Homotopy Theory of Finite Topological Spaces

Emily Clader

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Advisor: Robert Lipshitz

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CHAPTER 1

Introduction

In writing about finite topological spaces, one feels the need, as McCord did in his paper "Singular Homology Groups and Homotopy Groups of Finite Topological Spaces" [**8**], to begin with something of a disclaimer, a repudiation of a possible initial fear. What might occur as the homotopy groups of a topological space with only finitely many points? The naive answer would be to assume that these groups must be trivial. This is based, however, on a mistaken intuition, namely that a finite space is endowed with the discrete topology (as it would be, for example, if it were a finite subset of Euclidean space with the subspace topology). Upon a moment's further reflection, however, it is apparent that this is by no means necessarily the case, and there could be many nontrivial continuous maps into finite spaces with more interesting topologies. Indeed, the main theorem of McCord's paper provides a correspondence up to weak homotopy equivalence between finite topological spaces and finite simplicial complexes, proving that these two classes of spaces exhibit precisely the same homotopy groups.

The goal of this paper is to provide a thorough explication of McCord's results and prove a new extension of his main theorem. More specifically, Chapter Two contains the preliminary material on homotopy theory, beginning with a discussion of weak homotopy equivalence and the related notion of quasifibration. It also includes a sketch of the proof of the Dold-Thom theorem, perhaps the most remarkable application of quasifibrations, and it concludes with a brief introduction to inverse limits of topological spaces, the main ingredient in the generalization of McCord's theorem presented in Chapter Four.

McCord's paper is discussed in detail in Chapter Three. This chapter begins by establishing an equivalence between transitive, reflexive relations on a set and topologies under which the set is an A-space (that is, under which arbitrary intersections of open sets are open). This correspondence is used to prove that there is a natural association of a simplicial complex $\mathcal{K}(X)$ to each T_0 A-space X and a weak homotopy equivalence $|\mathcal{K}(X)| \to X$. With the help of one additional theorem, this implies that every finite topological space is weakly homotopy equivalent to a finite simplicial complex. Finally, by imposing a partial order on the set of barycenters of simplices of a simplicial complex, the same ideas are used to prove, conversely, that every finite simplicial complex is weakly homotopy equivalent to a finite topological space.

In Chapter Four, we prove that every finite simplicial complex is homotopy equivalent to an inverse limit of finite topological spaces, a generalization of McCord's theorem. The idea of the proof is to associate a sequence of progressively better finite approximations to a simplicial complex K; the first finite approximation is the space appearing in McCord's correspondence, while the rest are finite spaces with a strictly greater number of points but which retain the property of weak homotopy equivalence to K. By visualizing points in the inverse limit as convergent sequences of points in K, we obtain a homeomorphism between the geometric realization of K and a quotient space of the inverse limit. Finally, we prove that this quotient space is in fact a deformation retract of the entire inverse limit, thereby obtaining the result.

CHAPTER 2

Preliminaries

1. Weak Homotopy Equivalences

A map $f : X \to Y$ is called a **weak homotopy equivalence** if the induced maps

$$f_* : \pi_n(X, x_0) \to \pi_n(Y, f(x_0))$$

are isomorphisms for all $n \geq 0$ and all basepoints $x_0 \in X$. In the case where $n = 0$, the term "isomorphism" should be understood as simply "bijection", since there is no natural group structure on the sets $\pi_0(X, x_0)$ and $\pi_0(Y, f(x_0))$, which are the sets of path-components of X and Y respectively.

It should be noted that, as the terminology indicates, every homotopy equivalence is also a weak homotopy equivalence. This is obvious if we take a restricted notion of homotopy equivalence, considering only maps $f : X \to Y$ for which there exists a map $g : Y \to X$ and *based* homotopies $f \circ g \sim \mathrm{id}_Y$ and $g \circ f \sim \mathrm{id}_X$. For in this case, the existence of these based homotopies together with the functoriality of π_n implies that $f_* \circ g_* = \mathrm{id}$ and $g_* \circ f_* = \mathrm{id}$, so that f_* and g_* are inverse isomorphisms. But indeed, even if homotopies are not required to keep basepoints stationary, it is true that a homotopy equivalence is a weak homotopy equivalence. The proof is a straightforward generalization of Proposition 1.18 of [**6**], in which the statement is proved in the special case of π_1.

The converse, on the other hand, is not true: a weak homotopy equivalence is not necessarily a homotopy equivalence. Indeed, there are weak homotopy equivalences $X \to Y$ for which there does not exist a weak homotopy equivalence $Y \to X$, a phenomenon which is by definition impossible for the stronger notion of homotopy equivalence. One such example comes from the so-called "real line with two origins", the quotient space Y of $\mathbb{R} \times \{0, 1\}$ obtained via the identification $(x, 0) \sim (x, 1)$ if $x \neq 0$. There is a weak homotopy equivalence $f : S^1 \to Y$, so in particular Y has nontrivial fundamental group. However, since S^1 is Hausdorff, any map $g : Y \to S^1$ must agree on the two origins, and hence g factors through the map $Y \to \mathbb{R}$ which identifies the two origins. By composing with a contraction of $\mathbb{R}$ we see that g is nulhomotopic and hence induces the trivial homomorphism on all homotopy groups, so it cannot be a weak homotopy equivalence.

There is one situation, though, in which a weak homotopy equivalence is indeed a homotopy equivalence; namely, Whitehead's theorem states that if X and Y are CW complexes and $f : X \to Y$ is a weak homotopy equivalence, then f is a homotopy equivalence. Nevertheless, most of the weak homotopy equivalences that will be considered in this paper will be maps $K \to X$ where K is a CW complex and X is a finite topological space, so Whitehead's Theorem will not apply. Such maps will in general not be homotopy equivalences.

Indeed, a map from a CW complex K into a finite space X cannot be a homotopy equivalence unless K is a disjoint union of contractible spaces. To see this, consider first the case where

K is connected. If a map $f : K \to X$ is a homotopy equivalence, then there exists an inverse homotopy equivalence $g : X \to K$. The image of $g \circ f$ is a finite, connected subspace of K, and hence is a point. So, since $g \circ f \sim \mathrm{id}_K$, this homotopy provides a contraction of K. In the general case when K is not necessarily connected, the same reasoning shows that each connected component of K must be contractible.

2. Quasifibrations

A related notion to that of weak homotopy equivalence is the notion of a quasifibration. A map $p : E \to B$ over a path-connected base is said to be a **quasifibration** if the induced map

$$p_* : \pi_n(E, p^{-1}(b), x_0) \to \pi_n(B, b)$$

is an isomorphism for all $b \in B$, all $x_0 \in p^{-1}(b)$, and all $n \geq 0$.

First, as before, let us verify the logic of the terminology by observing that a fibration is also a quasifibration. Recall that a map $p : E \to B$ is said to be a **fibration** if for any space X and any homotopy $g_t : X \to B$ of maps from X into B, if we are given a lift $\tilde{g}_0 : X \to E$ of the first map in the homotopy, then there exists a lift $\tilde{g}_t : X \to E$ of the entire homotopy which extends $\tilde{g}_0$. This condition is sometimes expressed by saying that the map $p : E \to B$ has the **homotopy lifting property** with respect to all spaces X, or that the dotted map exists in the following commutative diagram:

$$\begin{tikzcd} X \times \{0\} \arrow[r, "\tilde{g}_0"] \arrow[d, hook] & E \arrow[d, "p"] \\ X \times I \arrow[r, "g_t"] \arrow[ur, dashed, "\tilde{g}_t"] & B. \end{tikzcd}$$

The proof that every fibration is a quasifibration uses a slightly different form of the homotopy lifting property. Namely, given a space X and a subspace $A \subset X$, a map $p : E \to B$ is said to have the **relative homotopy lifting property** with respect to the pair (X, A) if for any homotopy $g_t : X \to B$ of maps from X into B, if we are given both a lift $\tilde{g}_0 : X \to E$ of the first map in the homotopy *and* a lift $\tilde{g}_t : A \to E$ of the entire homotopy over the subspace A, then there exists a lift $\tilde{g}_t : X \to E$ of the entire homotopy over X extending these. One can show that the homotopy lifting property for cubes I^n implies the relative homotopy lifting property for pairs $(I^n, \partial I^n)$. In particular, any fibration has the relative homotopy lifting property for such pairs.

To show, now, that any fibration is also a quasifibration, we will begin by verifying surjectivity of $p_* : \pi_n(E, p^{-1}(b), x_0) \to \pi_n(B, b)$. Let $[f] \in \pi_n(B, b)$ be represented by a map $f : (I^n, \partial I^n) \to (B, b)$, which we can view as a based homotopy of maps $I^{n-1} \to B$. From this perspective, the relative homotopy lifting property for the pair $(I^{n-1}, \partial I^{n-1})$ says that we can extend any lift of f over the subspace $J^{n-1} \subset I^n$ (the union of all but one face of the cube) to a lift of f over all of I^n. In particular, suppose we lift f over J^{n-1} via the constant map to x_0 and extend this to a lift $\tilde{f} : I^n \to E$. Then $\tilde{f}$ represents a class in $\pi_n(E, p^{-1}(b), x_0)$ with $p_*([\tilde{f}]) = [f]$, so p_* is indeed surjective. The proof of injectivity is similar, for if $[f] = [g] \in \pi_n(B, b)$, then there exists a based homotopy from f to g, and we can use the homotopy lifting property again to obtain an appropriate homotopy from $\tilde{f}$ to $\tilde{g}$. Therefore, p_* is an isomorphism, so p is a quasifibration.

One sense in which quasifibrations are related to weak homotopy equivalences is illuminated by the following alternative definition of a quasifibration. Recall that, given an arbitrary map

$p : E \to B$, we define $E_p \subset E \times \mathrm{Map}(I, B)$ as the space of pairs (x, γ), where $x \in E$ and $\gamma : I \to B$ is a path starting at $p(x)$. This space is topologized as a subspace of $E \times \mathrm{Map}(I, B)$, where $\mathrm{Map}(I, B)$ is given the compact-open topology. The projection map $E_p \to B$ given by $(x, \gamma) \mapsto \gamma(1)$ is a fibration for any map p, and the fibers F_b of this map are called the homotopy fibers of p. There is an inclusion of each fiber $p^{-1}(b)$ into the homotopy fiber F_b of p over b, given by mapping x to the pair (x, γ) where γ is the constant path at x. An alternative definition of a quasifibration is given by requiring that each of these inclusions $p^{-1}(b) \hookrightarrow F_b$ be a weak homotopy equivalence.

To see that these two definition are equivalent, notice that since $F_b \to E_p \to B$ is always a fibration, the map $p_* : \pi_n(E_p, F_b, x_0) \to \pi_n(B, b)$ is an isomorphism for all $n \geq 0$. Therefore, in the following commutative diagram, the left-hand map is an isomorphism if and only if the upper map is an isomorphism:

$$\begin{tikzcd}
\pi_n(E, p^{-1}(b), x_0) \arrow[rr] \arrow[dr] & & \pi_n(E_p, F_b, x_0) \arrow[dl, "\cong"] \\
& \pi_n(B, b). &
\end{tikzcd}$$

Now, E is homeomorphic to the subspace of E_p consisting of pairs (x, γ) where γ is a constant path at x. Moreover, there is a deformation retraction from E_p to this subspace, given by progressively shrinking the paths γ to their basepoints; in particular, E_p is homotopy equivalent to E. Therefore, the upper map in the above diagram will be an isomorphism if and only if the inclusion $p^{-1}(b) \hookrightarrow F_b$ is a weak homotopy equivalence, that is if and only if the alternative definition of a quasifibration is fulfilled. Combining these observations, we see that the alternative definition is fulfilled if and only if the left-hand map in the above diagram is an isomorphism, which is precisely when p is a quasifibration by our original definition.

3. The Dold-Thom Theorem

A remarkable application of quasifibrations is given in the proof of the Dold-Thom theorem; for this reason, and because it may prove useful in the possible extension of our work discussed in the conclusion, we will state the theorem and sketch its proof here.

Given a space X, there is an action of the symmetric group S_n on the product X^n of n copies of X given by permuting the factors. The n-fold **symmetric product** $SP_n(X)$ is defined as the quotient space of X^n where we factor out this action of S_n; thus, it consists of unordered n-tuples of points in X.

If we fix a basepoint $e \in X$, we can identify each of the products X^n with the subspace $\{(e, x_1, \ldots, x_n) \mid x_i \in X\}$ of X^{n+1}. We thus obtain an inclusion $X^n \hookrightarrow X^{n+1}$ for each n, and since equivalent points under the action of S_n are mapped to equivalent points, this map descends to an inclusion $SP_n(X) \hookrightarrow SP_{n+1}(X)$.

The associations $X \mapsto SP_n(X)$ are functorial. That is, if $f : X \to Y$ is a basepoint-preserving map, there is an induced map $\tilde{f} : SP_n(X) \to SP_n(Y)$ given by $[(x_1, \ldots, x_n)] \mapsto [(f(x_1), \ldots, f(x_n))]$, and these maps satisfy $\widetilde{(f \circ g)} = \tilde{f} \circ \tilde{g}$ and $\widetilde{\mathrm{id}} = \mathrm{id}$. Moreover, a homotopy between two maps $X \to Y$ induces a homotopy between the corresponding maps $SP_n(X) \to SP_n(Y)$, so if X is homotopy equivalent to Y then $SP_n(X)$ is homotopy equivalent to $SP_n(Y)$.

The **infinite symmetric product** $SP(X)$ is defined as the increasing union

$$SP(X) = \bigcup_{n=1}^{\infty} SP_n(X),$$

or in other words the direct limit of the system

$$SP_1(X) \hookrightarrow SP_2(X) \hookrightarrow SP_3(X) \hookrightarrow \cdots,$$

equipped with the weak (or direct limit) topology, wherein $U \subset SP(X)$ is open if and only if $U \cap SP_n(X)$ is open for each $n \geq 1$. The association $X \mapsto SP(X)$ is still functorial; a map $f : X \to Y$ induces a map $\tilde{f} : SP_n(X) \to SP_n(Y)$ for each n, and the restriction of each such $\tilde{f}$ to the subspace $SP_{n-1}(X) \subset SP_n(X)$ is precisely the map $\tilde{f} : SP_{n-1}(X) \to SP_{n-1}(Y)$. Therefore, the maps $\tilde{f}$ are compatible with the direct systems defining $SP(X)$ and $SP(Y)$, so they induce a continuous map $\tilde{f} : SP(X) \to SP(Y)$. For the same reason as above, homotopic maps $X \to Y$ induce homotopic maps $SP(X) \to SP(Y)$.

We are now equipped to state the theorem of Dold and Thom:

Theorem 2.1 (Dold and Thom). *If X is a connected CW complex with basepoint, then $\pi_n(SP(X)) \cong H_n(X;\mathbb{Z})$.*

The proof of this result involves showing that the association $X \mapsto \pi_n(SP(X))$ defines a reduced homology theory on the category of basepointed CW complexes. By the functoriality of the associations $X \mapsto SP(X)$ and $SP(X) \mapsto \pi_n(SP(X))$, this candidate theory is indeed functorial. Thus, all that remains in order to show that it defines a homology theory is verification of the following three axioms:

(1) If $f : X \to Y$ and $g : X \to Y$ are homotopic maps between CW complexes, then they induce the same map $\pi_n(SP(X)) \to \pi_n(SP(Y))$.
(2) For CW pairs (X, A), there is a natural long exact sequence

$$\cdots \to \pi_n(SP(A)) \to \pi_n(SP(X)) \to \pi_n(SP(X/A)) \to \pi_{n-1}(SP(A)) \to \cdots.$$

(3) If $X = \bigvee_\alpha X_\alpha$ and $i_\alpha : X_\alpha \hookrightarrow X$ are the inclusions, then the homomorphism given by $\oplus_\alpha (\widetilde{i_\alpha})_* : \oplus_\alpha \pi_n(SP(X_\alpha)) \to \pi_n(SP(X))$ is an isomorphism for all $n \geq 0$.

The first axiom is evidently satisfied by our previous observations, since we already know that if $f \sim g : X \to Y$, then $\tilde{f} \sim \tilde{g} : SP(X) \to SP(Y)$, and hence $\tilde{f}$ and $\tilde{g}$ induce the same map on homotopy groups.

The wedge axiom is also essentially immediate, once we observe that if the basepoint in $\bigvee_\alpha X_\alpha$ and in each X_α is chosen to be the wedge point, then there is a homeomorphism

$$\varphi : \prod_\alpha{}' \; SP(X_\alpha) \to SP\left(\bigvee_\alpha X_\alpha\right),$$

where $\prod'$ denotes the 'weak' product, that is, the union of all the products of finitely many factors. The homeomorphism φ is given by simply concatenating all of the tuples in the various $SP(X_\alpha)$. We have a composition

$$SP(X_\alpha) \xrightarrow{\widetilde{i_\alpha}} SP\left(\bigvee_\alpha X_\alpha\right) \xrightarrow{\varphi^{-1}} \prod_\alpha{}' \; SP(X_\alpha),$$

which, under closer inspection, is simply the natural inclusion $j_\alpha : SP(X_\alpha) \hookrightarrow \prod'_\alpha SP(X_\alpha)$. Now, the map

$$\oplus_\alpha j_{\alpha*} : \bigoplus_\alpha \pi_n(SP(X_\alpha)) \to \pi_n\left(\prod_\alpha{}' \ SP(X_\alpha)\right)$$

is an isomorphism, since a map $f : Y \to \prod' SP(X_\alpha)$ is the same as a collection of maps $f_\alpha : Y \to SP(X_\alpha)$ with the property that for every $y \in Y$, $f_\alpha(y)$ is the basepoint for all but finitely-many values of α. Therefore, since $\oplus_\alpha j_{\alpha*} = \oplus_\alpha i_{\alpha*} \circ \varphi_*^{-1}$ and φ is a homeomorphism, we conclude that $\oplus_\alpha i_{\alpha*}$ is an isomorphism, as desired.

Thus, the real work of the theorem is in proving the existence of the long exact sequence. This is proved by showing that the map $p : SP(X) \to SP(X/A)$ induced by the quotient map $X \to X/A$ is a quasifibration with fiber $SP(A)$. To show that this map is a quasifibration, one uses a number of locality properties of quasifibrations, first reducing the problem to showing by induction that p is a quasifibration over each subspace $SP_n(X/A) \subset SP(X/A)$, and then reducing the inductive step of this latter claim to showing that if p is a quasifibration over $SP_{n-1}(X/A)$ then it is in fact a quasifibration over a neighborhood of $SP_{n-1}(X/A)$ and over $SP_n(X/A) \setminus SP_{n-1}(X/A)$.

Once we have that p is a quasifibration, the long exact sequence of the pair $(SP(X), SP(A))$ in homotopy yields:

$$\begin{tikzcd}
 & & & \pi_n(SP(X/A)) & & \\
\cdots \arrow[r] & \pi_n(SP(A)) \arrow[r, "i_*"] & \pi_n(SP(X)) \arrow[r, "j_*"] & \pi_n(SP(X), SP(A)) \arrow[u, "p_*"', "\cong"] \arrow[r, "\partial"] & \pi_{n-1}(SP(A)) \arrow[r] & \cdots .
\end{tikzcd}$$

This allows us to write down an exact sequence

$$\cdots \longrightarrow \pi_n(SP(A)) \xrightarrow{i_*} \pi_n(SP(X)) \xrightarrow{p_* \circ j_*} \pi_n(SP(X/A) \xrightarrow{\partial \circ p_*^{-1}} \pi_{n-1}(SP(A)) \longrightarrow \cdots ,$$

whose form is as required and whose naturality is clear. This, therefore, completes the proof that we have a reduced homology theory.

One can check that the coefficients of this homology theory, the groups $\pi_i(SP(S^n))$, are the same as the coefficients $H_i(S^n)$ for ordinary homology. To do so requires explicitly proving that $SP(S^2) \cong \mathbb{CP}^\infty$ and using the suspension property of reduced homology theories to deduce the result for all higher values of n. Thus, we conclude that this homology theory agrees with standard homology, which completes the proof of the theorem.

4. Inverse Limits of Topological Spaces

Given a countable sequence of spaces $X_0, X_1, X_2, \ldots$ and continuous maps $f_n : X_n \to X_{n-1}$ for each n, the **inverse limit** of this sequence, denoted $\varprojlim X_n$, is the set of all points $(x_0, x_1, x_2, \ldots) \in \prod_{n=0}^\infty X_n$ such that $f(x_n) = x_{n-1}$ for all n, topologized as a subspace of $\prod_{n=0}^\infty X_n$. The sequence of spaces and maps is sometimes called an **inverse limit sequence**, a special case of the more general notion of an **inverse system** of topological spaces.

It is possible, of course, that no point in the product $\prod_{n=0}^\infty X_n$ satisfies the requirements to belong to the inverse limit, so the inverse limit of the sequence of spaces is empty. For example, suppose we define a discrete space

$$X_n = \bigsqcup_{m=0}^{\infty} \{x_{n,m}\}$$

for each $n \geq 0$. Define continuous maps $f_n : X_n \to X_{n-1}$ by $f_n(x_{n,m}) = x_{n-1,m+1}$. The picture is the following, where the leftmost point in the row corresponding to X_i is the point $x_{i,0}$:

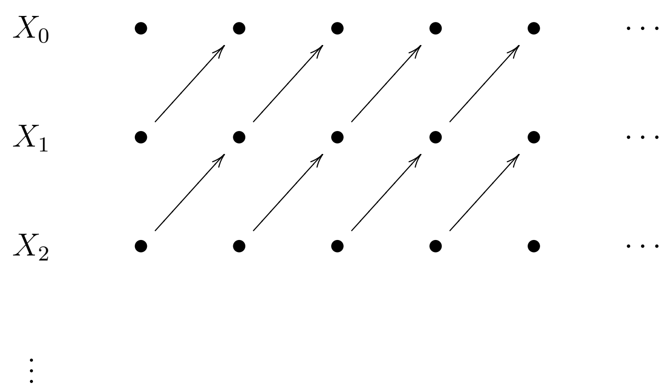


Figure 1: An inverse system for which the inverse limit is empty.

Suppose we attempt to construct a point $(x_{0,j_0}, x_{1,j_1}, x_{2,j_2}, \ldots)$ in the inverse limit. We will begin with the point x_{0,j_0} and observe that by the definition of the inverse limit, $x_{1,j_1} \in f_1^{-1}(x_0, j_0)$, which implies that $x_{1,j_1} = x_{1,j_0-1}$. Similarly $x_{2,j_2} = x_{2,j_0-2}$. Continuing in this way, we will find that we can only repeat the process up to the first j_0 iterations, at which point we are forced to conclude that there is no appropriate point x_{i,j_i} for $i > j_0$. Thus, the inverse limit of this sequence of spaces is empty.

One situation in which the inverse limit is never empty is when the spaces X_n are compact and Hausdorff (see [**7**]). This will not be the case in the inverse limits we consider in this paper, but it will nevertheless be clear in the cases under consideration that the inverse limit is nonempty.

There is a natural map

$$\lambda : \pi_i \left(\varprojlim X_n \right) \to \varprojlim \pi_i(X_n)$$

given by viewing a map $\varphi : S^i \to \varprojlim X_n$ as a collection of maps $\varphi_n : S^i \to X_n$ with the property that for each $x \in S^i$ and each n, $f_n(\varphi_n(x)) = \varphi_{n-1}(x)$. It can be shown (see [**6**], Proposition 4.67) that λ is surjective if the maps f_n are fibrations, and λ is injective if the induced maps $f_{n*} : \pi_{i+1}(X_n) \to \pi_{i+1}(X_{n-1})$ are surjective for n sufficiently large.

An interesting consequence of this fact comes from the notion of a Postnikov tower. Given a connected CW complex X, a commutative diagram of the form shown below is said to be a Postnikov tower if the map $X \to X_n$ induces an isomorphism on π_i for $i \leq n$ and $\pi_i(X_n) = 0$ for $i > n$. It is not difficult to construct a Postnikov tower for any connected CW complex X, and by the path-fibration construction mentioned in the previous section we can replace each of the maps $X_n \to X_{n-1}$ by a fibration without losing the defining properties of the tower.

A Postnikov tower for a space X gives one an inverse system of progressively better homotopy-theoretic models of X. Since the maps $\pi_i(X_{n+1}) \to \pi_i(X_n)$ are isomorphisms for n sufficiently

$$
\begin{tikzcd}
 & \vdots \arrow[d] \\
 & X_3 \arrow[d] \\
 & X_2 \arrow[d] \\
X \arrow[r] \arrow[ur] \arrow[uur] & X_1
\end{tikzcd}
$$

Figure 2: A Postnikov tower.

large, the result mentioned above implies that λ is an isomorphism. And since the maps $\pi_i(X) \to \pi_i(X_n)$ are isomorphisms for n sufficiently large, the same type of reasoning shows that the composition

$$\pi_i(X) \xrightarrow{f_*} \pi_i\left(\varprojlim X_n\right) \xrightarrow{\lambda} \varprojlim \pi_i(X_n)$$

is an isomorphism, where $f : X \to \varprojlim X_n$ is the natural map coming from the maps $X \to X_n$. This in particular implies that f_* is a weak homotopy equivalence, so X is a CW approximation for the inverse limit $\varprojlim X_n$.

CHAPTER 3

The Results of McCord

In his 1965 paper *Singular Homology Groups and Homotopy Groups of Finite Topological Spaces* [**8**], McCord established a correspondence up to weak homotopy equivalence between finite topological spaces and finite simplicial complexes. More precisely, he proved that to each finite simplicial complex K one can associate a finite topological space $\mathcal{X}(K)$ whose points are in one-to-one correspondence with the simplices of K, and if we topologize $\mathcal{X}(K)$ via the partial order induced by inclusion of simplices, then the natural map $K \to \mathcal{X}(K)$ is a weak homotopy equivalence. Conversely, he showed that to each finite topological space X one can associate a simplicial complex $\mathcal{K}(X)$ whose vertices are the points of X, and a weak homotopy equivalence $X \to \mathcal{K}(X)$.

Perhaps the first striking feature of this correspondence is that, as noted in the introduction, it alleviates any concern that finite spaces might be topologically uninteresting objects; in particular, these results imply that the class of finite topological spaces enjoys precisely the same homotopy groups as does the class of finite simplicial complexes. But more importantly, McCord's findings hint at the possibility that useful information about a topological space can be uncovered by studying its finite model. Recently, Barmak and Minian have explored this notion extensively. In [**3**] they introduced the notion of a collapse of finite spaces, which corresponds under McCord's association to a simplicial collapse, and they used this construction to develop an approach to simple homotopy theory based upon elementary moves on finite spaces. In [**2**] they defined a broader class of spaces, the so-called "h-regular CW complexes", to which McCord's theorems and their extensions apply. Aside from Barmak and Minian's work, the results of [**8**] were also recently used in an influential paper of Biss [**4**], which seeks to relate the finite poset $\mathrm{MacP}(k,n)$, a combinatorial analogue of the Grassmanian, to its associated simplicial complex and thereby to the Grassmanian $\mathrm{G}(k,n)$ of k-planes in $\mathbb{R}^n$.

This chapter is devoted to discussing McCord's paper, which will serve as the foundation for the generalization established in Chapter 4. Although we will generally restrict ourselves to finite topological spaces, the results of [**8**] apply more generally to A-spaces, a slightly broader class. An ***A*-space** is a topological space in which the intersection of an arbitrary collection of open sets is open.

Clearly every finite space is an A-space, since an arbitrary intersection of open sets is necessarily a finite intersection. However, not every A-space is finite. For example, there is a topology on $\mathbb{Z}^+$ generated by the sets of the form $\{m \mid m \geq n\}$ for each fixed $n \geq 1$. Since the intersection of an arbitrary collection of basis elements is another basis element, this topology makes $\mathbb{Z}^+$ into an A-space.

The crucial observation for McCord's correspondence is that given a set X, imposing a topology on X that makes it into an A-space is equivalent to imposing a transitive, reflexive relation on its elements. We discuss this equivalence below.

First, let X be an A-space; we will exhibit a transitive, reflexive relation on X. For each $x \in X$, the **open hull** of x, denoted U_x, is defined as the intersection of all open subsets of X containing x. This allows us to define a relation $\leq$ on X by declaring $x \leq y$ if $x \in U_y$. This is equivalent to the requirement that $U_x \subset U_y$, for it implies that every open set containing y also contains x, so the collection of open sets containing y is a subset of the collection of open sets containing x; therefore, the intersection of the latter is contained in the intersection of the former. This second description of $\leq$ makes it clear that it is a transitive, reflexive relation.

LEMMA 3.1. *Let X and Y be A-spaces, and let $\leq$ be the relation described above. Then a map $f : X \to Y$ is continuous if and only if it is order-preserving, that is, if and only if $x \leq y$ implies $f(x) \leq f(y)$.*

PROOF. Suppose $f : X \to Y$ is continuous, and let $x, y \in X$ be such that $x \leq y$. Then $x \in U_y$, and we must show that $f(x) \in U_{f(y)}$. To do so, let $V \subset Y$ be an open set containing $f(y)$. Then $f^{-1}(V)$ is an open subset of X containing y, and hence $x \in f^{-1}(V)$. Therefore, we have $f(x) \in V$, so that $f(x)$ lies in every open set containing $f(y)$. That is, $f(x) \in U_{f(y)}$, as required.

Conversely, suppose $f : X \to Y$ is order-preserving, and let $V \subset X$ be open. To show that $f^{-1}(V) \subset Y$ is open, it suffices to prove that for all $y \in f^{-1}(V)$, $U_y \subset f^{-1}(V)$. So, let $y \in f^{-1}(V)$, and let $x \in U_y$. Then by definition, $x \leq y$, and since f is order-preserving this implies that $f(x) \leq f(y)$, that is, $f(x)$ lies in every open set containing $f(y)$. In particular, V is an open set containing $f(y)$, so $f(x) \in V$. Therefore, $x \in f^{-1}(V)$. We have therefore shown that $U_y \subset f^{-1}(V)$ for each $y \in f^{-1}(V)$, and hence $f^{-1}(V)$ is open. □

Another useful observation about the relation $\leq$ is that it is antisymmetric (and hence a partial order) if and only if X is T_0. (Recall that a topological space X is said to be T_0 if, given any pair of distinct points in X, there exists an open set containing one but not the other.)

For the opposite direction of the equivalence between A-space structures and transitive, reflexive relations, suppose that we are given a set X with a transitive, reflexive relation $\leq$. Then we can define, for each $x \in X$, the set $U_x = \{y \in X \mid y \leq x\}$. This collection of sets forms the basis for a topology on X, with the reflexivity and transitivity corresponding precisely to the two axioms required to be a basis. And under this topology, X is an A-space; indeed, if $\{V_\alpha\}_{\alpha \in A}$ is a collection of open subsets of X and $V = \cap_{\alpha \in A} V_\alpha$, then for all $x \in V$ we have $x \in U_x \subset V_\alpha$ for each α, so that $x \in U_x \subset V$ and hence V is open.

Moreover, as the notation would suggest, U_x is the open hull of x in the topology we have defined on X. Since U_x is open by definition, it is clear that the open hull of x is contained in U_x. For the reverse inclusion, let V be any open set containing x. Then we can express V as a union of basis elements, say $V = \cup_{z \in B} U_z$. Since $x \in V$, x lies in some U_z, and therefore $x \leq z$. Therefore, for any $y \in U_x$ we have $y \leq x \leq z$, so $y \in U_z$ and hence $y \in V$. This shows that $U_x \subset V$, and thus establishes that U_x is contained in every open set that contains x. Hence, U_x is exactly the open hull of x, as claimed.

With this equivalence defined, we are prepared to prove the first direction of the correspondence between finite topological spaces (or, more generally, A-spaces) and simplicial complexes. This is encapsulated in the following theorem:

THEOREM 3.2 (McCord). *There exists a correspondence that assigns to each T_0 A-space X a simplicial complex $\mathcal{K}(X)$, whose vertices are the points of X, and a weak homotopy equivalence*

$f_X : |\mathcal{K}(X)| \to X$. *This correspondence is natural in* X*: a map* $\varphi : X \to Y$ *of* T_0 *A-spaces induces a simplicial map* $\mathcal{K}(X) \to \mathcal{K}(Y)$*, and* $\varphi \circ f_X = f_Y \circ |\varphi|$.

To prove Theorem 3.2, we will use the equivalence discussed above to notice that since X is a T_0 A-space, there is a partial order $\leq$ on the elements of X. With this in mind, define the simplicial complex $\mathcal{K}(X)$ by letting its vertices be the points of X and its simplices be the finite subsets of X that are totally ordered by $\leq$.

Example 3.3. We will define a finite topological space S^1_4, the 4-pointed finite model of the circle.

Let $S^1_4 = \{a, b, c, d\}$, and let $\mathcal{B} = \{\{a\}, \{c\}, \{a, b, c\}, \{a, d, c\}\}$ be the basis for the topology on S^1_4. Then the corresponding poset can be graphically depicted as follows, where an arrow from x to y indicates $x < y$:

$$\begin{tikzcd} b & d \\ a \arrow[u] \arrow[ur] & c. \arrow[u] \arrow[ul] \end{tikzcd}$$

The totally-ordered subsets of S^1_4 are $\{a, b\}$, $\{a, d\}$, $\{c, b\}$, $\{c, d\}$, and the four singletons. Therefore, the simplicial complex $\mathcal{K}(S^1_4)$ has four 0-simplices, four 1-simplices, and no other simplices. Its geometric realization is the circle S^1.

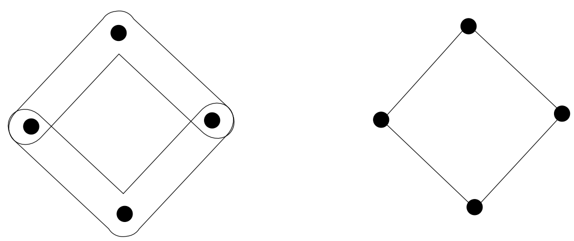


Figure 1: The Circle and Its Finite Model.

Notice that if X is a T_0 A-space and $Y \subset X$, then the open hull in Y of a point $y \in Y$ is simply the intersection of the open hull of y in X with the subspace Y; thus, the relation $\leq$ on Y is the restriction to Y of the relation $\leq$ on X. This implies that a totally-ordered subset of X is still totally-ordered as a subset of Y, and hence that if a collection of vertices belongs to $\mathcal{K}(Y) \subset \mathcal{K}(X)$ and the vertices span a simplex in $\mathcal{K}(X)$, then they span a simplex in $\mathcal{K}(Y)$. That is, $\mathcal{K}(Y)$ is a **full subcomplex** of $\mathcal{K}(X)$.

To define the map $f_X : |\mathcal{K}(X)| \to X$, we recall that each point $u \in |\mathcal{K}(X)|$ is contained in the interior of a unique simplex $\sigma = \{x_0, x_1, \ldots, x_k\}$, which corresponds to the totally-ordered subset $x_0 < x_1 < \cdots < x_k$ of X. We define $f_X(u) = x_0$. In other words, viewing the poset X as a directed graph as in Example 3.3, each totally-ordered subset consists of all vertices lying on an upward-pointing path starting at some fixed element $x_0 \in X$, and f_X maps the totally-ordered subset to its initial element, x_0.

Continuity of the map f_X will follow from the next lemma. Before stating the lemma, we require a definition: for a simplicial complex K with a subcomplex $L \subset K$, the **regular**

neighborhood of L in K is the set

$$\bigcup_{x\in L} \mathrm{st}_K(x) \subset |K|,$$

where $\mathrm{st}_K(x)$ denotes the open star of x in K, that is, the union of the interiors of the simplices of K that contain x as a vertex.

LEMMA 3.4. *Let X be a T_0 A-space, and let $Y \subset X$ be open. Then $(f_X)^{-1}(Y)$ is the regular neighborhood of $\mathcal{K}(Y)$ in $\mathcal{K}(X)$.*

PROOF. Before proving the lemma, let us observe that since regular neighborhoods are always open, this implies that f_X is continuous.

Let $u \in (f_X)^{-1}(Y)$. Then $f_X(u) \in Y$, that is, u lies in the interior of a simplex $\sigma = \{x_0, x_1, \ldots, x_k\}$ such that $x_0 < x_1 < \cdots < x_k$, and $x_0 \in Y$. The interior of σ is therefore a subset of $\mathrm{st}_{\mathcal{K}(X)}(x_0)$, so we have

$$u \in \mathrm{st}_{\mathcal{K}(X)}(x_0) \subset \bigcup_{y\in\mathcal{K}(Y)} \mathrm{st}_{\mathcal{K}(X)}(y),$$

that is, u lies in the regular neighborhood of $\mathcal{K}(Y)$ in $\mathcal{K}(X)$.

Conversely, suppose that u lies in the regular neighborhood of $\mathcal{K}(Y)$ in $\mathcal{K}(X)$. Then $u \in \mathrm{st}_{\mathcal{K}(X)}(y)$ for some $y \in \mathcal{K}(Y)$, and since the vertices of $\mathcal{K}(Y)$ are points in Y, we can view y as lying in Y. In particular, there is a simplex $\sigma = \{x_0, x_1, \ldots, x_k\}$ such that $x_0 < x_1 < \cdots < x_k$, $x_i = y$ for some i, and $u \in \mathrm{int}(\sigma)$. Then $x_0 \leq y$, so $x_0 \in U_y$, which says that x_0 is contained in every open set containing y. Thus, $x_0 \in Y$, which proves that f_X maps the interior of σ to Y, so $f_X(u) \in Y$. We conclude that $u \in f_X^{-1}(Y)$, as desired. □

At this point, let us digress to state (without proof) a theorem that will be used to show that the map f_X is a weak homotopy equivalence. This result appears in [**8**] as Theorem 6, and in [**6**] as Corollary 4K.2. The proof in [**8**] is essentially a sketch, since it closely follows the proof of an analogous result for quasifibrations (rather than weak homotopy equivalences) that appears in [**5**]. The main idea behind McCord's adaptation of the result for quasifibrations to the present situation is the observation that a surjective map $p : E \to B$ with the property that $\pi_n(p^{-1}(x), y) = 0$ for all $x \in E$, $y \in p^{-1}(x)$, and $n \geq 0$ is a quasifibration if and only if it is a weak homotopy equivalence. This follows immediately from the long exact sequence in homotopy of the pair $(E, p^{-1}(x))$.

THEOREM 3.5 (McCord). *Let $p : E \to B$ be any continuous map, and suppose that there exists an open cover $\mathcal{U}$ of B with the property that if $x \in U \cap V$ for some $U, V \in \mathcal{U}$, then there exists $W \in \mathcal{U}$ such that $x \in W \subset U \cap V$. Suppose further that for each $U \in \mathcal{U}$, the restriction $p|_{p^{-1}(U)} : p^{-1}(U) \to U$ is a weak homotopy equivalence. Then p is itself a weak homotopy equivalence.*

In our application of this result, we will take as the open cover $\mathcal{U}$ the collection of sets U_x for $x \in X$. In light of this theorem, we must show that $(f_X)|_{(f_X)^{-1}(U_x)} : (f_X)^{-1}(U_x) \to U_x$ is a weak homotopy equivalence for all $x \in X$. We will do so by proving that its domain and codomain are both contractible, so that the induced homomorphisms on homotopy groups are maps between trivial groups and hence clearly isomorphisms.

LEMMA 3.6. *If X is an A-space and $x \in X$, then U_x is contractible.*

Proof. Define a homotopy $F : U_x \times I \to U_x$ by

$$F(y,t) = \begin{cases} y & t \in [0,1) \\ x & t = 1. \end{cases}$$

If continuous, this clearly defines a contraction of U_x to x.

To see that F is continuous, let $V \subset U_x$ be open. If $x \in V$, then since no proper open subset of U_x contains x, we must have $U_x = V$ and hence $F^{-1}(V) = U_x \times I$, which is open. If $x \notin V$, then $F^{-1}(V) = V \times [0,1)$, which is also open. This completes the proof. $\square$

Lemma 3.7. *If X is a T_0 A-space and $x \in X$, then $(f_X)^{-1}(U_x)$ is contractible.*

Proof. Recall that the regular neighborhood of a full subcomplex deformation retracts onto the full subcomplex; in particular, $(f_X)^{-1}(U_x)$ deformation retracts onto $|\mathcal{K}(U_x)|$. So to prove the claim, it suffices to show that $|\mathcal{K}(U_x)|$ is contractible.

Let $V_x = U_x \setminus \{x\}$. We claim that $\mathcal{K}(U_x) = \mathrm{cone}(\mathcal{K}(V_x), x)$, or in other words that the simplices of $\mathcal{K}(U_x)$ consist precisely of the simplices of $\mathcal{K}(V_x)$, together with simplices of the form $\{x_0, x_1, \ldots, x_k, x\}$ where $\{x_0, x_1, \ldots, x_k\}$ is a simplex of $\mathcal{K}(V_x)$.

First, any such simplex is a simplex of $\mathcal{K}(U_x)$. To see this, observe that every simplex of $\mathcal{K}(V_x)$ is of the form $\{x_0, x_1, \ldots, x_k\}$ where $x_0 < x_1 < \cdots < x_k$, and where $x_i < x$ for all $i \in \{0, \ldots, k\}$. Therefore, for any simplex of $\mathcal{K}(V_x)$, the set $\{x_1, \ldots, x_k, x\}$ is a simplex of $\mathcal{K}(U_x)$.

Moreover, these are all the simplices of $\mathcal{K}(U_x)$. For if $\sigma = \{x_0, x_1, \ldots, x_k\}$ is a simplex of $\mathcal{K}(U_x)$ but not a simplex of $\mathcal{K}(V_x)$, then $x_i = x$ for some i.

Thus, we have established that $\mathcal{K}(U_x) = \mathrm{cone}(\mathcal{K}(V_x), x)$, and therefore its geometric realization is contractible to the cone point x. $\square$

The proof of Theorem 3.2 is now almost immediate:

Proof of Theorem 3.2. By the above two lemmas, the maps

$$(f_X)|_{(f_X)^{-1}(U_x)} : (f_X)^{-1}(U_x) \to U_x$$

are weak homotopy equivalences for all $x \in X$, and hence, by Theorem 3.5, f_X is itself a weak homotopy equivalence.

All that remains is to establish naturality of the association $X \mapsto \mathcal{K}(X)$. Let $\varphi : X \to Y$ be a map of T_0 A-spaces. Recall that since φ is continuous, it is order-preserving by Lemma 3.1. Thus, it maps totally-ordered sets onto totally-ordered sets, and so it maps simplices of $\mathcal{K}(X)$ onto simplices of $\mathcal{K}(Y)$. That is, the map $\varphi : \mathcal{K}(X) \to \mathcal{K}(Y)$ is simplicial.

To see that $\varphi \circ f_X = f_Y \circ |\varphi|$, let $u \in |\mathcal{K}(X)|$ lie in the interior of the simplex $\{x_0, x_1, \ldots x_k\}$, where $x_0 < x_1 < \cdots < x_k$. Then, since φ is simplicial, $|\varphi|(u)$ lies in the interior of the simplex $\{\varphi(x_0), \varphi(x_1), \ldots, \varphi(x_k)\}$, and since φ is order-preserving we have $\varphi(x_0) \le \varphi(x_1) \le \cdots \le \varphi(x_k)$. Hence, $(f_Y \circ |\varphi|)(u) = \varphi(x_0)$. On the other hand, since $f_X(u) = x_0$, we have $(\varphi \circ f_X)(u) = \varphi(x_0)$. So $f_Y \circ |\varphi| = \varphi \circ f_X$, as claimed. $\square$

It should be noted that Theorem 3.2 implies that every finite T_0 space is weakly homotopy equivalent to a finite simplicial complex. For the more general statement that every finite space (not necessarily T_0) is weakly homotopy equivalent to a finite simplicial complex, a further result is required:

Theorem 3.8 (McCord). *There exists a correspondence that assigns to each A-space X a quotient space $\hat{X}$ of X such that*

1. *The quotient map $\nu_X : X \to \hat{X}$ is a homotopy equivalence,*
2. *$\hat{X}$ is a T_0 A-space,*
3. *The construction is natural in X: for each map $\varphi : X \to Y$, there exists a unique map $\hat{\varphi} : \hat{X} \to \hat{Y}$ such that $\nu_Y \circ \varphi = \hat{\varphi} \circ \nu_X$.*

Proof. Let X be an A-space. Define an equivalence relation $\sim$ on X by declaring $x \sim y$ if $U_x = U_y$, and let $\hat{X} = X/\sim$. Since this is equivalent to the requirement that $x \leq y$ and $y \leq x$, we are essentially forcing that the relation $\leq$ is antisymmetric. To make this rigorous, though, we need to make two observations.

First of all, $\hat{X}$ is still an A-space, under the quotient topology from the quotient map $\nu_X : X \to \hat{X}$. For if $\{U_\alpha\}_{\alpha \in A}$ is a collection of open sets, then we have

$$\nu_X^{-1}\left(\bigcap_{\alpha \in A} U_\alpha\right) = \bigcap_{\alpha \in A} \nu_X^{-1}(U_\alpha),$$

and hence by the definition of the quotient topology and the fact that X is an A-space, we conclude that $\cap_{\alpha \in A} U_\alpha$ is open.

Second, the relation coming from this A-space structure on $\hat{X}$ is the same as the one induced via ν_X by the A-space structure on X. That is, if $x, y \in X$, then $\nu_X(x) \leq \nu_X(y)$ if and only if $x \leq y$. To see this, notice first that $U_x \subset \nu_X^{-1}(\nu_X(U_x))$. But moreover, the reverse containment is also true; for if $z \in \nu_X^{-1}(\nu_X(U_x))$ then $\nu_X(z) = \nu_X(w)$ for some $w \in U_x$, and thus $z \in U_z = U_w \subset U_x$. Therefore, we have

$$\nu_X^{-1}(\nu_X(U_x)) = U_x.$$

This in particular implies that $\nu_X(U_x)$ is open. Next, observe that

$$\nu_X(U_x) = U_{\nu_X(x)}.$$

To see why this is true, note that $\nu_X(U_x)$ is open and contains $\nu_X(x)$, so $U_{\nu_X(x)} \subset \nu_X(U_x)$; on the other hand, if $V \subset X$ is an open set such that $\nu_X(x) \in V$, then $\nu_X^{-1}(V)$ is an open set containing x and hence $U_x \subset \nu_X^{-1}(V)$. Therefore, $\nu_X(U_x) \subset V$. So $\nu_X(U_x)$ is contained in every open set that contains $\nu_X(x)$, that is, $\nu_X(U_x) \subset U_{\nu_X(x)}$.

At this point, the claim that $\nu_X(x) \leq \nu_X(y)$ if and only if $x \leq y$ follows immediately. For if $x \leq y$, then $\nu_X(x) \leq \nu_X(y)$ by continuity of ν_X; and if $\nu_X(x) \leq \nu_X(y)$, then the above implies that $\nu_X(x) \in \nu_X(U_y)$, so there exists a $z \in U_y$ such that $\nu_X(x) = \nu_X(z)$, and hence $x \leq z \leq y$.

Now, it is clear that the relation $\leq$ on $\hat{X}$ is antisymmetric (in addition to reflexive and transitive), so $\hat{X}$ is a T_0 A-space. This completes the proof of part (2) of the theorem.

To see that ν_X is a homotopy equivalence and thereby prove part (1) of the theorem, choose *any* right inverse $\mu : \hat{X} \to X$. While it is not *a priori* obvious that the map μ is continuous, continuity indeed follows because, by the above observations, μ is necessarily order-preserving. Therefore, we have $\nu_X \circ \mu = \mathrm{id}_{\hat{X}}$. To see that $\pi = \mu \circ \nu_X$ is homotopic to id_X, define a map $H : X \times I \to X$ by

$$H(x,t) = \begin{cases} x & t \in [0,1) \\ \pi(x) & t = 1. \end{cases}$$

To show that H is continuous, it suffices to verify that for each point $(x, s) \in X \times I$, there exists a neighborhood of (x, s) which is mapped by F into $U_{F(x,s)}$. We claim that $U_x \times I$ is such a neighborhood. To see this, notice first that $(\nu_X \circ \pi)(x) = (\nu_X \circ \mu \circ \nu_X)(x) = \nu_X(x)$, and hence by the definiton of ν_X we have $U_{\pi(x)} = U_x$ for all $x \in X$. This implies that $U_{F(x,s)} = U_x$. Take any $(y, t) \in U_x \times I$. Then, if $t < 1$, it is clear that $F(y, t) = y \in U_x$; if $t - 1$, then $F(y, t) = \pi(y) \in U_{\pi(y)} = U_y \subset U_x$. Therefore, it is indeed the case that $F(U_x \times I) \subset U_x = U_{F(x,s)}$, and by the above observations, this completes the proof that H is continuous.

Finally, the naturality statement in part (3) follows from the fact that if $\varphi : X \to Y$ is a continuous map of A-spaces, then it is order-preserving and hence maps equivalent points of X to equivalent points of Y. This implies that φ descends to a unique function $\hat{\varphi}$ such that the following diagram commutes:

$$\begin{tikzcd} \hat{X} \arrow[r, "\hat{\varphi}"] \arrow[d, "\nu_X"'] & \hat{Y} \arrow[d, "\nu_Y"] \\ X \arrow[r, "\varphi"'] & Y. \end{tikzcd}$$

And indeed, the function $\hat{\varphi}$ so defined is continuous by the universal property of the quotient map ν_X. This completes the proof. □

From Theorems 3.2 and 3.8 we see that if X is a finite topological space, then the homotopy inverse $\mu : \hat{X} \to X$ to ν_X is also a homotopy equivalence, and thus the composition $\mu \circ f_{\hat{X}} : |\mathcal{K}(\hat{X})| \to X$ gives a weak homotopy equivalence from a finite simplicial complex to X. That is, these two theorems together imply that every finite topological space is weakly homotopy equivalent to a finite simplicial complex.

We conclude this chapter by discussing the reverse direction of the correspondence established by McCord; since it follows with little extra work from what we have already shown, our coverage will be rather brief.

THEOREM 3.9. *There exists a correspondence that assigns to each simplicial complex K a T_0 space $\mathcal{X}(K)$ whose points are the barycenters of simplices of K, and a weak homotopy equivalence $f_K : |K| \to \mathcal{X}(K)$. To each simplicial map $\psi : K \to L$ is assigned a map $\psi' : \mathcal{X}(K) \to \mathcal{X}(L)$ such that $\psi \circ f_K$ is homotopic to $f_L \circ |\psi|$.*

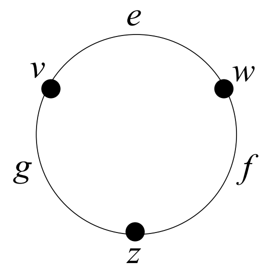
PROOF. Let K be a simplicial complex, and let K' denote its first barycentric subdivision. Define $\mathcal{X}(K)$ to be the set of barycenters of simplices of K, or in other words, the set of vertices of K'. We can make the set $\mathcal{X}(K)$ into an T_0 A-space by imposing the partial order given by $b(\sigma) \leq b(\sigma')$ whenever $\sigma \subset \sigma'$, where $b(\tau)$ denotes the barycenter of the simplex τ. It is clear that $\mathcal{K}(\mathcal{X}(K)) = K'$. Therefore, we can define $f_K : |K| \to \mathcal{X}(K)$ to be the map $f_{\mathcal{X}(K)}$ defined in Theorem 3.2, which indeed maps into $\mathcal{X}(K)$ since

$$f_K(|K|) = f_{\mathcal{X}(K)}(|K|) = f_{\mathcal{X}(K)}(|K'|) = f_{\mathcal{X}(K)}(|\mathcal{K}(\mathcal{X}(K))|) = \mathcal{X}(K).$$

Therefore, the fact that f_K is a weak homotopy equivalence follows from the proof of Theorem 3.2.

If $\psi : K \to L$ is a simplicial map, then we can define a simplicial map $\psi' : K' \to L'$ by setting $\psi'(b(\sigma)) = b(\psi(\sigma))$, and the maps $|\psi|$ and $|\psi'|$ are homotopic. We can view ψ' as a map $\mathcal{X}(K) \to \mathcal{X}(L)$, and from this perspective it is order-preserving and therefore continuous. It follows from Theorem 3.2 that $\psi' \circ f_K = f_L \circ |\psi'|$, and hence $\psi' \circ f_K$ is homotopic to $f_L \circ |\psi|$. □

EXAMPLE 3.10. Suppose we realize S^1 as a simplicial complex K with three 0-simplices (say v, w, and z) and three 1-simplices (say e, f, and g), arranged as follows:



Then the finite model of K will have six points, which we will denote by p_v, p_w, p_z, p_e, p_f, and p_g in accordance with the correspondence between points of $\mathcal{X}(K)$ and barycenters of simplices of K. Moreover, since $v \subset e$ and $v \subset g$, we will have $p_v \leq p_e$ and $p_v \leq p_g$. Similar orderings result from the inclusions of the other two vertices into the corresponding faces, so the collection of sets

$$\begin{aligned}
U_{p_v} &= \{p_v, p_e, p_g\} \\
U_{p_w} &= \{p_w, p_e, p_f\} \\
U_{p_z} &= \{p_z, p_f, p_g\} \\
U_{p_e} &= \{p_e\} \\
U_{p_f} &= \{p_f\} \\
U_{p_g} &= \{p_g\}
\end{aligned}$$

forms a basis for the topology on $\mathcal{X}(K)$.

CHAPTER 4

Inverse Limits of Finite Topological Spaces: An Extension of McCord

It follows from Theorem 3.9 that every finite simplicial complex K is weakly homotopy equivalent to a finite topological space $\mathcal{X}(K)$. The space $\mathcal{X}(K)$ is aptly termed the "finite model" of K, for it not only has isomorphic homotopy groups but, at least in the case of Example 3.3, also bears some intuitive resemblance to the original simplicial complex. This invites the question: might we model K better? Is it possible to obtain a finite topological space with strictly more points than $\mathcal{X}(K)$, which nevertheless retains the property of weak homotopy equivalence to K? If so, we might then wonder whether, by a process of iteratively refining our models, we might in the limit get the original space K back again.

These are the questions that motivate the following result:

THEOREM 4.1. *Any finite simplicial complex is homotopy equivalent to the inverse limit of a sequence of finite spaces.*

The first space in the inverse limit is essentially McCord's space $\mathcal{X}(K)$, and each of the others is indeed a larger finite space that is weakly homotopy equivalent to K. Although the inverse limit of these finite spaces is not, as we might have hoped, homeomorphic to the original simplicial complex, the homotopy equivalence mentioned in the theorem is in some sense "almost" a homeomorphism; it is a homeomorphism onto a quotient space of the inverse limit, and moreover onto a quotient space to which the entire inverse limit deformation retracts.

We will begin by discussing how the finite spaces are constructed.

Let K be a finite simplicial complex. To construct its finite models, we begin by letting X_0 be the finite space whose points are in one-to-one correspondence with the faces of simplices of K, just as in McCord's definition of $\mathcal{X}(K)$. Also analogously to McCord, we make X_0 into a poset by declaring that if $x, y \in X_0$ correspond to the faces σ_x and σ_y of K, then $x \leq y$ if and only if $\sigma_x \subseteq \sigma_y$. We topologize this finite space slightly differently than McCord's $\mathcal{X}(K)$, however; in our case, we endow X_0 with the topology generated by the sets

$$B_x = \{y \in X_0 \mid x \leq y\}$$

for $x \in X_0$, as opposed to the sets U_x defined in the previous section. The reason for this distinction involves the continuity of the maps q_n defined below.

For each $n \geq 0$, let K_n denote the n^{th} barycentric subdivision of K, and let X_n be the finite space whose points are in one-to-one correspondence with the faces of simplices of K_n. Using an analogous partial order on the points of X_n, we can endow each X_n with the topology generated by the sets B_x as above.

There is a natural map $p_n : |K| \to X_n$ for each n, since every point in K is contained in the interior of precisely one face of the n^{th} barycentric subdivision of K; indeed, this is precisely the map f_{K_n} appearing in Theorem 3.9. Moreover, there is a unique projection map $q_n : X_n \to X_{n-1}$

making the following diagram commute:

$$\begin{tikzcd}
& {|K|} \arrow[dl, "p_n"'] \arrow[dr, "p_{n-1}"] & \\
X_n \arrow[rr, "q_n"] & & X_{n-1}.
\end{tikzcd}$$

In light of the correspondence between points in X_n and faces of simplices in K_n, we will typically denote the simplex corresponding to $x \in X_n$ by σ_x^n. It is straightforward to check that for each $n \geq 0$ and each $x \in X_n$, one has

$$p_n^{-1}(B_x) = \mathrm{st}_n(\sigma_x^n),$$

where $\mathrm{st}_n(\sigma_x^n)$ is the open star of σ_x^n in K_n. This implies in particular that the maps p_n are all continuous, even in our modified topology on X_n. They are also open maps; for if $U \subset |K|$ is open and $x \in p_n(U)$, then there exists $z \in U$ such that $x = p_n(z)$, or in other words, $z \in \mathrm{int}(\sigma_x^n)$. So to say that $x \in p_n(U)$ is to say that $\mathrm{int}(\sigma_x^n) \cap U \neq \emptyset$, and it is easy to see that this implies that for any simplex σ_y^n of K_n such that $\sigma_x^n \subset \sigma_y^n$, we also have $\mathrm{int}(\sigma_y^n) \cap U \neq \emptyset$. That is, if $x \leq y$ then $y \in p_n(U)$, so $B_x \subset p_n(U)$. This says that for each $x \in p_n(U)$ one has $x \in B_x \subset p_n(U)$, and hence $p_n(U)$ is open.

By the commutativity of the above diagram, this implies that each q_n is continuous. Hence, we now have an inverse system:

$$X_0 \xleftarrow{q_1} X_1 \xleftarrow{q_2} X_2 \xleftarrow{q_3} X_3 \xleftarrow{q_4} \cdots$$

and we can define $\tilde{X}$ to be its inverse limit. The main work of the proof of Theorem 4.1 will be in showing that $|K|$ is homeomorphic to a quotient space of $\tilde{X}$.

Before doing so, however, it should be noted that the maps $p_n : |K| \to X_n$ are all still weak homotopy equivalences. To prove this, recall that for any basis element $B_x \subset X_n$, the set $p_n^{-1}(B_x) = \mathrm{st}_n(\sigma_x^n)$ is contractible. And, using the fact that B_x is the smallest open set containing x, it is readily verified that each B_x is also contractible. Hence the restriction $p_n|p_n^{-1}(B_x) : p_n^{-1}(B_x) \to B_x$ is a weak homotopy equivalence for each basis element B_x, and by Theorem 3.5 this is sufficient to conclude that p_n is a weak homotopy equivalence.

Lemma 4.2. *If K is a finite simplicial complex and the finite spaces X_n are defined as above, then $|K|$ is homeomorphic to a quotient space of $\varprojlim X_n$.*

Proof. Given $x = (x_0, x_1, x_2, \ldots) \in \tilde{X}$, we can associate to x a sequence of points in $|K|$ by choosing an arbitrary element $a_n \in p_n^{-1}(x_n)$ for each $n \geq 0$. Because these points lie in nested simplices of increasingly fine barycentric subdivisions of K, and since the maximum diameter of a geometric simplex of $|K_n|$ approaches zero as n approaches infinity, any sequence obtained in this way is Cauchy and therefore convergent.

Now, we could have chosen a different sequence $\{a_n\}$ corresponding to the same element $x \in \tilde{X}$. However, we claim that if $\{a_n\}$ and $\{b_n\}$ are any two sequences obtained in this way, then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$. To see this, let $a = \lim_{n\to\infty}$ and let $\varepsilon > 0$. By the convergence of $\{a_n\}$, there exists a natural number N such that $|a - a_n| < \frac{\varepsilon}{2}$ for all $n > N$. And because the diameters of the simplices $p_n^{-1}(x_n)$ approach zero as n approaches infinity, there also exists a natural number M such that $|a_n - b_n| < \frac{\varepsilon}{2}$ for all $n > M$. So for $n > \max\{N, M\}$, we have that $|a - b_n| < \varepsilon$, and hence $\{b_n\}$ converges to a, also.

We have thus established that there is a well-defined map

$$G : \tilde{X} \to |K|$$

given by sending $(x_0, x_1, x_2, \ldots)$ to the limit of any sequence $\{a_n\} \subset |K|$ where $p_n(a_n) = x_n$ for all n. To prove that G is continuous, let $U \subset |K|$ be any open set, and let $\overline{x} = (x_0, x_1, x_2, \ldots) \in G^{-1}(U)$. First, observe that

$$G^{-1}(U) \subset \prod_{n=0}^{\infty} p_n(U),$$

that is, $x_n \in p_n(U)$ for all n. For if there exists some n such that $x_n \notin p_n(U)$ and $\{a_n\}$ is as above, then the commutativity of the diagram on the previous page implies that $p_n(a_{n+1}) \notin p_n(U)$ and hence $a_{n+1} \notin U$. Similarly, we obtain that $a_i \notin U$ for all $i > n$, contradicting the assumption that the the sequence $\{a_n\}$ converges to $G(\overline{x}) \in U$. So, the above containment holds, and therefore we may as well assume that $a_n \in U$ for all n. Since U is open and $\{a_n\}$ converges to a point in U, there is an open set V such that each $a_n \in V$ and such that $\overline{V} \subset U$. Now, the set

$$\prod_{n=0}^{\infty} p_n(V)$$

is open since the p_n are open maps. Moreover, if $\overline{y} = (y_0, y_1, y_2, \ldots) \in \prod p_n(V)$, then $y_n \in p_n(V)$ for all n, so we can choose a sequence $\{b_n\} \subset V$ such that $p_n(b_n) = y_n$ for all n. Denote the limit of $\{b_n\}$ by b, so that $b = G(\overline{y})$. Then, since $\{b_n\} \subset V$, we have $b \in \overline{V} \subset U$, so $\overline{y} \in G^{-1}(U)$. Thus,

$$\overline{x} \in \prod_{n=0}^{\infty} p_n(V) \subset G^{-1}(U),$$

and hence $G^{-1}(U)$ is open.

Define an equivalence relation on $\tilde{X}$ by $x \sim y$ if and only if $G(x) = G(y)$, and denote by Y the corresponding quotient space of $\tilde{X}$. (In fact, one can check that this equivalence relation is simply the T_1 relation, wherein $x \sim y$ if and only if either every open set containing x also contains y or vise versa, since any open set in $\tilde{X}$ containing $(p_0(z), p_1(z), p_2(z), \ldots)$ necessarily contains every x such that $G(x) = z$. Thus, we might say that Y is the "T_1-ification" of $\tilde{X}$.)

We get an induced map $\tilde{G} : Y \to |K|$, which is by construction both well-defined and injective. Since $\tilde{G}([(p_0(x), p_1(x), p_2(x), \ldots)]) = x$ for any $x \in |K|$, it is also clearly surjective. Hence $\tilde{G}$ is a bijection. It is continuous by the universal property of quotient spaces, since if $\pi : \tilde{X} \to Y$ is the quotient map then $G = \tilde{G} \circ \pi$ is continuous. And its inverse is the map $\tilde{G}^{-1} : |K| \to Y$ defined by

$$x \mapsto [(p_0(x), p_1(x), p_2(x) \ldots)],$$

which is the composition of the quotient map π with the map $x \mapsto (p_0(x), p_1(x), p_2(x), \ldots)$, and hence is continuous. Thus, $\tilde{G}$ is a homeomorphism. $\square$

All that remains, now, is to show that in fact Y is homotopy equivalent to $|K|$. This will be achieved by way of the following lemma:

Lemma 4.3. *The quotient space Y is homeomorphic to a deformation retract of $\tilde{X}$.*

Proof. Let $\tilde{x} \in \tilde{X}$, and suppose that $G(\tilde{x}) = y$. We claim that every neighborhood of the point $(p_0(y), p_1(y), p_2(y), \ldots) \in \tilde{X}$ contains $\tilde{x}$. To see this, let $\tilde{x} = (x_0, x_1, x_2, \ldots)$, and let $\{a_n\}$ be a sequence of points converging to y such that $p_n(a_n) = x_n$ for all n. Then, for each $n \geq 0$, y lies in the interior of precisely one simplex σ_n of K_n, and moreover, it clearly must be the case that $a_n \in \mathrm{st}_n(\sigma_n)$. This implies that for each n, $x_n \in p_n(\mathrm{st}_n(\sigma_n)) = B_{p_n(y)} \subset X_n$. But since $B_{p_n(y)}$ is the smallest open subset of X_n containing $p_n(y)$, this implies that any open subset of X_n containing $p_n(y)$ also contains x_n. Hence, any open subset of $\tilde{X}$ containing $(p_0(y), p_1(y), p_2(y), \ldots)$ also contains $\tilde{x}$, as claimed.

Let E be any equivalence class under $\sim$, wherein every element defines a sequence converging to $x \in |K|$. Define a homotopy $h_E : E \times [0,1] \to h_E$ by

$$h_E(y,t) = \begin{cases} y & \text{if } t \in [0,1) \\ (p_0(x), p_1(x), \ldots) & \text{if } t = 1. \end{cases}$$

To see that this is continuous, let $U \subset Y$ be open. If the point $(p_0(x), p_1(x), \ldots)$ lies in U, then every element of the equivalence class also lies in U, so that $h_E^{-1}(U) = Y \times [0,1]$. If $(p_0(x), p_1(x), \ldots) \notin U$, then $h_E^{-1}(U) = U \times [0,1)$, which is open. Hence, h_E is continuous, and so we have a homotopy from the identity map on Y to a constant map. In particular, we have shown that every equivalence class is contractible.

Combining all of these homotopies on the various equivalence classes, we obtain a map $F : \tilde{X} \times [0,1] \to \tilde{X}$, which we claim is also continuous. To verify this, let $U \subset \tilde{X}$ be open, and define a subset $U^{BC} \subset U$ as follows:

$$U^{BC} = \{x \in U \mid (p_0(G(x)), p_1(G(x)), \ldots) \notin U\}.$$

These are the "boundary-convergent" points in U, those that we view as sequences of points in the open set U converging to a point that is not in U. The set U^{BC} is closed in U, since we can define a continuous map $p : \tilde{X} \to \tilde{X}$ by $p(x) = (p_0(G(x)), p_1(G(x)), \cdots)$, and

$$U \setminus U^{BC} = \{x \in U \mid p(x) \in U\} = U \cap p^{-1}(U),$$

which is an intersection of two open sets by the continuity of p and hence is open. Moreover:

$$F^{-1}(U) = (U \times [0,1]) \setminus (U^{BC} \times \{1\}),$$

so $F^{-1}(U)$ is open. Therefore, F is continuous, as claimed.

We have thus defined a deformation retraction of $\tilde{X}$ onto a subspace Z that contains exactly one element from each equivalence class. It is clear that if $i : Z \hookrightarrow \tilde{X}$ is the inclusion map and $\pi : \tilde{X} \to Y$ is the quotient map as above, then the map

$$f = \pi \circ i : Z \to Y,$$

is a bijection. Indeed, this map is a homeomorphism; for if $U \subset Z$ is open, then $U = V \cap Z$ for some open subset $V \subset \tilde{X}$, and $\pi(V) = f(U)$. And by the definition of Z, the set V is forced to contain every point in each equivalence class it intersects, so $\pi^{-1}(\pi(V)) = V$. In particular, $\pi(V)$ is open, so f is an open map. Conversely, if $U \subset Y$ is open, then $\pi^{-1}(V) \subset \tilde{X}$ is open, so $\pi^{-1}(V) \cap Z = f^{-1}(Z)$ is open in Z. Hence f is continuous.

Therefore, by the composition of the homotopy equivalence $\tilde{X} \to Z$ and the homeomorphism $f : Z \to Y$, we obtain the claim. $\square$

The proof of Theorem 4.1 is now immediate:

Proof of Theorem 4.1. Composing the homeomorphism from Lemma 4.2 and the homotopy equivalence from Lemma 4.3, we obtain the desired result. □

It is worth noting that, as observed by Barmak and Minian in [**2**], McCord's results apply more generally to regular CW complexes. Recall that a CW complex is **regular** if its attaching maps are all embeddings; it can be shown that this implies that the closure of each cell is a subcomplex. If K is a regular CW complex, one can define an associated simplicial complex K' whose vertices are the cells of K and whose n-simplices are the sets $\{e_1, e_2, \cdots, e_n\}$ of simplices where e_i is a face of e_{i+1} for each i. Moreover, $|K'|$ is homeomorphic to K. Therefore, the composition of this homeomorphism with the map $f_{K'} : |K'| \to \mathcal{X}(K')$ is a weak homotopy equivalence from K to a finite space.

For the same reasons, then, if K is a regular CW complex, we can apply the results of Theorem 4.1 to the simplicial complex K' to show that K is homotopy equivalent to an inverse limit of finite topological spaces.

CHAPTER 5

Conclusion: Symmetric Products of Finite Topological Spaces

A possible extension of this work could come from studying the symmetric products of finite topological spaces. In particular, if K is a simplicial complex, one can consider, on one hand, the spaces $SP_n(\mathcal{X}(K))$; on the other hand, each $SP_n(K)$ is naturally a Δ-complex (see [**6**]), and hence one can subdivide it to give it the structure of a simplicial complex and then consider $\mathcal{X}(SP_n(K))$. It seems plausible that $SP_n(\mathcal{X}(K))$ and $\mathcal{X}(SP_n(K))$ are closely related, perhaps weakly homotopy equivalent, so this topic may merit further investigation.

Moreover, the infinite symmetric product of a space is a monoid under the operation given by concatenating tuples, and under good conditions this operation is continuous as a map $\mu : SP(X)\times SP(X) \to SP(X)$. If μ is indeed continuous, then the set of based homotopy classes of maps into $SP(X)$ from some other basepointed topological space Y, denoted $\langle Y, SP(X)\rangle$, inherits the structure of a monoid, where we multiply maps pointwise. More generally, even if μ is not necessarily continuous, the set $\langle Y, SP(X)\rangle$ still forms a monoid as long as Y is compact. For if Y is compact and $f, g : Y \to SP(X)$ are basepoint-preserving maps, then $f(Y)$ and $g(Y)$ are compact subsets of $SP(X)$ and hence they lie in finite symmetric products $SP_n(X)$ and $SP_m(X)$, respectively. In particular, $(f \cdot g)(Y) \subset SP_{n+m}(X)$, so if $U \subset SP(X)$ is any open set, then

$$(f \cdot g)^{-1}(U) = (f \cdot g)^{-1}(U \cap SP_{n+m}(X)) = f^{-1}(U \cap SP_n(X)) \cap g^{-1}(U \cap SP_m(X)),$$

which is the intersection of two open sets (and hence open) since both f and g are continuous. So $f \cdot g$ is continuous, and we can conclude that $\langle Y, SP(X)\rangle$ is a monoid.

Given these monoids, one could take their group completions $\langle Y, SP(X)\rangle^*$. It might be interesting to check whether a cohomology theory arises in this way. For example, one could define

$$h^n(Y) = \langle Y, SP(\mathcal{X}(S^n))\rangle^*.$$

It is not clear whether this satisfies the necessary axioms required to define a cohomology theory, or indeed, even whether $\langle Y, SP(\mathcal{X}(S^n))\rangle$ is a monoid at all (unless, as noted above, we restrict ourselves to compact spaces Y). However, it would be interesting to explore whether this is the case, and if so, whether the resulting cohomology theory distinguishes between topological spaces (perhaps finite spaces) that singular cohomology does not.

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