MATH G4307 PROBLEM SET 5
DUE OCTOBER 11, 2011.

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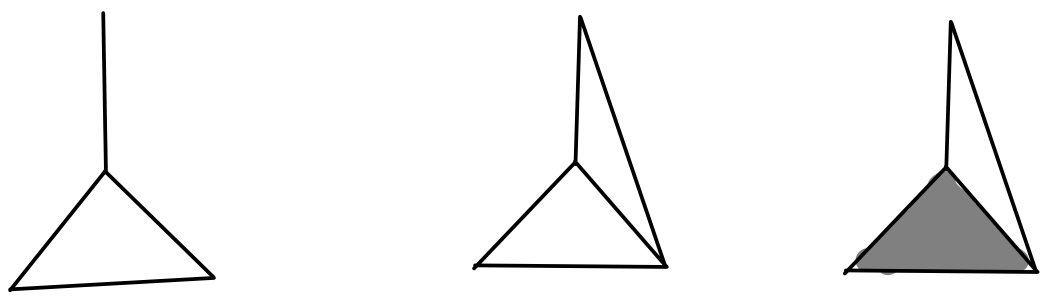
Exercises to turn in:

(E1) Find a simplicial complex $X_\bullet$ so that $|X_\bullet| \cong T^2$, the 2-dimensional torus. (Compare with the Δ-complex for T^2 in Hatcher, p. 102; they should be different.)

(E2) Compute the simplicial homology of the following simplicial complexes:



(a) (4 vertices, 4 edges) (b) (4 vertices, 5 edges) (c) (4 vertices, 5 edges, 1 face)

(E3) Compute the homology $(\ker(\partial_i)/\mathrm{Im}(\partial_{i+1})$ for each $i)$ for the following chain complexes:

(a) $C_1 = \mathbb{Z}^3$, $C_0 = \mathbb{Z}^3$, $\partial_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix}$, $C_i = \{0\}$ for $i \notin \{0,1\}$.

(b) $C_2 = \mathbb{Z}\langle f\rangle$, $C_1 = \mathbb{Z}\langle e_1, e_2\rangle$, $C_0 = \mathbb{Z}\langle v\rangle$, $C_i = \{0\}$ for $i > 2$, $\partial_1(e_i) = 0$ $(\forall i)$, $\partial_2(f) = e_1 + e_2$.

(c) $C_2 = \mathbb{Z}\langle f_1, f_2\rangle$, $C_1 = \mathbb{Z}\langle e_1, e_2\rangle$, $C_0 = \mathbb{Z}\langle v\rangle$, $C_i = \{0\}$ for $i > 2$, $\partial_1(e_i) = 0$ $(\forall i)$, $\partial_2(f_1) = e_1 + e_2$, $\partial_2(f_2) = e_1 - e_2$.

(E4) Hatcher 2.1.11 (p. 132)

(E5) Hatcher 2.1.12 (p. 132)

(E6) Hatcher 2.1.14 (p. 132). (You'll have to read the definition of a short exact sequence.)

Problems to think about but not turn in:

(P1) The *Euler characteristic* of a finite simplicial complex $X_\bullet = \{X_0, X_1, \dots\}$ is the alternating sum

$$\chi(X_\bullet) = \sum_{i=0}^{\infty} (-1)^i \# X_i.$$

(This generalizes the expression $v - e + f$ from Euler's formula.) We will show that this depends only on the geometric realization $|X_\bullet|$ the week after next.

(a) If $X_\bullet$ is a simplicial complex and Y is a covering space of the geometric realization $|X_\bullet|$ then Y inherits the structure of a simplicial complex (i.e., there's a simplicial complex $Y_\bullet$ so that $Y = |Y_\bullet|$). Explain how.

(b) Prove: if Y is an n-fold covering space of X then $\chi(Y) = n\chi(X)$.

(c) Now, consider the case of Riemann surfaces Σ. Generalize the previous part to *branched covers* $\Sigma_g \to \Sigma_h$. (Hint: choose a simplicial complex for Σ_h so that the branch points are vertices of the complex, and use this to induce a simplicial complex for Σ_g.)

This (fairly easy) result turns out to be quite important; it's called the Riemann-Hurwitz formula.

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