n)$ is evaluated as;

1. if $v_n^{(1)} < \eta^{(1)}$ and $v_n^{(2)} < \eta^{(2)}$

$$A(n) = 0. \tag{3.16}$$

2. if $\eta^{(1)} \leq v_n^{(1)} < q^{(1)}\eta^{(1)}$ and $v_n^{(2)} < \eta^{(2)}$

$$A(n) = \Phi\left(\frac{\left[\frac{v_n^{(1)}}{\eta^{(1)}}\right]\eta^{(1)} - \mu^{(1)}}{\sigma^{(1)}}\right). \tag{3.17}$$

3. if $v_n^{(1)} < \eta^{(1)}$ and $\eta^{(2)} \leq v_n^{(2)} < q^{(2)}\eta^{(2)}$

$$A(n) = \Phi\left(\frac{\left[\frac{v_n^{(2)}}{\eta^{(2)}}\right]\eta^{(2)} - \mu^{(2)}}{\sigma^{(2)}}\right). \tag{3.18}$$

4. if $\eta^{(1)} \le v_n^{(1)} < q^{(1)}\eta^{(1)}$ and $\eta^{(2)} \le v_n^{(2)} < q^{(2)}\eta^{(2)}$

$$A(n) = 1 - \Phi\left(-\frac{\left[\frac{v_n^{(1)}}{\eta^{(1)}}\right]\eta^{(1)} - \mu^{(1)}}{\sigma^{(1)}} \ , \ -\frac{\left[\frac{v_n^{(2)}}{\eta^{(2)}}\right]\eta^{(2)} - \mu^{(2)}}{\sigma^{(2)}} \ ; \ \delta \right). \tag{3.19}$$

5. if $v_n^{(1)} \ge q^{(1)}\eta^{(1)}$ or $v_n^{(2)} \ge q^{(2)}\eta^{(2)}$

$$A(n) = 1. \tag{3.20}$$

Here, $\Phi(\cdot)$ denotes the standard normal distribution function and $\Phi(\cdot,\cdot;\delta)$ denotes the 2-dimensional standard normal distribution function of $N\left(\mathbf{0}, \begin{pmatrix} 1 & \delta \\ \delta & 1 \end{pmatrix}\right)$.

Thus to value an MBS at $m = 0$ by (2.7), we have a formula

$$V(0,N) = \sum_{n=1}^{N} E_0^{*}\left[\Delta(0,n)\left\{a_n\left(1 - A(n)\right) + b_n\left(1 - A(n-1)\right)\right\}\right], \tag{3.21}$$

where $A(0) = 0$. Note that $A(n)$ is a function of $\{R_n\}$ and $\{P_n\}$ through $\{v_n^{(1)}\}$ and $\{v_n^{(2)}\}$. In this valuation, no past observation on prepayment is available and hence the parameters $(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \eta^{(1)}, \eta^{(2)})$, which we need to specify as a model for describing for prepayment behaviors, have to be specified in advance, probably based on past experiences. On the other hand, when we value an MBS at $m > 0$ issued at 0, we observe the paths $\{(r_n, R_n, P_n) : 0 \le n \le m\}$ and survival rate

$$\frac{\overline{MB}_n}{MB_n} = 1 - \frac{K}{L_n} \quad (1 \le n \le m). \tag{3.22}$$

Hence in this case we may estimate the parameters by minimizing

$$\sum_{j=1}^{n} \left\{ 1 - \frac{\overline{MB}_j}{MB_j} - A(j) \right\}^2, \tag{3.23}$$

where $v_j^{(i)} = \max\{u_j^{(i)}\} (i = 1, 2)$'s are given up to n.

4 Monte Carlo Simulation

4.1 Method

To value an MBS by Monte Carlo simulation at $m = 0$, first, we generate N 3-dimensional vectors of random numbers I times;

$$(\boldsymbol{\varepsilon}_1^{(i)}, \boldsymbol{\varepsilon}_2^{(i)}, \dots, \boldsymbol{\varepsilon}_N^{(i)}) \quad (i = 1, \dots, I), \tag{4.1}$$

where

(4.2)

$$\boldsymbol{\varepsilon}_j^{(i)} = \begin{pmatrix} \varepsilon_j^{(0)(i)} \\ \varepsilon_j^{(1)(i)} \\ \varepsilon_j^{(2)(i)} \end{pmatrix} \sim iid\ N\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho_{01} & \rho_{02} \\ \rho_{10} & 1 & \rho_{12} \\ \rho_{20} & \rho_{21} & 1 \end{pmatrix} \right) \qquad (j = 1, \ldots, N)$$

Through (3.3), (3.4) and (3.5), these innovations in turn generate I paths of spot rates, mortgage rates and house prices over N respectively;

$$(4.3) \qquad (r_1^{(i)}, r_2^{(i)}, \ldots, r_N^{(i)}), (R_1^{(i)}, R_2^{(i)}, \ldots, R_N^{(i)}), (P_1^{(i)}, P_2^{(i)}, \ldots, P_N^{(i)}).$$

From these paths we obtain I sets of N random discount factors, I paths of maxima $v_n^{(1)(i)}$ and $v_n^{(2)(i)}$ via (3.12) with (3.1) and (3.2):

$$(4.4) \qquad (\Delta_1^{(i)}, \Delta_2^{(i)}, \ldots, \Delta_N^{(i)}) \quad \text{with} \quad \Delta_n^{(i)} = \exp\left(-\sum_{j=0}^{n-1} r_j^{(i)} h \right), and$$

$$(4.5) \qquad (v_1^{(j)(i)}, v_2^{(j)(i)}, \ldots, v_N^{(j)(i)}) \quad (j = 1, 2).$$

From (3.16) through (3.20), we obtain the rates of borrowers who prepaid up to $n = 1, \ldots, N$ along the i-th realization of spot rates, mortgage rates and house prices in (4.3);

$$(4.6) \qquad (A(1)^{(i)}, A(2)^{(i)}, \ldots, A(N)^{(i)}),$$

Consequently, we can value an MBS numerically by $V(0, N) = \sum_{n=1}^{N} CF(0, n)$ with

$$(4.7) \quad CF(0, n) = \frac{1}{I} \sum_{i=1}^{I} \Delta_n^{(i)} \left\{ a_n \left(1 - A^{(i)}(n) \right) + b_n \left(1 - A^{(i)}(n-1) \right) \right\}.$$

In our simulation, we set $I = 1000$ for each case.

4.2 Numerical valuation and Comparison

Not only to demonstrate an evaluation but also to learn the capacity of our valuation model for an MBS, we run various Monte Carlo simulations and draw some important features of our model. In the simulation, we consider a 30-year MBS with \$100 face value and 6.5% coupon made of mortgage loans with 7% mortgage rate and equal monthly payment. Here, 0.5% is the servicing fee. Thus,

$$R_0 = 0.07, \ S = 0.005, \ C = 0.065, \ \text{and} \ N = 360.$$

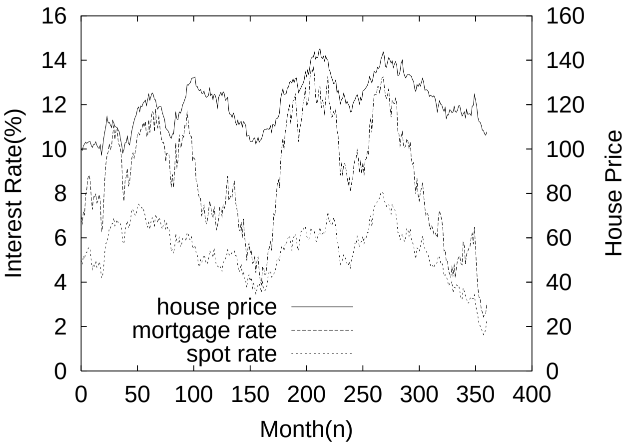

Figure 1: Sample paths of spot rate, mortgage rate and house price process

For the parameters in the spot rate model, mortgage rate model, and house price model, put as our standard case

$$\begin{aligned}
&\theta_0^{(1)} = 0.2,\ \theta_1^{(1)} = 0.05,\ \theta_2^{(1)} = 0.008,\ r_0 = 0.05, \\
&\theta_0^{(2)} = 0.2,\ \theta_1^{(2)} = 0.07,\ \theta_2^{(2)} = 0.016, \\
&P_0 = 100,\ \mu_0 = 0.00,\ \sigma_0 = 0.06,\ \phi = 0.5, \\
&\rho_{01} = 0.8,\ \rho_{02} = 0.5,\ \rho_{12} = 0.7,\ \text{and } h = 1/12 = 0.083.
\end{aligned}$$

In this standard case, the short-term rate $\{r_n\}$ and mortgage rate $\{R_n\}$ are distinguished in the two ways that the volatility $\theta_2^{(1)} = 0.016$ (i.e. 1.6% annual rate) of $\{R_n\}$ is twice bigger than the volatility $\theta_2^{(0)} = 0.008$ and the mean reversion levels are respectively $\theta_1^{(1)} = 0.07$ and $\theta_1^{(0)} = 0.05$. The adjustment speed to the mean reversion level is commonly 0.20. The two rates are highly correlated (0.8). On the other hand, the volatility in house price changes is 6% and the initial drift is set 0. The correlations of $\{\log P_n\}$ to $\{R_n\}$ and $\{r_n\}$ are respectively 0.7 and 0.5 in their innovations.

As demonstration, sample paths of spot rate, mortgage rate and house price process are graphed in Figure 1. Clearly the house prices move most volatilely.

Next, the parameters of the normal distribution of the thresholds are given as

$$\begin{aligned}
&q^{(1)} = 10,\ \mu^{(1)} = 0.04,\ \sigma^{(1)} = 0.0133,\ \eta^{(1)} = 0.008, \\
&q^{(2)} = 10,\ \mu^{(2)} = 0.4,\ \sigma^{(2)} = 0.133,\ \eta^{(2)} = 0.08,\ \text{and } \delta = 0.5.
\end{aligned}$$

In this specification, the threshold unit $\eta^{(2)} = 0.08$ for sale of house is 10 times bigger than that for refinance. This means that each 8% change

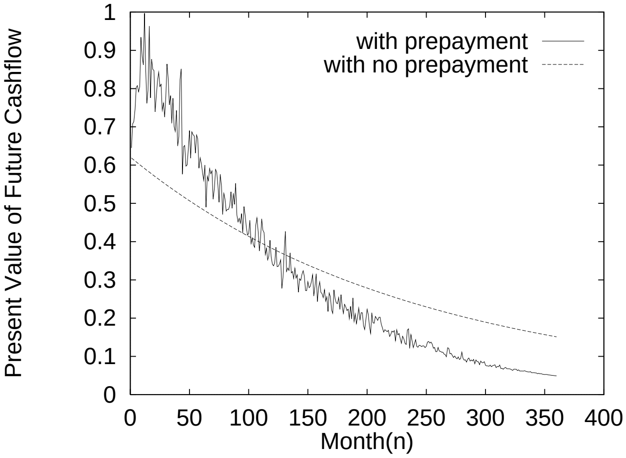

Figure 2: The values of cash flows without and with prepayments

in house price urges borrowers to sell houses while each 0.8% change in mortgage rate causes prepayments for refinance. The means and standard deviations are also specified in the same way. The group number is $10 \times 10 = 100$. The number of the paths we generate is $I = 1000$ for each valuation.

In this set-up, we obtained a theoretical value of the MBS as 111.948 dollars. The value of cash flows $\{CF(0, n)\}$ with and without prepayments in this Monte Carlo evaluation are graphed in Figure 2. The graph shows that the values of the cash flows in the first 50 months overwhelmingly dominate those in the rest. This implies severe prepayments in the early months. The average number of the groups of borrowers who prepaid due to either sale of house or refinance is about 66 groups in the thousand cases.

4.3 Comparisons among one-factor, two-factor and three-factor models

In this subsection, we compare the one-factor model that KK(2000) considered by assuming the linear relation $R_n = a + br_n$, the two-factor model for refinance in which the roles of the short-term rate $\{r_n\}$ and the mortgage rate $\{R_n\}$ are separated, and the three-factor model where the factor of house price $\{P_n\}$ is introduced in addition.

We specified the threshold unit as 0.004 in the case of one-factor model because the volatility of the short term rate is a half of that of the mortgage rate. We ran 5 independent simulations each of which consists 1000 paths of $\{r_n\}$, $\{R_n\}$ and $\{P_n\}$. The result is summarized in Table 1.

From the table it is observed that the separation of the roles of the discount factor by $\{r_n\}$ and the refinancing incentive factor by $\{R_n\}$ increases MBS prices, while the addition of the equity incentive to the two-factor

Table 1: price vs. model

model	price(1)	price(2)	price(3)	price(4)	price(5)
one-factor	111.836	111.761	111.557	111.962	111.912
two-factor	112.379	112.217	111.957	112.343	112.255
three-factor	111.948	111.752	111.506	111.877	111.779

model decreases MBS prices. The values of MBS's in one-factor and three factor models are close each other in these simulations.

4.4 Effect of (μ_0, σ, ϕ) on MBS prices

In the following manner, we investigate some effects of changes of parameter (μ_0, σ, ϕ) in the house price model on MBS values.

(1) Effect of changes in (μ_0, σ) with $\phi = 0.5$ (Table 2)

(2) Effect of changes in (σ, ϕ) with $\mu_0 = 0.0$ (Table 3)

(3) Effect of changes in (ϕ, μ_0) with $\sigma = 0.06$ (Table 4).

Both Table 2 and 3 show that the MBS prices decrease as σ increases for the other parameter held fixed. This is natural because an increase in volatility increases the possibility of more prepayments. On the other hand, the two tables also show that the prices are insensitive to changes of the other parameters (μ_0, ϕ) when the volatility is held fixed.

From Table 4 where $\sigma = 0.06$, it is observed that when ϕ is fixed, MBS prices go down as μ_0 increases. This is because an increase in μ_0 tends to appreciate an increase in house price, leading to more prepayments. This tendency is stronger when ϕ is larger as in Table 4 and Figure 3. Recall that the larger ϕ's introduce the less information on new changes of the house price into the drift movement and stabilize the drift movement. This is confirmed in Figure 4 where the graphs of the drift movements are demonstrated with $\phi = 0.1$ and $\phi = 0.9$ in the cases of $\mu_0 = 0.09$. Thus when ϕ is larger, the initial drift μ_0 affects more on the movement of house price, and hence on MBS prices as is shown in Figure 5.

4.5 Effect of the correlations $(\rho_{01}, \rho_{02}, \rho_{12})$ on MBS prices

The correlations $(\rho_{01}, \rho_{02}, \rho_{12})$ of the innovations of $\{r_n\}$, $\{R_n\}$ and $\{P_n\}$ also make an effect on MBS prices. To investigate the effect, simulations with proportional changes of the correlations and the other parameters fixed were carried out in Table 5. The table shows that as the correlations increase, the prices decrease rather greatly as we expect. When ρ_{12} is positive,

Table 2: price vs. μ_0 and σ ($\phi = 0.5$)

$\mu_0 \setminus \sigma$	0.02	0.04	0.06	0.08	0.1
-0.06	112.369	112.266	111.981	111.534	110.995
-0.03	112.369	112.258	111.967	111.507	110.963
0.00	112.367	112.249	111.948	111.470	110.919
0.03	112.364	112.237	111.924	111.436	110.872
0.06	112.362	112.227	111.902	111.400	110.829
0.09	112.359	112.214	111.880	111.365	110.785

Table 3: price vs. σ and ϕ ($\mu_0 = 0.0$)

$\sigma \setminus \phi$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.02	112.367	112.367	112.367	112.367	112.367	112.367	112.367	112.367	112.367
0.04	112.248	112.249	112.249	112.249	112.249	112.249	112.250	112.251	112.252
0.06	111.944	111.945	111.945	111.946	111.948	111.950	111.951	111.954	111.961
0.08	111.465	111.466	111.467	111.468	111.470	111.472	111.475	111.482	111.506
0.1	110.905	110.907	110.912	110.913	110.919	110.926	110.935	110.941	110.971

Table 4: price vs. ϕ and μ_0 ($\sigma = 0.06$)

$\phi \setminus \mu_0$	-0.06	-0.03	0	0.03	0.06	0.09
0.1	111.966	111.958	111.944	111.932	111.918	111.905
0.2	111.969	111.959	111.945	111.930	111.913	111.902
0.3	111.972	111.961	111.945	111.930	111.909	111.896
0.4	111.975	111.964	111.946	111.928	111.906	111.891
0.5	111.981	111.967	111.948	111.924	111.902	111.880
0.6	111.992	111.971	111.950	111.921	111.892	111.856
0.7	112.010	111.978	111.951	111.911	111.872	111.830
0.8	112.036	111.998	111.954	111.899	111.833	111.758
0.9	112.114	112.041	111.961	111.840	111.700	111.546

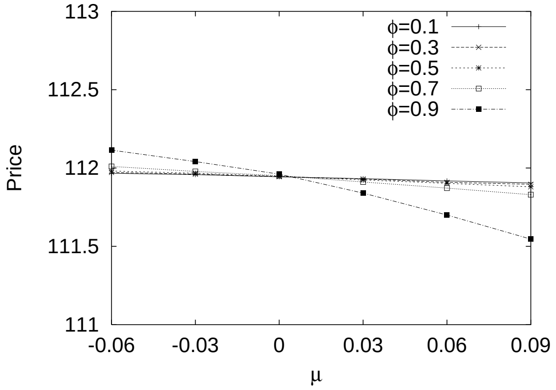

Figure 3: price vs. μ_0

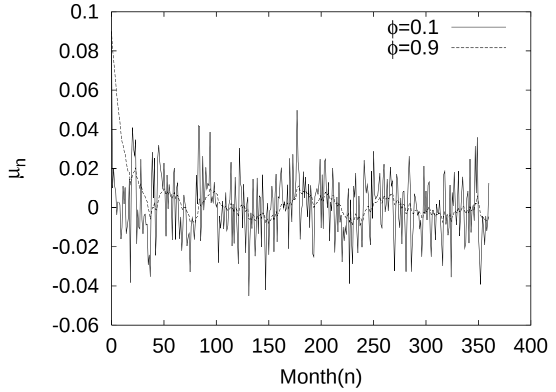

Figure 4: sample paths of $\{\mu_n\}(\mu_0 = 0.09)$

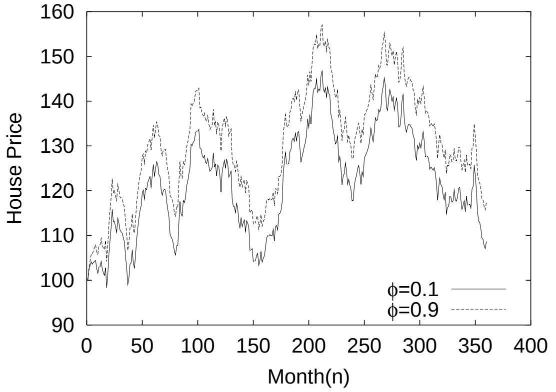

Figure 5: sample paths of house price process$(\mu_0 = 0.09)$

Table 5: price vs. correlations between spot rate, mortgage rate, and house price

ρ_{01}	ρ_{02}	ρ_{12}	price
0.0	0.0	0.0	113.425
0.2	0.125	0.175	113.049
0.4	0.25	0.35	112.658
0.6	0.375	0.525	112.290
0.8	0.5	0.7	111.948

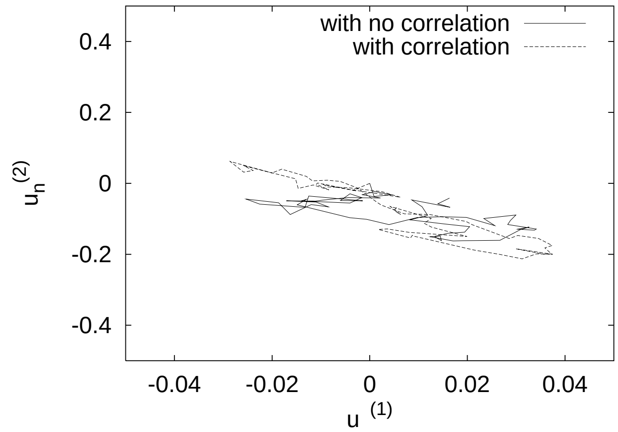

Figure 6: sample paths of $(u_n^{(1)}, u_n^{(2)})$ ($\rho_{12} = 0$ and $\rho_{12} = 0.7$)

the incentive variables $u_n^{(1)} = R_0 - R_n$ and $u_n^{(2)} = \log P_n - \log P_0$ are negatively correlated and when ρ_{12} is higher, the distribution of $(u_n^{(1)}, u_n^{(2)})$ tends to span more linearly leading to larger $v_n^{(1)} = \max_{j \le n}\{u_j^{(1)}\}$ and $v_n^{(2)} = \max_{j \le n}\{u_j^{(2)}\}$. This situation is demonstrated in Figure 6

4.6 Effect of threshold parameters and group number $(q^{(1)}, q^{(2)})$ on MBS prices

The threshold correlation δ also affects the prepayment ratios in the model and hence the MBS prices. As δ increases, the thresholds $(d_k^{(1)}, d_k^{(2)})$'s of borrowers get closer the 45 degree line (see Figure 7). But then paths of $(u_n^{(1)}, u_n^{(2)})$ have to move more widely to cause a certain level of prepayment. Hence when δ is higher, $(u_n^{(1)}, u_n^{(2)})$ tends to cause less prepayments, meaning higher MBS prices. This situation is confirmed in Table 6.

The effect of the group number $(q^{(1)}, q^{(2)})$ on MBS prices is investigated

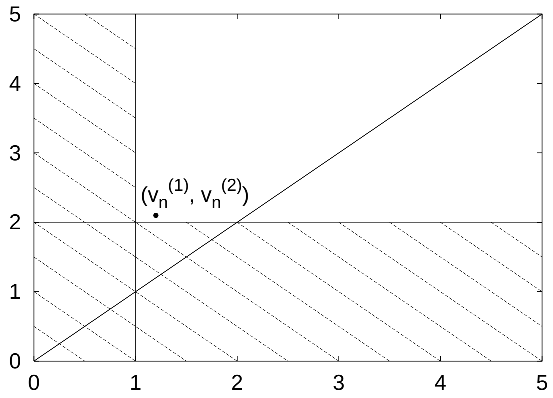

Figure 7: Prepayed area

Table 6: price vs. correlation between thresholds $(\delta)(\mu_0 = 0,\ \sigma_0 = 0.06,\ \phi = 0.5)$

δ	price
0.0	111.860
0.1	111.877
0.2	111.894
0.3	111.911
0.4	111.929
0.5	111.948
0.6	111.967
0.7	111.986
0.8	112.005
0.9	112.025
1.0	112.052

Table 7: price vs. $q^{(1)}$ and $q^{(2)}$

$q^{(1)} \ \backslash \ q^{(2)}$	10	20	30	40	50
10	111.948	111.864	111.834	111.817	111.810
20	111.437	111.358	111.331	111.314	111.308
30	111.262	111.185	111.158	111.142	111.135
40	111.165	111.089	111.062	111.046	111.040
50	111.120	111.044	111.018	111.002	110.996

Table 8: price vs. $\mu^{(1)}$ and $\mu^{(2)}$

$\mu^{(1)} \setminus \mu^{(2)}$	0.2	0.3	0.4	0.5	0.6
0.02	104.857	105.357	105.558	105.639	105.680
0.03	107.817	108.665	109.015	109.159	109.232
0.04	110.331	111.467	111.945	112.143	112.246
0.05	112.112	113.453	114.017	114.254	114.378
0.06	113.242	114.722	115.343	115.607	115.746

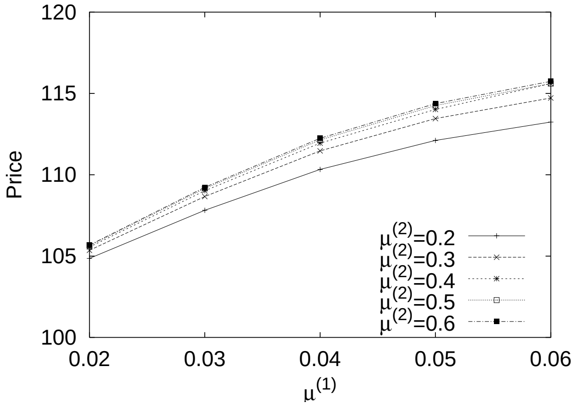

Figure 8: price vs. $\mu^{(1)}$ and $\mu^{(2)}$

in Table 7. A larger group number implies a finer division of borrower's incentive levels by threshold. In general, a finer division gives a closer approximation for the remaining balance, provided a loan pool is big and hence the number of borrowers is large. Table 7 shows that as group number increases, MBS prices increase but the changes in the increases decrease, implying a possibility of the convergence of MBS prices to a certain level as $q^{(i)} \to \infty$ $(i = 1, 2)$. Practically speaking, loan pool is not necessarily big and hence the group number may be chosen relatively to the pool size.

Thirdly, we change the mean level $(\mu^{(1)}, \mu^{(2)})$ of thresholds and see the effect on MBS prices. Clearly the higher the mean level is, the less prepayments occur, and hence the larger the MBS prices are. This is demonstrated in Table 8 and Figure 8. The prices are shown to be very sensitive to changes of the mean level.

4.7 Effect of interest model parameters on MBS prices

First, we here change the volatility parameters $(\theta_2^{(0)}, \theta_2^{(1)})$ in the interest model where the correlation $\rho_{01} = 0.8$ is held fixed. Since the maximum

Table 9: price vs. $\theta_2^{(0)}$ and $\theta_2^{(1)}$

$\theta_2^{(0)} \setminus \theta_2^{(1)}$	0.008	0.012	0.016	0.02
0.004	116.141	114.454	112.318	110.316
0.006	116.323	114.443	112.114	109.973
0.008	116.566	114.480	111.948	109.660
0.01	116.871	114.566	111.819	109.376

Table 10: price vs. $\theta_0^{(0)}$ and $\theta_0^{(1)}$

$\theta_0^{(0)} \setminus \theta_0^{(1)}$	0.1	0.2	0.3	0.4
0.1	110.516	111.684	112.717	113.678
0.2	111.016	111.948	112.773	113.540
0.3	111.326	112.154	112.885	113.564
0.4	111.526	112.298	112.978	113.608

$v_n^{(1)} = \max_{j \le n} u_j^{(1)}$ matters for pricing MBS's, an increase in the volatility $\theta_2^{(1)}$ tends to increase $v_n^{(1)}$ and hence decreases MBS values. On the other hand, since $\{r_n\}$ acts as a discount factor for cash flows, the effect of $\theta_2^{(0)}$ on MBS prices is indefinite. This is shown in Table 9. In the case of $\theta_2^{(1)} = 0.008$ in the table, as $\theta_2^{(0)}$ increases, the MBS prices gradually increase, while in case of $\theta_2^{(1)} = 0.012$, the MBS prices first decrease and then increase.

Next let us consider an effect of the adjustment speed parameters $(\theta_0^{(0)}, \theta_0^{(1)})$ on MBS prices where the mean reversion level $(\theta_1^{(0)}, \theta_1^{(1)}) = (0.05, 0.07)$ and the volatility $(\theta_2^{(0)}, \theta_2^{(1)}) = (0.008, 0.016)$ are fixed. Table 10 gives a result on this case. When the adjustment speed to the mean reversion level is bigger, the interest rates tend to stay around the mean reversion level, implying less prepayments, though the volatility is another factor. This phenomenon is demonstrated in Figure 9. As in Table 10, $\theta_0^{(0)}$ fixed, the MBS prices increase as the speed $\theta_0^{(1)}$ for $\{R_n\}$ increases, while for $\theta_0^{(1)}$ fixed the effect of the speed $\theta_0^{(0)}$ for discount factor $\{r_n\}$ on MBS prices is indefinite and does not change the prices much.

5 Concluding Remarks

In this paper, we extended KK(2000)'s framework and model in a discrete time setting to describe prepayment behaviors due to incentives of equity factor for sale of house and interest factor for refinance, where house price

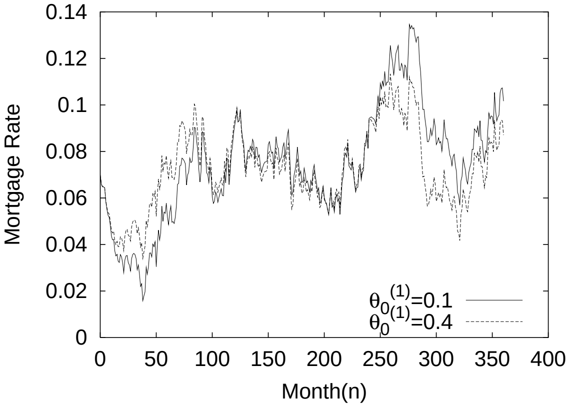

Figure 9: sample paths of mortgage rate processes

variable is a non-Markovian. The extension involves a two-dimensional specification as a boundary hitting problem for the two factors. In addition, we separated the role of short term interest as discount factor from that of mortgage interest as incentive to value an MBS. Furthermore, the prepayment behaviors are directly embedded into the cash flows of an MBS, which is important because cash flow patterns and hence values of MBS are changed by the distribution of time points of prepayments. Through various simulations we found that our model has a great capacity as a valuation model for an MBS. But in doing so, we left out the problems on partial prepayment, default, and sale of house caused by non-economic reasons. Further, we assumed that thresholds for incentive variables are constant. These problems call for further research.

References

[1] Deng, Y., J. M. Quigley, and R. Van Order (2000) Mortgage terminations, heterogeneity and the exercise of mortgage options, *Econometrica* 68, 275-307.

[2] Downing, C., Stanton, R., and Wallace N. (2001) An empirical test of a two-factor mortgage prepayment and valuation model: how much do house prices matter?, Working paper.

[3] Dunn, K. B., and J. J. McConnell (1981a) A comparison of alternative models for pricing GNMA mortgage-backed securities, *Journal of Finance* 36, 471-483.

[4] Dunn, K. B., and J. J. McConnell (1981b) Valuation of mortgage-backed securities, *Journal of Finance* 36, 599-617.

[5] Dunn, K. B., and C. S. Spatt (1986) The effect of refinancing costs and market imperfections on the optimal call strategy and the pricing of debt contracts, Working paper, Carnegie-Mellon University.

[6] Johnston, E., and L. Van Drunen (1988) Pricing mortgage pools with heterogeneous mortgagors: Empirical evidence, Working paper, University of Utah.

[7] Kariya, T. and M. Kobayashi (2000) Pricing Mortgage-Backed Securities (MBS) -A Model Describing the Burnout Effect-, *Asia-Pacific Financial Markets* 7, 189-204.

[8] Kau, J. B., and D. C. Keenan (1995) An overview of the option-theoretic pricing of mortgages, *Journal of Housing Research* 6, 217-244.

[9] Kau, J. B., D. C. Keenan, W. J. Muller, and J. F. Epperson (1992) A generalized valuation model for fixed-rate residential mortgages, *Journal of Money, Credit and Banking* 24, 279-299.

[10] Kau, J. B., D. C. Keenan, W. J. Muller, and J. F. Epperson (1995) The valuation at origination of fixed rate mortgages with default and prepayment, *Journal of Real Estate Finance and Economics* 11, 5-39.

[11] Nakamura, N. (2001) Valuation of mortgage-backed securities based upon a structural approach, Working paper.

[12] Schwartz, E. S., and W. N. Torous (1989) Prepayment and the valuation of mortgage-backed securities, *Journal of Finance* 44, 375-392.

[13] Stanton, R. (1995) Rational prepayment and the value of mortgage-backed securities, *The Review of Financial Studies* 8, 677-708.

[14] Timmis, G. C. (1985) Valuation of GNMA mortgage-backed securities with transaction costs, heterogeneous households and endogenously generated prepayment rates, Working paper, Carnegie-Mellon University.