

# BORDERED HEEGAARD FLOER HOMOLOGY

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ABSTRACT. We construct Heegaard Floer theory for 3-manifolds with connected boundary. The theory associates to an oriented two-manifold a differential graded algebra. For a three-manifold with specified boundary, the invariant comes in two different versions, one of which (type  $D$ ) is a module over the algebra and the other of which (type  $A$ ) is an  $\mathcal{A}_\infty$  module. Both are well-defined up to chain homotopy equivalence. For a decomposition of a 3-manifold into two pieces, the  $\mathcal{A}_\infty$  tensor product of the type  $D$  module of one piece and the type  $A$  module from the other piece is  $\widehat{HF}$  of the glued manifold.

As a special case of the construction, we specialize to the case of three-manifolds with torus boundary. This case can be used to give another proof of the surgery exact triangle for  $\widehat{HF}$ . We relate the bordered Floer homology of a three-manifold with torus boundary with the knot Floer homology of a filling.

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## 1. INTRODUCTION

**1.1. Background.** Since the pioneering work of Simon Donaldson, techniques from gauge theory have taken a central role in the study of smooth four-manifold topology [3]. His numerical invariants, associated to closed, smooth four-manifolds, have shed much light on our understanding of differential topology in dimension four. Moreover, these invariants, and the subsequent closely-related Seiberg-Witten invariants [44] and Heegaard Floer invariants [33] all fit into a formal framework reminiscent of the “topological quantum field theories” proposed by Witten [43]. Crudely speaking, these theories have the following form. To a closed three-manifold  $Y$  one associates a (suitably graded) Abelian group the *Floer homology of  $Y$* , and, to a four-manifold  $W$  with boundary identified with  $Y$ , a homology class in the Floer homology of the boundary. If a closed, smooth four-manifold  $X$  decomposes along  $Y$  into a union of two four-manifolds with boundary  $X_1$  and  $X_2$ , then the numerical (Donaldson, Seiberg-Witten, or Heegaard Floer) invariant of the closed four-manifold is obtained as a suitable pairing between the relative invariants coming from  $X_1$  and  $X_2$  in a corresponding version of Floer homology of  $Y$ .

As the name suggests, the first such construction was proposed by Andreas Floer (for a restricted class of three-manifolds) as a tool for studying Donaldson’s theory. A complete construction of the corresponding three-dimensional Floer theory for Seiberg-Witten invariants was given by Kronheimer and Mrowka [18].

The aim of the present paper is to perform a corresponding construction one dimension lower. Specifically, we produce an invariant which, loosely speaking, associates to a parametrized, closed, oriented surface  $F$  a differential graded algebra  $\mathcal{A}(F)$ , and to a three-manifold with boundary  $F$ , a differential graded module over  $\mathcal{A}(F)$ . When a closed, oriented three-manifold can be decomposed along  $F$  into two pieces  $Y_1$  and  $Y_2$ , a suitable variant of Floer homology is gotten as a pairing of the differential graded modules associated to  $Y_1$  and  $Y_2$ .

We give now a slightly more detailed version of this picture, starting with some more remarks about Heegaard Floer homology, and then a more precise sketch of the invariants constructed in the present paper.

Recall that there are several variants Heegaard Floer homology stemming from the fact that, in its most basic form, the Heegaard Floer homology of a three-manifold is the homology of a chain complex defined over a polynomial ring in an indeterminate  $U$ . The full theory can be promoted to construct invariants for closed, smooth four-manifolds [35], similar in character (and conjecturally equal) to Seiberg-Witten invariants. In the present paper, we focus on the specialization of Heegaard Floer homology for three-manifolds in the case where  $U = 0$ , giving the three-manifold invariant denoted  $\widehat{HF}(Y)$ . The corresponding simplified theory is not rich enough to construct interesting closed four-manifold invariants, but it does already contain interesting geometric information about the underlying three-manifold including, for example, full information about the minimal genera of embedded surfaces in an irreducible three-manifold  $Y$  [30].

With this background in place, we proceed as follows. Let  $F$  be a compact, oriented two-manifold. We associate to  $F$  a differential graded algebra  $\mathcal{A}(F)$ . For three-manifolds  $Y$  with boundary identified with  $F$ , we associate a chain complex over  $\mathcal{A}(F)$ . Indeed, there are two variants of the construction, the *type D-module of  $Y$* ,

denoted  $\widehat{CFD}(Y)$ , which is a left differential graded module over  $\mathcal{A}(F)$ ; and another, the *type A-module of Y*, denoted  $\widehat{CFA}(Y)$ , which is a more general type of object: a right  $\mathcal{A}_\infty$ -module over  $\mathcal{A}(F)$ . Both complexes are defined by counting pseudo-holomorphic curves in an associated picture. The definitions of both of these modules depend on a number of auxiliary choices, including compatible Heegaard diagrams and associated choices of almost complex structures. However, as is typical of Floer homology theories, the underlying quasi-isomorphism types of the modules are, in fact, independent of these choices, giving rise to topological invariants of our three-manifold-with-boundary. When speaking informally about these invariants, we refer to them collectively as the *bordered Heegaard Floer theory* of  $Y$ .

The relationship with closed invariants is encapsulated as follows. Suppose that  $Y_1$  and  $Y_2$  are two three-manifolds with boundary, where  $\partial Y_1$  is identified with  $F$  and  $\partial Y_2$  is identified with  $-F$ . Then we can form a closed three-manifold  $Y$  by identifying  $Y_1$  and  $Y_2$  along their boundaries. The pairing theorem for bordered Heegaard Floer homology states that the Heegaard Floer complex  $\widehat{HF}$  of  $Y$  is quasi-isomorphic to the  $\mathcal{A}_\infty$  tensor product of  $\widehat{CFA}(Y_1)$  with  $\widehat{CFD}(Y_2)$ .

Recall that  $\widehat{HF}$  can be calculated algorithmically for any closed three-manifold, thanks to the important work of Sarkar and Wang [39]. Nevertheless, the present work (and forthcoming extensions) makes it possible to compute it in infinite families: by cutting a 3-manifold into simpler pieces along surfaces and computing an invariant for each piece, we can reduce the computation to a computation for each piece and an algebra computation. Indeed, this might lead a more efficient algorithm for computation in general.

Moreover, bordered Heegaard Floer homology provides a conceptual framework for organizing the structure of  $\widehat{HF}$ . We give a few of these properties in the present paper, and return to further applications in forthcoming work.

**1.2. The bordered Floer homology package.** The formal algebraic setting for the constructions herein lie well outside the working toolkit for the typical low-dimensional topologist:  $\mathcal{A}_\infty$ -modules over differential graded algebras, and their  $\mathcal{A}_\infty$  tensor products (though, we should point out, these objects are now familiar in symplectic topology, cf. [10, 17, 40]). We will recall the basics before giving precise statements of our results. But before doing this, we sketch the outlines of the geometry which underpins the constructions for the benefit of our reader, giving informal statements of the basic results of this package.

Central to this geometric picture is the cylindrical reformulation of Heegaard Floer homology, as developed by the first author [20]. Recall that Heegaard Floer homology in its original incarnation is an invariant associated to a Heegaard diagram for a three-dimensional manifold  $Y$ . Starting from a Heegaard diagram for  $Y$ , specified by an oriented surface  $\Sigma$ , equipped with two  $g$ -tuples of attaching circles  $\{\alpha_1, \dots, \alpha_g\}$  and  $\{\beta_1, \dots, \beta_g\}$ , and an additional basepoint  $z \in \Sigma$  in the complement of these circles, one considers a version of Lagrangian Floer homology in the  $g$ -fold symmetric product of  $\Sigma$ . In the cylindrical reformulation, disks in the  $g$ -fold symmetric product are reinterpreted as holomorphic curves in the four-manifold  $\Sigma \times [0, 1] \times \mathbb{R}$ , with coordinates  $(x, s, t)$ , satisfying certain constraints (dictated by the attaching circles) at  $s \in \{0, 1\}$  and asymptotic constraints as  $t$  approaches  $\pm\infty$ .

Starting from this cylindrical reformulation, suppose that our Heegaard diagram for  $Y$  is equipped with a closed, separating curve  $Z$  in  $\Sigma$ , which does not meet any  $\beta$ -circle and meets each  $\alpha$ -circle minimally, in at most two points. The constructions in this paper emerge when one considers limits of holomorphic curves in  $\Sigma \times [0, 1] \times \mathbb{R}$  as the complex structure on  $\Sigma$  is pinched along  $Z$ .

In the limit as the  $Z$  circle is pinched to a node, the holomorphic curves limit to holomorphic curves in  $((\Sigma_1^+ \amalg \Sigma_2^+) \times [0, 1] \times \mathbb{R})$ , where here  $\Sigma_1$  and  $\Sigma_2$  are the two components of  $\Sigma - Z$ , and  $\Sigma_i^+$  denotes the corresponding surface equipped with cylindrical ends. The holomorphic curves from these two halves have constrained limiting behavior as they enter the ends of  $\Sigma_1^+$  and  $\Sigma_2^+$ : there are finitely many values of  $t$  where our holomorphic curves are asymptotic to arcs in  $\partial\Sigma_i$ . (In the present case, the contact manifold in question is the circle, and its symplectization, the end of  $\Sigma_i^+$ , is a cylinder.) The data of the asymptotics (the values of  $t$  and the arcs in  $\partial\Sigma_i$ ) must match in the limit. Thus, we may reconstruct holomorphic curve counts for  $\Sigma \times [0, 1] \times \mathbb{R}$  from holomorphic curve counts in  $\Sigma_i^+ \times [0, 1] \times \mathbb{R}$ .

The data on the Heegaard surface specified by  $Z$  has a more intrinsic, three-dimensional interpretation. The  $\alpha$ -circles meet  $Z$  generically in a collection of points, which come in pairs (those points which belong to the same  $\alpha_i$ ). We call this structure a *matched circle*. It specifies a closed surface  $F$ . This can be seen in terms of Morse functions, as follows. Think about the Heegaard diagram as the intersections of the ascending and descending disks of a self-indexing Morse function on  $M$  with the middle level in the usual way. We then construct  $F$  from  $Z$  by flowing  $Z$  backwards and forwards under the Morse flow. Concretely,  $F$  is obtained from a disk with boundary  $Z$  by attaching one-handles along the matched pairs of points, and then filling in the remaining disk with a two-handle. (Our assumptions guarantee there is only one two-handle to be added in the end.) The two halves of Heegaard diagram specify three-manifolds  $Y_1$  and  $Y_2$  meeting along their common boundary  $F$ .

In general terms, then, the differential graded algebra  $\mathcal{A}(F)$  associated to the surface  $F$  is constituted from the Reeb chords of a corresponding matched circle for  $F$ . The product on the algebra and its differential are induced from (relatively simple) holomorphic curve counts in the cylinder which interpolates between Reeb chords. The precise algebra is defined in Section 3.

The type  $D$  module for the component  $Y_2$  is defined as a chain complex over  $\mathcal{A}(F)$ , generated by  $g_2$ -tuples of intersection points of various of the  $\alpha_i$  and  $\beta_j$  supported in  $\Sigma_2$ , with a differential given as a weighted count of rigid holomorphic curves, where the coefficient in  $\mathcal{A}(F)$  of a rigid holomorphic curve measures the asymptotics of its Reeb chords at its boundary. The precise definition is given in Section 6. In that section, we also prove a more precise version (Theorem 6.23) of the following:

**Theorem 1.1.** *The homotopy equivalence class of the differential module  $\widehat{CFD}(Y_2)$  is a topological invariant of three-manifolds  $Y_2$  with boundary  $F$ .*

Algebraic operations on the type  $A$  module for the component of  $Y_1$  are defined by counting rigid holomorphic curves subject to certain height constraints on their corresponding Reeb chords. Specifically, we require that certain clumps of these Reeb chords occur at the same height. The proper algebraic set-up for these operations is that of an  $\mathcal{A}_\infty$ -module over  $\mathcal{A}(F)$ , and the precise definition of the type  $A$  module is

given in Section 7, where we also prove a more precise version (Theorem 7.16) of the following:

**Theorem 1.2.** *The  $\mathcal{A}_\infty$ -homotopy equivalence class of the  $\mathcal{A}_\infty$ -module  $\widehat{CFA}(Y_1)$  is a topological invariant of three-manifolds  $Y_1$  with boundary  $F$ .*

Of course, differential graded modules over a differential graded algebra are special cases of  $\mathcal{A}_\infty$ -modules. Moreover, there is a pairing, the  $\mathcal{A}_\infty$ -tensor product, between two  $\mathcal{A}_\infty$ -modules  $M_1$  and  $M_2$ , giving rise to a chain complex  $M_1 \widetilde{\otimes} M_2$ , whose quasi-isomorphism type depends only on the quasi-isomorphism types of the two factors.

The Heegaard Floer homology of  $Y$  can be reconstituted from the bordered Heegaard Floer homology of its two components according to the following:

**Theorem 1.3.** *Let  $Y_1$  and  $Y_2$  be two three-manifolds with parameterized boundary  $\partial Y_1 = F = -\partial Y_2$ , and fix corresponding bordered Heegaard diagrams for  $Y_1$  and  $Y_2$ . Let  $Y$  be the closed three-manifold obtained by gluing  $Y_1$  and  $Y_2$  along  $F$ . Then,  $\widehat{CF}(Y)$  is quasi-isomorphic to the  $\mathcal{A}_\infty$  tensor product of  $\widehat{CFA}(Y_1)$  and  $\widehat{CFD}(Y_2)$ . In particular,*

$$\widehat{HF}(Y) \cong H_* \left( \widehat{CFA}(Y_1) \widetilde{\otimes}_{\mathcal{A}(F)} \widehat{CFD}(Y_2) \right).$$

We give two proofs of the above theorem. One proof (in Section 8) makes use of the powerful technique of Sarkar and Wang [39]: we construct convenient Heegaard diagrams for  $Y_1$  and  $Y_2$ , where the holomorphic curve counts can be calculated combinatorially. For such diagrams, the  $\mathcal{A}_\infty$ -structure of the type  $A$  module simplifies immensely (higher multiplications all vanish),  $\mathcal{A}_\infty$  tensor products coincide with traditional tensor products, and the proof of the pairing theorem becomes quite simple. The other proof (in Section 9) involves a rescaling argument to identify  $\widehat{CF}(Y)$  with another model for the  $\mathcal{A}_\infty$  tensor product. This second proof gives geometric insight into the connection between the analysis and the algebra. Indeed, this method might prove to be useful in other holomorphic curve contexts where there is no analogue of the Sarkar-Wang method.

**1.3. On gradings.** One further surprising aspect of the theory deserves mention in this introduction, and that concerns the structure of gradings.

Gradings in the usual Floer theory have an unconventional form: they take values in a cyclic Abelian group. (Indeed, for the case of Floer homology of Seiberg-Witten monopoles [18], this has an elegant and more intrinsic formulation in terms of isotopy classes of non-vanishing vector fields over three-manifolds.)

Gradings in bordered Floer homology have a correspondingly even less conventional form. The algebra of a surface is graded by the Heisenberg group associated to its intersection form. The gradings for bordered Heegaard Floer modules take values in various quotients of this Heisenberg group. (The obvious bordered manifold analogue of the geometric grading on Seiberg-Witten Floer homology has the same form.)

**1.4. The case of three-manifolds with torus boundary.** As explained in Section 10, for three-manifolds with boundary the torus  $T$ , the algebra  $\mathcal{A}(T)$  is particularly simple: it is finite-dimensional (an eight-dimensional subalgebra of  $4 \times 4$  upper-triangular matrices) and has vanishing differential.

The bordered Heegaard Floer homology of a solid torus bounded by  $T$  can be easily calculated, as in Section 10, see also [21, Section 5.3]. A fundamental result in Heegaard Floer homology, the *surgery exact triangle*, gives a long exact sequence relating the Heegaard Floer homology groups of three three-manifolds which are obtained as different fillings of the same three-manifold with torus boundary. (The result was first proved in [32], but it has its origins in Floer’s theory of instanton homology, see [9], compare also [19] for a Seiberg-Witten analogue.) This result can now be seen as a consequence of the pairing theorem, together with a short exact sequence relating three type  $D$  modules associated to three different fillings of  $T$  by solid tori.

In a related vein, recall [31, 37] that there is a construction of Heegaard Floer homology theory for knots  $K$  in a three-manifold  $Y$ . Information about this knot Floer homology turns out to determine the bordered Floer homology of the knot complement. This is stated precisely and worked out in Theorem 10.17 in Subsection 10.7. This result provides a number of explicit, non-trivial examples of bordered Floer homology.

Indeed, using Theorem 10.17 to connect knot Floer homology with bordered Heegaard Floer homology, and combining it with an adaptation of Theorem 1.3 (stated in Theorem 10.12 below), we are able to calculate knot Floer homology groups of satellite knots in terms of the filtered chain homotopy type of the knot filtration of the pattern knot, together with some information (a type  $A$  module) associated to the pattern. We illustrate this in some concrete examples, where the type  $A$  module can be calculated explicitly.

**1.5. Previous work.** Some of the material in this paper first appeared in the first author’s Ph.D. thesis [21], particularly Sections 4 and 5. As in that work, we construct here an invariant of bordered manifolds that lives in a suitable category of chain complexes. By modifying the algebra and modules appropriately we are able to reconstruct the invariant of a closed 3-manifold. Related constructions in the case of manifolds with torus boundary have been worked out by Eftekhary [5]. For the case of satellite knots, there is extensive work by Hedden, including [11], which we used as a check of some of our work. In a different direction, recall that a different construction for manifolds-with-boundary, equipped with a suture on the boundary, has been given by András Juhász [15].

An introduction to some of the structures used in this paper, in the form of a toy model, is in a separate paper [25].

**1.6. Future directions.** The theory in this paper is in some ways rather limited as a theory of 3-manifolds with boundary: we only deal with a single, connected boundary component, for the somewhat limited  $\widehat{HF}$  theory, and do not deal with tangles. (The present theory does allow for knots that do not cross the boundary.) In future papers we will fix all these shortcomings.

- For a bordered Heegaard diagram for a 3-manifold  $M$  with  $\partial M$  consisting of two boundary components  $F_1$  and  $F_2$ , we associate a bimodule over  $\mathcal{A}(F_1) \otimes \mathcal{A}(F_2)$ , and similarly for more boundary components. These bimodules obey an appropriate version of the Pairing Theorem. This allows us to change the parametrization of the boundary, giving a representation of the mapping class group of  $F$  on the derived category of modules over  $F$ . (Concretely, for

torus boundary we can, for instance, change the framing on the torus.) This homotopy equivalence class of this bimodule is not, however, an invariant of  $M$ ; it depends on some extra choices.

- It may seem strange to have two different invariants, the type  $D$  and the type  $A$  modules, associated to the same bordered 3-manifold. In fact, each one can be computed with the other by tensoring with an appropriate bimodule.
- Decomposing a three-manifold into simple pieces, and calculating the bimodules associated to the pieces, one can calculate the Heegaard Floer homology for large classes of 3-manifolds, using the basic computations and a relevant version of the pairing theorem.
- András Juhász has constructed an invariant of sutured manifolds, called sutured Floer homology, which has been remarkably effective at studying geometric questions. One can recover that invariant from ours by tensoring with a module associated to the sutures.
- We extend the theory to tangles in 3-manifolds with boundary by suitably modifying the algebra. This theory becomes particularly simple in the setting of *grid diagrams*, which have previously been useful for computing  $\widehat{HFK}$  [26, 27]. In that context the complication of a non-abelian grading group disappears, and all holomorphic curve counts can be done explicitly.

**1.7. Organization.** In Section 2 we briefly recall the language of  $\mathcal{A}_\infty$ -modules. In that section, we also give a concrete description of the  $\mathcal{A}_\infty$  tensor product which holds when one of the two factors has a particular simple algebraic structure, which we call here a *type  $D$  structure*. In Section 3 we construct the differential graded algebra associated to a closed, oriented surface. In Section 4, we recall the construction of Heegaard diagrams for three-manifolds with boundary. In Section 5, we collect the technical tools for moduli spaces of holomorphic curves which are counted in the algebraic structure underlying bordered Floer homology. With this background in place, we proceed in Section 6 to the definition of the type  $D$  module, and establish its invariance properties sketched in Theorem 1.1. In Section 7, we treat the case of type  $A$  modules, establishing a precise version of Theorem 1.2. This gives an algorithm to calculate the bordered Floer homology of a three-manifold, and indeed it also can be used to prove Theorem 1.3. In Section 9, we turn our attention to an analytic proof of Theorem 1.3. In Section 10, we focus on the case of bordered three-manifolds with torus boundary, calculating the associated algebra, and also the bordered Floer homology for solid tori. In this section, we also relate this invariant with knot Floer homology, giving many concrete examples of bordered Floer homology.

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## 2. $\mathcal{A}_\infty$ STRUCTURES

In this paper, we use of the notion of an  $\mathcal{A}_\infty$  module. Although  $\mathcal{A}_\infty$  notions, first introduced by Stasheff in the study of  $H$ -spaces [42], have become commonplace now

in symplectic geometry, see for example [10, 17, 40], they might not be so familiar to low-dimensional topologists. In Sections 2.1 and 2.2, we review the notions of primary importance to us now, in particular sketching the  $\mathcal{A}_\infty$  tensor product. Keller has another pleasant exposition of this material [16].

The type  $D$  module  $\widehat{CFD}(\mathcal{H})$  defined in Section 6 is an ordinary differential graded module, not an  $\mathcal{A}_\infty$  module, so the reader may wish to skip this section and concentrate on the type  $D$  structure at first. Similarly, the type  $A$  module  $\widehat{CFA}(\mathcal{H})$  of a nice diagram  $\mathcal{H}$  is an ordinary differential graded module. However, the notion of grading by a non-commutative group (Section 2.4) is used in all cases.

In Section 2.3, we introduce a further algebraic structure which naturally gives rise to an  $\mathcal{A}_\infty$ -module, which we call a *type  $D$  structure*. We also give an explicit construction of the  $\mathcal{A}_\infty$  tensor product, when one of the two factors is induced from a type  $D$  structure. This material is apparently new. As the name suggests, the type  $D$  module of a bordered three-manifold comes from a type  $D$  structure.

Finally, in Section 2.4 we introduce gradings of differential graded algebras or  $\mathcal{A}_\infty$  algebras with values in a non-commutative group.

**2.1.  $\mathcal{A}_\infty$  algebras and modules.** Although we will be working over differential graded algebras, or DGAs, (which are less general than  $\mathcal{A}_\infty$  algebras) we recall the definition of  $\mathcal{A}_\infty$  algebras here as it makes a useful warm-up for defining for  $\mathcal{A}_\infty$  modules, which we will encounter in this paper.

To avoid complicating matters at first, we will work in the category of  $\mathbb{Z}$ -graded complexes over a fixed ground ring  $\mathbf{k}$ , which we assume to have characteristic two. In particular, we consider modules  $M$  over  $\mathbf{k}$  which are graded by the integers, so that

$$M = \bigoplus_{d \in \mathbb{Z}} M_d.$$

If  $M$  is a graded module and  $m \in \mathbb{Z}$ , we define  $M[n]$  to be the graded module defined by  $M[n]_d = M_{d-n}$ .

Note that since we are working over characteristic two, the following discussion also carries over readily to the ungraded setting. The modules in the present paper are graded, but not by  $\mathbb{Z}$ . We return to this point in Section 2.4.

**Definition 2.1.** Fix a ground ring  $\mathbf{k}$  with characteristic two. An  $\mathcal{A}_\infty$ -algebra  $\mathcal{A}$  over  $\mathbf{k}$  is a graded  $\mathbf{k}$ -module  $A$ , equipped with  $\mathbf{k}$ -linear multiplication maps

$$\mu_i: A^{\otimes i} \rightarrow A[2-i]$$

defined for all  $i \geq 1$ , satisfying the compatibility conditions

$$\sum_{i+j=n+1} \sum_{\ell=1}^{n-j+1} \mu_i(a_1 \otimes \cdots \otimes a_{\ell-1} \otimes \mu_j(a_\ell \otimes \cdots \otimes a_{\ell+j-1}) \otimes a_{\ell+j} \otimes \cdots \otimes a_n) = 0$$

for each  $n \geq 1$ . Here,  $A^{\otimes i}$  denotes the  $\mathbf{k}$ -module  $\overbrace{A \otimes_{\mathbf{k}} \cdots \otimes_{\mathbf{k}} A}^i$ . Note that throughout this section, all tensor products are over  $\mathbf{k}$  unless otherwise specified. We use  $\mathcal{A}$  for the  $\mathcal{A}_\infty$  algebra and  $A$  for its underlying  $\mathbf{k}$ -module. An  $\mathcal{A}_\infty$  algebra is *strictly unital* if there is an element  $1 \in A$  with the property that  $\mu_2(a, 1) = \mu_2(1, a) = a$  and  $\mu_i(a_1, \dots, a_i) = 0$  if  $i \neq 2$  and  $a_j = 1$  for some  $j$ .

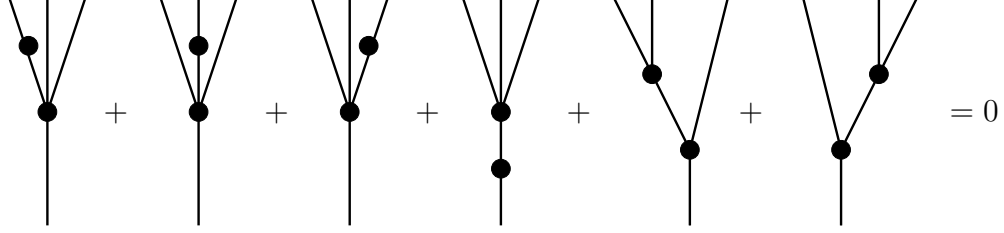


FIGURE 1. A graphical representation of the  $\mathcal{A}_\infty$  compatibility condition with  $n = 2$ . (It can be interpreted as stating that multiplication is associative up to homotopy.)

In particular, an  $\mathcal{A}_\infty$  algebra is a chain complex over  $\mathbf{k}$ , with differential  $\mu_1$ . In the case where all  $\mu_i = 0$  for  $i > 2$ , an  $\mathcal{A}_\infty$  algebra is just a differential graded algebra over  $\mathbf{k}$ , with differential  $\mu_1$  and (associative) multiplication  $\mu_2$ . We have assumed that  $\mathbf{k}$  has characteristic two; in the more general case, the compatibility equation must be taken with signs (see, e.g., [16]).

We can think of the algebraic operations graphically as follows. The module  $A^{\otimes i}$  is denoted by drawing  $i$  parallel, downward-oriented strands. The multiplication operation  $\mu_i$  is represented by an oriented tree with one vertex,  $i$  incoming strands and one outgoing strand. With this convention, the compatibility relation with  $n$  inputs can be visualized as follows. Fix a tree  $T$  with one vertex,  $n$  incoming edges, and one outgoing one. Consider next all the trees  $S$  (with two vertices) with the property that if we contract one edge in  $S$ , we obtain  $T$ . Each such tree  $S$  represents a composition of multiplication maps. The compatibility condition states that the sum of all the maps gotten by these resolutions  $S$  vanishes. See Figure 1 for an example.

Another way to think of the compatibility conditions uses the tensor algebra

$$\mathcal{T}^*(A[1]) := \bigoplus_{n=0}^{\infty} A^{\otimes n}[n].$$

The maps  $\mu_i$  can be combined into a single map

$$\mu: \mathcal{T}^*(A[1]) \rightarrow A[2]$$

with the convention that  $\mu_0 = 0$ . We can also construct a natural degree 1 endomorphism  $\overline{D}: \mathcal{T}^*(A[1]) \rightarrow \mathcal{T}^*(A[1])$  by

$$\overline{D}(a_1 \otimes \cdots \otimes a_n) = \sum_{j=1}^n \sum_{\ell=1}^{n-j+1} a_1 \otimes \cdots \otimes \mu_j(a_\ell \otimes \cdots \otimes a_{\ell+j-1}) \otimes \cdots \otimes a_n.$$

Then the compatibility condition is that

$$\mu \circ \overline{D} = 0$$

or equivalently

$$\overline{D} \circ \overline{D} = 0.$$

This condition can also be expressed graphically as in Figure 2. We draw a shaded band to represent some collection of parallel,  $\mathcal{A}$ -colored strands, i.e., an element of

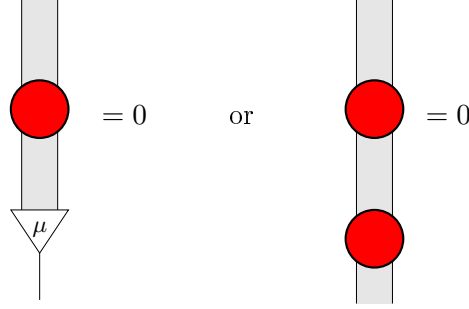


FIGURE 2. Two alternate graphical representation of the  $\mathcal{A}_\infty$  compatibility condition. The shaded bands represent terms from  $\mathcal{T}^*(\mathcal{A})$ . The triangle represents the collection of multiplication maps  $\mu$ , and the large dot represents the differential  $\overline{D}$  on  $\mathcal{T}^*(\mathcal{A})$ .

$\mathcal{T}^*(A[1])$ . A large dot on a gray strand represents  $\overline{D}$ , and a downward-pointing triangle represents  $\mu$ .

At some points later we will want to allow only finitely many nonzero  $\mu_i$ . We give this condition a name:

**Definition 2.2.** An  $\mathcal{A}_\infty$ -algebra  $(A, \{\mu_i\})$  is *operationally bounded* if  $\mu_i = 0$  for  $i$  sufficiently large.

**Definition 2.3.** A (right)  $\mathcal{A}_\infty$ -module  $\mathcal{M}$  over  $\mathcal{A}$  is a graded  $\mathbf{k}$ -module  $M$ , equipped with operations

$$m_i: M \otimes A^{\otimes(i-1)} \rightarrow M[2-i],$$

defined for all  $i \geq 1$ , satisfying the compatibility conditions

$$\begin{aligned} 0 = & \sum_{i+j=n+1} m_i(m_j(\mathbf{x} \otimes a_1 \otimes \cdots \otimes a_{j-1}) \otimes \cdots \otimes a_{n-1}) \\ & + \sum_{i+j=n+1} \sum_{\ell=1}^{n-j} m_i(\mathbf{x} \otimes a_1 \otimes \cdots \otimes a_{\ell-1} \otimes \mu_j(a_\ell \otimes \cdots \otimes a_{\ell+j-1}) \otimes \cdots \otimes a_{n-1}). \end{aligned}$$

An  $\mathcal{A}_\infty$  module  $\mathcal{M}$  over a strictly unital  $\mathcal{A}_\infty$ -algebra is said to be *strictly unital* if for any  $\mathbf{x} \in M$ ,  $m_2(\mathbf{x} \otimes 1) = \mathbf{x}$  and  $m_i(\mathbf{x} \otimes a_1 \otimes \cdots \otimes a_{i-1}) = 0$  if  $i > 2$  and some  $a_j = 1$ . The module  $\mathcal{M}$  is said to be *bounded* if  $m_i = 0$  for all sufficiently large  $i$ .

Graphically, the module  $M \otimes A^{\otimes i}$  is represented by  $i+1$  parallel strands, where the leftmost strand is colored with  $M$  (while the others are colored by  $A$ ). The multiplication map  $m_i$  is represented by the one-vertex tree with  $i+1$  incoming strands, the leftmost of which is  $M$ -colored, and whose output is also  $M$ -colored. The compatibility condition can be thought of the same as before, except now we have distinguished the leftmost strands with  $M$ . Thus, the compatibility conditions can be thought of as stating the vanishing of maps induced by sums of trees, with arbitrarily many inputs (and one output).

As before, we can also write this condition in terms of the tensor algebra, best expressed graphically. In addition to the previous conventions, a triangle against the  $M$ -strand represents the map  $m: M \otimes \mathcal{T}^*(A[1]) \rightarrow M$  obtained by collecting

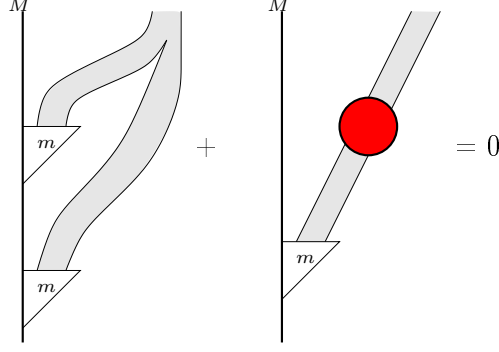


FIGURE 3. The relation of an  $\mathcal{A}_\infty$  module. In the left term, the left vertical strand represents our module  $M$ , and the shaded region represents terms from the tensor algebra of  $\mathcal{A}$ . This region is split, and some of the terms act on the module via the multiplication  $m$ ; the remaining strands are then multiplied again. This is related to the right term, where the large dot represents a differential  $\bar{D}$  on the tensor algebra, defined below.

all the  $m_i$ . We also allow merging and splitting of gray bands with their natural interpretations (tensor product and sum over all ways of dividing a clump into two, respectively). With these notational shorthands, the module compatibility condition can be conveniently illustrated as in Figure 3. One warning: splitting and then merging a band is *not* the identity: it acts on the  $A^{\otimes n}$  summand by multiplication by  $n + 2$ .

Let  $\mathcal{M}$  be a right  $\mathcal{A}_\infty$  module over  $\mathcal{A}$ . Suppose moreover that, for all  $i > 2$ ,  $m_i = 0$  and  $\mu_i = 0$ . Then not only is  $\mathcal{A}$  a differential graded algebra, but also  $\mathcal{M}$  is a module over  $\mathcal{A}$  (in the traditional sense), equipped with a differential satisfying the Leibniz rule with respect to the algebra action on the module.

**Definition 2.4.** Let  $\mathcal{M}$  and  $\mathcal{M}'$  be two right  $\mathcal{A}_\infty$  modules. A *strictly unital morphism* of  $\mathcal{A}_\infty$ -modules  $f$ , or simply an  $\mathcal{A}_\infty$ -morphism, is a collection of maps

$$f_i: M \otimes A^{\otimes(i-1)} \rightarrow M'[1-i]$$

indexed by  $i \geq 1$ , satisfying the compatibility conditions

$$\begin{aligned} 0 = & \sum_{i+j=n+1} m'_i(f_j(\mathbf{x} \otimes a_1 \otimes \cdots \otimes a_{j-1}) \otimes \cdots \otimes a_{n-1}) \\ & + \sum_{i+j=n+1} f_i(m_j(\mathbf{x} \otimes a_1 \otimes \cdots \otimes a_{j-1}) \otimes \cdots \otimes a_{n-1}) \\ & + \sum_{i+j=n+1} \sum_{\ell=1}^{n-j} f_i(\mathbf{x} \otimes a_1 \otimes \cdots \otimes a_{\ell-1} \otimes \mu_j(a_\ell \otimes \cdots \otimes a_{\ell+j-1}) \otimes \cdots \otimes a_{n-1}) \end{aligned}$$

and the unital condition

$$f_i(\mathbf{x} \otimes a_1 \otimes \cdots \otimes a_{i-1}) = 0$$

if  $i > 1$  and some  $a_j = 1$ .

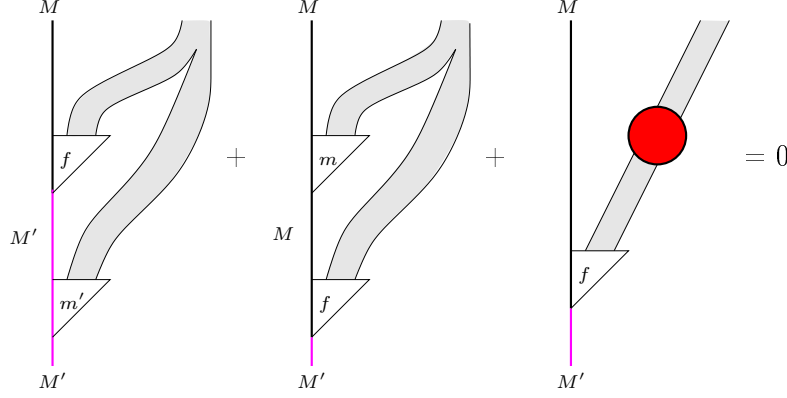


FIGURE 4. Illustration of the basic relation of an  $\mathcal{A}_\infty$  morphism of  $A$ -modules.

We call a strictly unital morphism  $\{f_i\}$  of  $\mathcal{A}_\infty$ -modules *bounded* if  $f_i = 0$  for  $i$  sufficiently large.

The compatibility condition is illustrated in Figure 4. Now the left strand can be colored by  $M$  or  $M'$ , and there are three kinds of triangles: those representing  $\mathcal{A}_\infty$ -module operations on  $\mathcal{M}$  or  $\mathcal{M}'$ , and those representing the map  $f$ .

For example, for any  $\mathcal{A}_\infty$  module  $\mathcal{M}$ , the *identity morphism*  $\mathbb{I}$  is the map with

$$\begin{aligned} \mathbb{I}_1(\mathbf{x}) &:= \mathbf{x} \\ \mathbb{I}_i(\mathbf{x} \otimes A^{\otimes i-1}) &:= 0 \quad (i > 0). \end{aligned}$$

If  $f$  is an  $\mathcal{A}_\infty$ -module morphism from  $\mathcal{M}$  to  $\mathcal{M}'$ , and  $g$  is an  $\mathcal{A}_\infty$ -module morphism from  $\mathcal{M}'$  to  $\mathcal{M}''$ , we can form their *composite morphism*  $g \circ f$ , defined by

$$(g \circ f)_n(\mathbf{x} \otimes (a_1 \otimes \cdots \otimes a_{n-1})) := \sum_{i+j=n+1} g_j(f_i(\mathbf{x} \otimes a_1 \otimes \cdots \otimes a_{i-1}) \otimes \cdots \otimes a_{n-1}).$$

**Definition 2.5.** Given any collection of maps

$$h_i: M \otimes A^{\otimes i-1} \rightarrow M'[-i]$$

with  $h_i(\mathbf{x} \otimes a_1 \otimes \cdots \otimes a_{i-1}) = 0$  if  $i > 1$  and some  $a_j = 1$ , we can construct a strictly unital  $\mathcal{A}_\infty$ -morphism  $f$  by the expression

$$\begin{aligned} f_n(\mathbf{x} \otimes a_1 \otimes \cdots \otimes a_{n-1}) &= \\ &\sum_{i+j=n+1} h_i(m_j(\mathbf{x} \otimes a_1 \otimes \cdots \otimes a_{j-1}) \otimes \cdots \otimes a_{n-1}) \\ &+ \sum_{i+j=n+1} m'_i(h_j(\mathbf{x} \otimes a_1 \otimes \cdots \otimes a_{j-1}) \otimes \cdots \otimes a_{n-1}) \\ &+ \sum_{i+j=n+1} \sum_{\ell=1}^{n-j} h_i(\mathbf{x} \otimes a_1 \otimes \cdots \otimes a_{\ell-1} \otimes \mu_j(a_\ell \otimes \cdots \otimes a_{\ell+j-1}) \otimes \cdots \otimes a_{n-1}). \end{aligned}$$

If an  $\mathcal{A}_\infty$ -morphism  $f$  can be obtained from a map  $h$  in this way, we call  $f$  *null homotopic*. Two  $\mathcal{A}_\infty$ -morphisms  $f, g: \mathcal{M} \rightarrow \mathcal{M}'$  are *homotopic* if their difference is null homotopic.

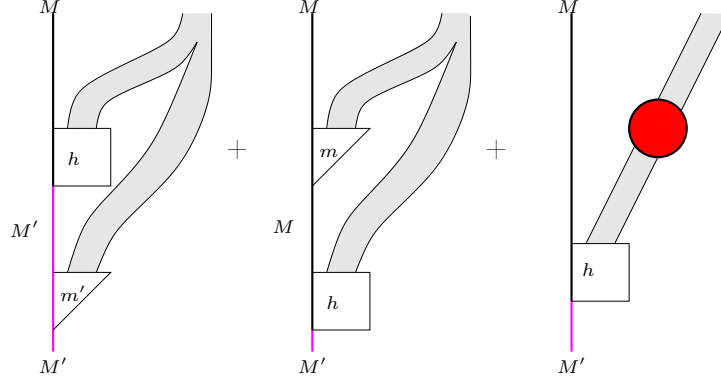


FIGURE 5. **Null homotopies.** Given any map of the form specified by the square labeled  $h$ , the above diagrams define a (null-homotopic) chain map.

Two  $\mathcal{A}_\infty$ -modules  $\mathcal{M}$  and  $\mathcal{M}'$  are  $(\mathcal{A}_\infty)$  *homotopy equivalent* if there are  $\mathcal{A}_\infty$ -module morphisms  $f$  from  $\mathcal{M}$  to  $\mathcal{M}'$  and  $g$  from  $\mathcal{M}'$  to  $\mathcal{M}$  such that  $f \circ g$  and  $g \circ f$  are homotopic to the identity.

The expression associated to a homotopy is illustrated in Figure 5, where the square represents the homotopy.

There are analogous definitions for left  $\mathcal{A}_\infty$  modules, maps, and homotopies.

**2.2.  $\mathcal{A}_\infty$  tensor products.** In the category of strictly unital modules and strictly unital morphisms, the usual naïve tensor product  $M \otimes_A N$  of a right module  $M$  and a left module  $N$  over a differential graded algebra  $A$  is not available; the quotient used to define the tensor product does not make sense. Even in the original setting, this tensor product is often not the right one: if  $M$  and  $M'$  are homotopy equivalent, the  $M \otimes_A N$  and  $M' \otimes_A N$  are not necessarily homotopy equivalent.

To produce a functor which works for  $\mathcal{A}_\infty$ -modules and respects homotopy equivalence, we pass instead to the  $\mathcal{A}_\infty$  tensor product, a generalization of the derived tensor product.

**Definition 2.6.** Let  $\mathcal{A}$  be an  $\mathcal{A}_\infty$ -algebra over  $\mathbf{k}$ ,  $\mathcal{M}$  be a right  $\mathcal{A}_\infty$ -module over  $\mathcal{A}$ , and  $\mathcal{N}$  be a left  $\mathcal{A}_\infty$ -module over  $\mathcal{A}$ . Then their  $\mathcal{A}_\infty$ -*tensor product* is the chain complex

$$\mathcal{M} \widetilde{\otimes}_{\mathcal{A}} \mathcal{N} := M \otimes T^*(A[1]) \otimes N$$

equipped with the boundary operator

$$\begin{aligned} \partial(\mathbf{x} \otimes a_1 \otimes \cdots \otimes a_n \otimes \mathbf{y}) := & \\ & \sum_{i=1}^{n+1} m_i(\mathbf{x} \otimes a_1 \otimes \cdots \otimes a_{i-1}) \otimes \cdots \otimes a_n \otimes \mathbf{y} \\ & + \sum_{i=1}^n \sum_{\ell=1}^{n-i+1} \mathbf{x} \otimes a_1 \otimes \cdots \otimes \mu_i(a_\ell \otimes \cdots \otimes a_{\ell+i-1}) \otimes \cdots \otimes a_n \otimes \mathbf{y} \\ & + \sum_{i=1}^{n+1} \mathbf{x} \otimes a_1 \otimes \cdots \otimes m_i(a_{n-i+2} \otimes \cdots \otimes a_n \otimes \mathbf{y}). \end{aligned}$$

**Lemma 2.7.** *The operator  $\partial$  on  $\mathcal{M} \tilde{\otimes} \mathcal{N}$  defined above satisfies  $\partial^2 = 0$ .*

*Proof.* This is a straightforward consequence of the relations defining  $\mathcal{A}_\infty$  algebras and modules.  $\square$

An important case is when  $\mathcal{N} = \mathcal{A}$ . The resulting complex is called the *bar resolution*  $\overline{\mathcal{M}}$  of  $\mathcal{M}$ . As a  $\mathbf{k}$ -module it is

$$\overline{\mathcal{M}} \cong M \otimes \mathcal{T}^*(A[1]) \otimes A \cong (M \otimes \mathcal{T}^+(A[1]))[-1]$$

where  $\mathcal{T}^+(A[1]) := \bigoplus_{i=1}^\infty (A[1])^{\otimes i}$  is  $\mathcal{T}^*(A[1])$  without the first summand of  $\mathbf{k}$ .  $\overline{\mathcal{M}}$  has the further structure of a right  $\mathcal{A}_\infty$  module over  $\mathcal{A}$ , with multiplications  $m_1 = \partial$  as defined above and, for  $i \geq 2$ ,

$$\begin{aligned} m_i((\mathbf{x} \otimes a_1 \otimes \cdots \otimes a_n) \otimes b_1 \otimes \cdots \otimes b_{i-1}) := & \\ & \sum_{\ell=1}^n \mathbf{x} \otimes a_1 \otimes \cdots \otimes a_{n-\ell} \otimes \mu_{i+\ell-1}(a_{n-\ell+1} \otimes \cdots \otimes a_n \otimes b_1 \otimes \cdots \otimes b_{i-1}). \end{aligned}$$

Note the range of summation: the multiplication  $\mu_{i+\ell-1}$  is applied to *all* of the  $b_j$ 's and *at least one*  $a_j$ .

The following lemma is again straightforward and standard.

**Lemma 2.8.**  *$\overline{\mathcal{M}}$  is an  $\mathcal{A}_\infty$  module over  $\mathcal{A}$ .*

Notice that if  $\mathcal{A}$  is an honest differential graded algebra then  $\overline{\mathcal{M}}$  is an honest differential graded module, i.e., all of the higher  $m_i$  vanish. Furthermore, if  $\mathcal{M}$  is also an honest differential graded module, this is the usual bar resolution. If instead  $\mathcal{N}$  is an honest differential graded module, then we could also define  $\mathcal{M} \tilde{\otimes} \mathcal{N}$  as  $\overline{\mathcal{M}} \otimes_{\mathcal{A}} \mathcal{N}$ , the naïve tensor product with the bar resolution, but this is not possible in general.

*Remark 2.9.* One way to understand the products on  $\overline{\mathcal{M}}$  and Lemma 2.8 is via  $\mathcal{A}_\infty$ -bimodules. If  $\mathcal{A}$  and  $\mathcal{B}$  are two  $\mathcal{A}_\infty$ -algebras, an  $\mathcal{A}$ - $\mathcal{B}$  bimodule  $\mathcal{N}$  is a  $\mathbf{k}$ -module  $N$  with maps

$$m_{i,j}: A^{\otimes i} \otimes N \otimes B^{\otimes j} \rightarrow N$$

satisfying a natural version of the compatibility conditions. In particular,  $\mathcal{A}$  is an  $\mathcal{A}$ - $\mathcal{A}$  bimodule in a natural way. A more general version of Lemma 2.8 is that if  $\mathcal{N}$  is an  $\mathcal{A}$ - $\mathcal{B}$  bimodule, then  $\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}$  is a right  $\mathcal{B}$  module.

**Proposition 2.10.** *If  $\mathcal{A}$  and  $\mathcal{M}$  are strictly unital, then  $\overline{\mathcal{M}}$  is homotopy equivalent to  $\mathcal{M}$ .*

*Proof.* We define an  $\mathcal{A}_\infty$ -module map  $\phi$  from  $\overline{\mathcal{M}}$  to  $\mathcal{M}$  by

$$\phi_i((\mathbf{x} \otimes a_1 \otimes \cdots \otimes a_n) \otimes b_1 \otimes \cdots \otimes b_{i-1}) := m_{i+n}(\mathbf{x} \otimes a_1 \otimes \cdots \otimes a_n \otimes b_1 \otimes \cdots \otimes b_{i-1}).$$

Similarly, we define an  $\mathcal{A}_\infty$  module map  $\psi$  from  $\mathcal{M}$  to  $\overline{\mathcal{M}}$  by

$$\psi_i(\mathbf{x} \otimes a_1 \otimes \cdots \otimes a_{i-1}) := \mathbf{x} \otimes a_1 \otimes \cdots \otimes a_{i-1} \otimes 1.$$

Also define maps for a homotopy  $h$  from  $\overline{\mathcal{M}}$  to  $\overline{\mathcal{M}}$  by

$$h_i((\mathbf{x} \otimes a_1 \otimes \cdots \otimes a_n) \otimes b_1 \otimes \cdots \otimes b_{i-1}) := \mathbf{x} \otimes a_1 \otimes \cdots \otimes a_n \otimes b_1 \otimes \cdots \otimes b_{i-1} \otimes 1.$$

It is straightforward to verify that  $\phi \circ \psi$  is the identity map, while  $\psi \circ \phi$  is homotopic, via  $h$ , to the identity map.  $\square$

**Proposition 2.11.** *An  $\mathcal{A}_\infty$  map  $f: \mathcal{M} \rightarrow \mathcal{M}'$  induces a chain map  $f \tilde{\otimes} \mathbb{I}: \mathcal{M} \tilde{\otimes} \mathcal{N} \rightarrow \mathcal{M}' \tilde{\otimes} \mathcal{N}$ . If  $f$  is null homotopic, so is  $f \tilde{\otimes} \mathbb{I}_{\mathcal{N}}$ . In particular, if  $\mathcal{M}_1$  and  $\mathcal{M}_2$  are homotopy equivalent right  $\mathcal{A}_\infty$  modules and  $\mathcal{N}_1$  and  $\mathcal{N}_2$  are homotopy equivalent left  $\mathcal{A}_\infty$  modules, then  $\mathcal{M}_1 \tilde{\otimes} \mathcal{N}_1$  is homotopy equivalent to  $\mathcal{M}_2 \tilde{\otimes} \mathcal{N}_2$ .*

*Proof.* For  $f$  a collection of maps  $f_i: M \otimes A^{\otimes(i-1)} \rightarrow M[c-i]$  for a fixed  $c \in \{0, 1\}$ , define  $f \tilde{\otimes} \mathbb{I}_{\mathcal{N}}$  by

$$(f \tilde{\otimes} \mathbb{I}_{\mathcal{N}})(\mathbf{x} \otimes a_1 \otimes \cdots \otimes a_n \otimes \mathbf{y}) := \sum_{i=1}^{n+1} f_i(\mathbf{x} \otimes a_1 \cdots \otimes a_{i-1}) \otimes a_i \otimes \cdots \otimes a_n \otimes \mathbf{y}.$$

If  $f$  is an  $\mathcal{A}_\infty$  map, so is  $f \tilde{\otimes} \mathbb{I}_{\mathcal{N}}$ , and if  $h$  is a null homotopy of  $f$ , then  $h \tilde{\otimes} \mathcal{N}$  is a null homotopy of  $f \tilde{\otimes} \mathbb{I}_{\mathcal{N}}$ . The same construction works, *mutatis mutandis*, for chain maps and homotopies on the right.  $\square$

**2.3. Another model for the  $\mathcal{A}_\infty$  tensor product.** The derived tensor product for ordinary differential graded algebras and modules agrees with the naïve tensor product when one of the two modules is projective. In the context of  $\mathcal{A}_\infty$  structures, although there is no general notion of naïve tensor product, for  $\mathcal{A}_\infty$  modules of a certain type, we can nonetheless define a new kind of tensor product  $\boxtimes$  which, when defined, agrees with the derived tensor product  $\tilde{\otimes}$ . As we will see in Section 9, this new tensor product matches better with the geometry and helps give a proof of the pairing theorem.

We first specialize to the case where  $\mathcal{A}$  is a differential graded algebra, which is all that we need in the present paper.

**Definition 2.12.** Let  $N$  be a graded  $\mathbf{k}$ -module, equipped with a map

$$\delta_1: N \rightarrow (A \otimes N)[1],$$

satisfying the compatibility condition that

$$(\mu_2 \otimes \mathbb{I}_N) \circ (\mathbb{I}_A \otimes \delta_1) \circ \delta_1 + (\mu_1 \otimes \mathbb{I}_N) \circ \delta_1: N \rightarrow A \otimes N$$

vanishes. We call the pair  $(N, \delta_1)$  a *type D structure over  $A$  with base ring  $\mathbf{k}$* . Let  $(N, \delta_1)$  and  $(N', \delta'_1)$  be two type D structures. A  $\mathbf{k}$ -module map  $\psi_1: N \rightarrow A \otimes N'$  is a *D-structure morphism* if

$$(\mu_2 \otimes \mathbb{I}_{N'}) \circ (\mathbb{I}_A \otimes \psi_1) \circ \delta_1 + (\mu_2 \otimes \mathbb{I}_{N'}) \circ (\mathbb{I}_A \otimes \delta'_1) \circ \psi_1 + (\mu_1 \otimes \mathbb{I}_{N'}) \circ \psi_1 = 0.$$

A homotopy  $h$  between two  $D$ -structure morphisms  $\psi_1$  and  $\phi_1$  from  $(N, \delta_1)$  to  $(N', \delta'_1)$  is a  $\mathbf{k}$ -module map  $h: N \rightarrow (A \otimes N')[-1]$  with

$$(\mu_2 \otimes \mathbb{I}_{N'}) \circ (\mathbb{I}_A \otimes h) \circ \delta_1 + (\mu_2 \otimes \mathbb{I}_{N'}) \circ (\mathbb{I}_A \otimes \delta'_1) \circ h + (\mu_1 \otimes \mathbb{I}_{N'}) \circ h = \psi_1 - \phi_1.$$

**Lemma 2.13.** *If  $(N, \delta)$  is a type  $D$  structure, then  $A \otimes N$  can be given the structure of a left  $A$  module, with*

$$\begin{aligned} m_1(a \otimes \mathbf{y}) &:= [(\mu_2 \otimes \mathbb{I}_N) \circ (\mathbb{I}_A \otimes \delta_1) + \mu_1 \otimes \mathbb{I}_N](a \otimes \mathbf{y}) \\ m_2(a_1 \otimes (a \otimes \mathbf{y})) &:= \mu_2(a_1 \otimes a) \otimes \mathbf{y}. \end{aligned}$$

Moreover, if  $\psi_1: N \rightarrow A \otimes N'$  is a  $D$ -structure morphism, then there is an induced chain map  $f$  from  $A \otimes N$  to  $A \otimes N'$ , defined by

$$f(a \otimes \mathbf{y}) := (m_2 \otimes \mathbb{I}_{N'}) \circ (\mathbb{I}_A \otimes \psi_1)$$

Similarly, a homotopy between two  $D$ -structure morphisms induces a chain homotopy between the associated chain maps.

*Proof.* The proof is straightforward. □

Given a type  $D$  structure  $(N, \delta)$  we denote the module from Lemma 2.13 by  $\mathcal{N}$ . (This conflicts slightly with earlier conventions, since here the underlying  $\mathbf{k}$ -module of  $\mathcal{N}$  is  $A \otimes N$  rather than just  $N$ .)

For completeness, we now define a type  $D$  structure over a general  $\mathcal{A}_\infty$  algebra  $\mathcal{A}$ ; although this is not necessary for this paper, the same constructions are used for the tensor product we define below. A map  $\delta_1: N \rightarrow (A \otimes N)[1]$  can naturally be used to construct maps

$$\delta_k: N \rightarrow (A^{\otimes k} \otimes N)[k]$$

inductively by

$$\begin{aligned} \delta_0 &= \mathbb{I}_N \\ \delta_i &= (\mathbb{I}_{A^{\otimes(i-1)}} \otimes \delta_1) \circ \delta_{i-1}. \end{aligned}$$

By construction, this maps satisfies the basic equation

$$(\mathbb{I}_{A^{\otimes j}} \otimes \delta_i) \circ \delta_j = \delta_{i+j}$$

for all  $i, j \geq 0$ , as shown on the top of Figure 6. Here we represent the collection maps  $\delta_i$  as a triangle with one  $N$ -input and outputs a band of  $A$ -colored strand and an  $N$ -strand.

**Definition 2.14.** A map  $\delta_1$  as above is *bounded* if for all  $\mathbf{x} \in N$ , there is an integer  $n$  so that for all  $i \geq n$ ,  $\delta_i(\mathbf{x}) = 0$ .

For a bounded map  $\delta_1$ , we can think of the  $\delta_k$  as fitting together to form a map

$$\begin{aligned} \delta: N &\rightarrow \mathcal{T}^*(A[1]) \otimes N, \\ \delta(\mathbf{x}) &:= \sum_{k=0}^{\infty} \delta_k(\mathbf{x}). \end{aligned}$$

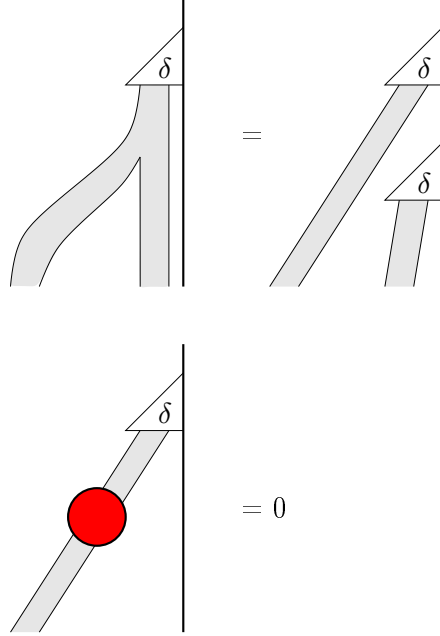


FIGURE 6. **Type  $D$  structures.** Relations between the  $\delta_k$  associated to a type  $D$  structure.

If  $\delta_1$  is not bounded then, instead, the  $\delta_k$  fit together to form a map

$$\delta: N \rightarrow \overline{\mathcal{T}}^*(A[1]) \otimes N = \prod_{i=0}^{\infty} (A[1])^{\otimes i},$$

$$\delta(\mathbf{x}) := \prod_{k=0}^{\infty} \delta_k(\mathbf{x}).$$

Here  $\overline{\mathcal{T}}^*(V)$  denotes the completion of  $\mathcal{T}^*(V)$  with respect to the number of tensor factors.

We now generalize Definition 2.12.

**Definition 2.15.** Fix an  $\mathcal{A}_{\infty}$  algebra  $\mathcal{A}$  and a pair  $(N, \delta_1)$  as above. Assume that either  $(N, \delta_1)$  is bounded or  $\mathcal{A}$  is operationally bounded. Then we say  $(N, \delta_1)$  defines a type  $D$  structure if

$$(m \otimes \mathbb{I}_N) \circ \delta = 0$$

or equivalently

$$(\overline{D} \otimes \mathbb{I}_N) \circ \delta = 0$$

as shown on the bottom of Figure 6. For two type  $D$  structures  $(N, \delta_1)$  and  $(N', \delta'_1)$  a map  $\psi_1: N \rightarrow A \otimes N'$  can be extended to maps

$$\psi_k: N \rightarrow (A^{\otimes k} \otimes N')[k-1]$$

$$\psi_k(\mathbf{x}) := \sum_{i+j=k-1} (\mathbb{I}_{A^{\otimes(i+1)}} \otimes \delta'_j) \circ (\mathbb{I}_{A^{\otimes i}} \otimes \psi) \circ \delta_i.$$

The map  $\psi_1$  is *bounded* if  $\psi_k = 0$  for  $k$  sufficiently large. We can define

$$\psi(\mathbf{x}) := \sum_{k=1}^{\infty} \psi_k(\mathbf{x}),$$

which maps to  $\mathcal{T}^*(A[1]) \otimes N$  if  $\psi_1$  is bounded and  $\overline{\mathcal{T}}^*(A[1]) \otimes N$  otherwise. Assuming either  $\psi_1$  is bounded or  $\mathcal{A}$  is operationally bounded, we say  $\psi$  is a *D-structure morphism* if

$$(\mu \otimes \mathbb{I}_{N'}) \circ \psi = 0.$$

Similarly,  $\psi$  is *homotopic* to  $\phi$  if

$$(\mu \otimes \mathbb{I}_{N'}) \circ h = \psi - \phi$$

for  $h$  constructed from a map  $h_1: N \rightarrow (A \otimes N')[-1]$  in the analogous way (and with  $h_1$  bounded if  $\mathcal{A}$  is not operationally bounded).

Note that if  $(N, \delta_1)$  and  $(N', \delta'_1)$  are both bounded then any morphism  $\psi_1: N \rightarrow A \otimes N'$  between them is automatically bounded.

For definiteness, we will restrict from now on to the case that  $(A, \{\mu_i\})$  is operationally bounded. Note that this includes the case of *dg* algebras.

There is a generalization of Lemma 2.13.

**Lemma 2.16.** *If  $(N, \delta)$  is a type D structure, then  $A \otimes N$  can be given the structure of a left  $\mathcal{A}_\infty$  module  $\mathcal{N}$ , with*

$$m_i := \sum_{k=0}^{\infty} (\mu_{i+k} \otimes \mathbb{I}_N) \circ (\mathbb{I}_A \otimes \delta_k).$$

Moreover, if  $\psi_1: N \rightarrow A \otimes N'$  is a D-structure morphism, then there is an induced  $\mathcal{A}_\infty$  morphism  $f$  from  $\mathcal{N}$  to  $\mathcal{N}'$ , with

$$f_i := \sum_{k=1}^{\infty} (\mu_{i+k+1} \otimes \mathbb{I}_N) \circ \psi_k.$$

Similarly, a homotopy between two D-structure morphisms induces a homotopy between the associated  $\mathcal{A}_\infty$  maps.

**Definition 2.17.** Let  $\mathcal{M}$  be a right  $\mathcal{A}_\infty$  module over  $\mathcal{A}$  and  $(N, \delta_1)$  a type D structure. Suppose moreover that either  $\mathcal{M}$  is a bounded  $\mathcal{A}_\infty$  module or  $(N, \delta_1)$  is a bounded type D structure. Then, we can form the  $\mathbf{k}$ -module  $M \boxtimes N = M \otimes_{\mathbf{k}} N$ , equipped with the endomorphism

$$\partial^{\boxtimes}(\mathbf{x} \otimes \mathbf{y}) = \sum_{k=0}^{\infty} (m_{k+1} \otimes \mathbb{I}_N)(\mathbf{x} \otimes \delta_k(\mathbf{y})).$$

See Figure 7 for an illustration of  $\partial^{\boxtimes}$ .

**Lemma 2.18.** *Let  $M$  be an  $\mathcal{A}_\infty$  module and  $N$  a type D structure. Assume that either  $M$  or  $N$  is bounded. Then the  $\mathbf{k}$ -module  $M \otimes N$  equipped with the endomorphism  $\partial^{\boxtimes}$  is a chain complex.*

*Proof.* See Figure 8. □

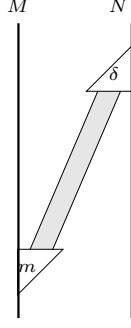
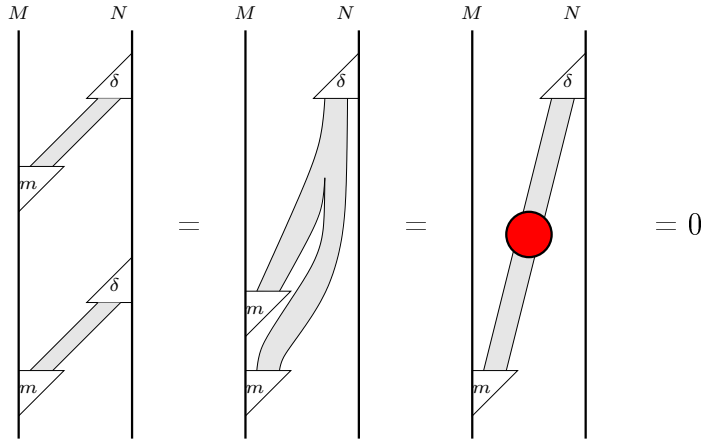
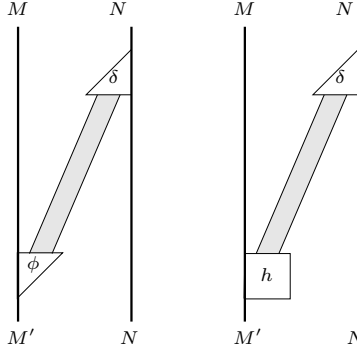


FIGURE 7. Definition of the boundary operator.

FIGURE 8. The proof that  $\partial^{\boxtimes} \circ \partial^{\boxtimes} = 0$  is illustrated above, using the definition of  $\partial^{\boxtimes}$  from Figure 7, along with the relations from Figures 3 and 6.

**Lemma 2.19.** *Let  $\phi: \mathcal{M} \rightarrow \mathcal{M}'$  be a morphism of  $\mathcal{A}_\infty$  modules and  $\psi: (N, \delta_1) \rightarrow (N', \delta'_1)$  a morphism of type D structures. Then:*

- *If either  $\mathcal{M}$ ,  $\mathcal{M}'$  and  $\phi$  are bounded or  $\delta_1$  is bounded then  $\phi$  induces a chain map  $\phi \boxtimes \mathbb{I}_N: \mathcal{M} \boxtimes N \rightarrow \mathcal{M}' \boxtimes N$ .*
- *If either  $\delta_1$  and  $\delta'_1$  are bounded or  $\mathcal{M}$  is bounded then  $\psi$  induces a chain map  $\mathbb{I}_M \boxtimes \psi: \mathcal{M} \boxtimes N \rightarrow \mathcal{M} \boxtimes N'$ .*
- *Suppose  $\phi': \mathcal{M} \rightarrow \mathcal{M}'$  is another morphism of  $\mathcal{A}_\infty$  modules which is homotopic to  $\phi$ . Suppose further that either  $\delta_1$  is bounded or  $\phi$ ,  $\phi'$  and the homotopy between them are all bounded. Then  $\phi \boxtimes \mathbb{I}_N$  is homotopic to  $\phi' \boxtimes \mathbb{I}_N$ .*
- *Suppose  $\psi': (N, \delta_1) \rightarrow (N', \delta'_1)$  is another morphism of type D structures which is homotopic to  $\psi$ . Suppose further that either  $\mathcal{M}$  is bounded or  $\psi$ ,  $\psi'$  and the homotopy between them are all bounded. Then  $\mathbb{I}_M \boxtimes \psi$  and  $\mathbb{I}_M \boxtimes \psi'$  are homotopic.*
- *Under boundedness conditions so that all of the  $\boxtimes$ -products are defined, given a third map  $\phi': \mathcal{M}' \rightarrow \mathcal{M}''$ ,  $(\phi' \circ \phi) \boxtimes \mathbb{I}_N$  is homotopic to  $(\phi' \boxtimes \mathbb{I}_N) \circ (\phi \boxtimes \mathbb{I}_N)$ .*

FIGURE 9. A graphical representation of  $\phi \boxtimes \mathbb{I}_N$  and  $h \boxtimes \mathbb{I}_N$ .

- Under boundedness conditions so that all of the  $\boxtimes$ -products are defined, given a third map  $\psi': (N', \delta'_1) \rightarrow (N'', \delta''_1)$ ,  $\mathbb{I}_M \boxtimes (\psi' \circ \psi)$  is homotopic to  $(\mathbb{I}_M \boxtimes \psi') \circ (\mathbb{I}_M \boxtimes \psi)$ .
- Under boundedness conditions so that all of the  $\boxtimes$ -products are defined,  $(\phi \boxtimes \mathbb{I}_{N'}) \circ (\mathbb{I}_M \boxtimes \psi)$  is homotopic to  $(\mathbb{I}_M \boxtimes \psi) \circ (\phi \boxtimes \mathbb{I}_N)$ .

*Proof.* Given  $\phi$ , we define the induced map  $\Phi \otimes \mathbb{I}_N$  by

$$(\phi \otimes \mathbb{I}_N)(\mathbf{x} \otimes \mathbf{y}) := \sum_{k=0}^{\infty} (\phi_{k+1} \otimes \mathbb{I}_N) \circ (\mathbf{x} \otimes \delta_k(\mathbf{y})),$$

where  $\phi_k$  are the components of  $\phi$ . Similarly, given a homotopy  $h$  with components  $h_k$  from  $\phi$  to  $\phi'$ , we define a homotopy  $h \boxtimes \mathbb{I}_N$  from  $\phi \boxtimes \mathbb{I}_N$  to  $\phi' \boxtimes \mathbb{I}_N$  by

$$(h \boxtimes \mathbb{I}_N)(\mathbf{x} \otimes \mathbf{y}) = \sum_{k=0}^{\infty} (h_{k+1} \otimes \mathbb{I}_N) \circ (\mathbf{x} \otimes \delta_k(\mathbf{y})).$$

See Figure 9 for an illustration.

Conversely, given  $\psi$ , we define  $\mathbb{I}_M \boxtimes \psi$  by

$$(\mathbb{I}_M \otimes \psi)(\mathbf{x} \otimes \mathbf{y}) = \sum_{k=0}^{\infty} (m_{k+1} \otimes \mathbb{I}_{N'}) \circ (\mathbf{x} \otimes \psi_k(\mathbf{y})),$$

with  $\psi_k$  constructed from the map  $\psi_1: N \rightarrow A \otimes N'$  as in Definition 2.15, and similarly for a homotopy from  $\psi$  to  $\psi'$ .

Verification of the properties of these maps is a straightforward modification of the argument that  $\partial^{\boxtimes} \circ \partial^{\boxtimes} = 0$ .  $\square$

**Proposition 2.20.** *Let  $\mathcal{M}$  be an  $\mathcal{A}_{\infty}$  module over an operationally bounded  $\mathcal{A}_{\infty}$  algebra  $\mathcal{A}$  and  $(N, \delta_1)$  be a type D structure. Suppose that either  $\mathcal{M}$  is a bounded  $\mathcal{A}_{\infty}$  module or  $\delta_1$  is bounded. In the former case, suppose moreover that  $(N, \delta_1)$  is homotopy equivalent to a bounded type D structure. Then the product  $M \boxtimes N$  is quasi-isomorphic to the  $\mathcal{A}_{\infty}$  tensor product  $\mathcal{M} \hat{\otimes} \mathcal{N}$ .*

*Proof.* The boundedness assumption implies that  $M \boxtimes N$  is defined. Moreover, since  $\boxtimes$  respects homotopy equivalences (by Lemma 2.19), we may assume that  $\mathcal{N}$  is bounded. By Proposition 2.10,  $\mathcal{M}$  is  $\mathcal{A}_{\infty}$ -homotopy equivalent to  $\overline{\mathcal{M}}$ , and hence by Lemma 2.19,

$\mathcal{M} \boxtimes \mathcal{N} \simeq \overline{\mathcal{M}} \boxtimes \mathcal{N}$ . But this is the same complex with the same boundary operator as  $\mathcal{M} \widetilde{\otimes} \mathcal{N}$ . For instance, as  $\mathbf{k}$ -modules we have

$$\overline{\mathcal{M}} \boxtimes \mathcal{N} \cong (M \otimes \mathcal{T}^+(A[1])[-1] \otimes N \cong M \otimes \mathcal{T}^*(A[1]) \otimes (A \otimes N) \cong \mathcal{M} \widetilde{\otimes} \mathcal{N}.$$

(Recall that the underlying space of  $\mathcal{N}$  is  $A \otimes N$ . Also, the  $\mathcal{A}_\infty$ -homotopy equivalence between  $\mathcal{M}$  and  $\overline{\mathcal{M}}$  is not bounded, so we need the boundedness assumption on  $\mathcal{N}$ .)  $\square$

*Remark 2.21.* Suppose that the ground ring  $\mathbf{k}$  is a field, and  $a_1, \dots, a_n$  are a basis for  $A$  over  $\mathbf{k}$ . We write

$$\delta_1 = \sum_i a_i \otimes D_i,$$

where  $D_i: N \rightarrow N$ . If  $M$  is a right  $\mathcal{A}_\infty$  module over  $A$  then we can write the differential  $\partial^\boxtimes$  more explicitly as follows:

$$(2.22) \quad \partial^\boxtimes(\mathbf{x} \otimes \mathbf{y}) = \sum m_{k+1}(\mathbf{x}, a_{i_1}, \dots, a_{i_k}) D_{i_k} \circ \dots \circ D_{i_1}(\mathbf{y}),$$

where here the sum is taken over all  $k$ -element sequences  $i_1, \dots, i_k$  of elements in  $\{1, \dots, n\}$  (including the empty sequence with  $k = 0$ ).

**2.4. Gradings by non-commutative groups.** In earlier sections, we considered the familiar case of  $\mathbb{Z}$ -graded modules. Here we record basic definitions of  $\mathcal{A}_\infty$  algebras (including differential algebras) that are graded over a possibly non-commutative group  $G$ . The definitions are straightforward generalizations of standard definitions. As before, we assume that our ground ring  $\mathbf{k}$  has characteristic 2 so that we need not worry about signs. In general, we need an absolute  $\mathbb{Z}/2$  grading to appropriately add signs, i.e., we need a homomorphism from  $G$  to  $\mathbb{Z}/2$  taking the distinguished central element  $\lambda$  to 1, but we will not further consider this here.

We start with ordinary differential graded algebras.

**Definition 2.23.** Let  $(G, \lambda)$  be a pair of a group  $G$  (written multiplicatively) with a distinguished element  $\lambda$  in the center of  $G$ . A *differential algebra graded by  $(G, \lambda)$*  is, first of all, a differential algebra  $A$  with a grading  $\text{gr}$  of  $A$  by  $G$  (as a set), i.e., a decomposition  $A = \bigoplus_{g \in G} A_g$ . We say an element  $a \in A_g$  is *homogeneous of degree  $g$*  and write  $\text{gr}(a) = g$ . For homogeneous elements  $a$  and  $b$ , we further require the grading to be

- compatible with the product, i.e.,  $\text{gr}(a \cdot b) = \text{gr}(a) \text{gr}(b)$ , and
- compatible with the differential, i.e.,  $\text{gr}(\partial a) = \lambda^{-1} \text{gr}(a)$ .

For an ordinary  $\mathbb{Z}$ -grading, we would take  $(G, \lambda) = (\mathbb{Z}, 1)$ . (We choose the differential to lower the grading to end up with homology rather than cohomology.) We require  $\lambda$  to be central because otherwise the identity

$$\partial(ab) = \partial a \cdot b + a \cdot \partial b$$

would not be homogeneous in general. The map  $k \mapsto \lambda^k$  is a homomorphism from  $G$  to  $\mathbb{Z}$ , but even if  $\lambda$  is of infinite order in  $G$ , there may not be a map to  $\mathbb{Z}$  taking  $\lambda$  to 1, so we do not in general get an ordinary  $\mathbb{Z}$ -grading.

There is a natural extension of this definition to  $\mathcal{A}_\infty$  algebras. More generally, if  $V$  and  $W$  are  $G$ -graded  $\mathbf{k}$ -modules, we can turn  $V \otimes W$  into a  $G$ -graded  $\mathbf{k}$ -module by

setting

$$(2.24) \quad \text{gr}(v \otimes w) := \text{gr}(v) \text{gr}(w)$$

for homogeneous elements  $v$  and  $w$  of  $V$  and  $W$ , respectively. If  $V$  is a  $G$ -graded module and  $n \in \mathbb{Z}$ , define  $V[n]$  to be the graded module with

$$(2.25) \quad V[n]_g := V_{\lambda^{-n}g}.$$

Since  $\lambda$  is central, we have  $V[n] \otimes W \cong (V \otimes W)[n]$ , etc. Now the previous definitions of the multiplications map  $m_i$  and their compatibility conditions carry over as written.

We now turn to differential modules.

**Definition 2.26.** Let  $A$  be a differential algebra graded by  $G$ , as in Definition 2.23. Let  $S$  be a set with a right  $G$  action. A *right differential  $A$ -module graded by  $S$* , or, simply, a right  $S$ -graded module, is a right differential  $A$ -module  $M$  with a grading  $\text{gr}$  of  $M$  by  $S$  (as a set) so that, for homogeneous elements  $a, b \in A$  and  $x, y \in M$ ,

- $\text{gr}(ax) = \text{gr}(a) \text{gr}(x)$  and
- $\text{gr}(\partial x) = \lambda^{-1} \text{gr}(x)$ .

If the action of  $G$  on  $S$  is transitive,  $S$  is determined by the stabilizer  $G_s$  of any point  $s \in S$ , and we can identify  $S$  with  $G_s \backslash G$ , the space of right cosets of  $G_s$ .

More generally, if  $V$  is graded by  $S$  and  $W$  is graded by  $G$ , then Equation (2.24) gives a natural grading of  $V \otimes W$  by  $S$  and Equation (2.25) defines a shift functor; then the previous definitions for  $\mathcal{A}_\infty$  modules, morphisms, and homotopies carry through. (For morphisms, the two modules should be graded by the same set.)

Left modules are defined similarly, but with gradings by a set  $T$  with a left action by  $G$ .

Finally we turn to tensor products.

**Definition 2.27.** Let  $A$  be a differential algebra graded by  $(G, \lambda)$ , let  $M$  be a right  $A$ -module graded by a right  $G$ -set  $S$ , and let  $N$  be a left  $A$ -module graded by a left  $G$ -set  $T$ . Define  $S \times_G T$  by

$$S \times_G T := (S \times T) / \{ (s, gt) \sim (sg, t) \mid s \in S, t \in T, g \in G \}.$$

Define a  $(S \times_G T)$  grading on  $M \otimes_{\mathbf{k}} N$  by

$$(2.28) \quad \text{gr}(m \otimes n) := [\text{gr}(m) \times \text{gr}(n)].$$

By inspection, this descends to the naïve tensor product

$$M \otimes_A N := M \otimes_{\mathbf{k}} N / \langle m \otimes an - ma \otimes n \mid m \in M, n \in N, a \in A \rangle.$$

Furthermore, there is a natural action of  $\lambda$  (or any element in the center of  $G$ ) on  $S \times_G T$  via

$$\lambda \cdot [s \times t] := [s\lambda \times t] = [s \times \lambda t].$$

(This definition fails to descend to the quotient for non-central elements in place of  $\lambda$ .) Again by inspection, the boundary operator

$$\partial(m \otimes_A n) := (\partial m) \otimes_A n + m \otimes_A (\partial n)$$

acts by  $\lambda^{-1}$  on the gradings. Thus  $M \otimes_A N$  is a chain complex with a grading by  $S \times_G T$ , a set with a  $\mathbb{Z}$ -action.

Using Equation (2.28) and the same shift functor as before, we can extend the definitions of both versions of the tensor products and chain maps and homotopies to this setting as well.

We now see a genuinely new feature of DGAs where the grading is non-commutative: even if  $\lambda$  has infinite order in  $G$ ,  $S$ , and  $T$ , it may not have infinite order on  $S \times_G T$ . We illustrate this with a simple, and pertinent, example. Let  $H$  be a variant of the Heisenberg group:

$$H = \langle \alpha, \beta, \lambda \mid \alpha\beta = \lambda^2\beta\alpha, \lambda \text{ is central} \rangle.$$

(The usual Heisenberg group over the integers has  $\lambda$  instead of  $\lambda^2$  in the first relation; with the exponent of 2, the group supports a  $\mathbb{Z}/2$  grading.) Let  $L$  be the subgroup of  $H$  generated by  $\alpha$ . Then  $\lambda$  has infinite order on  $H$  and on  $H/L$  and  $L \backslash H$ , the spaces of left and right cosets of  $L$ . However, the reduced product is

$$\begin{aligned} (L \backslash H) \times_H (H/L) &\simeq L \backslash H/L \\ &= \{ L\beta^k\lambda^l L \mid k, l \in \mathbb{Z} \}. \end{aligned}$$

These double cosets are not all distinct. In particular,

$$\begin{aligned} L\beta^k\lambda^l L &= L\alpha\beta^k\lambda^l L \\ &= L\beta^k\lambda^{l+2k}\alpha L \\ &= L\beta^k\lambda^{l+2k} L. \end{aligned}$$

Note that equalities like this generate all equalities between the double cosets. Thus the order of  $\lambda$  on  $L\beta^k\lambda^l L$  is  $2k$  for  $k \neq 0$ . This example is exactly what we see when we do 0-surgery on a knot in  $S^3$ .

Observe that one can extend the non-commutative gradings to  $\mathcal{A}_\infty$  algebras and modules, by interpreting the shift operator  $M[i]$  as multiplication by  $\lambda^{-i}$  on the grading set.

### 3. THE ALGEBRA ASSOCIATED TO A SURFACE

**3.1. The strands algebra  $\mathcal{A}(n, k)$ .** We now define the algebra associated to a parametrized boundary of a 3-manifold. We start by defining algebras  $\mathcal{A}(n, k)$ , which are simplified versions of the algebras we will use. The actual algebra we use is an appropriate subalgebra of  $\mathcal{A}(n, k)$ . We give three different descriptions of the same algebra, which will be useful for different purposes.

Throughout, we work over the field with two elements  $\mathbb{F}_2$ .

**3.1.1. Algebraic definition.** For non-negative integers  $n$  and  $k$  with  $n \geq k$ , let  $\mathcal{A}(n, k)$  be the  $\mathbb{F}_2$ -vector space generated by partial functions  $(S, T, \phi)$ , where  $S$  and  $T$  are  $k$ -element subsets of the set  $\{1, \dots, n\}$  (hereafter denoted  $[n]$ ) and  $\phi$  is a bijection from  $S$  to  $T$ , with  $i \leq \phi(i)$  for all  $i \in S$ . For brevity, we will also adopt a 2-line notation for these generators, so that, for instance,  $\langle \begin{smallmatrix} 1 & 2 & 4 \\ 5 & 3 & 4 \end{smallmatrix} \rangle$  denotes the algebra element  $(\{1, 2, 4\}, \{3, 4, 5\}, \{1 \mapsto 5, 2 \mapsto 3, 4 \mapsto 4\})$  inside  $\mathcal{A}(3, 5)$ .

For  $a = (S, T, \phi)$  a generator of  $\mathcal{A}(n, k)$ , let  $\text{inv}(a)$  or  $\text{inv}(\phi)$  be the number of *inversions* of  $\phi$ : the number of pairs  $i, j \in S$  with  $i < j$  and  $\phi(j) < \phi(i)$ . We make  $\mathcal{A}(n, k)$  into an algebra using composition. For  $a, b \in \mathcal{A}(n, k)$ , with  $a = (S, T, \phi_1)$ ,  $b = (T, U, \phi_2)$ , and  $\text{inv}(\phi_1 \circ \phi_2) = \text{inv}(\phi_2) + \text{inv}(\phi_1)$ , define  $a \cdot b$  to be  $(S, U, \phi_1 \circ \phi_2)$ .

In all other cases (i.e., if the range of  $\phi_1$  is not the domain of  $\phi_2$ , or if  $\text{inv}(\phi_1 \circ \phi_2) < \text{inv}(\phi_2) + \text{inv}(\phi_1)$ ), the product of two generators is zero.

*Remark 3.1.* Setting the product to be zero if the number of inversions in the composition  $\phi_1 \circ \phi_2$  decreases is reminiscent of the nilCoxeter algebra.

There is one idempotent  $I(S)$  in  $\mathcal{A}(n, k)$  for each  $k$ -element subset  $S$  of  $[n]$ ; it is  $(S, S, \text{id}_S)$ . It is easy to see that the  $I(S)$  are all the primitive idempotents in  $\mathcal{A}(n, k)$ . Let  $\mathcal{I}(n, k) \subset \mathcal{A}(n, k)$  be the subalgebra generated by the idempotents.

We furthermore define a (co)differential on  $\mathcal{A}(n, k)$ . For  $a = (S, T, \phi)$  in  $\mathcal{A}(n, k)$ , let  $\text{Inv}(\phi)$  be the set counted in  $\text{inv}(\phi)$ : the set  $\{(i, j) \mid i < j, \phi(j) < \phi(i)\}$ . For  $\sigma \in \text{Inv}(\phi)$ , let  $\phi_\sigma$  be defined by

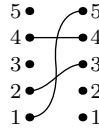
$$\phi_\sigma(k) = \begin{cases} \phi(j) & k = i \\ \phi(i) & k = j \\ \phi(k) & \text{otherwise.} \end{cases}$$

Note that  $\text{inv}(\phi_\sigma) < \text{inv}(\phi)$ , since the inversion  $(i, j)$  was removed from  $\text{Inv}(\phi)$  and none are added. (However, more inversions might be removed.) Then define

$$\partial a := \sum_{\substack{\sigma \in \text{Inv}(\phi) \\ \text{inv}(\phi_\sigma) = \text{inv}(\phi) - 1}} (S, T, \phi_\sigma).$$

Note that  $\text{inv}$  provides a  $\mathbb{Z}$ -grading on  $\mathcal{A}(n, k)$ . However, this grading is not the one we ultimately use.

**3.1.2. Geometric definition.** There is an alternate description of the algebra  $\mathcal{A}(n, k)$ , in terms of *strands*. We represent a triple  $(S, T, \phi)$  by a diagram with  $n$  dots on both the left and right (numbered from the bottom up), with a set of strands representing  $\phi$  connecting the subsets  $S$  on the left and  $T$  on the right. The condition that  $i \leq \phi(i)$  means that strands stay horizontal or move up as you move from left to right. For instance, the diagram



represents the algebra element  $\langle \begin{smallmatrix} 1 & 2 & 4 \\ 5 & 3 & 4 \end{smallmatrix} \rangle$ .

It is a standard fact that  $\text{inv}(a)$  is the minimal number of crossings in any strand diagram. Accordingly, we may view  $\mathcal{A}(n, k)$  as strand diagrams as above, with no pair of strands crossing more than once, considered up to homotopy.

We can define the product on  $\mathcal{A}(n, k)$  in these pictures using horizontal juxtaposition. The product  $a \cdot b$  of two algebra elements is 0 if the right side of  $a$  does not match the left side of  $b$ ; otherwise, we place the two diagrams next to each other and join them together. For instance, the algebra element above can be factored as

The condition that  $\text{inv}(a \cdot b) = \text{inv}(a) + \text{inv}(b)$  corresponds to declaring that if in the product two strands cross each other twice, we set the product to be 0:

$$(3.2) \quad \text{Diagram of two strands crossing twice} = 0.$$

The boundary operator  $\partial a$  also has a natural graphical description. For any crossing in a strand diagram, we can consider *smoothing* it:

$$\text{Crossing} \mapsto \text{Smoothing}.$$

The boundary of a strand diagram is the sum over all ways of smoothing one crossing of the diagram, where we again adopt the convention that a diagram with two crossings is 0. For instance, we have

$$(3.3) \quad \partial \left( \text{Diagram with 5 crossings} \right) = \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \text{Diagram 4} + \text{Diagram 5}.$$

It is straightforward to check that this agrees with the earlier description. In combinatorial terms, the terms in the boundary of an algebra element  $a$  correspond to the edges in the Hasse diagram of the Bruhat order on the symmetric group that start from the permutation corresponding to  $a$ ; the strand-smoothing description is standard in that language.

**3.1.3. Reeb chords.** There is one more description of  $\mathcal{A}(n, k)$ , relating it to the Reeb chords we will see from the moduli spaces. Fix an oriented circle  $Z$  with a set of  $n$  distinct points  $\mathbf{a} = \{a_1, \dots, a_n\} \subset Z$ , together with a basepoint  $z \in Z \setminus \mathbf{a}$ . We view  $Z$  as a contact 1-manifold, and  $\mathbf{a}$  as a Legendrian submanifold of  $Z$ . A *Reeb chord*  $\rho$  in  $(Z \setminus z, \mathbf{a})$  is an immersed (and consequently embedded) arc in  $Z \setminus z$  with endpoints in  $\mathbf{a}$ , with orientation induced by the orientation on  $Z$ . The initial endpoint of  $\rho$  is denoted  $\rho^-$  and the final endpoint  $\rho^+$ . A set  $\boldsymbol{\rho} = \{\rho_1, \dots, \rho_j\}$  of  $j$  Reeb chords is said to be *consistent* if  $\boldsymbol{\rho}^- = \{\rho_1^-, \dots, \rho_j^-\}$  and  $\boldsymbol{\rho}^+ = \{\rho_1^+, \dots, \rho_j^+\}$  are both sets with  $j$  elements (i.e., no two  $\rho_i$  share initial or final endpoints). We adopt a two-line notation for consistent sets of Reeb chords so that, for instance,  $\left\{ \begin{smallmatrix} 1 & 2 \\ 5 & 3 \end{smallmatrix} \right\}$  is the set of chords  $\{[1, 5], [2, 3]\}$ .

A consistent set of Reeb chords  $\boldsymbol{\rho}$  defines an algebra element  $a_0(\boldsymbol{\rho})$  as follows. By a *complement to  $\boldsymbol{\rho}$*  we mean a  $(k - |\boldsymbol{\rho}|)$ -element subset  $S_0$  of  $\{1, \dots, n\}$  which is disjoint from  $\boldsymbol{\rho}^- \cup \boldsymbol{\rho}^+$ . Define

$$a_0(\boldsymbol{\rho}) := \sum_{\substack{S_0 \text{ complement} \\ \text{to } \boldsymbol{\rho}}} (S_0 \cup \boldsymbol{\rho}^-, S_0 \cup \boldsymbol{\rho}^+, \phi_{S_0})$$

where  $\phi_{S_0}|_{S_0} = \text{id}$  and  $\phi_{S_0}(\rho_i^-) = \rho_i^+$ . Geometrically, this corresponds to taking the sum over all ways of consistently adding horizontal strands to the set of strands connecting each  $\rho_i^-$  to  $\rho_i^+$ . In particular,  $a_0(\boldsymbol{\rho})$  is defined to be zero if there is no consistent extension, for instance if  $|\boldsymbol{\rho}| > k$ , and  $a_0(\emptyset)$  is the identity. For convenience,

also define  $a_0(\boldsymbol{\rho})$  to be 0 if  $\boldsymbol{\rho}$  is not consistent. If  $a_0(\boldsymbol{\rho}) \neq 0$ , we say that  $\boldsymbol{\rho}$  is *completable* in  $\mathcal{A}(n, k)$ .

The strands algebra  $\mathcal{A}(n, k)$  is generated as an algebra by the elements  $a_0(\boldsymbol{\rho})$  and the idempotents  $I(S)$ , with  $S \subset \mathbf{a}$  and  $|S| = k$ . In fact, the non-zero elements of the form  $I(S)a_0(\boldsymbol{\rho})$  form a basis for  $\mathcal{A}(n, k)$  as a  $\mathbb{F}_2$ -vector space.

**Definition 3.4.** A pair of Reeb chords  $(\rho, \sigma)$  is said to be *interleaved* if  $\rho^- < \sigma^- < \rho^+ < \sigma^+$ . They are said to be *nested* if  $\rho^- < \sigma^- < \sigma^+ < \rho^+$ . An unordered pair of Reeb chords is said to be interleaved (respectively nested) if either possible ordering is interleaved (respectively nested).

We can now give a definition of the product on  $\mathcal{A}(n, k)$  in terms of these generators.

**Definition 3.5.** If  $\rho$  and  $\sigma$  are Reeb chords such that  $\rho^+ = \sigma^-$ , we say that  $\rho$  and  $\sigma$  are *composable* and define their *join*  $\rho \uplus \sigma$  to be the Reeb chord  $[\rho^-, \sigma^+]$  obtained by concatenating the two chords. We extend these notions to sets of Reeb chords by defining  $\boldsymbol{\rho} \uplus \boldsymbol{\sigma}$  to be obtained from the union  $\boldsymbol{\rho} \cup \boldsymbol{\sigma}$  by replacing every composable pair  $\rho_i \in \boldsymbol{\rho}$  and  $\sigma_j \in \boldsymbol{\sigma}$  by their join  $\rho_i \uplus \sigma_j$ . If  $\rho_i \in \boldsymbol{\rho}$ , let

$$\rho_i^{++} := \begin{cases} \sigma_j^+ & \rho_i, \sigma_j \text{ composable for some } \sigma_j \in \boldsymbol{\sigma} \\ \rho_i^+ & \text{otherwise.} \end{cases}$$

Similarly define  $\sigma_j^{--}$  for  $\sigma_j \in \boldsymbol{\sigma}$ . For  $\boldsymbol{\rho}$  and  $\boldsymbol{\sigma}$  consistent sets of Reeb chords, we say that  $\boldsymbol{\rho}$  and  $\boldsymbol{\sigma}$  are *composable* if there are no double crossings in  $\boldsymbol{\rho} \uplus \boldsymbol{\sigma}$ ; precisely, if there is no pair  $\rho_i \in \boldsymbol{\rho}$  and  $\sigma_j \in \boldsymbol{\sigma}$  so that  $\rho_i^- < \sigma_j^{--}$ ,  $\sigma_j^- < \rho_i^+$ , and  $\rho_i^{++} < \sigma_j^+$ .

**Lemma 3.6.** *For completable sets  $\boldsymbol{\rho}, \boldsymbol{\sigma}$  of Reeb chords, we have*

$$a_0(\boldsymbol{\rho})a_0(\boldsymbol{\sigma}) = \begin{cases} a_0(\boldsymbol{\rho} \uplus \boldsymbol{\sigma}) & \boldsymbol{\rho} \text{ and } \boldsymbol{\sigma} \text{ composable} \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* This follows directly from the definitions.  $\square$

On the other hand, there are two ways a term in the boundary operator can appear, depending on whether we smooth a crossing between two moving strands or between a moving strand and a horizontal strand. (Geometrically on the type  $A$  module, these appear respectively as odd shuffle curve ends and a join curve ends, as defined in Definition 5.39.)

**Definition 3.7.** For a Reeb chord  $\rho_1$ , a *splitting* of  $\rho_1$  is a pair of composable chords  $(\rho_2, \rho_3)$  so that  $\rho_1 = \rho_2 \uplus \rho_3$ . For a set  $\boldsymbol{\rho}$  of Reeb chords, a *weak splitting* of  $\boldsymbol{\rho}$  is a set obtained from  $\boldsymbol{\rho}$  by replacing a chord  $\rho_1 \in \boldsymbol{\rho}$  with two chords  $\{\rho_2, \rho_3\}$ , where  $(\rho_2, \rho_3)$  is a splitting of  $\rho_1$ . For a completable set  $\boldsymbol{\rho}$ , a *splitting* of  $\boldsymbol{\rho}$  is a weak splitting of  $\boldsymbol{\rho}$ , replacing  $\rho_1$  with  $\{\rho_2, \rho_3\}$ , which is completable and does not introduce double crossings, i.e., there is no chord  $\rho_4 \in \boldsymbol{\rho}$  nested in  $\rho_1$  and with  $\rho_4^- < \rho_2^+ = \rho_3^- < \rho_4^+$ .

For a set of Reeb chords  $\boldsymbol{\rho}$ , a *weak shuffle* of  $\boldsymbol{\rho}$  is a set obtained by replacing a nested pair  $(\rho_1, \rho_2)$  of chords in  $\boldsymbol{\rho}$  with the corresponding interleaved pair  $([\rho_1^-, \rho_2^+], [\rho_2^-, \rho_1^+])$ . For a completable set  $\boldsymbol{\rho}$ , a *shuffle* of  $\boldsymbol{\rho}$  is a weak shuffle with nested pair  $(\rho_1, \rho_2)$  that does not introduce double crossings, i.e., there is no chord  $\rho_3 \in \boldsymbol{\rho}$  with  $\rho_2$  nested in  $\rho_3$  and  $\rho_3$  nested in  $\rho_1$ .

**Lemma 3.8.** *For a completable set  $\rho$  of Reeb chords, we have*

$$\partial a_0(\rho) = \sum_{\substack{\rho' \text{ a splitting} \\ \text{of } \rho}} a_0(\rho') + \sum_{\substack{\rho' \text{ a shuffle} \\ \text{of } \rho}} a_0(\rho').$$

*Proof.* This follows directly from the definitions.  $\square$

See also Lemma 3.21 for alternate characterizations of composability, splittings, and shuffles in terms of grading.

**3.2. Matched circles and their algebra.** We now turn to the algebra associated to a parametrized surface. We represent our boundary surfaces by *matched circles*.

**Definition 3.9.** A *matched circle* is a triple  $(Z, \mathbf{a}, M)$  of an oriented circle  $Z$ ,  $4k$  points  $\mathbf{a} = \{a_1, \dots, a_{4k}\}$  in  $Z$  (as in Section 3.1.3), and a *matching*: a 2-to-1 function  $M: \mathbf{a} \rightarrow [2k]$ . We require that performing surgery along the  $2k$  pairs of points (viewed as 0-spheres) yields a single circle, rather than several disjoint circles. (For matched circles from Heegaard diagrams, this is a consequence of the homological linear independence condition from Section 4.)

A *pointed matched circle*  $\mathcal{Z}$  is a matched circle together with a basepoint  $z \in Z \setminus \mathbf{a}$ . To each matched circle  $\mathcal{Z}$  we associate an oriented surface  $F(\mathcal{Z})$ , or just  $F$ , of genus  $k$ , by taking a disk with boundary  $Z$  (respecting the orientation), attaching oriented 2-dimensional 1-handles along the pairs specified by  $M$ , and filling the resulting boundary circles with another disk. Alternately,  $F$  may be obtained from a polygon with  $4k$  sides, with the points  $\mathbf{a}$  in the middle of the sides, by gluing pairs of sides specified by  $M$  in an orientation-preserving way. The last disk glued in is a neighborhood of the unique resulting vertex.

The points  $\mathbf{a}$  in a matched circle  $(Z, \mathbf{a}, M)$  inherit a cyclic ordering from the orientation of  $Z$ ; in a pointed matched circle, this cyclic ordering is lifted to an honest ordering, which we denote by  $\prec$ .

We get pointed matched circles by splitting a pointed Heegaard diagram in two along a circle  $Z$  that does not intersect  $\beta$ -circles, intersects each  $\alpha$ -circle at most twice, and passes through the base point. The points  $\mathbf{a}$  are the intersections with the  $\alpha$ -circles, the matching  $M$  records which intersection points are on the same  $\alpha$ -circle, and the orientation on  $Z$  is that inherited from the left (west, type A) side. See Figure 10.

We will now define  $\mathcal{A}(\mathcal{Z})$ , our main algebra of interest. The algebra  $\mathcal{A}(\mathcal{Z})$  decomposes as a direct sum

$$\mathcal{A}(\mathcal{Z}) = \bigoplus_{i=-k}^k \mathcal{A}(\mathcal{Z}, i)$$

where  $\mathcal{A}(\mathcal{Z}, i)$  is a subalgebra of  $\mathcal{A}(4k, k-i)$ . For each subset  $s$  of  $[2k]$  we have one idempotent  $I(s)$  in  $\mathcal{A}(\mathcal{Z})$ , given by

$$(3.10) \quad I(s) := \sum \{ I(S) \mid S \text{ a section of } M \text{ over } s \},$$

where by a *section* of  $M$  over  $s$  we mean a subset of  $S$  that maps bijectively to  $s$  via  $M$ . That is, each element of  $s$  defines a pair of points in  $\mathbf{a}$ ;  $I(s)$  is the sum over

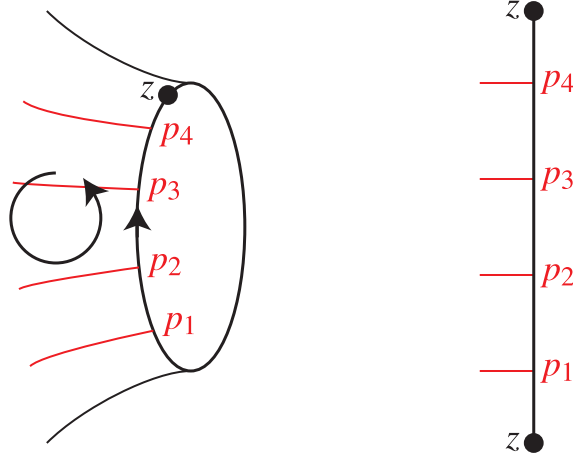


FIGURE 10. Left: the orientation of a Heegaard surface, and the induced orientation of its boundary. Right: the induced ordering of  $\mathbf{a}$ : in this figure,  $p_1 \prec p_2 \prec p_3 \prec p_4$

the  $2^{|s|}$  primitive idempotents in  $\mathcal{A}(n, k)$  that pick one point out of each of these pairs. Let  $\mathcal{I}(\mathcal{Z})$  be the algebra generated by the  $I(s)$ , and define

$$I := \sum_{s \subset [2k], |s|=k} I(s)$$

to be the maximal idempotent in  $\mathcal{I}(\mathcal{Z})$ . Then  $\mathcal{A}(\mathcal{Z})$  is the subalgebra of  $\bigoplus_{i=0}^{2k} \mathcal{A}(4k, i)$  generated (as an algebra) by  $\mathcal{I}(\mathcal{Z})$  and by  $Ia_0(\boldsymbol{\rho})I$  for every set of Reeb chords  $\boldsymbol{\rho}$ . Let  $a(\boldsymbol{\rho})$  denote  $Ia_0(\boldsymbol{\rho})I$ , the projection of  $a_0(\boldsymbol{\rho})$  to  $\mathcal{A}(\mathcal{Z})$ .

It is easy to verify that  $\mathcal{A}(\mathcal{Z})$  is closed under  $\partial$ . There is a convenient basis over  $\mathbb{F}_2$  with non-zero elements of the form  $I(s)a(\boldsymbol{\rho})$ , for  $I(s)$  a primitive idempotent and  $\boldsymbol{\rho}$  a consistent set of Reeb chords. In particular,  $\mathcal{I}(\mathcal{Z}) = \mathcal{A}(\mathcal{Z}) \cap \bigoplus_{i=0}^{2k} \mathcal{I}(4k, i)$ . We set

$$\begin{aligned} \mathcal{A}(\mathcal{Z}, i) &= \mathcal{A}(\mathcal{Z}) \cap \mathcal{A}(n, k - i) \\ \mathcal{I}(\mathcal{Z}, i) &= \mathcal{I}(\mathcal{Z}) \cap \mathcal{A}(n, k - i). \end{aligned}$$

In two-line notation, we express the basic generators  $I(s)a(\boldsymbol{\rho})$  as

$$\begin{bmatrix} x_1 & \cdots & x_k & z_1 & \cdots & z_l \\ y_1 & \cdots & y_k & & & \end{bmatrix} := I(\{M(x_1), \dots, M(x_k), M(z_1), \dots, M(z_l)\})a(\left\{\begin{smallmatrix} x_1 & \cdots & x_k \\ y_1 & \cdots & y_k \end{smallmatrix}\right\}).$$

For instance, the element in Figure 11 is denoted  $\begin{bmatrix} 3 & 5 \\ 8 & \end{bmatrix}$  or, equivalently,  $\begin{bmatrix} 3 & 7 \\ 8 & \end{bmatrix}$ .

Graphically, we will denote the matching  $M$  by drawing dotted arcs on the left and right of a strand diagram. We will draw the strands in an element  $\mathcal{A}(\mathcal{Z})$  as usual, except that when we need to sum over different sections of  $s_0$  we indicate strands that appear in half the terms with dashed lines. See Figure 11.

The summands  $\mathcal{A}(\mathcal{Z}, i) \subset \mathcal{A}(\mathcal{Z})$  for  $i \neq 0$  will act trivially on the modules  $\widehat{CFD}(Y)$  and  $\widehat{CFA}(Y)$  defined in Sections 6 and 7. So, for this paper, only the subalgebra  $\mathcal{A}(\mathcal{Z}, 0) \subset \mathcal{A}(\mathcal{Z})$  will be of interest. The other summands of  $\mathcal{A}(\mathcal{Z})$  do play an important role in the case of disconnected boundary; see [23].

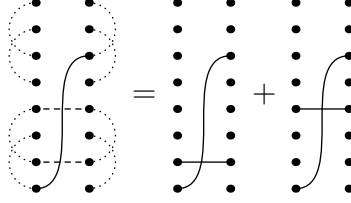


FIGURE 11. An element of  $\mathcal{A}(\mathcal{Z})$  and its expansion as an element of  $\mathcal{A}(4k, k)$ . Here  $k=2$  and so  $F(\mathcal{Z})$  is a surface of genus two.

*Remark 3.11.* One can think of the index  $i$  in  $\mathcal{A}(\mathcal{Z}, i)$  as corresponding to  $\text{spin}^c$ -structures on the surface  $F(\mathcal{Z})$ , compare [23]. For a 3-manifold  $Y$  with connected boundary  $\partial Y$ , only the middle  $\text{spin}^c$ -structure on  $\partial Y$  extends over  $Y$ .

**3.3. Gradings.** Although the algebra  $\mathcal{A}(n, k)$  has a  $\mathbb{Z}$ -grading, for instance by the crossing number, it turns out that the subalgebra  $\mathcal{A}(\mathcal{Z})$  has no  $\mathbb{Z}$ -grading for a surface of genus at least two.

*Example 3.12.* Consider the two algebra elements

$$x = \text{diagram} \quad \text{and} \quad y = \text{diagram}$$

which are the identity outside of the torus portion of the diagram shown. On one hand, we have

$$y \cdot x = \text{diagram}$$

but on the other hand,

$$\begin{aligned} dx &= \text{diagram} \\ dx \cdot y &= \text{diagram} \\ d(dx \cdot y) &= y \cdot x. \end{aligned}$$

This last equation cannot be coherent with any  $\mathbb{Z}$ -grading.

Instead,  $\mathcal{A}(\mathcal{Z})$  has a grading in a non-commutative group. Gradings of a differential algebra  $A$  by a non-commutative group  $G$  are straightforward but not very standard. Briefly, it is a grading of  $A$  as an algebra by  $G$  together with a distinguished central element  $\lambda \in G$  so that for every homogeneous element  $a \in A$ , we have  $\text{gr}(\partial a) = \lambda^{-1} \text{gr}(a)$ . See Subsection 2.4 for the details of the algebraic setting.

**3.3.1. Grading on  $\mathcal{A}(n, k)$ .** We start by giving a grading  $\text{gr}'$  of  $\mathcal{A}(n, k)$  with values in a non-commutative group  $G'(n)$ , with the property that  $\text{gr}'$  is unchanged by adding horizontal strands. Let  $Z' = Z \setminus \{z\}$ . The group  $G'(n)$  can be abstractly defined as a

central extension by  $\frac{1}{2}\mathbb{Z}$  of the relative homology group  $H_1(Z', \mathbf{a})$  (which is isomorphic to  $\mathbb{Z}^{n-1}$ ). In particular, there is an exact sequence of groups

$$\frac{1}{2}\mathbb{Z} \xrightarrow{\lambda} G'(n) \xrightarrow{[\cdot]} H_1(Z', \mathbf{a})$$

where in the first map 1 maps to  $\lambda$  (and  $1/2$  maps to an element  $\lambda^{1/2}$ ).

The  $H_1(Z', \mathbf{a})$  component of the grading of an element  $a$  is given by its *homology class*  $[a]$ , defined to be the element of the relative homology group  $H_1(Z', \mathbf{a})$  given by summing up the intervals corresponding to the strands:

$$[a] := \sum_{s \in S} [s, \phi(s)].$$

To define the central extension and the complete grading, we make some more definitions. For  $p \in \mathbf{a}$  and  $\alpha \in H_1(Z', \mathbf{a})$ , define the *multiplicity*  $m(\alpha, p)$  of  $p$  in  $\alpha$  to be the average multiplicity with which  $\alpha$  covers the regions on either side of  $p$ . Concretely,  $m([a], p)$  is given by counting with signs how many times a horizontal line at height  $p$  intersects the strand diagram for  $a$ , with the convention that an intersection at an endpoint counts as half of an intersection. We choose orientations so that  $m([a], p)$  is non-negative for any  $p \in \mathbf{a}$  and  $a \in \mathcal{A}(n, k)$ . Extend  $m$  to a map  $H_1(Z', \mathbf{a}) \times H_0(\mathbf{a}) \rightarrow \frac{1}{2}\mathbb{Z}$  bilinearly.

For  $\alpha_i \in H_1(Z', \mathbf{a})$ , define

$$(3.13) \quad L(\alpha_1, \alpha_2) := m(\alpha_2, \delta\alpha_1)$$

where  $\delta$  is the standard connecting homomorphism  $\delta: H_1(Z', \mathbf{a}) \rightarrow H_0(\mathbf{a})$ . The map  $L$  can be thought of as the linking number of  $\delta\alpha_1$  and  $\delta\alpha_2$ . Define  $G'(n)$  to be the group generated by pairs  $(k, \alpha)$ , where  $k \in \frac{1}{2}\mathbb{Z}$  and  $\alpha \in H_1(Z', \mathbf{a})$ , with multiplication given by

$$(3.14) \quad (k_1, \alpha_1) \cdot (k_2, \alpha_2) := (k_1 + k_2 + L(\alpha_1, \alpha_2), \alpha_1 + \alpha_2).$$

In an element  $g = (k, \alpha)$  of  $G'(n)$ , we refer to  $k$  as the *Maslov component* of  $g$  and  $\alpha$  as the *homological component* of  $g$ . In these coordinates on  $G'(n)$ , the distinguished central element is  $\lambda = (1, 0)$ .

To see that this is a good definition, we make  $L$  more explicit and prove some elementary properties.

**Lemma 3.15.** *If  $\alpha_1 = \sum u_i[p_i, p_{i+1}]$  and  $\alpha_2 = \sum v_i[p_i, p_{i+1}]$ , then*

$$L(\alpha_1, \alpha_2) = \sum_{i=1}^{n-1} \frac{u_i v_{i+1} - u_{i+1} v_i}{2}.$$

*In particular,  $L(\alpha_1, \alpha_2) = -L(\alpha_2, \alpha_1)$ .*

Note that  $u_i v_{i+1} - u_{i+1} v_i$  can be interpreted as the determinant of a  $2 \times 2$  submatrix of the matrix  $\begin{pmatrix} u_1 & u_2 & \cdots & u_n \\ v_1 & v_2 & \cdots & v_n \end{pmatrix}$ .

*Proof.* Recall that  $\delta[p_i, p_{i+1}] = p_{i+1} - p_i$ , from which it follows that  $\delta\alpha_1 = \sum_{i=1}^n (u_{i-1} - u_i)p_i$ , where we set  $u_0 = u_n = 0$ . Then

$$\begin{aligned} L(\alpha_1, \alpha_2) &= \sum_{i=1}^n \frac{(u_{i-1} - u_i)(v_{i-1} + v_i)}{2} \\ &= \sum_{i=1}^n \frac{u_{i-1}v_i - u_i v_{i-1}}{2} + \frac{u_{i-1}v_{i-1}}{2} - \frac{u_i v_i}{2} \\ &= \sum_{i=1}^n \frac{u_{i-1}v_i - u_i v_{i-1}}{2} \end{aligned}$$

since the sum telescopes.  $\square$

**Proposition 3.16.**  $G'(n)$  is a group.

*Proof.* The inverse of  $(k, \alpha)$  is easily seen to be  $(-k, -\alpha)$ . We must also check that the product is associative:

$$\begin{aligned} ((k_1, \alpha_1) \cdot (k_2, \alpha_2)) \cdot (k_3, \alpha_3) &= (k_1 + k_2 + k_3 + L(\alpha_1, \alpha_2) + L(\alpha_1 + \alpha_2, \alpha_3), \alpha_1 + \alpha_2 + \alpha_3) \\ &= (k_1 + k_2 + k_3 + L(\alpha_1, \alpha_2) + L(\alpha_1, \alpha_3) + L(\alpha_2, \alpha_3), \alpha_1 + \alpha_2 + \alpha_3) \end{aligned}$$

where we use bilinearity of  $L$ . Associating in the opposite order gives the same result.  $\square$

The following lemma will be useful later.

**Lemma 3.17.** The product of a sequence of elements in  $G'(n)$  is given by

$$(k_1, \alpha_1) \cdots (k_n, \alpha_n) = \left( \sum_i k_i + \sum_{i < j} L(\alpha_i, \alpha_j), \sum_i \alpha_i \right).$$

*Proof.* As in the proof of Proposition 3.16, expand the right-associated product on the left hand side of the statement and use bilinearity of  $L$ .  $\square$

The map  $L$  is a 1-cocycle on  $H_1(Z', \mathbf{a})$  which defines the central extension  $G'(n)$ .

**Definition 3.18.** The grading  $\text{gr}'(a)$  of an element  $a \in \mathcal{A}(n, k)$  with starting idempotent  $S$  is

$$\begin{aligned} \iota(a) &:= \text{inv}(a) - m([a], S) \\ \text{gr}'(a) &:= (\iota(a), [a]). \end{aligned}$$

Note that  $\delta[a] = T - S \in H_0(\mathbf{a})$ , so by Lemma 3.15, we have  $m([a], S) = m([a], T)$ .

**Proposition 3.19.** The function  $\text{gr}'$  defines a grading (in the sense of Definition 2.23) on  $\mathcal{A}(n, k)$ , with  $\lambda = (1, 0)$ .

*Proof.* For the behavior of  $\text{gr}'$  under the action of  $\partial$ , observe that  $\iota(\partial a) = \iota(a) - 1$ : neither  $S$  nor  $[a]$  is changed by  $\partial$ , so  $m([a], S)$  is unchanged, and in the definition of  $\iota$  only the  $\text{inv}$  term changes. Thus with  $\lambda = (1, 0)$ , we have  $\text{gr}'(\partial a) = \lambda^{-1} \text{gr}'(a)$ ,

as desired. We also check that  $\text{gr}'$  is compatible with multiplication: if  $a$  is given by  $(S, T, \phi_1)$  and  $b$  is given by  $(T, U, \phi_2)$ , then

$$\begin{aligned} [a \cdot b] &= [a] + [b] \\ \iota(a \cdot b) &= \text{inv}(a \cdot b) - m([a] + [b], S) \\ &= \text{inv}(a) + \text{inv}(b) - m([a], S) - m([b], T) + m([b], T - S) \\ &= \iota(a) + \iota(b) + m([b], \delta[a]) \end{aligned}$$

and so  $\text{gr}'(a \cdot b) = \text{gr}'(a) \cdot \text{gr}'(b)$ .  $\square$

**Proposition 3.20.** *For a consistent set  $\boldsymbol{\rho}$  of Reeb chords,  $a_0(\boldsymbol{\rho})$  is homogeneous. In particular, the grading  $\text{gr}'$  descends to a grading on the subalgebra  $\mathcal{A}(\mathcal{Z})$ .*

*Proof.* Observe that adding a horizontal strand at  $p \in \mathbf{a}$  to an algebra element  $a = (S, T, \phi)$  does not change  $[a]$  and changes  $\text{inv}(a)$  and  $m([a], S)$  by the same amount,  $m([a], p)$ . Thus  $\text{gr}'(a)$  is unchanged by adding a horizontal strand (moving from  $\mathcal{A}(n, k)$  to  $\mathcal{A}(n, k+1)$ ). In particular the terms in the definition of  $a_0(\boldsymbol{\rho})$  all have the same grading, proving the first statement. Since  $\mathcal{A}(\mathcal{Z})$  is generated by elements of the form  $I(s)a_0(\boldsymbol{\rho})$ , and  $I(s)$  is clearly homogeneous, the second statement follows.  $\square$

In light of Proposition 3.20, there is an alternate characterization of composability (Definition 3.5) and splittings and shuffles (Definition 3.7). Define a partial ordering on  $G'(n)$  by comparing the Maslov components:  $(k_1, \alpha_1) < (k_2, \alpha_2)$  if  $\alpha_1 = \alpha_2$  and  $k_1 < k_2$ .

**Lemma 3.21.** *Let  $\boldsymbol{\rho}$  and  $\boldsymbol{\sigma}$  be completable sets of Reeb chords such that  $\boldsymbol{\rho} \uplus \boldsymbol{\sigma}$  is also completable.*

- (1) *The pair  $(\boldsymbol{\rho}, \boldsymbol{\sigma})$  is composable if and only if  $\text{gr}'(a_0(\boldsymbol{\rho} \uplus \boldsymbol{\sigma})) = \text{gr}'(a_0(\boldsymbol{\rho})) \cdot \text{gr}'(a_0(\boldsymbol{\sigma}))$ .*
- (2) *A weak splitting  $\boldsymbol{\rho}'$  of  $\boldsymbol{\rho}$  is a splitting if and only if  $\text{gr}'(a_0(\boldsymbol{\rho}')) = \lambda^{-1} \text{gr}'(a_0(\boldsymbol{\rho}))$ .*
- (3) *A weak shuffle  $\boldsymbol{\rho}'$  of  $\boldsymbol{\rho}$  is a shuffle if and only if  $\text{gr}'(a_0(\boldsymbol{\rho}')) = \lambda^{-1} \text{gr}'(a_0(\boldsymbol{\rho}))$ .*

*In all cases, if equality does not hold then the left hand side has lower grading.*

*Proof.* Clear.  $\square$

3.3.2. *Refined grading on  $\mathcal{A}(\mathcal{Z})$ .* We can also construct a refined grading  $\text{gr}$  on  $\mathcal{A}(\mathcal{Z})$  with values in a smaller group  $G(\mathcal{Z})$ .

The group  $G(\mathcal{Z})$  can be abstractly defined as a central extension of  $H_1(F) \simeq \mathbb{Z}^{2k}$ . Again, there is a sequence of groups

$$\frac{1}{2}\mathbb{Z} \xhookrightarrow{\lambda} G(\mathcal{Z}) \xrightarrow{[\cdot]} H_1(F)$$

where  $1 \in \frac{1}{2}\mathbb{Z}$  maps to  $\lambda \in G(\mathcal{Z})$ . This central extension may be described either via an explicit 1-cocycle, which we will do a little later, or abstractly by giving commutators, which we do now. Let  $\lambda$  be the image of  $1 \in \frac{1}{2}\mathbb{Z}$  in  $G(\mathcal{Z})$ , and for  $g \in G(\mathcal{Z})$ , let  $[g]$  be its image in  $H_1(F)$ , as indicated above. Then, for  $g, h \in G(\mathcal{Z})$ , we require

$$(3.22) \quad gh = hg\lambda^{2([g] \cap [h])}.$$

where  $\cap$  is the homological intersection product  $H_1(F) \times H_1(F) \rightarrow H_0(F) \simeq \mathbb{Z}$ . Equation (3.22) determines  $G(\mathcal{Z})$  uniquely.

$G(\mathcal{Z})$  can also be given a topological interpretation as the group of homotopy classes of non-vanishing vector fields on  $F \times [0, 1]$  with fixed behavior on the boundary.

To define  $\text{gr}$ , first suppose that we have an element  $\alpha$  of  $H_1(Z', \mathbf{a})$  for which  $M_*(\delta\alpha) = 0$ ; that is, for each pair  $p, q$  of points in  $\mathbf{a}$  identified by  $M$ , the sum of the multiplicities of  $\delta\alpha$  at  $p$  and at  $q$  is 0. Then we can construct an element  $\bar{\alpha}$  of  $H_1(F(\mathcal{Z}))$  from  $\alpha$  by embedding  $Z$  in  $F(\mathcal{Z})$  to make  $\alpha$  a chain in  $F(\mathcal{Z})$ , and then adding multiples of the cores of the 1-handles we attached in the construction of  $F(\mathcal{Z})$  to form a cycle  $\bar{\alpha}$ . Indeed, we can identify  $H_1(F)$  as  $\ker(M_* \circ \delta)$  inside  $H_1(Z', \mathbf{a})$ .

Let  $i_*: H_1(F) \rightarrow H_1(Z', \mathbf{a})$  denote the inclusion map. Then,  $G(\mathcal{Z})$  is the subgroup of elements  $g \in G'(4k)$  for which  $[g] \in \text{Im}(i_*)$ : this subgroup is a  $\mathbb{Z}$ -central extension of  $H_1(F)$ . In fact,

$$[g] \cap [h] = m(i_*([h]), \delta i_*([g])),$$

which implies Equation (3.22).

To define  $\text{gr}$ , pick an arbitrary base idempotent  $I(s_0) \in \mathcal{A}(\mathcal{Z})$ , and for each other idempotent  $I(s)$ , pick a group element  $g(s, s_0) \in G'(4k)$  so that  $M_*\delta([g(s, s_0)]) = s - s_0$ . Then define

$$(3.23) \quad \text{gr}(I(s)a(\boldsymbol{\rho})I(t)) := g(s, s_0) \text{gr}'(a(\boldsymbol{\rho}))g(t, s_0)^{-1}.$$

Note that the homological part of  $\text{gr}$  is in the subgroup  $G(\mathcal{Z}) \subset G'(4k)$ :

$$\begin{aligned} M_*\delta[\text{gr}(I(s)a(\boldsymbol{\rho})I(t))] &= M_*\delta[g(s, s_0)] + M_*\delta[a(\boldsymbol{\rho})] - M_*\delta[g(t, s_0)] \\ &= (s - s_0) + (t - s) - (t - s_0) \\ &= 0. \end{aligned}$$

It is elementary to verify that  $\text{gr}$  is a grading.

*Remark 3.24.* An alternate way to construct the refined grading, which we employ in Section 10.1 for the case where  $F$  is a torus, is to use an equivariant map from  $G'(4k)$  to  $G(\mathcal{Z})$  that fixes  $G(\mathcal{Z})$  as a subset of  $G'(4k)$  (after extending scalars from  $\mathbb{Z}$  to  $\mathbb{Q}$ ).

#### 4. BORDERED HEEGAARD DIAGRAMS

Bordered three-manifolds can be described by a suitable type of Heegaard diagram, which we call here *bordered Heegaard diagrams* (see Definition 4.2 below). Bordered Heegaard Floer homology is the homology group of a chain complex, whose generators are combinatorially associated to the Heegaard diagram, and whose differentials count pseudo-holomorphic curves representing various homology classes connecting the generators. In the present section, we describe the bordered Heegaard diagrams for bordered three-manifolds. In Section 4.1, we define the notion, and discuss the relevant existence and uniqueness properties. In Section 4.2, we give some examples of bordered Heegaard diagrams, to give the reader a concrete picture. In Section 4.3, we describe the generators of the bordered Heegaard Floer complex. Moreover, we discuss the  $\text{spin}^c$  structures associated to generators, and the homology classes connecting them. As in the closed case, one cannot use an arbitrary bordered Heegaard diagram to define bordered Heegaard Floer homology: rather, one needs to use bordered diagrams satisfying certain combinatorial conditions ensuring that the sums encountered in the differentials are finite. In Subsection 4.4, we describe these criteria. Finally, in Subsection 4.5, we discuss how to glue bordered Heegaard diagrams to

obtain Heegaard diagrams for closed manifolds, and explain also how the generators of the theory transform.

#### 4.1. Bordered Heegaard diagrams: definition, existence, and uniqueness.

We recall first the definition relevant for Heegaard Floer homology of closed three-manifolds:

**Definition 4.1.** A *pointed Heegaard diagram*  $\mathcal{H}$  is a quadruple  $(\Sigma, \alpha, \beta, z)$  consisting of

- a compact, oriented surface  $\Sigma$  with no boundary, of some genus  $g$ ,
- two  $g$ -tuples  $\alpha = \{\alpha_1, \dots, \alpha_g\}$  and  $\beta = \{\beta_1, \dots, \beta_g\}$  of circles in  $\Sigma$ , with each the circles in each tuple disjoint, and
- a point  $z$  in  $\Sigma \setminus (\alpha \cup \beta)$ ,

such that the  $\alpha$ -circles and  $\beta$ -circles intersect transversally and  $\Sigma \setminus \alpha$  and  $\Sigma \setminus \beta$  are each connected.

When discussing pointed Heegaard diagrams (and their bordered variants), we slightly abuse notation, letting  $\alpha$  denote both the set or tuple of  $\alpha$ -curves and the union of the  $\alpha$ -curves, and similarly for  $\beta$ . This will not cause confusion. Note that the  $\alpha$ -circles and  $\beta$ -circles are each maximal collections of embedded circles with connected complement. An equivalent requirement to connectivity of the complement is that the  $\alpha_i$  are homologically linearly independent in  $H_1(\Sigma)$ , spanning a Lagrangian subspace, and likewise for the  $\beta_i$ .

The above definition has the following analogue for bordered three-manifolds:

**Definition 4.2.** For a *bordered* pointed Heegaard diagram, we modify the above definition to a quadruple  $(\bar{\Sigma}, \bar{\alpha}, \beta, z)$  consisting of

- a compact, oriented surface  $\bar{\Sigma}$  with one boundary component, of some genus  $g$ ;
- a  $g$ -tuple of pairwise-disjoint circles  $\beta = \{\beta_1, \dots, \beta_g\}$  in the interior of  $\bar{\Sigma}$ ;
- a  $(g + k)$ -tuple of pairwise-disjoint curves  $\bar{\alpha}$  in  $\bar{\Sigma}$ , split into  $g - k$  circles  $\alpha^c = (\alpha_1^c, \dots, \alpha_{g-k}^c)$  in the interior of  $\bar{\Sigma}$  and  $2k$  arcs  $\bar{\alpha}^a = (\bar{\alpha}_1^a, \dots, \bar{\alpha}_{2k}^a)$  in  $\bar{\Sigma}$  with boundary on  $\partial\bar{\Sigma}$  (and transverse to  $\partial\bar{\Sigma}$ ); and
- a point  $z$  in  $(\partial\bar{\Sigma}) \setminus (\alpha \cap \partial\bar{\Sigma})$ ,

such that the intersections are transverse and  $\bar{\Sigma} \setminus \bar{\alpha}$  and  $\bar{\Sigma} \setminus \beta$  are connected.

The conditions on the numbers of  $\beta$ -circles and  $\alpha$ -circles and arcs again come naturally from requiring that the tuples be maximal collections of curves of the appropriate type with connected complement. Again an equivalent requirement to connectivity of the complement is that the  $\beta_i$  be homologically linearly independent in  $H_1(\bar{\Sigma})$  and that the  $\alpha_i$  be homologically linearly independent in  $H_1(\bar{\Sigma}, \partial\bar{\Sigma})$ . An Euler characteristic argument shows that  $\bar{\Sigma} \setminus \bar{\alpha}$  is a planar surface with  $2(g - k) + 1$  boundary components, exactly one of which meets the original boundary of  $\bar{\Sigma}$ . Thus if we let  $N$  be the union of a collar neighborhood of  $\partial\bar{\Sigma}$  and a tubular neighborhood of  $\bar{\alpha}^a$ ,  $\partial N$  consist of two connected components (one of which is, of course,  $\partial\bar{\Sigma}$ ). Let  $F$  denote the surface  $N \cup \mathbb{D}^2 \cup \mathbb{D}^2$  obtained by filling in the two boundary components; it will be the boundary of the 3-manifold.

As in the Introduction, a curve  $Z$  in a pointed Heegaard diagram  $\mathcal{H}$  that

- intersects no  $\beta$ -circles,

- intersects each  $\alpha$ -circle at most twice, and
- passes through the basepoint

splits  $\mathcal{H}$  into two bordered pointed Heegaard diagrams.

The boundary of a bordered pointed Heegaard diagram naturally has the structure of a pointed matched circle  $(\partial\bar{\Sigma}, \bar{\alpha} \cap \partial\bar{\Sigma}, M, z)$ , in the sense of Section 3. For convenience later, set  $\mathbf{a} = \bar{\alpha} \cap \partial\bar{\Sigma}$ .

The data  $(\bar{\Sigma}, \bar{\alpha}, \beta)$  specifies a 3-manifold with boundary. One way to see this is to double  $(\bar{\Sigma}, \bar{\alpha}, \beta)$ , i.e., take two copies and glue them along their common boundary. The result is a Heegaard diagram with a  $\mathbb{Z}/2$ -action, fixing a circle. It specifies a closed 3-manifold with a  $\mathbb{Z}/2$ -action, the fixed set of which is a null-homologous surface. The three-manifold with boundary  $Y = Y(\bar{\Sigma}, \bar{\alpha}, \beta)$  specified by  $(\bar{\Sigma}, \bar{\alpha}, \beta)$  is a fundamental domain for the  $\mathbb{Z}/2$ -action. The boundary of  $Y$  is homeomorphic to the surface  $F$  defined above. In fact, the homeomorphism between  $\partial Y$  and  $F$  is canonical up to isotopy. Since there is a canonical (up to isotopy) identification of  $F$  with a model surface of genus  $k$ ,  $(\bar{\Sigma}, \bar{\alpha}, \beta)$  specifies not just a 3-manifold with boundary but rather a *bordered 3-manifold*, i.e., a 3-manifold with parameterized boundary.

Notice that  $\partial Y(\bar{\Sigma}, \bar{\alpha}, \beta)$  is also identified canonically (up to homotopy) with the surface specified by the matched circle  $(\partial\bar{\Sigma}, \bar{\alpha} \cap \partial\bar{\Sigma}, M)$ .

Conversely, any 3-manifold with connected boundary is specified by some bordered Heegaard diagram. One way to see this is in terms of Morse theory (cf. [21, Section 2.2]).

**Definition 4.3.** Let Riemannian metric on  $Y$  and  $f$  a self-indexing Morse function. We say that the metric and the Morse function  $f$  are *boundary compatible*

- The boundary of  $Y$  is geodesic.
- $\nabla f|_{\partial Y}$  is tangent to  $\partial Y$ .
- $f$  has a unique index 0 and a unique index 3 critical point, both of which lie on  $\partial Y$ , and are the unique index 0 and 2 critical points of  $f|_{\partial Y}$ , respectively.
- The index 1 critical points of  $f|_{\partial Y}$  are also index 1 critical points of  $f$ .

(To construct such  $f$ , construct  $f$  in a collar neighborhood of  $\partial Y$ , extend arbitrarily over the interior of  $Y$ , and then perturb in the interior of  $Y$  to obtain a Morse function.) Then take  $\Sigma = f^{-1}(3/2)$ ,  $\bar{\alpha}$  the intersection of the ascending disks of the index 1 critical points of  $f$  with  $\Sigma$ , and  $\beta$  the intersection of the descending disks of the index 2 critical points of  $f$  with  $\Sigma$ . Note that  $f$  induces a  $\mathbb{Z}/2$ -equivariant Morse function on  $Y \cup_{\partial} (-Y)$ .

We can furthermore pick the metric and Morse function on  $\partial Y$  initially and then extend it to all of  $Y$  by the argument above. This allows us to construct a bordered Heegaard diagram for a bordered 3-manifold, not just a 3-manifold with boundary.

It follows from the Morse theory description that any two bordered Heegaard diagrams for the same bordered 3-manifold can be connected by (bordered) Heegaard moves:

**Proposition 4.4.** *Any two bordered Heegaard diagrams for the same bordered 3-manifold differ by a sequence of*

- *isotopies of the  $\alpha$ -curves and  $\beta$ -circles, not crossing  $\partial\Sigma$ ,*
- *handleslides of  $\alpha$ -curves over  $\alpha$ -circles, and  $\beta$ -circles over  $\beta$ -circles, and*

- *stabilizations and destabilizations in the interior of  $\bar{\Sigma}$ .*

Before turning to the proof of the Proposition 4.4, for which we follow closely the proof of [33, Proposition 2.2], we include two lemmas which will be useful for excluding stabilizing by index zero (or three) critical points (compare [33, Lemmas 2.3 and 2.4]).

**Lemma 4.5.** *Let  $F$  be a compact surface of genus  $h$  with a single boundary component. Let  $\gamma = \{\gamma_1, \dots, \gamma_h\}$  be an  $h$ -tuple of embedded, pairwise disjoint, simple closed curves so that  $F \setminus \gamma$  is a punctured disk. Suppose moreover that  $\delta$  is a simple closed curve in  $F$  which is disjoint from the  $\gamma$ . Then either  $\delta$  is null-homologous or there is some  $\gamma_i$  with the property that  $\delta$  is isotopic to a curve obtained by handlesliding  $\gamma_i$  across some collection of the  $\gamma_j$  with  $j \neq i$ .*

*Proof.* If we surger out  $\gamma_1, \dots, \gamma_h$ , we replace  $F$  by the disk  $D$  with  $2h$  marked points  $\{p_1, q_1, \dots, p_h, q_h\}$  (where the pair of points  $\{p_i, q_i\}$  corresponds to the zero-sphere which replaced the circle  $\gamma_i$ ). Now,  $\delta$  can be viewed as a Jordan curve in the disk  $D$ . If  $\delta$  separates some  $p_i$  from its corresponding  $q_i$ , then it is easy to see that  $\gamma$  is isotopic to the curve gotten by handlesliding  $\gamma_i$  over some collection of the  $\gamma_j$  with  $j \neq i$ . Otherwise,  $\delta$  was null-homologous.  $\square$

**Lemma 4.6.** *Let  $F$  be a compact surface of genus  $h$  with a single boundary component and  $\{\gamma_1, \dots, \gamma_d\}$  a collection of embedded, pairwise disjoint, simple closed curves. Then any two  $h$ -tuples linearly-independent  $\gamma_i$ 's are related by a sequence of isotopies and handleslides.*

*Proof.* This is proved by induction on  $h$ . The case where  $h = 0$  is vacuously true. Next, suppose we have two subsets with an element in common, which we label  $\gamma_1$ . Surger out  $\gamma_1$  to obtain a surface  $F'$  with one lower genus. Isotopies in  $F'$  which cross the two surgery points correspond to handleslides in  $F$  across  $\gamma_1$ . Thus, in this case, the proof follows from the induction hypothesis. Finally, consider the case where we have two disjoint subsets  $\{\gamma_1, \dots, \gamma_h\}$  and  $\{\gamma_{h+1}, \dots, \gamma_{2h}\}$ . The curve  $\gamma_{h+1}$  cannot be not null-homologous, so after renumbering, we can obtain  $\gamma_{h+1}$  by handlesliding  $\gamma_1$  over some of the  $\{\gamma_2, \dots, \gamma_h\}$ , according to Lemma 4.5. Thus, we have reduced to the case where the two subsets are not disjoint.  $\square$

*Proof of Proposition 4.4.* The proof involves adapting standard handle calculus as in [33, Section 2.1].

Fix a Riemannian metric on  $Y$  and Morse function  $f$  on a collar neighborhood  $N(\partial Y)$  of  $\partial Y$  satisfying the conditions of Definition 4.3 on the open collar neighborhood. Any two bordered Heegaard diagrams  $\mathcal{H}_0$  and  $\mathcal{H}_1$  come from extensions  $f_0$  and  $f_1$  of  $f$  to a self-indexing Morse function on all of  $Y$  with no extra index 0 or index 3 critical points. We may connect  $f_0$  and  $f_1$  by a generic path  $f_t$  of smooth functions, which are Morse functions for generic  $t$ .

Although the  $f_t$  will not in general be self-indexing, there are still no flow lines from higher-index critical points to lower-index. To  $f_t$  at generic  $t$  we can still associate a generalized Heegaard diagram  $\mathcal{H}_t$ , possibly with extra  $\alpha$ - and  $\beta$ -circles.

While  $f_t$  remains a Morse function, the Heegaard diagram changes by isotopy of the  $\alpha$ - and  $\beta$ -curves. When  $f_t$  fails to be a Morse function, there is either a birth-death singularity involving a pair of critical points of adjacent index, or there is a flow line

between two critical points of the same index. The possibilities, and the way the generalized Heegaard diagram changes, are:

**Index 0 and 1 birth–death:** A disjoint  $\alpha$ -circle is added or deleted, without changing the span of the  $\alpha$ -circles.

**Index 1 and 2 birth–death:** The Heegaard diagram is stabilized or destabilized.

**Index 2 and 3 birth–death:** A disjoint  $\beta$ -circle is added or deleted, without changing the span of the  $\beta$ -circles.

**Flow line between index 1 critical points:** A handleslide of an  $\alpha$ -circle or  $\alpha$ -arc over an  $\alpha$ -circle. Since our conditions imply that the descending disk of the index 1 critical points corresponding to  $\alpha$ -arcs lie entirely on the boundary, we cannot have a handleslide over an  $\alpha$ -arc.

**Flow line between index 2 critical points:** A handleslide of a  $\beta$ -circle over a  $\beta$ -circle.

Note also that the critical points on the boundary cannot be involved in any cancellations, since  $f$  was fixed on all of  $N(\partial Y)$ .

Now, given any two bordered Heegaard diagrams  $\mathcal{H}_0$  and  $\mathcal{H}_1$  for  $Y$ , in view of the above remarks, we can stabilize  $\mathcal{H}_0$  and  $\mathcal{H}_1$  to obtain a pair of bordered Heegaard diagrams  $\mathcal{H}'_0$  and  $\mathcal{H}'_1$  with of the same genus and with the following property. Each of  $\mathcal{H}'_0$  and  $\mathcal{H}'_1$  can be extended by adding new  $\alpha$  and  $\beta$  circles if necessary to form a pair of generalized bordered Heegaard diagrams  $\mathcal{H}''_0$  and  $\mathcal{H}''_1$  respectively, which in turn can be connected by isotopies and handleslides (with the above constraints: i.e.,  $\alpha$ -arcs are allowed to slide over  $\alpha$ -circles, but not vice versa). The proposition is proved then once we show that for each generalized Heegaard diagram  $\mathcal{H}''$  of genus  $g$ , if  $\mathcal{H}_1$  and  $\mathcal{H}_2$  are any two ordinary bordered Heegaard diagrams obtained from  $\mathcal{H}''$  by picking out  $g - k$  of the  $\alpha$ -circles and  $g$  of the  $\beta$ -circles, then  $\mathcal{H}_1$  and  $\mathcal{H}_2$  can be connected by handleslides and isotopies.

To this end, consider any two  $(g - k)$ -tuples of the  $\alpha$ -circles of  $\mathcal{H}''$  chosen so that, they, along with the  $2k$   $\alpha$ -arcs, span  $H_1(\overline{\Sigma}, \partial\overline{\Sigma})$ . The fact that these two  $(g - k)$ -tuples can be connected by isotopies and handleslides follows directly from Lemma 4.6, applied to the surface  $F = \overline{\Sigma} \setminus \{\alpha_1^a, \dots, \alpha_{2k}^a\}$ . It follows similarly that any two  $g$ -tuples of the  $\beta$  circles which span  $H_1(\overline{\Sigma})$  can be connected by handleslides and isotopies. This completes the proof.  $\square$

Let  $(\Sigma, \alpha, \beta, z)$  be the result of attaching a *cylindrical end* to  $(\overline{\Sigma}, \overline{\alpha}, \beta, z)$ , in the sense of symplectic field theory [6]. Topologically  $\Sigma = \overline{\Sigma} \setminus \partial\overline{\Sigma}$ , and  $\alpha_i^a = \overline{\alpha}_i^a \setminus (\partial\overline{\alpha}_i^a)$ . In due course, we will choose a conformal structure on  $\Sigma$  making it a punctured Riemann surface, and so that each  $\alpha_i^a$  is radial near the puncture. Abusing terminology, we will often also refer to  $(\Sigma, \alpha, \beta, z)$  as a pointed bordered Heegaard diagram.

**4.2. Examples of bordered Heegaard diagrams.** Before proceeding to the Heegaard Floer homology, we give here a few examples of bordered Heegaard diagrams.

Fix an oriented surface  $\Sigma$ , equipped with a  $g$ -tuple of pairwise disjoint, homologically independent curves,  $\beta$ , and a  $g - 1$ -tuple of pairwise disjoint, homologically independent curves  $\alpha^c = \{\alpha_1^c, \dots, \alpha_{g-1}^c\}$ . Then  $(\Sigma, \alpha^c, \beta)$  specifies a Heegaard diagram for a three-manifold with torus boundary, and indeed any such three-manifold  $Y$  can be described by a Heegaard diagram of this type. To turn such a diagram into a

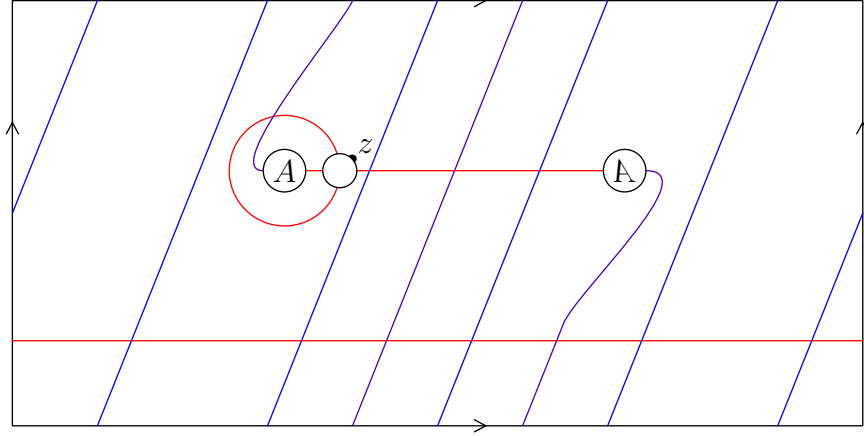


FIGURE 12. Bordered Heegaard diagram for the trefoil complement. This is based on the observation that  $+5$  surgery on the trefoil is the lens space  $L(5, -1)$ , with the knot forming the core of the solid torus a Berge knot. The diagram is on a genus 2 surface, represented as a torus with a handle attached (at  $A$ ).

bordered Heegaard diagram, we proceed as follows. Fix an additional pair of circles  $\gamma_1$  and  $\gamma_2$  in  $\Sigma$  so that:

- $\gamma_1$  and  $\gamma_2$  intersect, transversally, in a single point  $p$  and
- both of the homology classes  $[\gamma_1]$  and  $[\gamma_2]$  are homologically independent from  $[\alpha_1^c], \dots, [\alpha_{g-1}^c]$ .

Let  $D$  be a disk around  $p$  which is disjoint from all the above curves, except for  $\gamma_1$  and  $\gamma_2$ , each of which it meets in a single arc. Then, the complement of  $D$  specifies a bordered Heegaard diagram for  $Y$ , for some parametrization of  $\partial Y$ . A bordered Heegaard diagram for the trefoil complement is illustrated in Figure 12. (See also Section 10 for a further discussion of bordered three-manifolds with torus boundary.)

It is easy to construct examples where the boundary has higher genus by taking boundary connect sums. Specifically, fix bordered, pointed Heegaard diagrams  $\mathcal{H}_i = (\bar{\Sigma}_i, \bar{\alpha}_i, \beta_i, z_i)$  for  $Y_i$  with  $i = 1, 2$ . Take the boundary connect sum of  $\bar{\Sigma}_1$  and  $\bar{\Sigma}_2$  along  $z_1$  and  $z_2$ , i.e., attach a rectangle to  $\bar{\Sigma}_1 \cup \bar{\Sigma}_2$  by gluing two opposite sides to  $\partial \Sigma_1$  and  $\partial \Sigma_2$  along intervals containing  $z_1$  and  $z_2$ . Introduce a new basepoint  $z$  supported in the rectangle. This procedure gives a bordered Heegaard diagram for the boundary connect sum of  $Y_1$  with  $Y_2$ . We illustrate this in Figure 13, where we form the boundary connect sum of two bordered Heegaard diagrams for genus one handlebodies, to obtain a bordered Heegaard diagram for a genus two handlebody.

In fact, it is easy to see that any bordered Heegaard diagram in which there are no  $\alpha$  circles necessarily represents a genus  $g$  handlebody. In Figure 14, we have illustrated a genus two handlebody with respect to a parameterization of the genus two surface which is different from the one used in Figure 13.

**4.3. Generators, homology classes and  $\text{spin}^c$  structures.** Fix a pointed bordered Heegaard diagram, specifying some three-manifold  $Y$ .

**Definition 4.7.** By a *generator* we mean a  $g$ -element subset  $\mathbf{x} = \{x_1, \dots, x_g\}$  so that

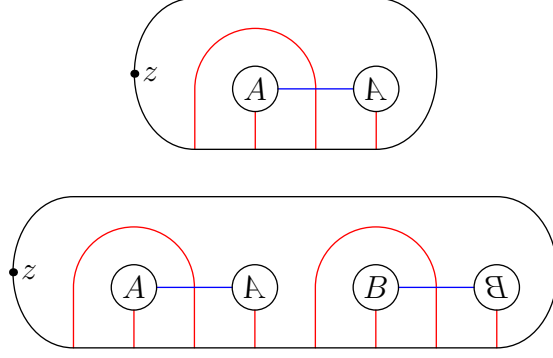


FIGURE 13. Top: a bordered genus one handlebody. Bottom: the boundary connect sum of two copies of the handlebody on top.

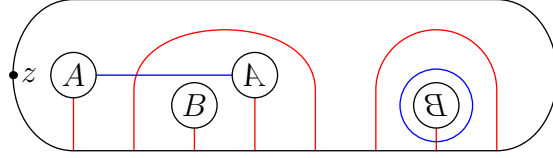


FIGURE 14. A bordered Heegaard diagram for a genus two handlebody, using a different parameterization of the boundary.

- exactly one  $x_i$  lies on each  $\beta$ -circle,
- exactly one  $x_i$  lies on each  $\alpha$ -circle and
- at most one  $x_i$  lies on each  $\alpha$ -arc.

Let  $\mathfrak{S}(\mathcal{H})$  or  $\mathfrak{S}(\Sigma, \alpha, \beta)$  denote the set of generators. Given a generator  $\mathbf{x}$ , let  $o(\mathbf{x})$  denote the set of  $\alpha$ -arcs which are occupied by  $\mathbf{x}$ , i.e.,  $\{i \mid \mathbf{x} \cap \alpha_i^a \neq \emptyset\} \subset [2k]$ .

Soon (Section 5 and later) we will be interested in holomorphic curves in  $\Sigma \times I_s \times \mathbb{R}_t$ , where  $I_s = [0, 1]$  is the unit interval with parameter  $s$  and  $\mathbb{R}_t$  is  $\mathbb{R}$  with parameter  $t$ . These curves have boundary on  $\alpha \times \{1\} \times \mathbb{R}_t$  and  $\beta \times \{0\} \times \mathbb{R}_t$ . These will be asymptotic to  $g$ -tuples of chords  $\mathbf{x} \times I_s$  and  $\mathbf{y} \times I_s$  at  $\pm\infty$ , where  $\mathbf{x}$  and  $\mathbf{y}$  are generators. Each such curve carries a relative homology class, and we let  $\pi_2(\mathbf{x}, \mathbf{y})$  denote the set of these relative homology classes. More precisely, we make the following definition.

**Definition 4.8.** Fix generators  $\mathbf{x}$  and  $\mathbf{y}$ . Let  $I_s = [0, 1]$  and  $I_t = [-2, 2]$  be intervals. We work in the relative homology group

$$H_2(\overline{\Sigma} \times I_s \times I_t, ((S_\alpha \cup S_\beta \cup S_\partial) \times I_t) \cup (G_\mathbf{x} \times \{-2\}) \cup (G_\mathbf{y} \times \{2\})),$$

where

$$\begin{aligned} S_\alpha &= \alpha \times \{1\} \\ S_\beta &= \beta \times \{0\} \\ S_\partial &= (\partial\overline{\Sigma} \setminus z) \times I_s \\ G_\mathbf{x} &= \mathbf{x} \times I_s \\ G_\mathbf{y} &= \mathbf{y} \times I_s. \end{aligned}$$

Let  $\pi_2(\mathbf{x}, \mathbf{y})$ , the *homology classes* from  $\mathbf{x}$  to  $\mathbf{y}$ , denote those elements of this group which map to the relative fundamental class of  $\mathbf{x} \times I_s \cup \mathbf{y} \times I_s$  under the composition of the boundary homomorphism and collapsing the remainder of the boundary.

*Remark 4.9.* We use the notation  $\pi_2$  for this object to agree with the original conventions [33]. In the cylindrical setting of Heegaard Floer homology this is a homology class rather than a homotopy class.

Given a homology class  $B \in \pi_2(\mathbf{x}, \mathbf{y})$ , projecting to  $\bar{\Sigma}$  gives a well-defined element of  $H_2(\bar{\Sigma}, \alpha \cup \beta \cup \partial\bar{\Sigma})$ , i.e., a linear combination of the regions of  $\bar{\Sigma} \setminus (\alpha \cup \beta)$ , which we call the *domain* of  $B$ . It is not hard to see that a homology class  $B$  is uniquely determined by its domain. Given any point  $p \in \bar{\Sigma} \setminus (\alpha \cup \beta)$ , the *local multiplicity* of  $B$  at  $p$  is the local degree of the two-chain at the point  $p$ .

Notice that concatenation (in the  $t$  factor) gives a product  $*$ :  $\pi_2(\mathbf{x}, \mathbf{y}) \times \pi_2(\mathbf{y}, \mathbf{w}) \rightarrow \pi_2(\mathbf{x}, \mathbf{w})$ ; this operation corresponds to addition of domains. If  $\pi_2(\mathbf{x}, \mathbf{y})$  is nonempty then concatenation makes  $\pi_2(\mathbf{x}, \mathbf{y})$  into a free, transitive  $\pi_2(\mathbf{x}, \mathbf{x})$ -set. Elements of  $\pi_2(\mathbf{x}, \mathbf{x})$  are called *periodic classes*; their domains are *periodic domains*.

**Lemma 4.10.** *There is a natural group isomorphism  $\pi_2(\mathbf{x}, \mathbf{x}) \cong H_2(Y, \partial Y)$ .*

To see this, note that we can identify  $H_2(Y, \partial Y)$  with  $H_2(\bar{\Sigma} \times I_s, S_\alpha \cup S_\beta \cup S_\partial)$ .

We can use Lemma 4.10 to construct an action of  $H_2(Y, \partial Y)$  on  $\pi_2(\mathbf{x}, \mathbf{y})$  for any  $\mathbf{x}$  and  $\mathbf{y}$  via inclusion at the  $t = 0$  level. It is straightforward to see that this is a free, transitive action and gives a group homomorphism if we map the identity to the vertical strip. For details, see [21, Lemma 2.6.1]. We sometimes denote the image of  $B \in \pi_2(\mathbf{x}, \mathbf{x})$  in  $H_2(Y, \partial Y)$  by  $[B]$ .

We split up the boundary of the domain of  $B$  into three pieces,  $\partial^\alpha(B)$ , contained in  $\alpha$ ,  $\partial^\beta(B)$ , contained in  $\beta$ , and  $\partial^\partial(B)$ , contained in  $\partial\bar{\Sigma}$ . We orient them so that the oriented boundary of the domain of  $B$  is  $\partial^\alpha(B) - \partial^\beta(B) + \partial^\partial(B)$ ; in particular, this implies that  $\partial(\partial^\alpha(B))$  and  $\partial(\partial^\beta(B))$  are both equal to  $\mathbf{y} - \mathbf{x}$  relative to  $\partial\bar{\Sigma}$ . We will think of  $\partial^\partial(B)$  as an element of  $H_1(\partial\bar{\Sigma}, \mathbf{a})$ .

**Definition 4.11.** Let  $\pi_2^\partial(\mathbf{x}, \mathbf{y})$  denote the kernel of the map  $\partial^\partial$ . Elements of  $\pi_2^\partial(\mathbf{x}, \mathbf{y})$  are called *provincial homology classes*.

In other words, a homology class  $B$  belongs to  $\pi_2^\partial(\mathbf{x}, \mathbf{y})$  if and only if the domain of  $B$  does not include any of the regions adjacent to  $\partial\bar{\Sigma}$ .

The concatenation maps obviously restrict to maps  $\pi_2^\partial(\mathbf{x}, \mathbf{y}) \times \pi_2^\partial(\mathbf{y}, \mathbf{w}) \rightarrow \pi_2^\partial(\mathbf{x}, \mathbf{w})$ , again making  $\pi_2^\partial(\mathbf{x}, \mathbf{y})$  into a free, transitive  $\pi_2^\partial(\mathbf{x}, \mathbf{x})$ -set (when nonempty). Elements of  $\pi_2^\partial(\mathbf{x}, \mathbf{x})$  are called *provincial periodic classes*.

**Lemma 4.12.** *There is a natural isomorphism  $\pi_2^\partial(\mathbf{x}, \mathbf{x}) \cong H_2(Y)$ .*

Again, the proof is straightforward and based on the fact that  $H_2(Y) \cong H_2(\bar{\Sigma}, S_\alpha \cup S_\beta)$ ; see [21, Lemma 2.6.4]. Abusing notation, we may also sometimes denote the image of  $B \in \pi_2^\partial(\mathbf{x}, \mathbf{x})$  in  $H_2(Y)$  by  $[B]$ .

Next we ask when  $\pi_2(\mathbf{x}, \mathbf{y})$  and  $\pi_2^\partial(\mathbf{x}, \mathbf{y})$  are nonempty. As in the closed case, the answer relates to  $\text{spin}^c$  structures on  $Y$ .

Choose a self-indexing Morse function  $f$  (and metric) on  $Y$  inducing the bordered Heegaard diagram  $(\bar{\Sigma}, \bar{\alpha}, \bar{\beta})$ . Let  $\mathbf{x}$  be a generator. Using  $\nabla f$ ,  $z$  and  $\mathbf{x}$  we will

construct a non-vanishing vector field  $\vec{v}_z(\mathbf{x})$ ; this reduces the structure group of  $TY$  from  $SO(3)$  to  $SO(2) = U(1)$ , and composing with the inclusion  $U(1) \hookrightarrow U(2) = \text{spin}^c(3)$  defines a  $\text{spin}^c$  structure on  $Y$ .

Each  $x_i \in \mathbf{x}$  lies on a flow line  $\gamma_i$  of  $\nabla f$  from an index 1 critical point of  $f$  to an index 2 critical point of  $f$ . The point  $z$  lies on a flow line  $\gamma$  from the index 0 critical point of  $f$  to the index 3 critical point of  $f$ . Let  $B$  denote a regular neighborhood of  $\gamma_1 \cup \dots \cup \gamma_g \cup \gamma$ . Since each flow line  $\gamma_i$  or  $\gamma$  runs between critical points of opposite parity, it is possible to extend  $\nabla f|_{Y \setminus B}$  to a vector field  $\vec{v}_z^0(\mathbf{x})$  on  $Y$  so that  $\vec{v}_z^0(\mathbf{x})$  is non-vanishing on  $B$ . The vector field  $\vec{v}_z^0(\mathbf{x})$  still has exactly  $k$  zeroes, one for each  $\bar{\alpha}_i^a$  not containing an  $x_i$ . All of these zeroes lie on  $\partial Y$ , however, so one can modify  $\vec{v}_z^0(\mathbf{x})$  in a neighborhood of these zeroes to produce a non-vanishing vector field  $\vec{v}_z(\mathbf{x})$ . This vector field  $\vec{v}_z(\mathbf{x})$  induces the  $\text{spin}^c$  structure  $\mathfrak{s}_z(\mathbf{x})$ . It is routine to verify that  $\mathfrak{s}_z(\mathbf{x})$  does not depend on the choices made in its construction.

Notice that the restriction of  $\mathfrak{s}_z(\mathbf{x})$  to a collar neighborhood of  $\partial Y$  depends only on  $o(\mathbf{x})$ , and not on  $\mathbf{x}$  itself. That is, for  $o$  a  $k$ -element subset of  $\{\bar{\alpha}_1^a, \dots, \bar{\alpha}_{2k}^a\}$ , there is an induced  $\text{spin}^c$  structure  $\mathfrak{s}_z(o)$  on a collar neighborhood of  $\partial Y$ . Given  $\mathbf{x}$ , let  $\mathfrak{s}_z^{\text{rel}}(\mathbf{x})$  denote the relative  $\text{spin}^c$  structure on  $(Y, \text{nbd}(\partial Y))$  induced by  $(z, \mathbf{x})$ , relative to  $\mathfrak{s}_z(o)$ .

**Lemma 4.13.** *Given generators  $\mathbf{x}$  and  $\mathbf{y}$ ,  $\pi_2(\mathbf{x}, \mathbf{y}) \neq \emptyset$  if and only if  $\mathfrak{s}_z(\mathbf{x}) = \mathfrak{s}_z(\mathbf{y})$ . Further,  $\pi_2^\partial(\mathbf{x}, \mathbf{y}) \neq \emptyset$  if and only if  $o(\mathbf{x}) = o(\mathbf{y})$  and  $\mathfrak{s}_z^{\text{rel}}(\mathbf{x}) = \mathfrak{s}_z^{\text{rel}}(\mathbf{y})$ .*

See [21, Lemmas 2.6.2 and 2.6.5] for the proof, which is a straightforward adaptation of the closed case. For  $\mathfrak{s}$  a  $\text{spin}^c$  structure on  $Y$ , define  $\mathfrak{S}(\mathcal{H}, \mathfrak{s}) \subset \mathfrak{S}(\mathcal{H})$  to be those generators  $\mathbf{x}$  so that  $\mathfrak{s}_z(\mathbf{x}) = \mathfrak{s}$ . Similarly for  $\mathfrak{s}^{\text{rel}}$  a relative  $\text{spin}^c$  structure, define  $\mathfrak{S}(\mathcal{H}, \mathfrak{s}^{\text{rel}})$  to be those  $\mathbf{x}$  so that  $\mathfrak{s}_z^{\text{rel}}(\mathbf{x}) = \mathfrak{s}^{\text{rel}}$ .

**4.4. Admissibility criteria.** As in [33], we need certain additional hypothesis to ensure the definitions of the differentials involve only finite sums. We will have two different analogues of the traditional “weak admissibility” criterion, one of which we call *provincial admissibility* and the other of which we call simply *admissibility*.

In Sections 6 and 7, provincial admissibility will be sufficient to define the invariants of bordered 3-manifolds. However, in order for the tensor product involved in gluing two bordered 3-manifolds to make sense, one of the two manifolds has to be admissible and not just provincially admissible.

**Definition 4.14.** A bordered Heegaard diagram is called *provincially admissible* if every provincial periodic domain has both positive and negative coefficients.

**Definition 4.15.** A pointed bordered Heegaard diagram is called *admissible* if every periodic domain has both positive and negative coefficients.

**Proposition 4.16.** *Every bordered Heegaard diagram is isotopic to an admissible bordered Heegaard diagram. Further, any two admissible bordered Heegaard diagrams can be connected through a sequence of Heegaard moves in which every intermediate Heegaard diagram is admissible. The same statements hold if “admissible” is replaced by “provincially admissible.”*

*Proof.* This follows by winding transverse to the  $\beta$ -circles, as in the case of closed manifolds [33].  $\square$

As in the closed case, there are reformulations of the two admissibility criteria:

**Lemma 4.17.** *A pointed bordered Heegaard diagram  $(\Sigma, \alpha, \beta)$  is admissible if and only if there is an area function  $A$  on  $\Sigma$  such that  $A(P) = 0$  for any periodic domain  $P$ . A bordered Heegaard diagram  $(\Sigma, \alpha, \beta)$  is provincially admissible if and only if there is an area function  $A$  on  $\Sigma$  such that  $A(P) = 0$  for any provincial periodic domain  $P$ .*

*Proof.* The “only if” direction is immediate. For the “if” direction, proof follows exactly as in the closed case, see Lemma [33, 4.12]. For the reader’s benefit, we recast the result here in terms of Farkas’s lemma from the theory of convex sets [7]:

**Lemma 4.18.** *Let  $V$  be a vector space and  $\{p_i\}_{i=1}^N \subset V$  a finite set of vectors. Then either*

- *there is a nonzero linear functional  $\ell \in V^*$  such that  $\ell(p_i) \geq 0$  for all  $i$  or*
- *there are constants  $c_i > 0$  such that  $\sum_{i=1}^N c_i p_i = 0$ .*

Now, to prove the first statement, enumerate the regions  $R_1, \dots, R_N$  of  $\Sigma$ . Take  $V^*$  to be the vector space generated by the periodic domains in  $\Sigma$ , and  $p_i \in V = V^{**}$  to be evaluation at the  $i^{\text{th}}$  region  $R_i$ . By assumption, there is no nonzero  $P \in V^*$  such that  $p_i(V) \geq 0$  for all  $i$ . Consequently, there are constants  $c_i > 0$  such that  $\sum_{i=1}^N c_i p_i = 0$ . But then setting the area of  $R_i$  to be  $c_i$  is the desired area function.

The second statement follows by just the same reasoning, using only the provincial periodic domains.  $\square$

**Proposition 4.19.** *Suppose that  $\mathcal{H}$  is provincially admissible, as in Definition 4.14. Fix generators  $\mathbf{x}, \mathbf{y} \in \mathfrak{S}(\mathcal{H})$  and an element  $h \in H_1(Z', \mathbf{a})$ . Then there are finitely many  $B \in \pi_2(\mathbf{x}, \mathbf{y})$  such that  $\partial^\partial B = h$  and all coefficients of  $B$  are non-negative.*

*Proof.* Let  $A$  be an area form as guaranteed by Lemma 4.17. If  $B, B' \in \pi_2(\mathbf{x}, \mathbf{y})$  with  $\partial^\partial B = \partial^\partial B' = h$  then  $B - B'$  is a provincial periodic domain. Consequently the areas  $A(B)$  and  $A(B')$  are equal. But there are only finitely many positive domains of any given area.  $\square$

**Proposition 4.20.** *Suppose that  $\mathcal{H}$  is admissible, as in Definition 4.15. There are finitely many generators  $\mathbf{x}, \mathbf{y} \in \mathfrak{S}(\mathcal{H})$  and  $B \in \pi_2(\mathbf{x}, \mathbf{y})$  such that all coefficients of  $B$  are non-negative.*

*Proof.* There are only finitely many generators, so we may first fix  $\mathbf{x}$  and  $\mathbf{y}$ . Let  $A$  be an area form as guaranteed by Lemma 4.17. If  $B, B' \in \pi_2(\mathbf{x}, \mathbf{y})$  then  $B - B'$  is a periodic domain. Consequently the areas  $A(B)$  and  $A(B')$  are equal. But there are only finitely many positive domains of any given area.  $\square$

**4.5. Closed diagrams.** Let  $\mathcal{H}_1 = (\Sigma_1, \alpha_1, \beta_1, z)$  and  $\mathcal{H}_2 = (\Sigma_2, \alpha_2, \beta_2, z)$  be two bordered Heegaard diagrams for two bordered 3-manifolds with boundary,  $(Y_1, \partial Y_1)$  and  $(Y_2, \partial Y_2)$ , with compatible marked boundaries:  $F = \partial Y_1 = -\partial Y_2$ . Then we can form the closed three-manifold  $Y = Y_1 \cup_F Y_2$ . This manifold naturally comes with a Heegaard diagram  $\mathcal{H} = \mathcal{H}_1 \cup_\partial \mathcal{H}_2$ , whose underlying surface is obtained as  $\Sigma = \bar{\Sigma}_1 \cup_\partial \bar{\Sigma}_2$ , equipped with curves  $\alpha = \bar{\alpha}_1 \cup_\partial \bar{\alpha}_2$  and  $\beta = \beta_1 \cup \beta_2$  and base point  $z$ .

Fix generators  $\mathbf{x}_1 \in \mathfrak{S}(\mathcal{H}_1)$  and  $\mathbf{x}_2 \in \mathfrak{S}(\mathcal{H}_2)$ . If the corresponding sets of occupied  $\alpha$ -arcs are disjoint, i.e.,  $o(\mathbf{x}_1) \cap o(\mathbf{x}_2) = \emptyset$ , then the union  $\mathbf{x}_1 \cup \mathbf{x}_2$  can be naturally viewed as a generator in  $\mathfrak{S}(\mathcal{H})$  for the Heegaard Floer complex of the closed manifold  $Y$ .

**Lemma 4.21.** *Let  $(\mathcal{H}_1, z) = (\Sigma_1, \alpha_1, \beta_1, z)$  and  $(\mathcal{H}_2, z) = (\Sigma_2, \alpha_2, \beta_2, z)$  be two bordered Heegaard diagrams for the bordered three-manifolds  $(Y_1, \partial Y_1)$  and  $(Y_2, \partial Y_2)$ , with compatible marked boundaries:  $F = \partial Y_1 = -\partial Y_2$ . Given  $\mathbf{x}_1, \mathbf{y}_1 \in \mathfrak{S}(\mathcal{H}_1)$  and  $\mathbf{x}_2, \mathbf{y}_2 \in \mathfrak{S}(\mathcal{H}_2)$ , we have a natural identification of  $\pi_2(\mathbf{x}_1 \cup \mathbf{x}_2, \mathbf{y}_1 \cup \mathbf{y}_2)$  with the subset of  $\pi_2(\mathbf{x}_1, \mathbf{y}_1) \times \pi_2(\mathbf{x}_2, \mathbf{y}_2)$  consisting of  $B_1, B_2$  with  $\partial^\partial(B_1) + \partial^\partial(B_2) = 0$ . Moreover, given  $p \in \Sigma_2 \setminus (\alpha \cup \beta)$  and  $B_i \in \pi_2(\mathbf{x}_i, \mathbf{y}_i)$  for  $i = 1, 2$  which agree along the boundary, the local multiplicity of  $B_2$  at  $p$  coincides with the local multiplicity of the glued homology class in  $\pi_2(\mathbf{x}_1 \cup \mathbf{x}_2, \mathbf{y}_1 \cup \mathbf{y}_2)$  (under the above identification) at the point  $p$ , now thought of as a point in  $\Sigma = \Sigma_1 \cup \Sigma_2$ .*

*Proof.* The proof is straightforward.  $\square$

For  $B_1, B_2$  which agree along the boundary, let  $B_1 \natural B_2$  be the homology class in  $\pi_2(\mathbf{x}_1 \cup \mathbf{x}_2, \mathbf{y}_1 \cup \mathbf{y}_2)$  guaranteed by Lemma 4.21.

Finally, we give a criterion for when a closed Heegaard diagram is admissible.

**Lemma 4.22.** *Let  $\mathcal{H}_1$  and  $\mathcal{H}_2$  be two bordered Heegaard diagrams with  $\partial \mathcal{H}_1 = -\partial \mathcal{H}_2$ . If  $\mathcal{H}_1$  is admissible and  $\mathcal{H}_2$  is provincially admissible, then  $\mathcal{H} = \mathcal{H}_1 \cup_\partial \mathcal{H}_2$  is admissible.*

*Proof.* Let  $B_1 \natural B_2$  be a positive periodic domain for  $\mathcal{H}$ . Since  $\mathcal{H}_1$  is admissible,  $B_1 = 0$  and so  $\partial^\partial B_2 = 0$ . Then since  $\mathcal{H}_2$  is provincially admissible,  $B_2 = 0$ .  $\square$

## 5. MODULI SPACES

Let  $\mathcal{H} = (\bar{\Sigma}, \alpha, \beta, z)$  be a pointed bordered Heegaard diagram representing a bordered 3-manifold  $Y$ . As mentioned in the Introduction, we want to count  $J$ -holomorphic curves in  $\Sigma \times [0, 1] \times \mathbb{R}$  to get invariants of  $Y$ . In this section we give the technical results concerning the moduli spaces of holomorphic curves that we use to construct the invariants  $\widehat{CFD}(\mathcal{H})$  and  $\widehat{CFA}(\mathcal{H})$ . We begin with an outline of the main results that we will need and the ideas in the proof; the precise mathematics begins in Section 5.1 below.

We count  $J$ -holomorphic curves<sup>1</sup>

$$u: (S, \partial S) \rightarrow (\Sigma \times [0, 1] \times \mathbb{R}, (\alpha \times \{1\} \times \mathbb{R}) \cup (\beta \times \{0\} \times \mathbb{R}))$$

where  $S$  is a Riemann surface with boundary and punctures. Here we view  $\Sigma$ , the interior of  $\bar{\Sigma}$ , as a Riemann surface with one puncture, denoted  $p$ . As in Section 4.3, the coordinate on the  $[0, 1]$  factor is denoted  $s$  and the coordinate on the  $\mathbb{R}$  factor is denoted  $t$ .

We view the manifold  $\Sigma \times [0, 1] \times \mathbb{R}$  as having three different infinities:  $\Sigma \times [0, 1] \times \{+\infty\}$ ,  $\Sigma \times [0, 1] \times \{-\infty\}$  and  $p \times [0, 1] \times \mathbb{R}$ , which we refer to as  $+\infty$ ,  $-\infty$  and  $e\infty$  respectively. (Here,  $e$  stands for “east”.) The asymptotics of the holomorphic curves  $u$  we consider are:

- At  $\pm\infty$ ,  $u$  is asymptotic to a  $g$ -tuple of chords of the form  $x_i \times [0, 1] \times \{\pm\infty\}$ , where  $\mathbf{x} = \{x_i\}_{i=1}^g$  is a generator in the sense of Definition 4.7.
- At  $e\infty$ ,  $u$  is asymptotic to a finite collection of Reeb chords  $\{\rho_i \times \{(1, t_i)\}\}$ . Here  $\rho_i$  is a Reeb chord in the ideal contact boundary  $\partial \bar{\Sigma}$  (also called  $Z$ ) with endpoints on  $\mathbf{a} = \alpha \cap Z$  (see Section 3.1.3). We refer to  $t_i$  as the *height* of the chord  $\rho_i$ .

<sup>1</sup>With respect to an appropriate  $J$ ; see Definition 5.1 below.

The set of east punctures of  $u$  is partially ordered by the heights of the corresponding Reeb chords. This induces an *ordered partition*  $\vec{P} = (P_1, P_2, \dots, P_\ell)$  of the east punctures, where each  $P_i$  consists of those punctures occurring at one particular height. (For  $\widehat{CFD}(\mathcal{H})$  we only need to consider discrete partitions  $P$ , where each  $P_i$  has only one puncture.)

With this background, we consider the moduli space

$$\widetilde{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P})$$

(Definition 5.8), which consists of curves from the *decorated source*  $S^\triangleright$ , asymptotic to  $\mathbf{x}$  at  $-\infty$  and  $\mathbf{y}$  at  $+\infty$ , in the homology class  $B$  (as in Section 4.3) and respecting the ordered partition  $\vec{P}$  of the Reeb chords. Here the source surface  $S$  is decorated with labels describing its asymptotics (Definition 5.2). We also consider the reduced moduli space, its quotient

$$\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P}) := \widetilde{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P}) / \mathbb{R}$$

by translation in the  $t$  coordinate.

These moduli spaces are smooth manifolds (Proposition 5.5) whose dimensions are easy to compute (Equation (5.7)). In fact, for an *embedded* curve the dimension is independent of the source  $S$  and depends only on the asymptotics and homology class of the curve (Section 5.5); this gives rise to the gradings on the algebra and on the modules.

As is typical in symplectic geometry, our invariants are defined by looking only at those  $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P})$  which are 0-dimensional. To show that  $\partial^2 = 0$  and for proofs of invariance, we also need to consider 1-dimensional moduli spaces, and, in particular, we need appropriate compactifications. These are provided by the spaces  $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P})$  which include *holomorphic combs*. The idea is that, as in symplectic field theory, some parts of a holomorphic curve may go to infinity relative to other parts. In our case, this can result in a degeneration at  $\pm\infty$  or at  $e\infty$ . When a curve degenerates at  $\pm\infty$ , the result is two or more curves of the kind we have already discussed. A degeneration at  $e\infty$  creates a new kind of object: a curve in  $Z \times \mathbb{R} \times [0, 1] \times \mathbb{R}$ . Section 5.2 is devoted to discussing these curves at east infinity. This compactification is defined in Definition 5.18, and a schematic illustration of a holomorphic comb can be found in Figure 19.

The main result we need for our purposes is a classification of the codimension 1 degenerations (Section 5.4). Proposition 5.32 and Lemma 5.37 say that for the curves we consider, these degenerations come in four kinds:

- Degenerating into a two-story building at  $\pm\infty$ . These are exactly the kinds of degenerations that occur in Heegaard Floer homology for closed three-manifolds.
- A boundary branch point of  $\pi_\Sigma \circ u$  can go off to  $e\infty$ , in the process splitting a Reeb chord in two. Below, this is referred to as degenerating a “join curve” at  $e\infty$ . (From the point of view of the curve at  $e\infty$ , two Reeb chords come together and merge as we travel further away from the main surface.)
- A collapse in the ordering of  $\vec{P}$ , i.e., the heights of two parts of  $\vec{P}$  coming together. In the process, some “split curves” can degenerate at  $e\infty$ . These

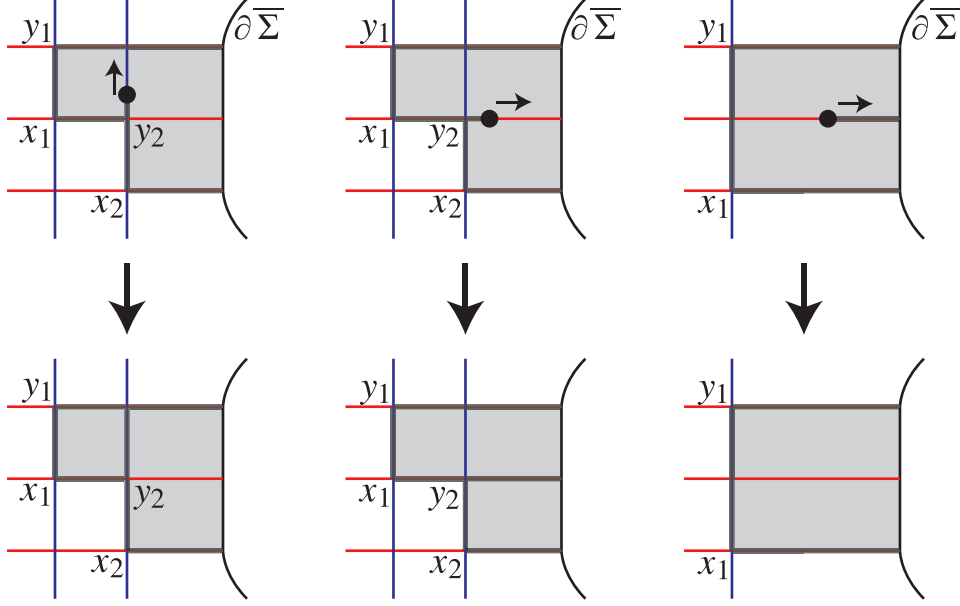


FIGURE 15. Examples of three kinds of codimension 1 degenerations. The large black dot represents a boundary branch point of  $\pi_\Sigma \circ u$ . Left: degenerating into a two-story building. Center: degenerating a join curve. Right: degenerating a split curve.

again correspond to boundary branch points of  $\pi_\Sigma \circ u$  approaching  $e\infty$ , but with different topology.

- Degenerating a “shuffle curve” at east  $\infty$ . This degeneration involves either an interior branch point or two boundary branch points approaching east  $\infty$ , and is explained in more detail at the end of Section 5.2; see in particular Figure 18 for a domain in which a shuffle curve degenerates. This type of degeneration is not relevant for  $\widehat{CFD}$ , only for  $\widehat{CFA}$ .

Figure 5 shows examples in which the first three kinds of degenerations occur.

One would now like to assert that if  $\text{ind}(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P}) = 2$ , then the space  $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P})$  is a compact one-manifold with boundary given by all holomorphic combs with the correct combinatorics to occur as one of the degenerations listed above, and use the fact that a one-manifold has an even number of endpoints.

Unfortunately, available gluing technology seems ill equipped to prove this assertion, the difficulty being with split and shuffle curve degenerations. (See Remarks 5.30 and 5.38 for more discussion of this point.) Instead, we prove that the sum of the numbers of combs in moduli spaces with correct combinatorics to occur as one of the four types of degenerations listed above is zero modulo 2 (Theorem 5.41). This is all that we will need later to define the invariants.

Fortified by this overview, we turn to the mathematics itself.

**5.1. Holomorphic curves in  $\Sigma \times [0, 1] \times \mathbb{R}$ .** In this subsection, we will define various moduli spaces of holomorphic curves in  $\Sigma \times [0, 1] \times \mathbb{R}$  with names like  $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright)$ ,

$\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$  and  $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P})$ . In Sections 6 and 7 we will use some of these moduli spaces (namely  $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P})$ ) to define our invariants of bordered 3-manifolds. These are, consequently, the spaces in which we will ultimately be interested. However, it is easier to formulate several technical results by describing the spaces  $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P})$  as subspaces of  $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright)$  and  $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$ , which we introduce first.

As before, let  $\mathcal{H}$  be a pointed bordered Heegaard diagram of genus  $g$  representing a manifold  $Y$  with boundary of genus  $k$ . Choose a symplectic form  $\omega_\Sigma$  on  $\Sigma$ , with respect to which the boundary of  $\Sigma$  is a cylindrical end. Let  $j_\Sigma$  be a complex structure on  $\Sigma$  compatible with  $\omega_\Sigma$ . With respect to  $j_\Sigma$ , the boundary is a puncture, which we denote  $p$ . We will assume that the  $\alpha$ -arcs are cylindrical near  $p$  in the following sense. Fix a punctured neighborhood  $U_p$  of  $p$ , and a symplectic identification  $\phi : U_p \xrightarrow{\cong} S^1 \times (0, \infty) \subset T^*S^1$ . Then we assume that both  $j_\Sigma$  and  $\phi(\alpha_i^a \cap U_p)$  are invariant with respect to the  $\mathbb{R}$ -translation action in  $S^1 \times (0, \infty)$ .

Let  $\Sigma_{\bar{e}}$  be the result of filling in the puncture of  $\Sigma$ , so  $j_\Sigma$  induces an almost complex structure on  $\Sigma_{\bar{e}}$ .

We consider moduli spaces of holomorphic curves in  $\Sigma \times [0, 1] \times \mathbb{R}$ . Let

$$\begin{aligned} \pi_\Sigma &: \Sigma \times [0, 1] \times \mathbb{R} \rightarrow \Sigma, \\ \pi_\mathbb{D} &: \Sigma \times [0, 1] \times \mathbb{R} \rightarrow [0, 1] \times \mathbb{R}, \\ s &: \Sigma \times [0, 1] \times \mathbb{R} \rightarrow [0, 1], \text{ and} \\ t &: \Sigma \times [0, 1] \times \mathbb{R} \rightarrow \mathbb{R} \end{aligned}$$

denote the projections. We equip  $[0, 1] \times \mathbb{R}$  with the symplectic form  $\omega_\mathbb{D} := ds \wedge dt$  and the complex structure  $j_\mathbb{D}$  with  $j_\mathbb{D}(\frac{\partial}{\partial s}) = \frac{\partial}{\partial t}$ , and equip  $\Sigma \times [0, 1] \times \mathbb{R}$  with the split symplectic form  $\pi_\Sigma^*(\omega_\Sigma) + \pi_\mathbb{D}^*(\omega_\mathbb{D})$ . Note also that there is an  $\mathbb{R}$ -action on  $\Sigma \times [0, 1] \times \mathbb{R}$  by translation on the  $t$  coordinate.

**Definition 5.1.** We say an almost complex structure  $J$  on  $\Sigma \times [0, 1] \times \mathbb{R}$  is *admissible* if the following conditions are satisfied:

- (1) The projection map  $\pi_\mathbb{D}$  is  $J$ -holomorphic.
- (2) For  $\frac{\partial}{\partial s}$  and  $\frac{\partial}{\partial t}$  the vector fields tangent to the fibers of  $\pi_\Sigma$  induced by  $s$  and  $t$  respectively,  $J\frac{\partial}{\partial s} = \frac{\partial}{\partial t}$  (and consequently the fibers of  $\pi_\Sigma$  are  $J$ -holomorphic).
- (3) The  $\mathbb{R}$ -action is  $J$ -holomorphic.
- (4) The complex structure is split, i.e.,  $J = j_\Sigma \times j_\mathbb{D}$ , near  $p \times [0, 1] \times \mathbb{R}$ .

For example, the split complex structure  $j_\Sigma \times j_\mathbb{D}$  is admissible in the above sense. Fix now an admissible  $J$ ; later, we will assume  $J$  is generic in certain senses.

**Definition 5.2.** By a *decorated source*  $S^\triangleright$  we mean:

- (1) A topological type of smooth (not nodal) Riemann surface  $S$  with boundary and punctures on the boundary.
- (2) A labeling of each puncture of  $S$  by one of  $+$ ,  $-$ , or  $e$ .
- (3) A labeling of each  $e$  puncture of  $S$  by a Reeb chord in  $(Z, \mathbf{a})$ , as defined in Section 3.1.3.

Given a decorated source  $S^\triangleright$ , let  $S_{\bar{e}}$  denote the result of filling in the  $e$  punctures of  $S$ . We will consider maps

$$u: (S, \partial S) \rightarrow (\Sigma \times [0, 1] \times \mathbb{R}, (\boldsymbol{\alpha} \times \{1\} \times \mathbb{R}) \cup (\boldsymbol{\beta} \times \{0\} \times \mathbb{R}))$$

such that

- (1) The map  $u$  is  $(j, J)$ -holomorphic with respect to some almost complex structure  $j$  on  $S$ .
- (2) The map  $u: S \rightarrow \Sigma \times [0, 1] \times \mathbb{R}$  is proper.
- (3) The map  $u$  extends to a proper map  $u_{\bar{e}}: S_{\bar{e}} \rightarrow \Sigma_{\bar{e}} \times [0, 1] \times \mathbb{R}$ .
- (4) The map  $u_{\bar{e}}$  has finite energy in the sense of Bourgeois, Eliashberg, Hofer, Wysocki and Zehnder [2].
- (5)  $\pi_{\mathbb{D}} \circ u$  is a  $g$ -fold branched cover. In particular,  $\pi_{\mathbb{D}} \circ u$  is non-constant on every component of  $S$ .
- (6) At each  $-$ -puncture  $q$  of  $S$ ,  $\lim_{z \rightarrow q} (t \circ u)(z) = -\infty$ .
- (7) At each  $+$ -puncture  $q$  of  $S$ ,  $\lim_{z \rightarrow q} (t \circ u)(z) = +\infty$ .
- (8) At each  $e$  puncture  $q$  of  $S$ ,  $\lim_{z \rightarrow q} (\pi_{\Sigma} \circ u)(z)$  is the Reeb chord  $\rho$  labeling  $q$ .
- (9)  $\pi_{\Sigma} \circ u$  does not cover the region of  $\Sigma$  adjacent to  $z$ .
- (10) For each  $t \in \mathbb{R}$  and each  $i = 1, \dots, g$ ,  $u^{-1}(\beta_i \times \{0\} \times \{t\})$  consists of exactly one point. Similarly, for each  $t \in \mathbb{R}$  and each  $i = 1, \dots, g - k$ ,  $u^{-1}(\alpha_i^c \times \{1\} \times \{t\})$  consists of exactly one point.

We call Condition (10) *weak boundary monotonicity*. Sometimes later we will impose the following additional condition, which we call *strong boundary monotonicity*:

- (11) For each  $t \in \mathbb{R}$  and each  $i = 1, \dots, 2k$ ,  $u^{-1}(\alpha_i^a \times \{1\} \times \{t\})$  consists of *at most* one point.

However, our moduli spaces are easiest to define if we do not initially impose this condition.

It follows from the Conditions (1)–(10) that at  $-\infty$ ,  $u$  is asymptotic to a  $g$ -tuple of chords of the form  $x_i \times [0, 1]$ , where  $x_i \in \boldsymbol{\alpha} \cap \boldsymbol{\beta}$ . By weak boundary monotonicity, the set  $\mathbf{x} = \{x_i\}$  has the property that exactly one point of  $\mathbf{x}$  lies on each  $\alpha$ -circle, and one point on each  $\beta$ -circle. However, more than one point of  $\mathbf{x}$  may lie on the same  $\alpha$ -arc. We call such a  $g$ -tuple a *generalized generator*. If the strong boundary monotonicity condition is also satisfied then at most one  $x_i$  lies on each  $\alpha$ -arc, and hence  $\mathbf{x}$  is a generator in the sense of Section 4.3. Exactly analogous statements hold at  $+\infty$ .

Any curve  $u$  satisfying Conditions (1)–(10) belongs to some homology class  $[u] \in \pi_2(\mathbf{x}, \mathbf{y})$ . (Here,  $\pi_2(\mathbf{x}, \mathbf{y})$  is the obvious generalization of the notion from Section 4.3 to the case of generalized generators.) We collect these curves into moduli spaces:

**Definition 5.3.** Given generalized generators  $\mathbf{x}$  and  $\mathbf{y}$  and a homology class  $B \in \pi_2(\mathbf{x}, \mathbf{y})$ , let

$$\widetilde{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright)$$

denote the moduli space of curves satisfying Conditions (1)–(10), with decorated source  $S^\triangleright$ , asymptotic to  $\mathbf{x}$  at  $-\infty$  and  $\mathbf{y}$  at  $+\infty$ , in the homology class  $B$ .

Given  $u \in \widetilde{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright)$ , for each puncture  $q$  of  $S$  labeled by  $e$ , let  $\text{ev}_q(u) = t \circ u_{\bar{e}}(q)$ . This gives an evaluation map  $\text{ev}_q: \widetilde{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright) \rightarrow \mathbb{R}$ . Set

$$\text{ev} = \prod_{q \in E(S^\triangleright)} \text{ev}_q: \widetilde{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright) \rightarrow \mathbb{R}^{E(S^\triangleright)},$$

where  $E(S^\triangleright)$  (or just  $E$ , when unambiguous) is the set of east punctures of  $S^\triangleright$ .

We next use the evaluation maps to define certain subspaces of the  $\widetilde{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright)$ . Fix a partition  $P = \{P_i\}$  of  $E$ . By the *partial diagonal*  $\Delta_P$  in  $\mathbb{R}^E$  we mean the subspace of  $\mathbb{R}^E$  defined by the set of equations  $\{x_p = x_q \mid P_i \in P, p, q \in P_i\}$ . That is,  $\Delta_P$  is obtained by requiring all coordinates in each part  $P_i$  of  $P$  to be equal. Now, we use the cycle  $\Delta_P$  to cut down the moduli spaces:

**Definition 5.4.** Given generalized generators  $\mathbf{x}$  and  $\mathbf{y}$ ,  $B \in \pi_2(\mathbf{x}, \mathbf{y})$ , a decorated source  $S^\triangleright$ , and a partition  $P$  of the  $e$  punctures of  $S^\triangleright$ , let

$$\widetilde{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P) := \text{ev}^{-1}(\Delta_P) \subset \widetilde{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright).$$

Notice that  $\widetilde{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright)$  is the special case of  $\widetilde{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$  when  $P$  is the discrete partition, with one element per part.

We turn now to the meaning of genericity for  $J$ .

**Proposition 5.5.** *There is a dense set of admissible  $J$  with the property that the moduli spaces  $\widetilde{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright)$  are transversally cut out by the  $\bar{\partial}$ -equations. Indeed, for any countable set  $\{M_i\}$  of submanifolds of  $\mathbb{R}^E$ , there is a dense set of admissible  $J$  which satisfy the further property that  $\text{ev}: \widetilde{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright) \rightarrow \mathbb{R}^E$  is transverse to all of the  $M_i$ .*

The proof is standard. See [28, Chapter 3] for a nice explanation in a slightly different context, or [20, Proposition 3.8] for the proof of the first half of the statement in this setting.

In particular, by Proposition 5.5, we can choose  $J$  so that the moduli spaces  $\widetilde{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$  are transversely cut out for all partitions  $P$ . We do so from now on.

The expected dimension of  $\widetilde{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$  is not hard to calculate:

**Proposition 5.6.** *The expected dimension  $\text{ind}(B, S^\triangleright, P)$  of  $\widetilde{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$  is*

$$(5.7) \quad \text{ind}(B, S^\triangleright, P) = g - \chi(S) + 2e(B) + |P|.$$

Here  $|P|$  denotes the number of parts in  $P$  and  $e(B)$  is the Euler measure of the domain of  $B$ .

Here the *Euler measure* of a region in  $\Sigma \setminus (\alpha \cup \beta)$  is its Euler characteristic minus  $1/4$  the number of corners (intersections of  $\alpha$ -curves and  $\beta$ -curves or  $\alpha$ -arcs and  $\partial\Sigma$ ), and is defined to be additive under union. It can be viewed as the spherical area or negative hyperbolic area of the region where the corners are right angles.

Proposition 5.6 follows from the corresponding index formula for closed Heegaard Floer homology, [20, Formula (6)], by a standard doubling argument at the east punctures, cf. [1, Section 5]. (The formula [20, Formula (6)] itself follows from the Riemann-Roch theorem by another doubling argument.) See also [21, Proposition 4.5.1] for more details.

Suppose  $u \in \widetilde{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$ . Then the  $t$ -coordinate of  $u$  induces a partial ordering on the partition  $P$ . This combinatorial data allows us to divide  $\widetilde{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$  into different strata.

**Definition 5.8.** Given  $\mathbf{x}$ ,  $\mathbf{y}$ , and  $S^\triangleright$  as above and an ordered partition  $\vec{P}$  of the  $e$  punctures of  $S^\triangleright$ , with  $P$  the corresponding unordered partition, let  $\widetilde{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P})$  denote the (open) subset of  $\widetilde{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$  consisting of those holomorphic curves for which the ordering of  $P$  induced by  $t$  agrees with the ordering given in  $\vec{P}$ .

That is, for  $u \in \widetilde{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$  and  $\vec{P} = (P_1, \dots, P_m)$ , we have  $u \in \widetilde{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P})$  if and only if for all  $i < i'$ ,  $q \in P_i$  and  $q' \in P_{i'}$  implies  $t \circ u(q) < t \circ u(q')$ .

Note that given a partition  $P$  of  $E(S^\triangleright)$ , there is an associated set  $[P]$  of multi-sets of Reeb chords, given by replacing each puncture of  $S^\triangleright$  in  $P$  by the associated Reeb chord. Similarly, for ordered partition  $\vec{P}$  of  $E(S^\triangleright)$  there is an associated sequence  $[\vec{P}]$  of multi-sets of Reeb chords.

The  $\mathbb{R}$ -action on  $\Sigma \times [0, 1] \times \mathbb{R}$  by translation on  $\mathbb{R}$  induces an  $\mathbb{R}$ -action on each  $\widetilde{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$  and  $\widetilde{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P})$ . We denote this action by  $\tau_R$  for  $R \in \mathbb{R}$ . The action is free except in the trivial case that  $S^\triangleright$  consists of  $g$  disks with two boundary punctures each, and  $B = 0$ . We say that a curve is *stable* if it is not this trivial case. We are primarily interested in the quotient by this action:

**Definition 5.9.** Given  $\mathbf{x}$ ,  $\mathbf{y}$ ,  $B$ ,  $S^\triangleright$ , and  $P$  or  $\vec{P}$  as above, let

$$\begin{aligned} \mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P) &:= \widetilde{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P) / \mathbb{R} \\ \mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P}) &:= \widetilde{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P}) / \mathbb{R}. \end{aligned}$$

The expected dimension of  $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$  (except in the trivial case  $B = 0$ ) is  $\text{ind}(B, S^\triangleright, P) - 1$ .

Although the evaluation maps  $\text{ev}_q$  (for  $q \in E$ ) do not descend to the quotient by translation, the difference between any two of them,  $\text{ev}_p - \text{ev}_q$ , does descend. We denote this difference by

$$\text{ev}_{p,q}: \mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P}) \longrightarrow \mathbb{R}.$$

We can also combine all the evaluation maps into a single map

$$\text{ev}: \mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P}) \longrightarrow \mathbb{R}^E / \mathbb{R}$$

where  $\mathbb{R}$  acts diagonally by translation on  $\mathbb{R}^E$ .

**5.2. Holomorphic curves in  $Z \times \mathbb{R} \times [0, 1] \times \mathbb{R}$ .** In order to discuss the compactifications of the moduli spaces  $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$  and  $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P})$ , we will need to consider certain holomorphic curves “at least  $\infty$ ,” i.e., in  $Z \times \mathbb{R} \times [0, 1] \times \mathbb{R}$ . We endow  $Z \times \mathbb{R} \times [0, 1] \times \mathbb{R}$  with the obvious split symplectic form. Inside this space, we have  $4k$  Lagrangian planes  $\mathbf{a} \times \mathbb{R} \times \{1\} \times \mathbb{R}$ . The space  $Z \times \mathbb{R} \times [0, 1] \times \mathbb{R}$  has four different ends: the ends  $\pm\infty$  in the first  $\mathbb{R}$ -factor, which we call east and west infinity respectively, and the ends  $\pm\infty$  in the second  $\mathbb{R}$ -factor, which we call  $\pm\infty$ . Note that there is an  $(\mathbb{R} \times \mathbb{R})$ -action on  $Z \times \mathbb{R} \times [0, 1] \times \mathbb{R}$ , by translation in the two  $\mathbb{R}$ -factors. There are projection maps  $\pi_\Sigma$ ,  $\pi_{\mathbb{D}}$ ,  $s$ , and  $t$ , just as before.

Fix a split complex structure  $J = j_\Sigma \times j_{\mathbb{D}}$  on  $Z \times \mathbb{R} \times [0, 1] \times \mathbb{R}$ .

**Definition 5.10.** By a *bi-decorated source*  $T^\diamond$  we mean:

- (1) A topological type of smooth (not nodal) Riemann surface  $T$  with boundary and punctures on the boundary.
- (2) A labeling of each puncture of  $T$  by either  $e$  or  $w$  (east or west), and
- (3) A labeling of each puncture of  $T$  by a Reeb chord  $\rho$  in  $(Z, \mathbf{a})$ .

Given a bi-decorated source  $T^\diamond$ , we will consider maps

$$v: (T, \partial T) \rightarrow ((Z \setminus z) \times \mathbb{R} \times [0, 1] \times \mathbb{R}, \mathbf{a} \times \mathbb{R} \times \{1\} \times \mathbb{R})$$

satisfying the following conditions:

- (1) The map  $v$  is  $(j, J)$ -holomorphic with respect to some almost complex structure  $j$  on  $S$ .
- (2) The map  $v$  is proper.
- (3) The map  $s \circ v$  maps  $\partial T$  to 1.
- (4) At each west puncture  $q$  of  $S$  labeled by a Reeb chord  $\rho$ ,  $\lim_{z \rightarrow q} \pi_\Sigma \circ u(z)$  is  $\rho \subset Z \times \{-\infty\}$ .
- (5) At each east puncture  $q$  of  $S$  labeled by a Reeb chord  $\rho$ ,  $\lim_{z \rightarrow q} \pi_\Sigma \circ u(z)$  is  $\rho \subset Z \times \{+\infty\}$ .

Note that each component of such a holomorphic curve necessarily maps to a single point under  $\pi_\mathbb{D}$  by the maximum principle, since its boundary maps entirely to  $s = 1$ .

Again, we collect these holomorphic curves into moduli spaces:

**Definition 5.11.** For  $T^\diamond$  a bi-decorated source let  $\tilde{\mathcal{N}}(T^\diamond)$  denote the moduli space of holomorphic maps from  $T$  satisfying properties (1)–(5) above.

A holomorphic map  $v$  is called *stable* if there are no infinitesimal automorphisms of  $v$ . (If  $\pi_\Sigma \circ v$  is non-constant on each component of  $T$ , one can verify that  $T$  is stable iff it has a component which is not a twice-punctured disk.) We shall be interested in  $\tilde{\mathcal{N}}(T^\diamond)$  only in the cases that the corresponding maps are stable.

Suppose that  $q$  is a puncture of  $T^\diamond$ . Then there is an evaluation map  $\text{ev}_q: \tilde{\mathcal{N}}(T^\diamond) \rightarrow \mathbb{R}$  given by  $\text{ev}_q(v) = \lim_{z \rightarrow q} t \circ v(q)$ . (Here, writing  $\lim$  is somewhat silly, since  $t \circ v$  is constant on each connected component of  $T^\diamond$ .) There are, consequently, evaluation maps

$$\begin{aligned} \text{ev}_w &:= \prod_{q \in W(S^\diamond)} \text{ev}_q: \tilde{\mathcal{N}}(T^\diamond) \rightarrow \mathbb{R}^{W(T^\diamond)} \quad \text{and} \\ \text{ev}_e &:= \prod_{q \in E(S^\diamond)} \text{ev}_q: \tilde{\mathcal{N}}(T^\diamond) \rightarrow \mathbb{R}^{E(T^\diamond)} \end{aligned}$$

where  $W(T^\diamond)$  or  $W$  is the set of west punctures of  $T^\diamond$  and  $E(T^\diamond)$  or  $E$  is the set of east punctures.

We define sub-moduli spaces of  $\tilde{\mathcal{N}}(T^\diamond)$  by cutting-down by partial diagonals.

**Definition 5.12.** Given a bi-decorated source  $T^\diamond$  and partitions  $P_w$  and  $P_e$  of the west and east punctures of  $T^\diamond$ , respectively, let

$$\tilde{\mathcal{N}}(T^\diamond; P_w, P_e) = (\text{ev}_w \times \text{ev}_e)^{-1}(\Delta_{P_w} \times \Delta_{P_e}).$$

If  $P_w$  is the discrete partition, we denote  $\tilde{\mathcal{N}}(T^\diamond; P_w, P_e)$  by  $\tilde{\mathcal{N}}(T^\diamond; P_e)$ .

The moduli spaces  $\tilde{\mathcal{N}}(T^\diamond)$  can be understood concretely. Indeed, they are determined by the  $t$ -coordinates of the components of  $T$ , and a holomorphic map from  $T^\diamond$  to  $\Sigma$ , which gives  $\pi_\Sigma \circ v$  (which in turn is determined by local branching data over  $\Sigma$ ). However, in general the moduli spaces  $\tilde{\mathcal{N}}(T^\diamond)$  are not transversally cut out. One situation in which they are—and indeed this is the case which is relevant for our applications—is the following:

**Proposition 5.13.** *Suppose  $T^\diamond$  is a decorated source such that all components of  $T$  are topological disks. Then  $\tilde{\mathcal{N}}(T^\diamond)$  is transversally cut out by the  $\bar{\partial}$  equation for any split complex structure on  $Z \times \mathbb{R} \times [0, 1] \times \mathbb{R}$ .*

*Proof.* This is proved by explicit computation of the cokernel of the  $\bar{\partial}$ -operator, which is identified with a certain sheaf cohomology group. See McDuff and Salamon [28, Section 3.3] for a nice exposition of these ideas in the absolute case. (The relative case can be deduced from the absolute case via a doubling argument; see, for instance, Hofer, Lizan, and Sikorav [14, Section 4].) More details can also be found in [21, Lemma 4.1.2].  $\square$

The  $(\mathbb{R} \times \mathbb{R})$ -translation action on  $Z \times \mathbb{R} \times [0, 1] \times \mathbb{R}$  induces an  $(\mathbb{R} \times \mathbb{R})$ -action on  $\tilde{\mathcal{N}}(T^\diamond)$  for any  $T^\diamond$ ; if  $\tilde{\mathcal{N}}(T^\diamond)$  is stable then (almost by definition) this  $(\mathbb{R} \times \mathbb{R})$ -action is free. As for the earlier moduli spaces, we are primarily interested in the quotient space:

**Definition 5.14.** Given a stable decorated source  $T^\diamond$ , let  $\mathcal{N}(T^\diamond) = \tilde{\mathcal{N}}(T^\diamond)/(\mathbb{R} \times \mathbb{R})$ .

Certain holomorphic curves in  $Z \times \mathbb{R} \times [0, 1] \times \mathbb{R}$  will play a special role below. By a *trivial component* we mean a component of  $T^\diamond$  which is a topological disk with exactly two boundary punctures, one east and one west, and both labeled by the same Reeb chord. Holomorphic maps from such components are uninteresting. In particular, they are preserved by the  $\mathbb{R}$ -translation action on  $Z \times \mathbb{R}$ .

A more interesting kind of component is a *join component*. This is a component of  $T^\diamond$  which is a topological disk with two west punctures and one east puncture. In counterclockwise order, suppose the punctures are labeled by  $(e, \rho_e)$ ,  $(w, \rho_1)$  and  $(w, \rho_2)$ . (We will sometimes refer to the  $(w, \rho_1)$  puncture as the *top puncture* of the join component, and the  $(w, \rho_2)$  puncture as the *bottom puncture*.) Then there is a holomorphic map from this component if and only if  $\rho_e = \rho_2 \uplus \rho_1$ . If such a holomorphic map exists, it is unique up to translation. A curve consisting entirely of one join component and some number of trivial components is called a *join curve*.

Symmetrically, a *split component* is a component of  $T^\diamond$  which is a topological disk with two punctures labeled  $e$  and one labeled  $w$ . In counterclockwise order, suppose the punctures are labeled by  $(w, \rho_w)$ ,  $(e, \rho_1)$  and  $(e, \rho_2)$ . (We will sometimes refer to the  $(e, \rho_2)$  puncture as the *top puncture* of the join component, and the  $(e, \rho_1)$  puncture as the *bottom puncture*.) Then there is a holomorphic map from such this component if and only if  $\rho_w = \rho_1 \uplus \rho_2$ . Again, if such a holomorphic map exists, it is unique up to translation. A stable curve consisting entirely of some number of split components and some number of trivial components is called a *split curve*. (Note the definition is not symmetric with that of join curves.)

The domains of these components are illustrated in Figure 16. Split and join components can arise from the degenerations in Figure 5.

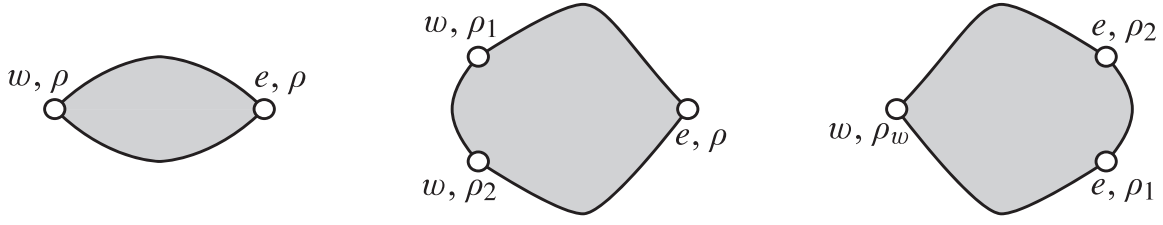


FIGURE 16. Left: a trivial component. Center: a join component. Right: a split component.

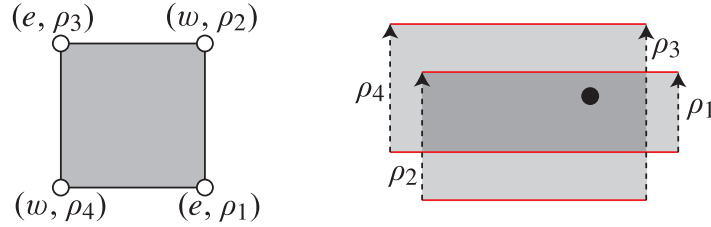


FIGURE 17. Left: a shuffle-component. Right: a schematic of the image of an odd shuffle-component in  $\partial\bar{\Sigma} \times \mathbb{R}$ .

Finally, there is one more complicated kind of curve which we must consider, a *shuffle curve*. A *shuffle component* is a topological disk with two  $e$  punctures and two  $w$  punctures, with cyclic ordering  $(e, w, e, w)$  around the boundary. (A shuffle component is pictured in Figure 17, and an example where one degenerates off is shown in Figure 18.) The map from a shuffle component to  $Z \times \mathbb{R}$  has either one interior branch point or two boundary branch points. Suppose that in counterclockwise order, the punctures of a shuffle component are labeled  $(e, \rho_1), (w, \rho_2), (e, \rho_3), (w, \rho_4)$ . Then the moduli space is nonempty if and only if  $\rho_1^+ = \rho_2^+, \rho_2^- = \rho_3^-, \rho_3^+ = \rho_4^+, \rho_4^- = \rho_1^-$ . If a join curve admits a representative, exactly one of the following cases occurs:

- $\{\rho_1, \rho_3\}$  is nested and  $\{\rho_2, \rho_4\}$  is interleaved, or
- $\{\rho_2, \rho_4\}$  is nested and  $\{\rho_1, \rho_3\}$  is interleaved.

(Recall the definition of interleaved and nested pairs of Reeb chords from Definition 3.4.) In fact, we will see in Proposition 5.31 that the two cases are *not* symmetric for our purposes: when the first occurs in a degeneration, it occurs an odd number of times. When the second occurs, it occurs an even number of times. We will call the first case an *odd shuffle component* and the second case an *even shuffle component*, to remember the distinction.

A *shuffle curve* is the union of a shuffle component and some number of trivial strips.

**5.3. Holomorphic combs.** Just as the compactifications of moduli spaces in Morse theory involve not just flow lines but broken flow lines, in order to discuss the compactifications of the moduli spaces  $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$  we need to introduce a more general object than holomorphic curves. The idea is that, because of the non-compactness of

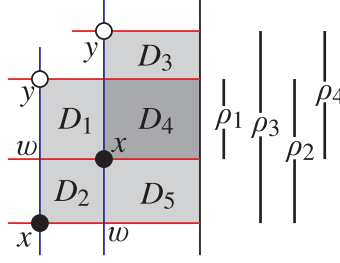


FIGURE 18. A domain in a Heegaard diagram where a shuffle component degenerates off. The domain is  $B = D_1 + D_2 + D_3 + 2D_4 + D_5 \in \pi_2(\mathbf{x}, \mathbf{y})$ , as indicated by the shading. There is a 1-parameter family of holomorphic curves in  $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\flat; P)$  where  $P$  is the partition  $\{\{\rho_1, \rho_3\}\}$ . One end of this moduli space consists of a two-story holomorphic building connecting  $\mathbf{x}$  to  $\mathbf{w}$  via  $D_2$  and  $\mathbf{w}$  to  $\mathbf{y}$  via  $D_1 + D_3 + 2D_4 + D_5$ . At the other end, either an interior branch point of  $\pi_\Sigma \circ u$  (in region  $D_4$ ) or two boundary branch points of  $\pi_\Sigma \circ u$  (between  $D_3$  and  $D_4$  or between  $D_4$  and  $D_5$ ) approach  $\partial\bar{\Sigma}$ , resulting in the degeneration of a shuffle curve.

the target space  $\Sigma \times [0, 1] \times \mathbb{R}$ , holomorphic curves can degenerate into “holomorphic buildings” with several stories, as described by Eliashberg, Givental, and Hofer [6]. In our setting, however, there are two different kinds of infinities:  $\pm\infty$  and  $e\infty$ . This leads us to the notion of *holomorphic combs*.

By a *simple holomorphic comb* we mean a pair of holomorphic maps  $(u, v)$  where  $u$  maps to  $\Sigma \times [0, 1] \times \mathbb{R}$  and  $v$  maps to  $Z \times \mathbb{R} \times [0, 1] \times \mathbb{R}$ , and such that the asymptotics of  $u$  at east  $\infty$  match with the asymptotics of  $v$  at west  $\infty$ . More precisely:

**Definition 5.15.** A *simple holomorphic comb* is a pair  $(u, v)$  with  $u \in \mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\flat)$  and  $v \in \mathcal{N}(T^\diamond)$  for some sources  $S^\flat$  and  $T^\diamond$ , together with a one-to-one correspondence between  $E(S^\flat)$  and  $W(T^\diamond)$ , preserving the labeling by Reeb chords, and such that  $\text{ev}(u) = \text{ev}_w(v)$  inside  $\mathbb{R}^{E(S^\flat)}/\mathbb{R} \cong \mathbb{R}^{W(T^\diamond)}/\mathbb{R}$ .

More generally, we also want to allow the components at east  $\infty$  to degenerate further:

**Definition 5.16.** A *holomorphic story* is a sequence  $(u, v_1, \dots, v_k)$  (for some  $k \geq 0$ ) where, for some  $B$ ,  $S^\flat$ , and  $T_i^\diamond$ ,  $u \in \mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\flat)$ ,  $v_i \in \mathcal{N}(T_i^\diamond)$ ,  $(u, v_1)$  is a simple holomorphic comb (if  $k \geq 1$ ), there is a correspondence between  $E(T_i^\diamond)$  and  $W(T_{i+1}^\diamond)$  for  $i = 1, \dots, k-1$  which preserves the labelings by Reeb chords, and  $\text{ev}_e(v_i) = \text{ev}_w(v_{i+1})$  in  $\mathbb{R}^{E(T_i^\diamond)}/\mathbb{R} \cong \mathbb{R}^{W(T_{i+1}^\diamond)}/\mathbb{R}$ .

We also need to allow degeneration at  $\pm\infty$ :

**Definition 5.17.** A *holomorphic comb* of height  $N$  is a sequence  $(u_j, v_{j,1}, \dots, v_{j,k_j})$ ,  $j = 1, \dots, N$ , of holomorphic stories with  $u_j$  a stable curve in  $\mathcal{M}^{B_j}(\mathbf{x}_j, \mathbf{x}_{j+1}; S_j^\flat)$  for some sequence of generalized generators  $\mathbf{x}_1, \dots, \mathbf{x}_{N+1}$ . We admit the case  $N = 0$ , the *trivial holomorphic comb* from  $\mathbf{x}_1$  to  $\mathbf{x}_1$ , which corresponds to a trivial (unstable) holomorphic curve.

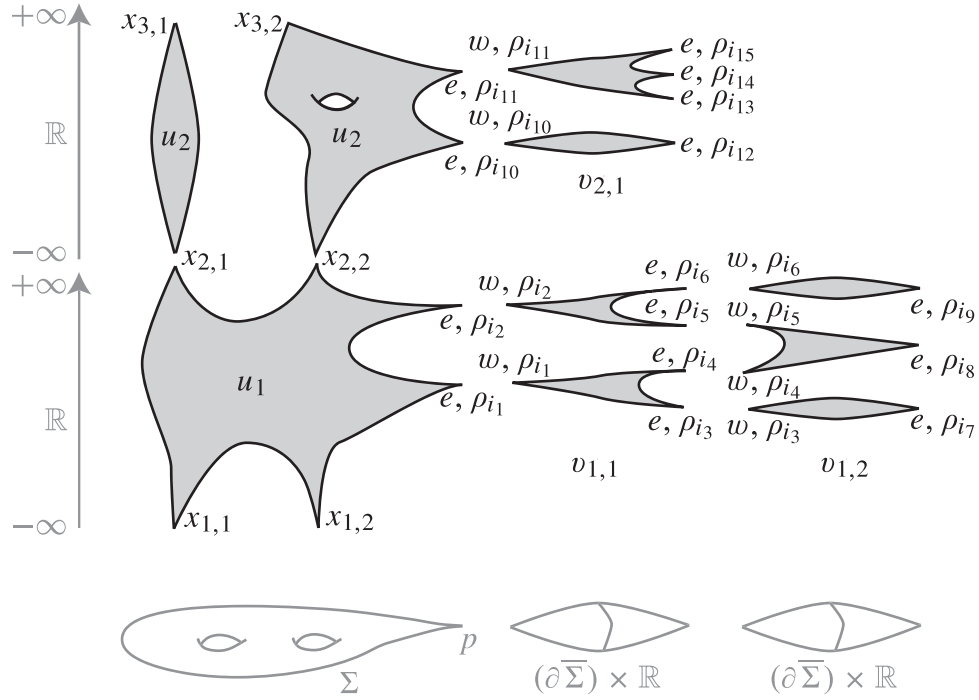


FIGURE 19. A schematic of a height 2 holomorphic comb.

See Figure 19 for a schematic illustration of a height 2 holomorphic comb.

Finally, to compactify our moduli spaces, we will also need to drop Condition (1) of Definition 5.2, and allow our holomorphic curves to have nodal sources. If we want to explicitly allow nodes, we will refer to such combs as *nodal holomorphic combs*; if we want to explicitly rule out nodal combs, we will refer to *smooth combs*. When the distinction is unimportant, we will simply talk about holomorphic combs.

Note that a (nodal) holomorphic comb  $U = \{(u_j, v_{j,1}, \dots, v_{j,k_j})\}_{j=1}^N$  naturally represents a homology class  $B$  in  $\pi_2(\mathbf{x}_1, \mathbf{x}_{N+1})$ , where  $\mathbf{x}_1$  and  $\mathbf{x}_{N+1}$  are the first and last generators involved. (Specifically,  $B = B_1 * \dots * B_N$ , where  $B_j$  is the domain of  $u_j$ . The  $v_{i,j}$  are irrelevant to this homology class.) Further,  $U$  is asymptotic to a well-defined set of Reeb chords at (far) east  $\infty$ : these are the asymptotics at east infinity of the  $v_{j,k_j}$ .

Next, we turn to what it means for a sequence of holomorphic curves to converge to a holomorphic comb or, more generally, for a sequence of holomorphic combs to converge to another holomorphic comb. This is a simple adaptation of definitions for holomorphic curves converging to holomorphic buildings in [6]; the reader is referred to [21, Section 4.2] for some more discussion of this point.

Given a simple holomorphic comb  $(u, v)$  with  $S^\triangleright$  and  $T^\diamond$  the decorated sources of  $u$  and  $v$  respectively, there is a natural way to (pre)glue  $S^\triangleright$  and  $T^\diamond$  to form a decorated source  $S^\triangleright \natural T^\diamond$ : the surface  $S \natural T$  is obtained by identifying small neighborhoods of the east punctures of  $S$  with neighborhoods of the corresponding west punctures of  $T$ . (Note that this is a purely topological operation, not involving any differential

equations. We use the term “preglue” to distinguish from gluing of holomorphic curves.) The  $e\infty$  punctures of  $T^\diamond$  and  $\pm\infty$  punctures of  $S^\triangleright$  carry over to decorate  $S^\triangleright \natural T^\diamond$ . The pregluing of sources extends in an obvious way to general smooth holomorphic combs. The construction also extends to nodal holomorphic combs: there is a canonical way to deform away the nodes in the source to produce a smooth Riemann surface.

From these constructions we can produce a compactification of  $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright)$ .

**Definition 5.18.** We make the following definitions.

- $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright)$  is the space of all (possibly nodal) holomorphic combs whose preglued surfaces are  $S^\triangleright$ , in the homology class  $B$ , with asymptotics  $\mathbf{x}$  at  $-\infty$ ,  $\mathbf{y}$  at  $+\infty$ .
- $\overline{\mathcal{M}}^B(s, y; S^\triangleright)$  is the closure of  $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright)$  in  $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright)$ .
- For  $p, q \in E(S^\triangleright)$ ,  $\overline{\text{ev}}_{p,q}: \overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright) \rightarrow [-\infty, \infty]$  is the extension of  $\text{ev}_{p,q}$ .
- $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$  is the space of all holomorphic combs respecting the partition  $P$ . Formally, we set

$$\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P) := \bigcap_{\substack{P_i \in P \\ p, q \in P_i}} \overline{\text{ev}}_{p,q}^{-1}(0).$$

- $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$  is the closure of  $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$  in  $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$ .
- $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P})$  is the closure of  $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P})$  in  $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$ .

The space  $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$  is often a proper subset of  $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$ , as the next example illustrates. (Because of difficulties with transversality at east  $\infty$ ,  $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright)$  may also be a proper subset of  $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright)$ .)

*Example 5.19.* Consider the portion of a Heegaard diagram shown in the left of Figure 20, and let  $S^\triangleright$  be the decorated source shown on the right of Figure 20. (We number the east punctures of  $S^\triangleright$  for convenience in referring to them.) Let  $P$  denote the partition  $\{\{1, 3\}, \{2, 4\}\}$ . The space  $\widetilde{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright)$  is then homeomorphic to  $\mathbb{R} \times (0, \infty) \times \mathbb{R} \times (0, \infty)$ , via the map

$$(\text{ev}_1, \text{ev}_1 - \text{ev}_2, \text{ev}_3, \text{ev}_3 - \text{ev}_4).$$

The space  $\widetilde{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$  is the subspace  $x_1 = x_3, x_2 = x_4$  of  $\mathbb{R} \times (0, \infty) \times \mathbb{R} \times (0, \infty)$ .

Now, consider the compactifications. In the compactification, when  $(\text{ev}_1 - \text{ev}_2) \rightarrow 0$  the curve degenerates a split component at east  $\infty$ , and similarly when  $(\text{ev}_3 - \text{ev}_4) \rightarrow 0$ . If  $(\text{ev}_1 - \text{ev}_2) \rightarrow 0$  and  $(\text{ev}_3 - \text{ev}_4) \rightarrow 0$ , there are two split components at east  $\infty$ . There is a one-parameter family of these curves *at east*  $\infty$ , given by the relative  $\mathbb{R}$ -coordinate of the branch point under  $\pi_\Sigma$ . Consequently the part of the space  $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright)$  with  $(\text{ev}_1 - \text{ev}_2)$  and  $(\text{ev}_3 - \text{ev}_4)$  small has the form shown in Figure 21, and  $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$  is the subspace pictured. By contrast,  $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$  contains the entire stratum in which two split components have degenerated.

Next, we turn to the technical results justifying the definitions of the moduli spaces  $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$ .

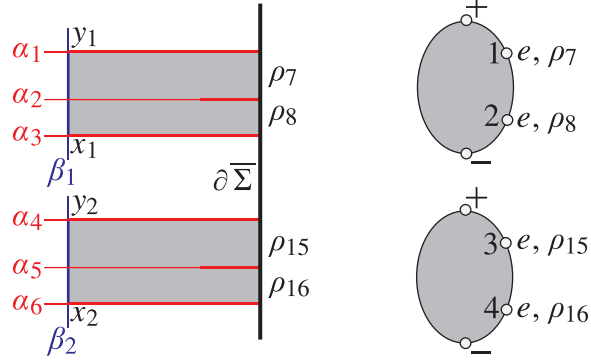


FIGURE 20. An example illustrating that  $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\diamond; P)$  may be a proper subset of  $\overline{\overline{\mathcal{M}}}^B(\mathbf{x}, \mathbf{y}; S^\diamond; P)$ . Left: a portion of a Heegaard diagram. The domain of interest is shaded in gray. Right: the decorated source  $S^\diamond$ .

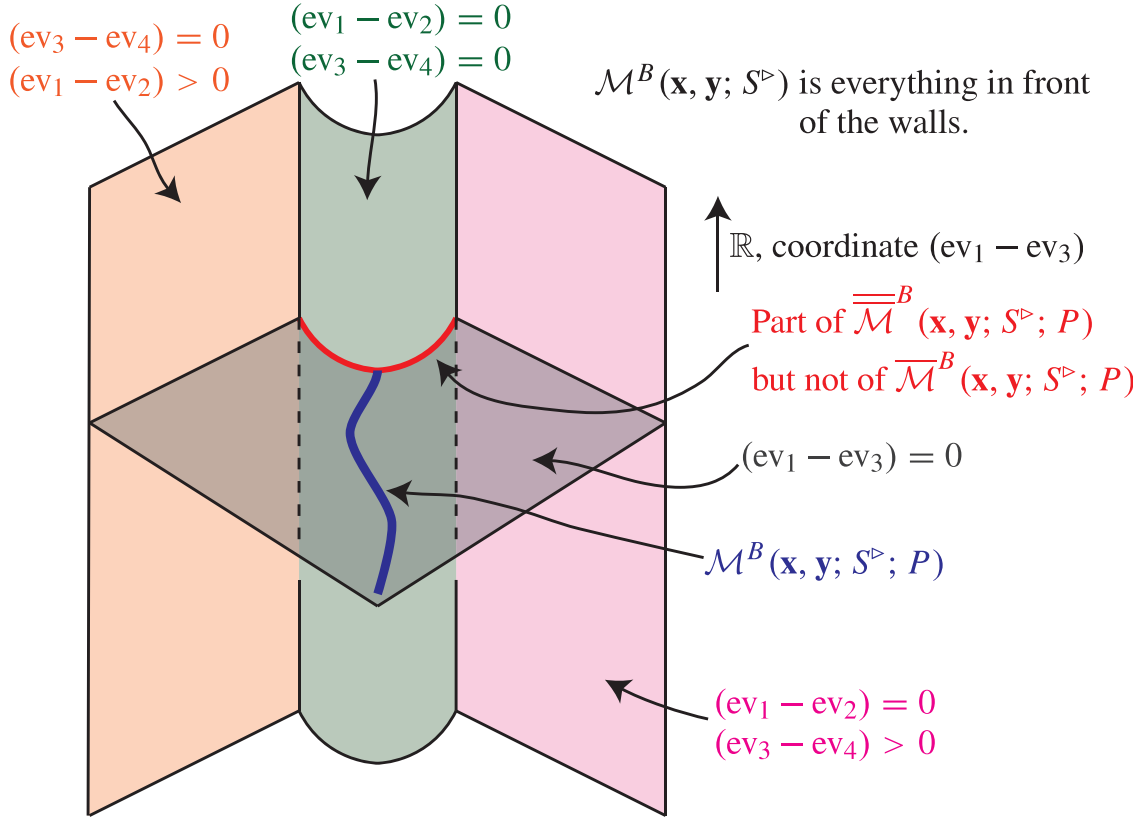


FIGURE 21. An illustration of various moduli spaces for the Heegaard diagram from Figure 20. The colored “walls” form the boundary of  $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\diamond)$ , the open part of which one visualizes as the empty space in front of the walls. The subspace where  $\text{ev}_1 = \text{ev}_3$  is the gray horizontal plane. The moduli space  $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\diamond; P)$  is a thick, squiggly line segment; the space  $\overline{\overline{\mathcal{M}}}^B(\mathbf{x}, \mathbf{y}; S^\diamond; P)$  is a “T” formed by that line segment and an arc in  $\partial \overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\diamond)$ .

**Proposition 5.20.** *The spaces  $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright)$  are compact. That is, suppose that  $\{U_n\}$  is a sequence of holomorphic combs in a fixed homology class, with a fixed preglued topological source. Then  $\{U_n\}$  has a subsequence which converges to a (possibly nodal) holomorphic comb  $U$ , in the same homology class as the  $U_n$ . Similarly,  $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$  and  $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P})$  are compact.*

*Proof.* Compactness for  $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright)$  is proved in [21, Proposition 4.2.1]; for the reader's convenience, we repeat the proof here, more or less verbatim. The argument is essentially soft; we deduce compactness in our setting from results of Bourgeois, Eliashberg, Hofer, Wysocki, and Zehnder [2].

To ease notation, we will restrict our attention to the case of a sequence  $\{u^n: S^n \rightarrow \Sigma \times [0, 1] \times \mathbb{R}\}_{n=1}^\infty$  of holomorphic curves in  $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright)$ , rather than a sequence of general holomorphic combs.

By [2, Theorem 10.1], we can find a subsequence of  $\{u^n\}$  for which  $\pi_{\mathbb{D}} \circ u^n$  converges to some multi-story holomorphic building. (Actually, [2, Theorem 10.1] is stated in the absolute case, i.e., for curves without boundary. See Section 10.3 of that paper for discussion of generalization to the relative case.) From now on, let  $\{u^n\}$  denote this subsequence. Note that in the process of finding the subsequence, marked points are added to the sources  $S^n$  to stabilize each component of the limit source. Call the limit of  $\pi_{\mathbb{D}} \circ u^n$  by

$$\{\pi_{\mathbb{D}} \circ u_{a,i}: S_{a,i} \rightarrow [0, 1] \times \mathbb{R}\}_{i=1}^\ell;$$

here,  $u_i$  denotes the  $i^{\text{th}}$  story of the limit. (Note that  $u_{a,i}$  is not defined yet, only  $\pi_{\mathbb{D}} \circ u_{a,i}$ .)

Fix a neighborhood  $U_p$  of the puncture  $p$  in  $\Sigma$  so that the almost complex structure  $J$  is split,  $J = j_\Sigma \times j_{\mathbb{D}}$ , over the closure  $\overline{U}_p$  of  $U_p$ . Also, choose  $U_p$  small enough that  $U_p \cap \beta = \emptyset$ . Then  $\pi_\Sigma \circ u^n$  restricted to  $(\pi_\Sigma \circ u^n)^{-1}(\overline{U}_p)$  form a sequence of holomorphic curves with Lagrangian boundary conditions on  $\partial \overline{U}_p \cup (\alpha \cap \overline{U}_p)$ . Consequently, by [2, Theorem 10.2], we can find a convergent subsequence. (Again, more marked points are added to stabilize the sources.) Again, call the subsequence  $\{u^n\}$ . Let

$$\{\pi_\Sigma \circ u_{b,i}: S_{b,i} \rightarrow \overline{U}_p \subset \Sigma, \pi_\Sigma \circ v_{b,i,j}: T_{b,i,j} \rightarrow \partial \overline{\Sigma} \times \mathbb{R}\}$$

denote the limit. We explain the indexing. Forgetting the new marked points and collapsing unstable components gives a map from the source of the limit curve  $(\bigcup_i S_{b,i}) \cup (\bigcup_{i,j} T_{b,i,j})$  to  $\bigcup_i S_{a,i}$ . This defines the index  $i$  on  $S_{b,i}$  and  $T_{b,i,j}$ . The index  $j$  comes from the story structure on  $\Sigma$ .

So far, the maps  $u_{b,i}$  and  $v_{b,i,j}$  are not defined, only their projections  $\pi_\Sigma \circ u_{b,i}$  and  $\pi_\Sigma \circ v_{b,i,j}$ . However, by the same considerations used to define the index  $i$ , the map  $\pi_{\mathbb{D}} \circ u_{a,i}$  is naturally defined on  $\{S_{b,i}, T_{b,i,j}\}$ . Set  $u_{b,i} = \pi_\Sigma \circ u_{b,i} \times \pi_{\mathbb{D}} \circ u_{a,i}$ , and similarly for  $v_{b,i,j}$ .

Turning to the rest of  $\Sigma$ , let

$$L_n := \left\{ \pi_{\mathbb{D}} \circ u^n \left( (\pi_\Sigma \circ u^n)^{-1}(\partial \overline{U}_p) \right) \right\}.$$

By considering a slightly larger neighborhood  $V_p \supset \overline{U}_p$ , one sees that  $\{L_n\}$  forms a convergent sequence of smooth (real) curves in  $[0, 1] \times \mathbb{R}$ . Then,  $\{u^n|_{(\pi_\Sigma \circ u^n)^{-1}(\Sigma \setminus U_p)}\}$  forms a sequence of holomorphic curves in  $\Sigma \times [0, 1] \times \mathbb{R}$  with Lagrangian boundary conditions  $(\alpha \times \{1\} \times \mathbb{R}) \cup (\beta \times \{0\} \times \mathbb{R}) \cup (\partial \overline{U}_p \times L_n)$ . The compactness theorem [2,

Theorem 10.2] now applies to give a convergent subsequence of the  $u^n|_{(\pi_\Sigma \circ u)^{-1}(\Sigma \setminus U_p)}$ , with limit  $u_{c,i}$ .

Now observe that the map  $u_{b,i}$  (which lives over  $\overline{U_p}$ ) and the map  $u_{c,i}$  (which lives over  $\Sigma \setminus U_p$ ) fit together to form a smooth curve  $u_{d,i}$ . (Again, this follows by considering the neighborhood  $V_p \supset \overline{U_p}$ .) We are left, then, with the subsequence  $u^n$  converging to the holomorphic comb  $\{u_{d,i}, v_{b,i,j}\}$ , as desired.

This proves compactness of  $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright)$ . Finally, the other spaces,  $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright)$ ,  $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$ , and  $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P})$ , are closed subspaces of  $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright)$  by definition, and hence compact.  $\square$

We next turn to gluing results, which show that the space  $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$  is well-behaved near some of the simplest holomorphic combs.

**Proposition 5.21.** *Let  $(u_1, u_2)$  be a two-story holomorphic building with  $u_1 \in \mathcal{M}^{B_1}(\mathbf{x}, \mathbf{y}; S_1^\triangleright; P_1)$  and  $u_2 \in \mathcal{M}^{B_2}(\mathbf{y}, \mathbf{w}; S_2^\triangleright; P_2)$ . Then for sufficiently small open neighborhoods  $U_1$  of  $u_1$  and  $U_2$  of  $u_2$ , there is an open neighborhood of  $(u_1, u_2)$  in  $\overline{\mathcal{M}}^{B_1 * B_2}(\mathbf{x}, \mathbf{w}; S_1^\triangleright \natural S_2^\triangleright; P_1 \cup P_2)$  which is homeomorphic to  $U_1 \times U_2 \times [0, 1)$ .*

**Proposition 5.22.** *Let  $(u, v)$  be a simple holomorphic comb with  $u \in \mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright)$  and  $v \in \mathcal{N}(T^\diamond; P_e)$ . Let  $m = |E(S^\triangleright)| = |W(T^\diamond)|$ . Assume that the moduli spaces  $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright)$  and  $\mathcal{N}(T^\diamond; P_e)$  are transversally cut out at  $u$  and  $v$  respectively, and that  $\text{ev}: \mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright) \rightarrow \mathbb{R}^m/\mathbb{R}$  and  $\text{ev}_w: \mathcal{N}(T^\diamond; P_e) \rightarrow \mathbb{R}^m/\mathbb{R}$  are transverse at  $(u, v)$ . Then, for sufficiently small open neighborhoods  $U_u$  of  $u$  and  $U_v$  of  $v$ , there is an open neighborhood of  $(u, v)$  in  $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright \natural T^\diamond; P_e)$  which is homeomorphic to  $(U_u \times_{\text{ev}} U_v) \times [0, 1)$ .*

*Proof of Propositions 5.21 and 5.22.* The results follow by standard arguments. Indeed, since each puncture of a holomorphic curve in our setting is mapped either to  $\pm\infty$  or to east or west  $\infty$ , the fact that we have two kinds of infinities is irrelevant to a local statement of this kind. The only complication to the usual arguments, then, is that the asymptotics at east and west infinity are degenerate. Thus, the proof can be obtained by adapting [1, Section 5.3] to the relative case or [20, Proposition A.1] to the Morse-Bott case.  $\square$

In the proof of Proposition 5.29 we will use the following generalization of Proposition 5.22 to non-simple one-story holomorphic combs:

**Proposition 5.23.** *Let  $(u, v_1, \dots, v_k)$  be a height 1 holomorphic comb, with  $u \in \mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright)$ ,  $v_i \in \mathcal{N}(T_i^\diamond)$  for  $i = 1, \dots, k-1$ , and  $v_k \in \mathcal{N}(T_k^\diamond; P_e)$ . Let  $m_i = |W(T_i^\diamond)|$ . Assume that all the moduli spaces are transversally cut out, and that the map*

$$\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright) \times \mathcal{N}(T_1^\diamond) \times \dots \times \mathcal{N}(T_k^\diamond; P_e) \rightarrow \mathbb{R}^{m_1} \times \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \times \mathbb{R}^{m_2} \times \dots \times \mathbb{R}^{m_k} \times \mathbb{R}^{m_k} \\ (u, v_1, \dots, v_k) \mapsto (\text{ev}(u), \text{ev}_w(v_1), \text{ev}_e(v_1), \text{ev}_w(v_2), \dots, \text{ev}_e(v_{k-1}), \text{ev}_w(v_k))$$

*is transverse to the diagonal  $\{(x_1, x_1, \dots, x_k, x_k) \mid x_i \in \mathbb{R}^{k_i}\}$ . Then, for sufficiently small open neighborhoods  $U_u$  of  $u$  and  $U_{v_i}$  of  $v_i$ , there is an open neighborhood of  $(u, v_1, \dots, v_k)$  in  $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright \natural T_1^\diamond \natural \dots \natural T_k^\diamond; P_e)$  which is homeomorphic to  $(U_u \times_{\text{ev}} U_{v_1} \times_{\text{ev}} \dots \times_{\text{ev}} U_{v_k}) \times [0, 1)^k$ .*

Again, the proof is essentially standard.

As mentioned at the beginning of this section, we cannot prove that the compactified moduli spaces are manifolds with corners in general. We will get around this difficulty by proving that certain evaluation maps are degree one, in an appropriate sense, at the corners; this will be enough for the results in Section 5.4. Before stating the propositions, we make some definitions.

**Definition 5.24.** Let  $v \in \mathcal{N}(T^\diamond)$ , and  $(u, v)$  a simple holomorphic comb. A *smeared neighborhood* of  $(u, v)$  in  $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$  is an open neighborhood of

$$\{(u, v') \mid v' \in \mathcal{N}(T^\diamond), (u, v') \in \overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)\}.$$

There is an exactly analogous notion of a *smeared neighborhood* of  $(u, v)$  in  $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P})$ .

**Definition 5.25.** Given a continuous map  $f: X \rightarrow Y$  of topological spaces and a point  $p \in Y$ , we say  $f$  is *proper near*  $p$  if there is an open neighborhood  $U \ni p$  such that  $f|_{f^{-1}(U)}: f^{-1}(U) \rightarrow U$  is proper.

We will use the following weak notion of stratified spaces.

**Definition 5.26.** An *n-dimensional stratified space* is a topological space  $X$  written as a union of strata  $\{X_i\}_{i=0}^n$ , where  $X_i$  is a smooth  $i$ -dimensional manifold, and the closure of  $X_k$  is contained in  $\bigcup_{i \leq k} X_i$ . We typically suppress the strata from the notation. Let  $X$  and  $Y$  be stratified spaces. A *stratified map*  $f: X \rightarrow Y$  is a continuous map with the property that the preimage of any stratum in  $Y_i \subset Y$  is a union of connected components of strata  $X_j$  of  $X$ , and the restriction of  $f$  to each stratum of  $X_j$ , thought of as a map into  $Y_i$ , is a smooth map.

**Definition 5.27.** Let  $X$  be a stratified space so that the top stratum is a smooth  $m$ -manifold, and let  $f: X \rightarrow \mathbb{R}_+^m$  be a stratified map so that  $f^{-1}((0, \infty)^m)$  is the top stratum of  $X$ . Let  $p \in \mathbb{R}_+^m$ , and assume  $f$  is proper near  $p$ . We say that  $f$  is *odd degree near*  $p$  if there is an open neighborhood  $U$  of  $p$  such that for any regular value  $p' \in U \cap (0, \infty)^m$ ,  $f^{-1}(p')$  consists of an odd number of points.

**Lemma 5.28.** *Let  $X$  be a stratified space such that:*

- *Each stratum of  $X$  is a smooth manifold.*
- *The union of the top two strata of  $X$  form an  $m$ -manifold with boundary.*

*Let  $f: X \rightarrow \mathbb{R}_+^m$  be a stratified map which is proper near 0. Assume that  $f^{-1}((0, \infty)^m)$  is the top stratum of  $X$ . Then*

- (1) *If  $p, p' \in (0, \infty)^m$  near 0 are regular values of  $f$  then  $|f^{-1}(p)| \equiv |f^{-1}(p')| \pmod{2}$ .*
- (2) *If the restriction of  $f$  to the preimage of some facet  $\mathbb{R}_+^{m-1}$  of  $\mathbb{R}_+^m$  is odd degree near  $0 \in \mathbb{R}_+^{m-1}$  then  $f$  is odd degree near  $0 \in \mathbb{R}_+^m$ .*

*Proof.* Let  $U$  be a neighborhood of 0 over which  $f$  is proper. Let  $V = f^{-1}(U \cap \mathbb{R}_+^m)$ . Then  $f|_V: V \rightarrow (U \cap (0, \infty)^m)$  is a proper map. So part (1) follows from standard degree theory.

For part (2), let  $p \in \mathbb{R}_+^{m-1}$  be a regular value of the restriction of  $f$ , and let  $B \subset \mathbb{R}_+^{m-1}$  be a ball neighborhood of  $p$  small enough that  $f|_{f^{-1}(B)}$  is a covering map.

Write  $f^{-1}(B) = B_1 \amalg \cdots \amalg B_{2\ell+1}$ ; each  $B_i$  is a  $(k-1)$ -ball. For each  $i$  choose a  $(k-1)$ -ball  $B'_i$  in the top stratum of  $X$  with  $\partial B'_i = \partial B_i$ . Note that by our assumptions on  $f$ ,  $f(\text{Int}(B_i))$  is entirely contained in  $(0, \infty)^m$ . So  $f(B_i \cup B'_i)$  is a (singular)  $(k-1)$ -sphere in  $\mathbb{R}_+^k$ , and for  $p' \in (0, \infty)^m$  sufficiently close to  $p$ , the sphere  $f(B_i \cup B'_i)$  has winding number 1 around  $p$ . It follows that  $f$  has degree  $2\ell + 1$  at regular values near  $p$ . Choosing  $p$  close enough to 0 gives the result.  $\square$

**Proposition 5.29.** *Suppose that  $(u, v)$  is a simple holomorphic comb in  $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$ . Assume that the source  $T^\diamond$  of  $v$  is a split curve and that there are two parts  $P_1$  and  $P_2$  of  $P$  such that, for each split component of  $T^\diamond$ , its bottom puncture belongs to  $P_1$  and its top puncture belongs to  $P_2$ . Assume that  $\text{ind}(B, S^\triangleright, P) = 2$ .*

*Let  $q_1 \in P_2$  and  $q_2 \in P_1$  be the top and bottom punctures, respectively, on one of the split components of  $T^\diamond$ . Then, for generic  $J$ , there is a smeared neighborhood  $U$  of  $(u, v)$  in  $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$  so that the evaluation map  $\overline{\text{ev}}_{q_1, q_2}: U \rightarrow \mathbb{R}_+$  is proper near 0 and odd degree near 0.*

*Proof.* To see the  $\overline{\text{ev}}$  is proper near 0 on  $U$ , note that  $\overline{\text{ev}}_{q_1, q_2}$  is clearly proper on the closure  $\overline{U}$  of  $U$ . By examining the topology of  $S^\triangleright$ , we see that  $\overline{U} \setminus U$  is mapped to strictly positive values by  $\overline{\text{ev}}_{q_1, q_2}$ .

Let the split components of  $T^\diamond$  be  $T_i^\diamond$ , for  $i = 1, \dots, N$ . For notational convenience, we will assume that  $T^\diamond$  has no trivial strips, so that the only parts of  $P$  are  $P_1$  and  $P_2$ . Let  $P'$  be the partition  $P$  with  $P_2$  split into the discrete partition; that is,  $P'$  has  $N+1$  parts, one of which is  $P_1$  and all others consisting of a single puncture. We will show that there is a smeared neighborhood  $U'$  of  $(u, v)$  in  $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P')$  so that the evaluation map  $\overline{\text{ev}}: U' \rightarrow \mathbb{R}_+^N$  is odd degree near 0 in the sense of Definition 5.27. (We see that  $\overline{\text{ev}}$  is proper near 0 as before.) This suffices, since for generic  $J$ , the map  $\text{ev}$  from  $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P')$  is transverse to the diagonal in  $(0, \infty)^N$  by Proposition 5.5.

By Proposition 5.23, we can make a smeared neighborhood  $U$  of  $(u, v)$  inside  $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P')$  into a stratified space of a particularly nice kind; in particular, the union of the top two strata is an  $N$ -manifold with boundary. The codimension 1 strata of  $U$  (of dimension  $N-1$ ) consist of simple holomorphic combs  $(u', v')$ , where  $v'$  is a split curve with no more than  $N$  split components. Higher codimension strata of  $U$  are more complicated one-story holomorphic combs.

Let us call such a neighborhood inside an  $N$ -dimensional compactified moduli space  $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P')$  a *split neighborhood*: namely, a split neighborhood is an open set containing the entire stratum of simple combs  $(u, v)$ , where  $v$  is a split curve with  $N$  split components, with all bottom punctures of  $v$  belonging to the same part of  $P'$  and all top punctures belonging to different parts.

We will show by induction on  $N$  that  $\overline{\text{ev}}$  on a split neighborhood has odd degree near 0. For  $N = 1$ , by Proposition 5.22 we have a map from  $[0, 1)$  to  $\mathbb{R}_+$  mapping 0 to 0 which is proper near 0; such a map automatically has odd degree near 0. For  $N > 1$ , note that the preimage of the interior of a facet of  $\mathbb{R}_+^N$  consists of holomorphic curves  $(u', v')$ , where  $v$  has a single split component; thus,  $u'$  lives in a split neighborhood  $U'$  of a smaller moduli space, of dimension  $N-1$ . By induction,  $\overline{\text{ev}}$  on  $U'$  has odd degree near 0, and so by Lemma 5.28,  $\overline{\text{ev}}$  on  $U$  also has odd degree near 0.  $\square$

*Remark 5.30.* It would be convenient to extend Proposition 5.5 to the compactified moduli spaces, and in particular to assert that for height 1 holomorphic combs, say,

the extension of  $\text{ev}$  to  $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright)$  is transverse to the partial diagonals  $\Delta_P$ . In particular, this would eliminate the need for the cumbersome Propositions 5.29 and 5.31. Such a statement would require a smooth structure on the  $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright)$ , which we have not constructed. Moreover, the validity of the statement would depend strongly on the smooth structure: in the language of Hofer [13], it depends on the *gluing profile*. (In fact, for the exponential gluing profile advocated there, the extension of  $\text{ev}$  would generally not be transverse to  $\Delta_P$ .)

We conclude this section with a result on shuffle curves. It states that, when they occur, odd shuffle curves appear in the boundary of  $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$  (with  $\text{ind}(B, S^\triangleright, P) = 2$ ) an odd number of times, and even shuffle curves appear an even number of times.

**Proposition 5.31.** *Fix a generic  $J$ . Let  $(u, v) \in \overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$  be a simple holomorphic comb with  $v$  a shuffle curve. Assume that the two  $e$  punctures  $q_i$  and  $q_j$  of the shuffle component belong to the same part  $P_i$  of  $P$ . Assume also that  $\text{ind}(B, S^\triangleright, P) = 2$ .*

- (1) *If  $v$  is an odd shuffle curve then there is a smeared neighborhood  $U$  of  $(u, v)$  in  $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$  so that  $\overline{U} \setminus \partial \overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$  is homeomorphic to  $\coprod_{i=1}^{2m-1} [0, 1]$  (for some  $m \in \mathbb{N}$ ).*
- (2) *If  $v$  is an even shuffle curve then there is a smeared neighborhood  $U$  of  $(u, v)$  in  $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$  so that  $\overline{U} \setminus \partial \overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$  is homeomorphic to  $\coprod_{i=1}^{2m-2} [0, 1]$  (for some  $m \in \mathbb{N}$ ).*

*Proof.* First some notation: let  $a_1, \dots, a_4$  be the endpoints of the Reeb chords on the shuffle component, ordered so that  $a_1 \leq \dots \leq a_4$ . Let  $\rho_{ij}$  ( $\{i, j\} \subset \{1, \dots, 4\}$ ) denote the Reeb chord from  $a_i$  to  $a_j$ . Let  $T^\diamond$  denote the source of  $v$ .

The two cases are similar; we first prove case (1). In this case, the  $e$  punctures of the shuffle component of  $T^\diamond$  are labeled by  $\rho_{14}$  and  $\rho_{23}$ . The  $w$  punctures are labeled by  $\rho_{13}$  and  $\rho_{24}$ . The moduli space  $\mathcal{N}(T^\diamond)$  has a natural compactification  $\overline{\mathcal{N}}(T^\diamond)$  which is homeomorphic to  $[0, 1]$ . (The parameter is obtained by moving the branch point(s); see Figure 22 for a schematic illustration.) By our transversality (Propositions 5.5 and 5.13) and gluing (Proposition 5.22) results, any curve in  $\mathcal{N}(T^\diamond)$  can be glued to  $u$  to obtain a curve in  $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright)$ .

To see when there exist such curves in  $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$  we consider the ends of the moduli space  $\overline{\mathcal{N}}(T^\diamond)$ . The two ends correspond to holomorphic combs  $(u, v_1, v_2)$  and  $(u, v_3, v_4)$ . The curves  $v_1, \dots, v_4$  are as follows:

- $v_1$  is a split curve with one split component, labeled by  $(w, \rho_{13})$ ,  $(e, \rho_{12})$  and  $(e, \rho_{23})$ , together with a trivial component labeled by  $(e, \rho_{24})$  and  $(w, \rho_{24})$ .
- $v_2$  is a join curve, with a join component labeled by  $(w, \rho_{24})$ ,  $(w, \rho_{12})$  and  $(e, \rho_{14})$ , together with a trivial component labeled by  $(e, \rho_{23})$  and  $(w, \rho_{23})$ .
- $v_3$  is a split curve with one split component, labeled by  $(w, \rho_{24})$ ,  $(e, \rho_{23})$  and  $(e, \rho_{34})$ , together with a trivial component labeled by  $(e, \rho_{13})$  and  $(w, \rho_{13})$ .
- $v_4$  is a join curve, with a join component labeled by  $(w, \rho_{13})$ ,  $(w, \rho_{34})$  and  $(e, \rho_{14})$ , together with a trivial component labeled by  $(e, \rho_{23})$  and  $(w, \rho_{23})$ .

See Figure 22.

Suppose one glues  $v_1$  to  $u$ , with some gluing parameter  $\epsilon_1$ . Then in the resulting curve the Reeb chord  $\rho_{23}$  will be above  $\rho_{12}$ . If one then glues  $v_2$  to the result, with

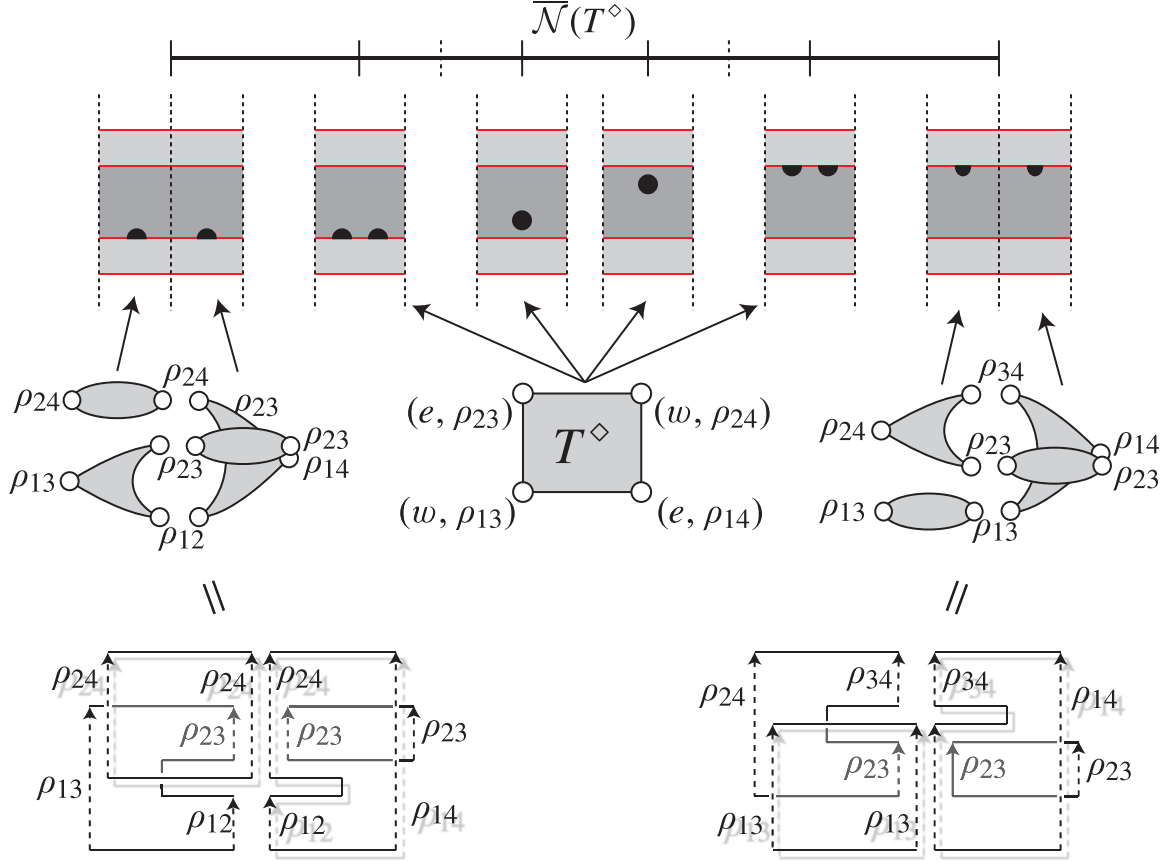


FIGURE 22. The one-parameter family of maps  $v$  from an odd shuffle component. The dark dots denote interior branch points, and the dark semi-dots denote boundary branch points. The two ends of the moduli space are shown on the left and the right, respectively. At the bottom, the two ends are sketched in the style of Figure 17. The corresponding pictures for an even shuffle component are obtained by reflection.

some gluing parameter  $\epsilon_2 \ll \epsilon_1$ , in the resulting curve the Reeb chord  $\rho_{23}$  will be above  $\rho_{14}$ .

By contrast, suppose one glues  $v_3$  to  $u$ , with some gluing parameter  $\epsilon_1$ . Then in the resulting curve the Reeb chord  $\rho_{34}$  will be above  $\rho_{23}$ . If one then glues  $v_2$  to the result, with some gluing parameter  $\epsilon_2 \ll \epsilon_1$ , in the resulting curve the Reeb chord  $\rho_{14}$  will be above  $\rho_{23}$ .

So, since the two Reeb chords switched order as we moved from one end of  $\overline{\mathcal{N}}(T^\diamond)$  to the other, for given  $\epsilon$  there is algebraically one curve  $v' \in \mathcal{N}$  so that gluing  $u$  with  $v'$  with gluing parameter  $\epsilon$  gives a curve with  $\rho_{14}$  and  $\rho_{23}$  at the same height—i.e., an element of  $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$ . Part (1) of the proposition follows.

Part (2) follows by a similar analysis. In this case, the  $e$  punctures of the shuffle component are labeled by  $\rho_{13}$  and  $\rho_{24}$ , and the  $w$  punctures are labeled by  $\rho_{14}$  and  $\rho_{23}$ . A schematic illustration of  $\overline{\mathcal{N}}(T^\diamond)$  may be obtained by turning Figure 22 upside down. An analysis as above shows that at both ends of  $\overline{\mathcal{N}}$ , the Reeb chord  $\rho_{24}$  is above  $\rho_{13}$ .

This implies that, algebraically, there are zero curves in  $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$  for a given gluing parameter.  $\square$

**5.4. Degenerations of holomorphic curves.** To construct our invariants we need the following result on degenerations of 1-dimensional moduli spaces. Let

$$\partial \overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P) := \overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P) \setminus \mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P).$$

**Proposition 5.32.** *Suppose that  $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$  is 1-dimensional. Then for generic  $J$  every holomorphic comb in  $\partial \overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$  has one of the following forms:*

- (1) *a two-story holomorphic building  $(u_1, u_2)$ ;*
- (2) *a simple holomorphic comb  $(u, v)$  where  $v$  is a join curve;*
- (3) *a simple holomorphic comb  $(u, v)$  where  $v$  is a shuffle curve;*
- (4) *a one-story holomorphic comb  $(u, v_1, \dots, v_k)$  such that each  $v_i$  is a split curve, and, furthermore, the result  $T_1^\diamond \natural \dots \natural T_k^\diamond$  of pregluing the sources of  $v_1, \dots, v_k$  is also a split curve; or*
- (5) *a boundary degeneration, i.e., a nodal holomorphic curve obtained by collapsing an arc or arcs in  $S^\triangleright$  with boundary on  $\partial S^\triangleright$ , so that every connected component of the complement of the collapsing arcs contains a puncture labeled  $+$  or  $-$ .*

(Recall the definitions of join curves, split curves and shuffle curves from the end of Section 5.2; in particular, a join curve or a shuffle curve has only one non-trivial component, while a split curve may have arbitrarily many.)

Note that in the case of boundary degenerations it follows further that each component of the resulting curve has  $\pi_{\mathbb{D}} \circ u$  non-constant on each connected component—and hence, for generic  $J$ , is transversally cut out.

*Proof of Proposition 5.32.* To begin with, some general remarks. Since  $\pi_2(\Sigma) = 0$ ,  $\pi_2(\Sigma, \alpha) = 0$  and  $\pi_2(\Sigma, \beta) = 0$ , no disks or spheres can bubble off. As usual, Deligne-Mumford type degenerations, in which a closed circle or circles degenerate, are codimension 2, and hence do not occur in these moduli spaces. By contrast, boundary degenerations, in which an arc with boundary on  $\partial S$  pinches, can occur in codimension 1; this is case (5) above. The statement that every component contains a puncture labeled  $+$  or  $-$  follows from the homological linear independence of the  $\alpha$ -curves and of the  $\beta$ -curves.

Next, suppose we have a simple holomorphic comb  $(u, v)$  in  $\partial \overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$ . Let  $S^{\triangleright'}$  and  $T^{\triangleright'}$  denote the sources of  $u$  and  $v$  respectively, and let  $m$  be  $|W(T^{\triangleright'})|$ .

The curve  $v$  induces a partition  $P'$  of  $W(T^{\triangleright'})$ , by specifying that  $p$  and  $q$  belong to the same part if they have the same  $t$ -coordinate in  $v$ . Then,  $u \in \mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^{\triangleright'}; P')$  (using the identification of  $E(S^{\triangleright'})$  and  $W(T^{\triangleright'})$ ).

Notice that  $|P'| \leq |P|$ . Also notice that  $\chi(S') = \chi(S) - \chi(T') + m$ . Since every component of  $T'$  must have a west puncture,  $\chi(T') \leq m$ , with equality if and only if  $T'$  is a disjoint union of topological disks with one west puncture per component.

Since, by hypothesis, the expected dimension of  $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$  is 1, by Formula (5.7) we have

$$\text{ind}(B, S^\triangleright, P) = g - \chi(S) + 2e(B) + |P| = 2.$$

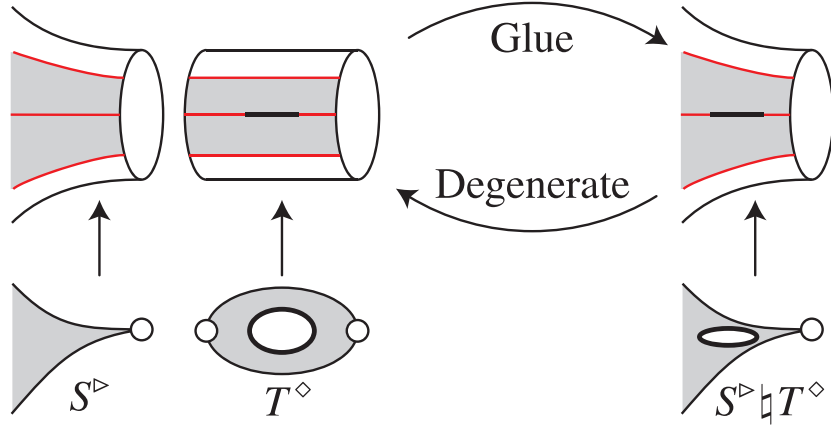


FIGURE 23. A putative annulus at east  $\infty$ . Any near enough curve has a boundary component with no punctures, which violates the maximum modulus principle.

Now,

$$\begin{aligned}
 \text{ind}(B, S^{\triangleright'}, P') &= g - \chi(S') + 2e(B) + |P'| \\
 &= g - \chi(S) + \chi(T') - m + 2e(B) + |P'| \\
 &= 2 - |P| + \chi(T') - m + |P'|.
 \end{aligned}$$

Since  $|P'| \leq |P|$  and  $\chi(T') \leq m$ , we therefore have  $\text{ind}(B, S^{\triangleright'}, P') \leq 2$ .

Suppose that  $\text{ind}(B, S^{\triangleright'}, P') = 2$ . Then  $\chi(T') = m$  and  $|P| = |P'|$ . Consequently,  $T'$  consists entirely of topological disks, each with a single puncture labeled  $w$ . It follows from Propositions 5.13 and 5.22 that  $v$  can be glued to  $u$ , and so  $\text{ind}(B, S^{\triangleright}, P) > \text{ind}(B, S^{\triangleright'}, P')$ , contradicting the hypothesis.

Next, suppose that  $\text{ind}(B, S^{\triangleright'}, P') = 1$ . There are now two cases:

- $|P| = |P'|$  and  $\chi(T') = m - 1$ . In this case, either  $T'$  consists of  $m - 1$  topological disks and one topological annulus, with two boundary punctures each, or just  $m - 1$  topological disks. It is easy to see that the former does not occur: the image in  $Z \times \mathbb{R}$  of the annular component of  $T'$  is a strip with a slit in the middle. (See Figure 23.) Any curve close enough to this comb will still have this slit and so would be forced, by the maximum modulus theorem, to have a component on which  $\pi_{\mathbb{D}} \circ u$  is constant.

By contrast, the case that  $T'$  consists of  $m - 1$  topological disks can occur. In this case, all but one of these disks has one  $w$  puncture, and one has two  $w$  punctures. We claim that none of the disks has two consecutive  $e$  punctures (not separated by a  $w$  puncture). Indeed, any two consecutive  $e$  punctures will have different heights in any curve with source  $S^{\triangleright'} \natural T^{\triangleright'}$ . But this contradicts the fact that  $|P| = |P'|$ . It follows that  $m - 2$  of the disks in  $T^{\triangleright'}$  are trivial components and the remaining disk is either a join component or a shuffle component.

- $|P'| = |P| - 1$  and  $\chi(T') = m$ . In this case,  $T'$  consists of  $m$  topological disks, each with a single west puncture. On first glance, it appears each disk may have an arbitrary number of east punctures. However, by Propositions 5.5, 5.13 and 5.22, we can glue  $u$  to  $v$  in  $\mathcal{M}(\mathbf{x}, \mathbf{y}; S^\triangleright)$  (though perhaps not in  $\overline{\mathcal{M}}(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$ ). It is then easy to see that the east punctures on a single component of  $T'$  will have different  $t$ -coordinates in all of the glued curves. It follows that if any component of  $T'$  has more than two east punctures then  $|P| - |P'| > 1$ , a contradiction. Therefore  $T^{\diamond'}$  is a split curve, as in Case (4) of the statement.

If  $\text{ind}(B, S^{\triangleright'}, P') \leq 0$  then, by Proposition 5.5, the moduli space  $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^{\triangleright'}; P')$  is empty.

Next, consider a general one-story holomorphic comb  $(u, v_1, \dots, v_\ell)$ . The arguments above then imply that the bi-decorated source  $T_1^\diamond \natural \dots \natural T_\ell^\diamond$  obtained by pregluing the sources of  $v_1, \dots, v_\ell$  is a join curve or a split curve.

Finally, consider a general holomorphic comb  $U$  in  $\partial \overline{\mathcal{M}}(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$ . It follows from Formula (5.7) that each story of  $U$  drops the expected dimension by 1, so there are at most two stories. Further, each story individually has index 1, so by the arguments above cannot have any components at east  $\infty$ .  $\square$

The first four kinds of degenerations will be involved in the definitions of the invariants, in Sections 6 and 7. So we now show that boundary degenerations can not occur in certain parts of the moduli spaces. We first show that strong boundary monotonicity depends only on the asymptotics of the moduli space.

**Lemma 5.33.** *Let  $u \in \mathcal{M}(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$ . Let  $a$  be an arc in  $\partial S$  mapped by  $u$  to  $\alpha_i \times \{1\} \times \mathbb{R}$  for some  $i$ . Then  $t \circ u|_a : a \rightarrow \mathbb{R}$  is monotonic.*

*Proof.* This is immediate from the fact that  $\pi_{\mathbb{D}} \circ u$  is holomorphic and has image contained in  $[0, 1] \times \mathbb{R}$ .  $\square$

**Definition 5.34.** Let  $\mathbf{x}$  be a generalized generator and let  $\vec{\rho} = (\rho_1, \dots, \rho_n)$  be a sequence of multi-sets of Reeb chords. Define

$$o(\mathbf{x}, \vec{\rho}) := (o(\mathbf{x}) \cup M(\vec{\rho}^+)) \setminus M(\vec{\rho}^-)$$

where  $o(\mathbf{x})$  and the union and difference are interpreted in terms of multi-sets, i.e., sets whose members may be repeated. (The idea is that  $o(\mathbf{x}, \vec{\rho})$  represents the occupied  $\alpha$ -arcs, with multiplicities, starting with  $\mathbf{x}$  and passing through the Reeb chords  $\vec{\rho}$ .) Let  $\vec{\rho}_{[i,j]}$  be the subsequence  $(\rho_i, \dots, \rho_j)$  of  $\vec{\rho}$ . (If  $i > j$  the subsequence is empty.) We say that the pair  $(\mathbf{x}, \vec{\rho})$  is *strongly boundary monotonic* if the following two conditions are satisfied:

- For each  $i = 0, \dots, n$ , the multi-set  $o(\mathbf{x}, \vec{\rho}_{[1,i]})$  is a  $k$ -element subset of  $[2k]$  with no repeated elements.
- For each  $i$ ,  $M(\rho_i^-)$  and  $M(\rho_i^+)$  (as multi-sets) have no elements with multiplicity bigger than 1.

This is equivalent to the following conditions.

- The generalized generator  $\mathbf{x}$  is a generator.
- $M(\rho_{i+1}^-) \subset o(\mathbf{x}, \rho_{[1,i]})$  and  $M(\rho_{i+1}^+)$  is disjoint from  $o(\mathbf{x}, \rho_{[1,i]}) \setminus M(\rho_{i+1}^-)$ .

Since this definition depends only on  $o(\mathbf{x})$ , we can also speak of the pair  $(o, \vec{\rho})$  being strongly boundary monotonic.

*Remark 5.35.* It is clear from the original definition in terms of occupied  $\alpha$ -arcs that strong boundary monotonicity is a closed condition on the compactified moduli spaces.

**Lemma 5.36.** *If  $(\mathbf{x}, \vec{\rho})$  is strongly boundary monotonic and  $u \in \mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P})$  with  $[\vec{P}] = \vec{\rho}$ , then  $u$  is strongly boundary monotonic.*

*Proof.* This is immediate from Lemma 5.33.  $\square$

**Lemma 5.37.** *Suppose that  $(\mathbf{x}, \vec{\rho})$  is strongly boundary monotonic. Then the moduli spaces  $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P})$  with  $[\vec{P}] = \vec{\rho}$  do not contain any boundary degenerations.*

*Proof.* Suppose that a sequence  $u_j \in \mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P})$  converges to nodal curve where an arc  $a \subset S$  has pinched. Let  $\partial a = \{\partial_+ a, \partial_- a\}$ ,  $u' = \lim_{j \rightarrow \infty} u_j$ , and  $S^{\triangleright'}$  the source of  $u'$ . Certainly  $\pi_\Sigma \circ u_j(\partial_+ a)$  and  $\pi_\Sigma \circ u_j(\partial_- a)$  are contained in either the same  $\alpha$ -curve or the same  $\beta$ -curve; for definiteness, assume  $\pi_\Sigma \circ u_j(\partial a) \subset \alpha_i^a$ . Also,  $\lim_{j \rightarrow \infty} t \circ u_j(\partial_+ a) = \lim_{j \rightarrow \infty} t \circ u_j(\partial_- a)$ . If  $\partial_+ a$  and  $\partial_- a$  do not lie on the same component of  $\partial S$  then for  $j$  large enough,  $u_j$  would violate the strong boundary monotonicity condition. So, let  $b$  denote the arc on  $\partial S$  connecting  $\partial_+ a$  and  $\partial_- a$ , and  $b'$  the corresponding arc in  $S'$ . Then  $t \circ u'$  takes the same value on the two endpoints of  $b'$ , so  $t \circ u'|_{b'}$  must assume a maximum or minimum value somewhere in the interior of  $b'$ . It follows that  $t \circ u'|_{b'}$  is constant. Hence,  $\pi_{\mathbb{D}} \circ u'$  is constant on the component of  $S'$  containing  $b'$ . But this contradicts the fact from Proposition 5.32 that every component of  $S^{\triangleright'}$  has a puncture labeled  $+$  or  $-$ .  $\square$

*Remark 5.38.* At this point, we would like to assert that if strong boundary monotonicity is satisfied then  $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P})$  is a compact 1-manifold with boundary, and  $\partial \overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P})$  consists of exactly the following pieces: two-story holomorphic buildings, degenerating a join curve, or degenerating a split curve. The difficulty with this statement is the case of shuffle curves and split curves with many split components: in these cases, the transversality needed by Proposition 5.22 is absent. One would like to assert, for instance, that simple combs  $(u', v')$  in  $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P}) \setminus \mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P})$  are isolated. The maps  $u'$  are, indeed, isolated, but this is not obvious for the components  $v'$  at east  $\infty$ . (This is related to the fact that the evaluation map from  $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P})$  collapses strata coming from nontrivial moduli at east  $\infty$ .)

However, all we will need later is that the sum of the number of elements of the moduli spaces occurring as the boundaries of  $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P})$  (when the latter space is one dimensional) is zero modulo two. This weaker statement is Theorem 5.41 below.

We now define formally various moduli spaces that can appear at the end of one-dimensional spaces, as illustrated in Figure 25.

**Definition 5.39.** Fix a one dimensional moduli space  $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P})$  (or  $\mathcal{M}$  for short) satisfying strong boundary monotonicity.

A *two-story end* of  $\mathcal{M}$  is an element of  $\mathcal{M}^{B_1}(\mathbf{x}, \mathbf{w}; S_1^\triangleright; \vec{P}_1) \times \mathcal{M}^{B_2}(\mathbf{w}, \mathbf{y}; S_2^\triangleright; \vec{P}_2)$ , where  $\mathbf{w} \in \mathfrak{S}(\mathcal{H})$ ,  $B_1 \in \pi_2(\mathbf{x}, \mathbf{w})$  and  $B_2 \in \pi_2(\mathbf{w}, \mathbf{y})$  are such that  $B = B_1 * B_2$ , and  $S^\triangleright = S_1^\triangleright \natural S_2^\triangleright$  is a way of splitting  $S^\triangleright$  in two which divides the ordered partition  $\vec{P}$  as  $\vec{P}_1 < \vec{P}_2$ .

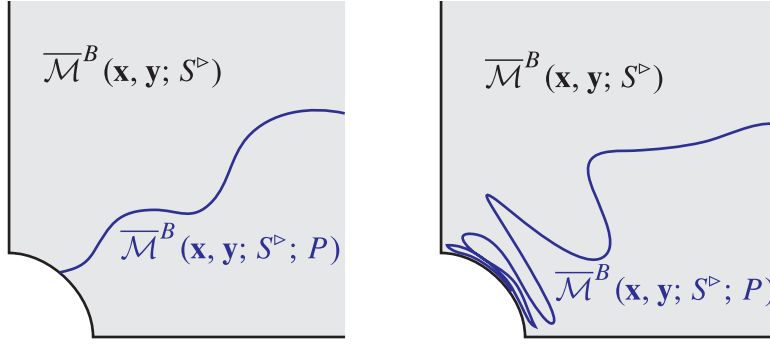


FIGURE 24. A picture of the expected behavior of  $\overline{\mathcal{M}}(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P}) \subset \overline{\mathcal{M}}(\mathbf{x}, \mathbf{y}; S^\triangleright)$  (left) and a behavior of  $\overline{\mathcal{M}}(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P}) \subset \overline{\mathcal{M}}(\mathbf{x}, \mathbf{y}; S^\triangleright)$  which we have not ruled out (right). The reader might compare with Figure 21.

A *join curve end* of  $\mathcal{M}$  at level  $i$  is an element of  $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^{\triangleright'}; \vec{P}')$ , where  $S^{\triangleright'}$  and  $\vec{P}'$  are obtained in the following way. Pick an east puncture  $q$  of  $S^\triangleright$  in the  $i^{\text{th}}$  part of  $\vec{P}$  and a decomposition  $\rho_q = \rho_a \uplus \rho_b$ . Then the decorated source  $S^{\triangleright'}$  is any source with a pair of punctures  $a$  and  $b$  labeled by  $\rho_a$  and  $\rho_b$ , and such that  $S^\triangleright$  is obtained from  $S^{\triangleright'}$  by pregluing a join component to  $S^{\triangleright'}$  at the punctures  $a$  and  $b$ . The partition  $\vec{P}'$  is obtained from  $\vec{P}$  by replacing  $q$  with  $\{a, b\}$  in the  $i^{\text{th}}$  part.

A *odd* (respectively *even*) *shuffle curve end* of  $\mathcal{M}$  at level  $i$  is an element of  $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^{\triangleright'}; \vec{P}')$ , where  $S^{\triangleright'}$  and  $\vec{P}'$  are obtained in the following way. Pick east punctures  $q_1$  and  $q_2$  of  $S^\triangleright$  contained in the  $i^{\text{th}}$  part of  $\vec{P}$ , and so that the corresponding Reeb chords are nested (respectively interleaved). Then  $S^{\triangleright'}$  is any source together with punctures  $q'_1$  and  $q'_2$  such that  $S^\triangleright$  is obtained from  $S^{\triangleright'}$  by pregluing an odd (respectively even) shuffle component to  $S^{\triangleright'}$  at the punctures  $q'_1$  and  $q'_2$ . The partition  $\vec{P}'$  is obtained from  $\vec{P}$  by replacing  $\{q_1, q_2\}$  with  $\{q'_1, q'_2\}$  in the  $i^{\text{th}}$  part.

A *collision of levels  $i$  and  $i + 1$  in  $\mathcal{M}$*  is an element of  $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^{\triangleright'}; (P_1, \dots, P_i \uplus P_{i+1}, \dots, P_n))$  with  $i$  satisfying  $1 \leq i < n$  and  $S^{\triangleright'}$  is obtained from  $S^\triangleright$  by contracting arcs on  $\partial S^\triangleright$  connecting punctures labeled by composable pairs of Reeb chords from  $P_i$  and  $P_{i+1}$ . (We are using here the notion of composable and the definition of  $\uplus$  from Definition 3.5.)

We will see that homological linear independence of the  $\alpha$ -curves often prevents collisions of two levels:

**Definition 5.40.** We say that two sets of Reeb chords  $\rho_i$  and  $\rho_{i+1}$  are *weakly composable* if for all  $\rho_j \in \rho_i$  and  $\rho_k \in \rho_{i+1}$ , if  $M(\rho_j^+) = M(\rho_k^-)$  then  $\rho_j^+ = \rho_k^-$ .

Note that if two sets of Reeb chords are composable in the sense of Section 3.1.3 then they are weakly composable, but the converse is false.

We are now ready to state the theorem we will use to prove all the algebraic properties of our modules.

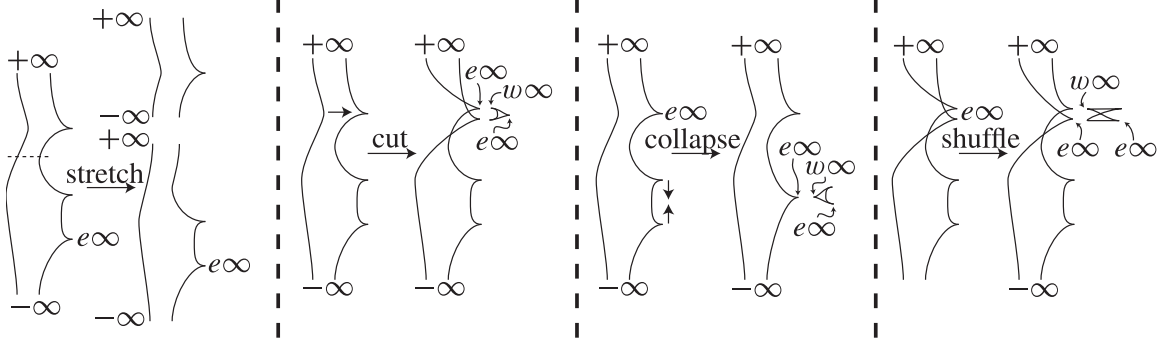


FIGURE 25. Schematics and examples of the four kinds of degenerations of Theorem 5.41. Far left: degenerating into a two-story building. Center left: degenerating a join curve. Center right: degenerating a split curve. Far right: degenerating a shuffle curve.

**Theorem 5.41.** *Suppose that  $(\mathbf{x}, \vec{\rho})$  is strongly boundary monotonic. Fix  $\mathbf{y}$ ,  $B \in \pi_2(\mathbf{x}, \mathbf{y})$ ,  $S^\triangleright$ , and  $\vec{P}$  so that  $[\vec{P}] = \vec{\rho}$  and  $\text{ind}(B, S^\triangleright, P) = 2$ . Let  $\mathcal{M} = \mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P})$ . Then the total number of all*

- (1) *two-story ends of  $\mathcal{M}$ ,*
- (2) *join curve ends of  $\mathcal{M}$ ,*
- (3) *odd shuffle curve ends of  $\mathcal{M}$ , and*
- (4) *collision of levels  $i$  and  $i + 1$  in  $\mathcal{M}$*

*is even. Moreover, in Case (4) the parts  $P_i$  and  $P_{i+1}$  are weakly composable.*

*Proof.* Let  $U_{<\epsilon}$  denote the open subset of  $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P})$  where two parts of  $\vec{P}$  have height difference  $< \epsilon$ . Let  $U_{\text{shuf}}$  denote the union of a smeared neighborhood of each shuffle curve end in  $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P})$ . It follows from the results above that

$$\overline{\mathcal{M}}_{\text{cropped}}^B = \overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P}) \setminus (U_{<\epsilon} \cup U_{\text{shuf}})$$

is a compact 1-manifold with boundary (smoothness following from Proposition 5.5, compactness from Proposition 5.20), the boundary of which consists of the following pieces:

- Two-story holomorphic buildings  $(u_1, u_2)$ . For  $\epsilon$  small enough, the number of these correspond exactly to the number of two-story ends of  $\mathcal{M}$ . (This is ensured by Proposition 5.21).
- Simple holomorphic combs  $(u, v)$  with  $v$  a join curve. For  $\epsilon$  small enough the number of these boundary components is the number of join curve ends of  $\mathcal{M}$ . (This is an application of Proposition 5.22, with transversality hypotheses ensured by Propositions 5.5 and 5.13)
- An even number of points for each even shuffle curve end, and an odd number of points for each odd shuffle curve end. (This is an application of Proposition 5.31.)
- The subspace of  $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; \vec{P})$  where two parts differ in height by  $\epsilon$ . By Propositions 5.29 (if at least one split curve degenerates) and 5.5 (if no split curves degenerate), for  $\epsilon$  small enough the number of boundary points of this form agrees, modulo 2, with the number of collisions of two levels from  $S^\triangleright$ . The

fact that the levels must be weakly composable follows from strong boundary monotonicity.

We have accounted for all the boundary components, in view of Proposition 5.32, combined with Lemma 5.37 (which excludes possible boundary degenerations).

In the above list, in the first two cases,  $\epsilon$  should be small enough that no degenerations of this kind occur in  $\overline{\mathcal{M}}^B \setminus \overline{\mathcal{M}}_{cropped}^B$ . In the third case,  $\epsilon$  should be small enough that  $U_{<\epsilon}$  is disjoint from  $U_{\text{shuf}}$ . In the fourth case,  $\epsilon$  should be small enough for Proposition 5.29 to apply.  $\square$

*Remark 5.42.* We will see in Lemma 5.56 that for moduli spaces of embedded curves, the only collisions which occur are ones in which the colliding parts are composable (as defined in Section 3.1.3) and not merely weakly composable, as well as similar restrictions on the join curve and shuffle curve ends.

**5.5. More on expected dimensions.** Recall from Proposition 5.6 (Formula (5.7)) that the expected dimension of  $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P})$  depends on the topology of the source curve. Our aim here is to study an index formula which depends on the source curve only through its homology class  $B$  and its asymptotics  $\vec{P}$ . We establish two key properties of this index:

- In cases where we have an embedded source curve representing the homology class, this formula agrees with our earlier index formula (according to Proposition 5.47 below).
- The index is additive under juxtapositions (Proposition 5.53).

A fundamental consequence of the index formula is the following:

- For collisions of levels occurring in the boundaries of two-dimensional, strongly boundary monotonic moduli spaces, the colliding levels are composable, rather than just weakly composable (Lemma 5.56).

This last property is the justification for setting double crossings of strands to zero in the algebra associated to a surface.

Before stating the index formula, we recall and extend some definitions from Section 3.3.1. For  $\alpha_1, \alpha_2 \in H_1(Z', \mathbf{a})$ , recall that  $L(\alpha_1, \alpha_2) = m(\alpha_2, \delta\alpha_1)$  is the “linking” of  $\alpha_1$  and  $\alpha_2$ . Concretely, if the  $\alpha_i$  are represented by single Reeb chords  $\rho_i$ , we have

$$L([\rho_1], [\rho_2]) = \begin{cases} 1/2 & \text{if } \rho_1^+ = \rho_2^- \\ -1/2 & \text{if } \rho_2^+ = \rho_1^- \\ 0 & \text{if } \rho_1 \cap \rho_2 = \emptyset \text{ or } \rho_1 \subset \rho_2 \text{ or } \rho_2 \subset \rho_1 \text{ or } \rho_1 = \rho_2 \\ 1 & \text{if } \rho_1^- \leq \rho_2^- \leq \rho_1^+ \leq \rho_2^+ \\ -1 & \text{if } \rho_2^- \leq \rho_1^- \leq \rho_2^+ \leq \rho_1^+. \end{cases}$$

(Recall the definition of  $\leq$  from Section 3.2.) Also recall that for  $a \in \mathcal{A}(n, k)$  with starting idempotent  $S$ ,  $\iota(a) = \text{inv}(a) - m([a], S)$  is the Maslov component of the grading of  $a$ . We can again express this more concretely for algebra elements expressed as Reeb chords.

**Lemma 5.43.** *For  $\rho$  a set of Reeb chords so that  $a(\rho) \neq 0$ ,*

$$\iota(a(\rho)) = - \sum_{\{\rho_1, \rho_2\} \subset \rho} |L([\rho_1], [\rho_2])| - \frac{|\rho|}{2}.$$

*Proof.* Recall from the proof of Proposition 3.20 that  $\iota(a)$  is unchanged under adding horizontal strands; thus we may pretend that the only strands in  $a(\boldsymbol{\rho})$  are those coming from the Reeb chords in  $\boldsymbol{\rho}$ . Both terms in  $\iota(a(\boldsymbol{\rho}))$ , namely  $\text{inv}(a(\boldsymbol{\rho}))$  and  $m([\boldsymbol{\rho}], \boldsymbol{\rho}^-)$ , can be written as sums over pairs of Reeb chords or single chords. A single chord contributes  $-1/2$  to  $m([\boldsymbol{\rho}], \boldsymbol{\rho}^-)$ , while a straightforward case analysis shows that the contribution to the sum from a pair of chords is  $-1$  if they are interleaved,  $-1/2$  if the endpoints abut, and  $0$  otherwise, as in the statement.  $\square$

We will shorten notation and write  $L(\rho_1, \rho_2)$  for  $L([\rho_1], [\rho_2])$  and  $\iota(\boldsymbol{\rho})$  for  $\iota(a(\boldsymbol{\rho}))$ . Also, for a strongly boundary monotonic sequence of sets of Reeb chords  $\vec{\boldsymbol{\rho}} = (\boldsymbol{\rho}_1, \dots, \boldsymbol{\rho}_\ell)$ , define  $\iota(\vec{\boldsymbol{\rho}})$  by

$$(5.44) \quad \iota(\vec{\boldsymbol{\rho}}) := \sum_i \iota(\boldsymbol{\rho}_i) + \sum_{i < j} L(\boldsymbol{\rho}_i, \boldsymbol{\rho}_j).$$

**Lemma 5.45.** *For  $\vec{\boldsymbol{\rho}}$  a strongly boundary monotonic sequence of Reeb chords,  $\iota(\vec{\boldsymbol{\rho}})$  is the Maslov component of  $\text{gr}'(a(\boldsymbol{\rho}_1)) \cdots \text{gr}'(a(\boldsymbol{\rho}_\ell))$ .*

*Proof.* Clear from the definitions and Lemma 3.17.  $\square$

Finally, for  $x \in \boldsymbol{\alpha} \cap \boldsymbol{\beta}$  and  $B$  a domain, define  $n_x(B)$  to be the average of the local multiplicities of  $B$  in the four regions surrounding  $x$ , and for  $\mathbf{x} \in \mathfrak{S}(\mathcal{H})$ , define  $n_{\mathbf{x}}(B) = \sum_{x \in \mathbf{x}} n_x(B)$ .

**Definition 5.46.** We say that a pair  $(B, \vec{\boldsymbol{\rho}})$  with  $B \in \pi_2(\mathbf{x}, \mathbf{y})$  and  $\vec{\boldsymbol{\rho}}$  a sequence of sets of Reeb chords is *compatible* if the homology classes on the boundary agree,  $[\vec{\boldsymbol{\rho}}] = \partial^\partial B$ , and  $(\mathbf{x}, \vec{\boldsymbol{\rho}})$  is strongly boundary monotonic. For such a pair, define the *embedded Euler characteristic*, *embedded index*, and *embedded moduli space* by

$$\begin{aligned} \chi_{\text{emb}}(B, \vec{\boldsymbol{\rho}}) &:= g + e(B) - n_{\mathbf{x}}(B) - n_{\mathbf{y}}(B) - \iota(\vec{\boldsymbol{\rho}}) \\ \text{ind}(B, \vec{\boldsymbol{\rho}}) &:= e(B) + n_{\mathbf{x}}(B) + n_{\mathbf{y}}(B) + |\vec{\boldsymbol{\rho}}| + \iota(\vec{\boldsymbol{\rho}}) \\ \mathcal{M}^B(\mathbf{x}, \mathbf{y}; \vec{\boldsymbol{\rho}}) &:= \bigcup_{\substack{[\vec{P}] = \vec{\boldsymbol{\rho}} \\ \chi(S^\triangleright) = \chi_{\text{emb}}(B, \vec{\boldsymbol{\rho}})}} \mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P}). \end{aligned}$$

Define  $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; \vec{\boldsymbol{\rho}})$  similarly.

Justification for this terminology is given by the following proposition.

**Proposition 5.47.** *Suppose that  $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P})$  admits a holomorphic representative  $u$ . Then*

$$(5.48) \quad \chi(S) = \chi_{\text{emb}}(B, [\vec{P}])$$

*if and only if  $u$  is embedded. When  $u$  is embedded, we also have*

$$(5.49) \quad \text{ind}(B, S^\triangleright, \vec{P}) = \text{ind}(B, [\vec{P}]).$$

*Proof.* We imitate the proof of [20, Proposition 4.2], to which the reader is referred for a more leisurely account. We prove that if  $u$  is embedded, the Euler characteristic is as stated; the other direction, and the index formula, are immediate consequences.

Fix an embedded holomorphic curve  $u \in \mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P})$ . Let  $\text{br}(u)$  denote the ramification number of  $\pi_\Sigma \circ u$ . (Here, boundary branch points contribute  $1/2$ .) By the Riemann-Hurwitz formula,

$$\chi(S) = e(S) + g/2 + c/2 = e(B) - \text{br}(u) + g/2 + c/2$$

where  $c = \sum_i |P_i|$  is the total number of Reeb chords ( $e$  punctures). So, to compute  $\chi(S)$  we only need to compute  $\text{br}(u)$ .

Let  $\tau_R(u)$  denote a copy of  $u$  translated by  $R$  units in the  $\mathbb{R}$ -direction. We will prove the result by comparing  $u \cdot \tau_R(u)$  for  $R$  small and  $R$  large.

Assume now that the partition  $\vec{P}$  is discrete, i.e., each part  $P_i$  consists of a single puncture labeled by  $\rho_i$ ; we will return to the general case at the end of the proof. Since  $u$  is embedded (including near the boundary), for small  $\epsilon$  the curves  $u$  and  $\tau_\epsilon(u)$  intersect only near where  $u$  is tangent to  $\partial/\partial t$ , i.e., at branch points of  $u$ . Since  $u$  and  $\tau_\epsilon(u)$  are  $J$ -holomorphic, their algebraic intersection number is  $\text{br}(u)$ . (Here and later, intersections along the boundary count for  $1/2$ .)

On the other hand, for  $R$  large, the number of intersections of  $u$  and  $\tau_R(u)$  is  $u \cdot \tau_R(u) = n_{\mathbf{x}}(B) + n_{\mathbf{y}}(B) - g/2$ . We will show that

$$(5.50) \quad u \cdot \tau_\epsilon(u) - u \cdot \tau_R(u) = \sum_{i < j} L(\rho_i, \rho_j).$$

and so

$$\begin{aligned} \chi(S) &= e(B) - \text{br}(u) + g/2 + |\vec{P}|/2 \\ &= e(B) - u \cdot \tau_\epsilon(u) + g/2 + |\vec{P}|/2 \\ &= e(B) - u \cdot \tau_R(u) + g/2 - \sum_{i < j} L(\rho_i, \rho_j) + |\vec{P}|/2 \\ &= g + e(B) - n_{\mathbf{x}}(B) - n_{\mathbf{y}}(B) - \iota(\vec{P}) \end{aligned}$$

as desired.

It follows from a simple Schwartz reflection argument that the intersection number  $u \cdot \tau_r(u)$  can only change when an  $e$  puncture of  $\tau_r(u)$ , mapped to  $\rho_1$ , say, passes an  $e$  puncture of  $u$ , mapped to  $\rho_2$ , say. Call this change  $-L'(\rho_1, \rho_2)$ ; that is,  $L'(\rho_1, \rho_2)$  is the number of double points which disappear when  $\rho_1$  goes from below  $\rho_2$  to above  $\rho_2$ . We claim that  $L'(\rho_1, \rho_2)$  is universal, depending only on  $\rho_1$  and  $\rho_2$ , and not on  $u$  itself. Equation (5.50) will then follow from a few model computations.

To see that  $L'(\rho_1, \rho_2)$  is universal we employ a doubling argument, which also computes  $L'(\rho_1, \rho_2)$ . That is, we construct a new Heegaard diagram with boundary  $(\Sigma', \boldsymbol{\alpha}', \boldsymbol{\beta}', z)$ , which we glue to  $\Sigma$  along the boundary. We also construct holomorphic curves  $u_2$  and  $u_1$  which can be glued to  $u$  and  $\tau_r(u)$  respectively, yielding curves  $uu_2$  and  $\tau_r(u)u_1$ , respectively. Since the number of intersections between  $uu_2$  and  $\tau_r(u)u_1$  is topological, remaining constant under deformations of  $uu_2$  and  $\tau_r(u)u_1$ , the number  $L'(\rho_1, \rho_2)$  is equal to the number of double points which *appear* when  $u_1$  is slid up relative to  $u_2$ .

In fact, it suffices to construct  $\Sigma'$ ,  $u_1$ ,  $u_2$  and so on locally near  $\rho_1 \cup \rho_2$ . The constructions of  $\Sigma'$ ,  $u_1$  and  $u_2$  can then be made to depend only on  $\rho_1$  and  $\rho_2$ , not on the curve  $u$  itself. This implies that  $L'(\rho_1, \rho_2)$  depends only on  $\rho_1$  and  $\rho_2$ .

Our model curves will have the domains shown in Figure 26. We distinguish 6 different cases of how  $\rho_1$  and  $\rho_2$  can overlap.

- (1)  $\rho_1^+ = \rho_2^-$ . In this case,  $L'(\rho_1, \rho_2) = 1/2$ . We choose the curves  $u_1$  and  $u_2$  to consist of one strip each, as indicated in Figure 26 (upper left). When sliding  $u_1$  up past  $u_2$ , one boundary double point appears.
- (2)  $\rho_1^- = \rho_2^+$ . In this case,  $L'(\rho_1, \rho_2) = -1/2$ . This case is the same as case (1), with the roles of  $u_1$  and  $u_2$  exchanged.
- (3)  $\rho_1$  is nested inside  $\rho_2$  or vice-versa. In this case,  $L' = 0$ . Indeed, taking  $u_1$  and  $u_2$  to consist of one strip each, as indicated in Figure 26 (upper right), there are no intersection points between  $u_1$  and  $u_2$  no matter how they are slid. (A degenerate sub-case is when  $\rho_1 = \rho_2$ .)
- (4)  $\rho_1$  and  $\rho_2$  are disjoint. In this case,  $L'(\rho_1, \rho_2) = 0$ . Taking  $u_1$  and  $u_2$  to each consist of a single strip, as in Figure 26 (lower left) again gives no intersection points, no matter how they are slid.
- (5)  $(\rho_1, \rho_2)$  is interleaved, with  $\rho_1^- \leq \rho_2^- \leq \rho_1^+ \leq \rho_2^+$ . In this case,  $L'(\rho_1, \rho_2) = 1$ . Take  $u_1$  and  $u_2$  to consist of a single strip each, as in Figure 26 (lower right). Then if  $u_2$  is slid much higher than  $u_1$ , there are no intersection points, while if  $u_2$  is much lower than  $u_1$ , there is a single interior intersection point.
- (6)  $(\rho_2, \rho_1)$  is interleaved, with  $\rho_2^- \leq \rho_1^- \leq \rho_2^+ \leq \rho_1^+$ . In this case,  $L' = -1$ . This is the same as case the previous case, with the roles of  $u_1$  and  $u_2$  reversed.

This completes the proof in the case that  $\vec{P}$  is a discrete partition. In the general case, it is not true that  $\text{br}(u)$  is equal to the intersection number of  $u$  with  $\tau_\epsilon(u)$  for  $\epsilon$  small. That is, there may be intersection points of  $u$  and  $\tau_\epsilon(u)$  which run off to east  $\infty$  as  $\epsilon \rightarrow 0$ . This phenomenon is a simple variant of the change in number of double points studied above, and it follows from the model computations used above that

$$u \cdot \tau_\epsilon(u) - \text{br}(u) = - \sum_i \sum_{\{\rho_a, \rho_b\} \subset P_i} |L(\rho_a, \rho_b)|.$$

Otherwise, the argument proceeds as before. The result follows from Lemma 5.43.  $\square$

Since Proposition 5.47 assumed the existence of a holomorphic curve, it is not obvious that  $\text{ind}(B, \vec{\rho})$  is additive. To prove this, we will adapt Sarkar's proof of the same fact in the non-bordered case [38, Theorem 3.2] to our situation. We start with a definition and some lemmas.

Suppose  $a$  is an oriented sub-arc of  $\alpha$ , with endpoints in  $\alpha \cap \beta$ , and  $b$  is an oriented sub-arc of  $\beta$ , with endpoints in  $\alpha \cap \beta$ . We define the *jittered intersection number*  $a \cdot b$  as follows. Let  $a_{SW}$ ,  $a_{SE}$ ,  $a_{NW}$ , and  $a_{NE}$  denote the four translates of  $a$  shown in Figure 27. The endpoints of these translates are disjoint from  $\beta$ , and the translates do not intersect  $\alpha \cap \beta$ . The translates inherit orientations from the orientations of  $a$  and  $b$ . Consequently, the intersection numbers  $a_{SW} \cdot b, \dots, a_{NE} \cdot b$  are well defined. Define

$$a \cdot b = \frac{1}{4} (a_{SW} \cdot b + a_{SE} \cdot b + a_{NW} \cdot b + a_{NE} \cdot b).$$

Note that, instead of jittering  $a$  we could have jittered  $b$  with the same result. We extend the definition of  $a \cdot b$  bilinearly to cellular 1-chains  $a$  contained in  $\alpha$  and  $b$  contained in  $\beta$ . We define  $b \cdot a$  analogously. We also define  $a \cdot a'$  for  $a$  and  $a'$  both

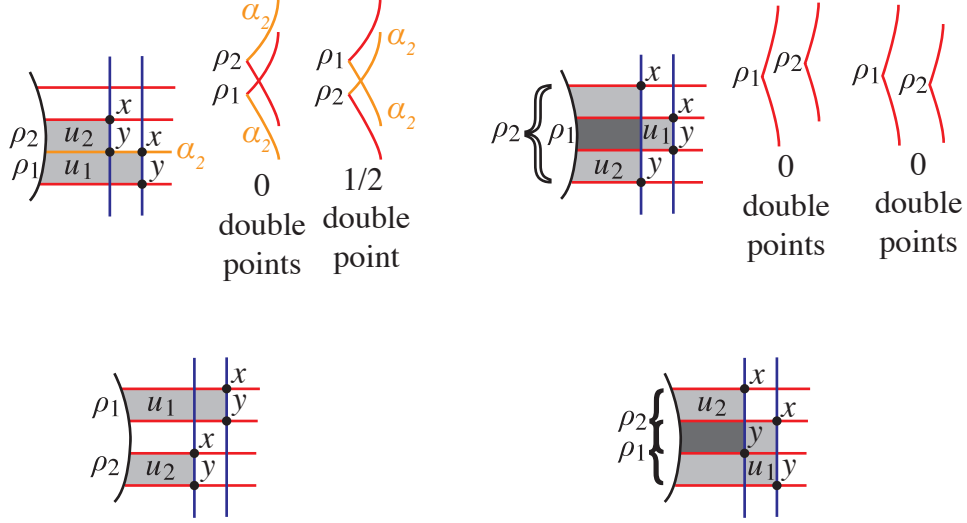


FIGURE 26. The domains of model curves used to compute  $\text{inv}'$ . Top left: case (1). Case (2) is obtained by switching  $\rho_1$  and  $\rho_2$ . Top right: case (3). Bottom left: case (4). Bottom right: case (5). Case (6) is obtained by switching  $\rho_1$  and  $\rho_2$ . In the top two pictures, a schematic for  $u_1(\partial S_1), u_2(\partial S_2) \subset \alpha \times \{1\} \times \mathbb{R}$  is also shown.

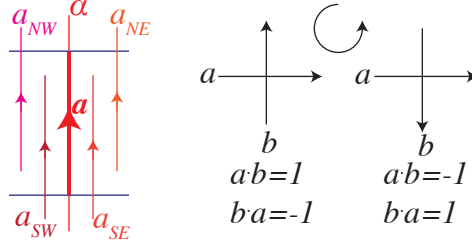


FIGURE 27. Left: the four translates  $a_{SW}$ ,  $a_{SE}$ ,  $a_{NW}$ , and  $a_{NE}$  of an arc  $a$ . Right: conventions for intersection numbers.

contained in  $\alpha$  (or  $\beta$ ) in exactly the same way. (This definition is analogous to the definitions of  $n_{\mathbf{x}}(B)$  for Proposition 5.47 and  $m(\alpha, P)$  in Section 3.3.1.)

The following lemma is due to Sarkar [38, Section 3]. For the reader's convenience we repeat the proof here.

**Lemma 5.51.** *The jittered intersection number  $a \cdot b$  has the following properties:*

- (1)  $a \cdot b = -b \cdot a$ .
- (2) If  $a$  and  $a'$  are both contained in  $\alpha$  then  $a \cdot a' = 0$ .

For the remaining items, let  $B \in \pi_2(\mathbf{x}, \mathbf{y})$ ,  $a = \partial^\alpha(B)$ ,  $b = \partial^\beta(B)$ ,  $B' \in \pi_2(\mathbf{y}, \mathbf{w})$ ,  $a' = \partial^\alpha(B')$ , and  $b' = \partial^\beta(B')$ .

- (3)  $n_{\mathbf{x}}(B) - n_{\mathbf{y}}(B) = a \cdot b$ .
- (4)  $n_{\mathbf{x}}(B') - n_{\mathbf{y}}(B') = a' \cdot b$  and  $n_{\mathbf{y}}(B) - n_{\mathbf{w}}(B) = a \cdot b'$ .
- (5) If  $B$  and  $B'$  are provincial then  $a \cdot b' + b \cdot a' = 0$ .

*Proof.* Properties (1) and (2) are obvious.

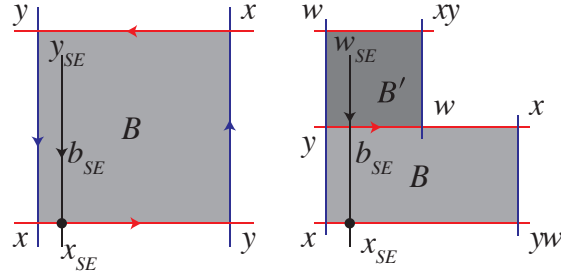


FIGURE 28. Parts (3) and (4) of the proof of Lemma 5.51. Left: part (3). The arrows show the orientations of  $a$  and  $b$  from  $\partial B$ . Notice that  $a \cdot b_{SE} = -1$  and  $n_{\mathbf{y}_{SE}} = n_{\mathbf{x}_{SE}} - (a \cdot b_{SE})$ . Right: part (4). The horizontal arrow indicates the orientation induced by  $\partial B'$ , the vertical arrow the orientation induced by  $\partial B$ . Notice that  $a' \cdot b_{SE} = -1 = n_{\mathbf{x}_{SE}} - n_{\mathbf{w}_{SE}}$ .

To understand Property (3), assume for simplicity that  $a$  and  $b$  are embedded arcs, and consider the path  $b_{SE}$ . The multiplicity of  $B$  at the starting point of  $b_{SE}$  is  $n_{\mathbf{y}_{SE}}(B)$ . Each time  $b_{SE}$  crosses  $a$ , the multiplicity of  $B$  increases by 1, for intersections contributing positively to  $a \cdot b$ , or decreases by 1, for intersections contributing negatively to  $a \cdot b$ . Consequently, at the ending point of  $b_{SE}$ , the multiplicity  $n_{\mathbf{x}_{SE}}$  is  $n_{\mathbf{y}_{SE}} + a \cdot b_{SE}$  and so  $n_{\mathbf{x}_{SE}}(B) - n_{\mathbf{y}_{SE}}(B) = a \cdot b_{SE}$ . Averaging over the four possible directions gives the result. Property (4) is proved similarly to Property (3), using  $b'$  instead of  $b$ . See Figure 28 for an example of these two properties.

Finally, Property (5) follows from the fact that  $a - b$  and  $a' - b'$  are null-homologous 1-chains, so  $(a - b) \cdot (a' - b') = 0$ . But by bilinearity and Property (2), this reduces to  $a \cdot b' + b \cdot a' = 0$ .  $\square$

We can extend Property (5) of Lemma 5.51 to non-provincial domains, as well.

**Lemma 5.52.** *With notation as in Lemma 5.51, for arbitrary  $B$  and  $B'$ , we have  $a \cdot b' + b \cdot a' = L(\partial^\partial B, \partial^\partial B')$ .*

*Proof.* We now have that  $a - b + \partial^\partial B$  and  $a' - b' + \partial^\partial B'$  are null-homologous 1-chains, so their cap product in homology is 0. We can relate this to an extension of the jittered intersection product by, for instance, treating  $\partial\bar{\Sigma}$  as an extra  $\beta$ -circle. The terms that do not vanish identically are

$$(a - b + \partial^\partial B) \cdot (a' - b' + \partial^\partial B') = -a \cdot b' - b \cdot a' + a \cdot \partial^\partial B' + \partial^\partial B \cdot a' = 0.$$

Almost by definition,  $a \cdot \partial^\partial B' = \frac{1}{2}m(\partial^\partial B', \delta(\partial^\partial B))$  and  $\partial^\partial B \cdot a' = -\frac{1}{2}m(\partial^\partial B, \delta(\partial^\partial B'))$ . Both of these terms are equal to  $\frac{1}{2}L(\partial^\partial B, \partial^\partial B')$ , as desired.  $\square$

**Proposition 5.53.** *The index is additive in the following sense. Let  $B \in \pi_2(\mathbf{x}, \mathbf{y})$  and  $B' \in \pi_2(\mathbf{y}, \mathbf{w})$ , and let  $\vec{\rho}$  and  $\vec{\rho}'$  be sequences of sets of Reeb chords with  $[\vec{\rho}] = \partial^\partial B$  and  $[\vec{\rho}'] = \partial^\partial B'$ . Then  $\text{ind}(B, \vec{\rho}) + \text{ind}(B', \vec{\rho}') = \text{ind}(B * B', (\vec{\rho}, \vec{\rho}'))$*

(Here,  $(\vec{\rho}, \vec{\rho}')$  denotes the sequence obtained by simply concatenating  $\vec{\rho}$  and  $\vec{\rho}'$ .)

*Proof.* It is clear that terms  $e(B)$  and  $|\vec{\rho}|$  of Definition 5.46 are additive. Thus, we must show that

$$\begin{aligned} n_{\mathbf{x}}(B) + n_{\mathbf{y}}(B) + n_{\mathbf{y}}(B') + n_{\mathbf{w}}(B') + \iota(\vec{\rho}) + \iota(\vec{\rho}') \\ = n_{\mathbf{x}}(B + B') + n_{\mathbf{w}}(B + B') + \iota((\vec{\rho}, \vec{\rho}')). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} [n_{\mathbf{x}}(B') - n_{\mathbf{y}}(B')] - [n_{\mathbf{y}}(B) - n_{\mathbf{w}}(B)] &= a' \cdot b - a \cdot b' \\ &= a' \cdot b + b \cdot a' - L(\partial^\partial B, \partial^\partial B') \\ &= \iota(\vec{\rho}) + \iota(\vec{\rho}') - \iota((\vec{\rho}, \vec{\rho}')), \end{aligned}$$

where the first equality uses property (4) of Lemma 5.51, the second uses Lemma 5.52, and the third uses property (1) and the definition of  $\iota$ . This is equivalent to what we were trying to show.  $\square$

We finish this section with two lemmas relating to which holomorphic curves occur in the boundary of moduli spaces of embedded curves.

The first lemma states that, as usual, the maximal index parts of the moduli space come from embedded curves. Together with additivity of  $\text{ind}(B, \vec{\rho})$  (Proposition 5.53) this implies that if a sequence of embedded holomorphic curves converges to a multi-story holomorphic building then all of the stories are embedded—a fact we will use in Propositions 6.6 and 7.8, for instance.

**Lemma 5.54.** *Suppose that  $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P})$  is nonempty for some generic almost complex structure  $J$ . Then  $\chi(S^\triangleright) \geq \chi_{\text{emb}}(B, [\vec{P}])$ , and consequently  $\text{ind}(B, S^\triangleright, \vec{P}) \leq \text{ind}(B, [\vec{P}])$ . The same is true if  $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P})$  admits a holomorphic representative in some generic 1-parameter family of almost complex structures.*

*Proof.* It follows from standard gluing arguments that interior double points can be resolved, decreasing the Euler characteristic by 2 and increasing the expected dimension by 2. (Alternately, this lemma can be proved by imitating the proof of Proposition 5.47 for a curve with double points.)  $\square$

**Corollary 5.55.** *Fix a generic almost complex structure  $J$ . Suppose that  $(B, \vec{\rho})$  has  $\text{ind}(B, \vec{\rho}) = 2$ . Then the two-story building which occur in the boundary of the embedded moduli space  $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; \vec{\rho})$  (as in Theorem 5.41, say) are embedded. Equivalently, if  $(u_1, u_2)$  is such a two-story building, with  $S_i^\triangleright$  the source of  $u_i$ ,  $B_i$  the homology class of  $B_i$  and  $\vec{P}_i$  the ordered partition associated to  $u_i$  then:*

- $\chi(S_i^\triangleright) = \chi_{\text{emb}}(B_i, \vec{P}_i)$
- $\text{ind}(B_i, \vec{P}_i) = 1$ .

*Proof.* This follows from Lemma 5.54 and the fact that moduli spaces of negative expected dimension ( $\text{ind} \leq 0$ ) are empty (Proposition 5.5).  $\square$

The second lemma states that when two levels collide in a sequence of embedded curves, the corresponding parts of a partition must be composable (as in Section 3.1.3), and not merely weakly composable (Definition 5.40). Similar statements hold for splittings and shuffles.

**Lemma 5.56.** *Let  $(\mathbf{x}, \vec{\rho})$  satisfy the strong boundary monotonicity condition, and  $B \in \pi_2(\mathbf{x}, \mathbf{y})$  be compatible with  $\vec{\rho}$ . Suppose that  $\text{ind}(B, \vec{\rho}) = 2$ . Suppose there is a holomorphic comb  $(u, v)$  in  $\partial \overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; \vec{\rho})$ , and let  $\vec{\rho}'$  be the sequence of sets of Reeb chords appearing as asymptotics of  $u$ . Then*

- (1) *if two levels  $\rho_i$  and  $\rho_{i+1}$  collide in  $u$ ,  $\rho_i$  and  $\rho_{i+1}$  are composable;*
- (2) *if  $u$  is a join curve end at level  $i$ ,  $\rho'_i$  is a splitting of  $\rho_i$ ; and*
- (3) *if  $u$  is an odd shuffle curve end at level  $i$ ,  $\rho'_i$  is a shuffle of  $\rho_i$ .*

*Proof.* For the first statement, by Theorem 5.41 we know that  $\rho_i$  and  $\rho_{i+1}$  are weakly composable. By Lemma 3.21, if  $\rho_i$  and  $\rho_{i+1}$  are weakly composable but not composable, then  $\text{gr}'(a(\rho_i \uplus \rho_{i+1})) < \text{gr}'(a(\rho_i)) \text{gr}'(a(\rho_{i+1}))$ , so, by Lemma 5.45,  $\iota(\vec{\rho}') < \iota(\vec{\rho})$ . Therefore  $\text{ind}(B, \vec{\rho}') < \text{ind}(B, \vec{\rho}) + (|\vec{\rho}'| - |\vec{\rho}|) = 1$ . Thus the embedded moduli space  $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; \vec{\rho}')$  has negative expected dimension and is therefore empty.

Similarly, a join curve end at level  $i$  gives a weak splitting of  $\rho_i$  by definition. By Lemma 3.21, for a weak splitting that is not a splitting we have  $\text{gr}'(a(\rho'_i)) < \text{gr}'(a(\rho_i)) - 1$ . This implies that  $\text{ind}(B, \vec{\rho}') < 1$ , so again  $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; \vec{\rho}')$  has negative expected dimension.

The case of odd shuffle curve ends is similar.  $\square$

*Remark 5.57.* A similar analysis shows that *even* shuffle curve ends on the boundary of a 1-dimensional embedded moduli space all have negative expected dimension. We do not use that, as we already showed in case 2 of Proposition 5.31 that such curves appear an even number of times on the boundary of any moduli space.

## 6. TYPE $D$ MODULES

We now turn to defining the type  $D$  module  $\widehat{CFD}(\mathcal{H})$  associated to  $\mathcal{H}$ , a bordered Heegaard diagram, using the work on the structure of the moduli spaces from Section 5.

**6.1. Definition of type  $D$  module.** Let  $\mathcal{H}$  be a bordered Heegaard diagram for a three-manifold  $Y$  (in the sense of Definition 4.2), and let  $\mathcal{Z}$  be the *reverse* of the pointed matched circle (in the sense of Definition 3.9) appearing on the boundary of  $\mathcal{H} = (\Sigma, \alpha, \beta, z)$ . That is,  $\mathcal{Z}$  is  $(Z, \overline{\alpha} \cap \partial \overline{\Sigma}, M, z)$ , where  $Z := -\partial \overline{\Sigma}$  has its orientation reversed (and in particular, the Reeb chords run in the opposite direction). Let  $\mathcal{A}(\mathcal{Z})$  be the associated algebra, as in Section 3. We will also abbreviate  $\mathcal{A}(\mathcal{Z})$  by just  $\mathcal{A}$  when it causes no confusion.

Our aim here is to define  $\widehat{CFD}(\mathcal{H})$  as a left module over  $\mathcal{A} = \mathcal{A}(\mathcal{Z})$ .

Let  $\mathfrak{S}(\mathcal{H})$  be the set the generators of  $\mathcal{H}$ , as defined in Definition 4.7: subsets of  $\alpha \cap \beta$  with

- $g$  elements in all,
- exactly one element on each  $\beta$  circle,
- exactly one element on each  $\alpha$  circle, and
- at most one element on each  $\alpha$  arc.

Let  $X(\mathcal{H})$  be the  $\mathbb{F}_2$ -vector space spanned by  $\mathfrak{S}(\mathcal{H})$ . Recall from Section 4.3 that, for  $\mathbf{x} \in \mathfrak{S}(\mathcal{H})$ ,  $o(\mathbf{x}) \subset [2k]$  is the set of  $\alpha$ -arcs occupied by  $\mathbf{x}$ . Define  $I_D(\mathbf{x})$  to be  $I([2k] \setminus o(\mathbf{x}))$ ; that is, the idempotent corresponding to the *complement* of  $o(\mathbf{x})$ . We

can now define an action of the sub-algebra of idempotents  $\mathcal{I}$  inside  $\mathcal{A}$  on  $X(\mathcal{H})$  via

$$(6.1) \quad I(s) \cdot \mathbf{x} := \begin{cases} \mathbf{x} & I(s) = I_D(\mathbf{x}) \\ 0 & \text{otherwise,} \end{cases}$$

where  $s$  is a  $k$ -element subset of  $[2k]$ , as in Formula (3.10). In particular, the summands  $\mathcal{A}(\mathcal{Z}, i)$  of  $\mathcal{A}(\mathcal{Z})$  act trivially on  $\widehat{CFD}(\mathcal{H})$  for  $i \neq 0$ .

As an  $\mathbb{F}_2$  vector space,  $\widehat{CFD}(\mathcal{H})$  is

$$\widehat{CFD}(\mathcal{H}) := \mathcal{A} \otimes_{\mathcal{I}} X(\mathcal{H}).$$

The module structure on  $\widehat{CFD}(\mathcal{H})$  is quite simple; it is defined by

$$(6.2) \quad a \cdot (b \otimes \mathbf{x}) := (ab) \otimes \mathbf{x}.$$

The differential on  $\widehat{CFD}(\mathcal{H})$ , on the other hand, involves counting moduli spaces  $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; \vec{\rho})$ , where  $\vec{\rho}$  is a sequence  $(\rho_1, \dots, \rho_k)$  of Reeb chords. (Here we are using  $\vec{\rho}$  both for a sequence of Reeb chords and for the corresponding sequence of one-element sets of Reeb chords.) Recall that  $a(\rho_i)$  is the element of  $\mathcal{A}$  associated to  $\rho_i$ , as in Section 3.2. For  $\vec{\rho}$  a sequence of Reeb chords, let  $a(\vec{\rho})$  be the product  $a(\rho_1) \cdots a(\rho_k)$ , and let  $-\vec{\rho}$  be the sequence  $(-\rho_1, \dots, -\rho_k)$  of Reeb chords with reversed orientation, i.e., if  $\rho$  is a Reeb chord on  $\partial\mathcal{H}$ ,  $-\rho$  is a Reeb chord on  $\mathcal{Z}$ , which is  $-\partial\mathcal{H}$ .

**Definition 6.3.** For  $\mathbf{x}, \mathbf{y} \in \mathfrak{S}(\mathcal{H})$  and  $B \in \pi_2(\mathbf{x}, \mathbf{y})$ , define

$$a_{\mathbf{x}, \mathbf{y}}^B := \sum_{\{\vec{\rho} \mid \text{ind}(B, \vec{\rho})=1\}} \#(\mathcal{M}^B(\mathbf{x}, \mathbf{y}; \vec{\rho})) a(-\vec{\rho}).$$

Here there is also an implicit condition that  $(B, \vec{\rho})$  is compatible (i.e., that  $(x, \vec{\rho})$  is strongly boundary monotonic and  $\partial^\partial B = [\vec{\rho}]$ ); we will often omit such conditions. Compactness of the moduli spaces  $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; \vec{\rho})$  implies that the sum defining  $a_{\mathbf{x}, \mathbf{y}}^B$  is finite. We now complete the definition of  $\widehat{CFD}(\mathcal{H})$  by defining

$$\partial \mathbf{x} := \sum_{\mathbf{y} \in \mathfrak{S}(\mathcal{H})} \sum_{B \in \pi_2(\mathbf{x}, \mathbf{y})} a_{\mathbf{x}, \mathbf{y}}^B \mathbf{y},$$

and extend via linearity and the Leibniz rule to all of  $\widehat{CFD}(\mathcal{H})$  by  $\partial(a\mathbf{x}) := (\partial a)\mathbf{x} + a(\partial \mathbf{x})$ .

*Remark 6.4.* The map  $\delta_1: X(\mathcal{H}) \rightarrow \mathcal{A} \otimes_{\mathcal{I}} X(\mathcal{H})$  given by

$$\delta_1(x) := \partial(\mathbb{I} \otimes x)$$

defines a type  $D$  structure over  $\mathcal{A}$  with base ring  $\mathcal{I}$  on  $X(\mathcal{H})$ , in the sense of Definition 2.12. (The compatibility relation is equivalent to Proposition 6.6 below.) Similarly Theorem 6.23 shows that this type  $D$  structure is homotopy invariant as a type  $D$  structure, not just as a module.

**Lemma 6.5.** *If  $\mathcal{H}$  is provincially admissible, then the sum in the definition of  $\partial \mathbf{x}$  on  $\widehat{CFD}(\mathcal{H})$  is finite for every  $\mathbf{x}$ . If  $\mathcal{H}$  is admissible, the resulting type  $D$  structure is bounded in the sense of Definition 2.14.*

*Proof.* Since the fibers of  $\pi_\Sigma$  are holomorphic, the intersections of any pseudo-holomorphic curve with a fiber are positive and so the domain of a holomorphic curve is positive. From Proposition 4.19 we then see that if  $\mathcal{H}$  is provincially admissible, for a given  $\mathbf{x}$ ,  $\mathbf{y}$ , and  $\vec{\rho}$  the union over  $B$  of 0-dimensional moduli spaces  $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; \vec{\rho})$  is finite. There are only finitely many possible generators  $\mathbf{y}$  and algebra elements  $a \in \mathcal{A}$ ; furthermore, for any given  $a$  there are only finitely many ways to write it as a product of Reeb chords, and so only finitely many possible  $\vec{\rho}$ . The first statement then follows.

If  $\mathcal{H}$  is admissible, then by Proposition 4.20, there are only finitely many non-empty  $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; \vec{\rho})$ ; in particular the dimension of any such moduli space is bounded, say by  $N$ . Then, in the language of Definition 2.14,  $\delta_{N+2}(\mathbf{x}) = 0$  for all  $\mathbf{x}$ .  $\square$

**Proposition 6.6.** *The boundary operator  $\partial$  on  $\widehat{CFD}(\mathcal{H})$  satisfies  $\partial^2 = 0$ .*

The proof of Proposition 6.6 involves a number of pieces, some of which we give as lemmas. The impatient reader is encouraged to skip to the examples after the proof, which illustrate the most important cases which arise in the verification that  $\partial^2 = 0$ .

In our first lemma, we look at which ordered lists of Reeb chords  $\vec{\rho}$  actually contribute to the boundary operator.

**Lemma 6.7.** *Suppose that  $(\mathbf{x}, \vec{\rho})$  is strongly boundary monotonic and  $I_D(\mathbf{x})a(-\vec{\rho}) \neq 0$ . Then for each  $i$ ,  $M(\rho_i^-) \neq M(\rho_i^+)$ .*

(See also Lemma 9.6.)

*Proof.* Recall from Definition 5.34 that strong boundary monotonicity says that there is a sequence of subsets  $o_i = o(\mathbf{x}, \vec{\rho}_{[1,i]})$  so that  $o_i = (o_{i-1} \setminus \{M(\rho_i^-)\}) \cup \{M(\rho_i^+)\}$ . On the other hand, let  $b_i = I_D(\mathbf{x})a(-\vec{\rho}_{[1,i]})$ . If  $b_i \neq 0$ , there is a unique idempotent  $I(s_i)$  so that  $b_i I(s_i) \neq 0$ ; in particular  $s_0 = I_D(\mathbf{x}) = [2k] \setminus o(\mathbf{x})$ . We also have  $s_i = (s_{i-1} \setminus \{M(\rho_i^+)\}) \cup \{M(\rho_i^-)\}$ , using the fact that reversing the orientation switches the endpoints  $\rho_i^-$  and  $\rho_i^+$ , so  $M((-\rho_i)^-) = M(\rho_i^+)$ . From this it follows by induction that  $s_i$  is the complement of  $o_i$  for all  $i$ . But then we have  $M(\rho_i^-) \in o_i$  (by strong boundary monotonicity) and  $M(\rho_i^+) \in s_i$  (since  $b_i a(-\rho_i) \neq 0$ ), so  $M(\rho_i^-) \neq M(\rho_i^+)$ .  $\square$

The next two lemmas identify how some of the terms in Theorem 5.41 relate to the algebraic framework. Specifically, a term in  $\partial^2 \mathbf{x}$  can come either from  $a_1 \mathbf{y}$  appearing in  $\partial \mathbf{x}$  and  $a_2 \mathbf{z}$  appearing in  $\partial \mathbf{y}$ , contributing  $a_1 a_2 \mathbf{z}$  to  $\partial^2 \mathbf{x}$ , or from  $a \mathbf{y}$  appearing in  $\partial \mathbf{x}$  and  $a'$  appearing in  $\partial a$ , contributing  $a' \mathbf{y}$  to  $\partial^2 \mathbf{x}$ . In either case there is a geometric interpretation.

**Lemma 6.8.** *For  $\mathbf{x}, \mathbf{w}, \mathbf{y} \in \mathfrak{S}(\mathcal{H})$ ,  $B_1 \in \pi_2(\mathbf{x}, \mathbf{w})$ , and  $B_2 \in \pi_2(\mathbf{w}, \mathbf{y})$ , we have*

$$a_{\mathbf{x}, \mathbf{w}}^{B_1} a_{\mathbf{w}, \mathbf{y}}^{B_2} = \sum_{\{\rho \mid \text{ind}(B, \vec{\rho})=2\}} \sum_{\vec{\rho}=(\vec{\rho}_1, \vec{\rho}_2)} \#[\mathcal{M}^{B_1}(\mathbf{x}, \mathbf{w}; \vec{\rho}_1) \times \mathcal{M}^{B_2}(\mathbf{w}, \mathbf{y}; \vec{\rho}_2)] a(-\vec{\rho}).$$

*Proof.* Clear.  $\square$

**Lemma 6.9.** *For  $\mathbf{x}, \mathbf{y} \in \mathfrak{S}(\mathcal{H})$  and  $B \in \pi_2(\mathbf{x}, \mathbf{y})$ , we have*

$$\partial a_{\mathbf{x}, \mathbf{y}}^B = \sum_{\{\vec{\rho} \mid \text{ind}(B, \vec{\rho})=2\}} \sum_{\substack{\rho_i, \rho_{i+1} \\ \text{composable}}} \# \mathcal{M}^B(\mathbf{x}, \mathbf{y}; (\rho_1, \dots, \rho_i \uplus \rho_{i+1}, \dots, \rho_n)) a(-\vec{\rho}).$$

*Proof.* By definition,

$$a_{\mathbf{x}, \mathbf{y}}^B = \sum_{\{\vec{\rho} \mid \text{ind}(B, \vec{\rho})=1\}} [\#\mathcal{M}^B(\mathbf{x}, \mathbf{y}; \vec{\rho})] a(-\rho_1) \cdots a(-\rho_m).$$

and so

$$\begin{aligned} \partial a_{\mathbf{x}, \mathbf{y}}^B &= \sum_{\{\vec{\rho} \mid \text{ind}(B, \vec{\rho})=1\}} \sum_i [\#\mathcal{M}^B(\mathbf{x}, \mathbf{y}; \vec{\rho})] a(-\rho_1) \cdots (\partial(a(-\rho_i))) \cdots a(-\rho_m) \\ &= \sum_{\{\vec{\rho} \mid \text{ind}(B, \vec{\rho})=1\}} \sum_i \sum_{\rho_i = \rho_j \uplus \rho_k} [\#\mathcal{M}^B(\mathbf{x}, \mathbf{y}; \vec{\rho})] a(-\rho_1) \cdots a(-\rho_j) a(-\rho_k) \cdots a(-\rho_m). \end{aligned}$$

To see this, note that if  $\rho_j$  and  $\rho_k$  are composable (in that order), then  $-\rho_j$  and  $-\rho_k$  are *not* composable; thus,

$$\partial(a(-\rho_i)) = \sum_{\{a, b \mid \rho_i = \rho_j \uplus \rho_k\}} a(-\rho_j) a(-\rho_k).$$

Now, from the definition of  $\text{ind}(B, \vec{\rho})$  (Definition 5.46), if  $\rho_i = \rho_j \uplus \rho_k$  then

$$\text{ind}(B, (\rho_1, \dots, \rho_j, \rho_k, \dots, \rho_m)) = \text{ind}(B, (\rho_1, \dots, \rho_i, \dots, \rho_m)) + 1 = 2.$$

(By contrast,  $\text{ind}(B, \vec{\rho}) = \text{ind}(B, (\rho_1, \dots, \rho_k, \rho_j, \dots, \rho_m))$ .) The result follows by reindexing the sum to run over  $\vec{\rho}' = (\rho_1, \dots, \rho_j, \rho_k, \dots, \rho_m)$ .  $\square$

*Proof of Proposition 6.6.* Roughly, the proof proceeds by considering the boundaries of the index 1 moduli spaces. More precisely, we appeal to Theorem 5.41.

Observe that

$$\begin{aligned} \partial^2(a\mathbf{x}) &= \partial \left[ (\partial a)\mathbf{x} + a \left( \sum_{\mathbf{w}} a_{\mathbf{x}, \mathbf{w}} \mathbf{w} \right) \right] \\ &= (\partial^2 a)\mathbf{x} + 2(\partial a)(\partial \mathbf{x}) + a \left[ \left( \sum_{\mathbf{w}} (\partial a_{\mathbf{x}, \mathbf{w}}) \mathbf{w} \right) + \left( \sum_{\mathbf{w}} \sum_{\mathbf{y}} a_{\mathbf{x}, \mathbf{w}} a_{\mathbf{w}, \mathbf{y}} \mathbf{y} \right) \right], \end{aligned}$$

where here  $a_{\mathbf{x}, \mathbf{y}} = \sum_B a_{\mathbf{x}, \mathbf{y}}^B$ . So it suffices to show that for all  $\mathbf{x}$  and  $\mathbf{y}$ ,

$$\partial(a_{\mathbf{x}, \mathbf{y}}) + \sum_{\mathbf{w}} a_{\mathbf{x}, \mathbf{w}} a_{\mathbf{w}, \mathbf{y}} = 0.$$

(This is already implicit in Lemma 2.13.)

Fix generators  $\mathbf{x}$  and  $\mathbf{y}$ . Fix also  $B \in \pi_2(\mathbf{x}, \mathbf{y})$ , and an algebra element  $a$ . Fix a factorization  $a = a(-\vec{\rho})$  of  $a$  such that  $\text{ind}(B, \vec{\rho}) = 2$ . (This index condition in fact depends only on  $a$ .) By Theorem 5.41, applied to  $(\mathbf{x}, \mathbf{y}, B, \vec{\rho})$  and the union, over all sources  $S^\triangleright$  with  $\chi(S) = \chi_{\text{emb}}(B, \vec{\rho})$ , of  $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P})$ , the sum of the following terms is equal to zero:

- (1) The number of two-story ends, i.e., the number of elements of

$$\mathcal{M}^{B_1}(\mathbf{x}, \mathbf{w}; \vec{\rho}_1) \times \mathcal{M}^{B_2}(\mathbf{w}, \mathbf{y}; \vec{\rho}_2)$$

where  $B = B_1 * B_2$  and  $\vec{\rho} = (\vec{\rho}_1, \vec{\rho}_2)$ .

- (2) The number of join curve ends, i.e., the number of elements of

$$\mathcal{M}^B(\mathbf{x}, \mathbf{y}; (\rho_1, \dots, \rho_{i-1}, \{\rho_j, \rho_k\}, \rho_{i+1}, \dots, \rho_n))$$

where  $\rho_i = \rho_j \uplus \rho_k$ .

- (3) The number of split curve ends (necessarily with one split component), i.e., the number of elements of

$$\mathcal{M}^B(\mathbf{x}, \mathbf{y}; (\rho_1, \dots, \rho_{i-1}, \rho_i \uplus \rho_{i+1}, \rho_{i+2}, \dots, \rho_n))$$

where  $\rho_i^+ = \rho_{i+1}^-$ .

- (4) The number of other collisions of levels, i.e., the number of elements of

$$\mathcal{M}^B(\mathbf{x}, \mathbf{y}; (\rho_1, \dots, \rho_{i-1}, \{\rho_i, \rho_{i+1}\}, \rho_{i+2}, \dots, \rho_n))$$

where  $\rho_i^+ \neq \rho_{i+1}^-$ .

(Both the third and fourth contributions come from the fourth case of Theorem 5.41. Shuffle curve ends cannot happen when the partition is discrete.)

By Lemma 6.8, the first term corresponds to  $\sum_{\mathbf{w}, B_1, B_2} a_{\mathbf{x}, \mathbf{w}}^{B_1} a_{\mathbf{w}, \mathbf{y}}^{B_2}$ , and by Lemma 6.9 the third term corresponds to  $\sum_B \partial a_{\mathbf{x}, \mathbf{y}}^B$ . We will show that the remaining two terms cancel against each other.

For the fourth term, there are several cases that cannot contribute:

- $M(\rho_i^-) = M(\rho_{i+1}^-)$  or  $M(\rho_i^+) = M(\rho_{i+1}^+)$ . This never occurs by Lemma 6.7 and strong boundary monotonicity.
- $M(\rho_i^+) = M(\rho_{i+1}^-)$  and  $\rho_i^+ \neq \rho_{i+1}^-$ . This would have a boundary degeneration, violating Lemma 5.37.
- $M(\rho_i^-) = M(\rho_{i+1}^+)$  and  $\rho_i^- \neq \rho_{i+1}^+$ . In this case,  $a(-\rho_i)a(-\rho_{i+1}) = 0$ .
- $(\rho_i, \rho_{i+1})$  are interleaved (in that order). This degeneration does not occur in codimension one for embedded curves. More precisely, then the sets  $\{\rho_i\}$  and  $\{\rho_{i+1}\}$  are not composable by Definition 3.5, so by Lemma 5.56 this degeneration cannot occur.
- $(\rho_{i+1}, \rho_i)$  are interleaved. Then  $a(-\rho_i)a(-\rho_{i+1}) = 0$  as there are double crossing strands.

The remaining possibilities are

- (4a)  $\rho_i^- = \rho_{i+1}^+$ . These are exactly the moduli spaces which occur in Case (2), for the factorization  $a(-\rho_1) \cdots a(-(\rho_i \uplus \rho_{i+1})) \cdots a(-\rho_n)$  of  $a$ . So, these two contributions to  $\partial^2$  cancel.
- (4b)  $\{M(\rho_i^-), M(\rho_i^+)\} \cap \{M(\rho_{i+1}^-), M(\rho_{i+1}^+)\} = \emptyset$ , with  $\{\rho_i, \rho_{i+1}\}$  nested (in either order) or disjoint. In this case, the same degeneration also occurs for the factorization  $a(-\rho_1) \cdots a(-\rho_{i+1})a(-\rho_i) \cdots a(-\rho_n)$  of  $a$  (also in Case (4b)). So, these two contributions to  $\partial^2$  cancel.

This completes the proof.  $\square$

The most interesting points in the proof of Proposition 6.6 are seen in the following examples, illustrated in Figure 29. We put the boundary of  $\overline{\Sigma}$  on the right to visually indicate that the orientation of  $\partial \overline{\Sigma}$  is reversed in the algebra.

*Example 6.10.* On the left of Figure 29 is a piece of a diagram with four generators. The complex  $\widehat{CFD}$  of this piece satisfies the relations

$$\begin{aligned}\partial\{a, c\} &= \begin{bmatrix} 1 \\ 2 \end{bmatrix} \{a, d\} + \begin{bmatrix} 1 \\ 3 \end{bmatrix} \{e, c\}. \\ \partial\{e, c\} &= \{b, d\}. \\ \partial\{a, d\} &= \begin{bmatrix} 2 \\ 3 \end{bmatrix} \{b, d\}.\end{aligned}$$

The fact that  $\partial^2 = 0$  follows from the relations in the algebra, more specifically

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

This illustrates Case (1) in the proof of Proposition 6.6 (at the two two-story ends of the moduli space), as well as the cancellation of Case (2) and Case (4a) (when the boundary branch point goes out to  $\partial\Sigma$ ). The reader may also want to compare with Figure 5.

*Example 6.11.* In the center of Figure 29, we have

$$\begin{aligned}\partial\{a, c\} &= \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \{b, c\} + \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix} \{a, d\}. \\ \partial\{b, c\} &= \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix} \{b, d\}. \\ \partial\{a, d\} &= \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \{b, d\}.\end{aligned}$$

The fact that  $\partial^2 = 0$  follows from the fact that  $a(\{\frac{3}{4}\})$  and  $a(\{\frac{1}{2}\})$  commute in the algebra—or, more precisely, because

$$\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \cdot \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}.$$

This is the self-cancellation of Case (4b) of the proof of Proposition 6.6.

*Example 6.12.* On the right of Figure 29, we have

$$\begin{aligned}\partial\{a\} &= \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \{b\} + \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} \{c\}. \\ \partial\{b\} &= \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \{c\}.\end{aligned}$$

Here,  $\partial^2 = 0$  because of a differential in the algebra:

$$\partial\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}.$$

This illustrates Case (3) of the proof of Proposition 6.6, or equivalently Lemma 6.9.

**6.2. Twisted coefficients.** As with the original Heegaard Floer homology [32, Section 8], there are versions of  $\widehat{CFD}(\mathcal{H})$  with twisted coefficients. We describe the twist by  $H_2(Y, \partial Y)$ . Recall from Lemma 4.10 that, for any  $\mathbf{x} \in \mathfrak{S}(\mathcal{H})$ ,  $\pi_2(\mathbf{x}, \mathbf{x}) \cong H_2(Y, \partial Y)$ . For each  $\text{spin}^c$  structure  $\mathfrak{s}$  on  $Y$ , pick a base generator  $\mathbf{x}_0 \in \mathfrak{S}(\mathcal{H}, \mathfrak{s})$ . (For  $\text{spin}^c$  structures  $\mathfrak{s}$  without any generators, the twisted chain complex is 0.) Let  $\underline{\mathfrak{S}}(\mathcal{H}, \mathfrak{s})$  be the set of elements of the form  $e^B \mathbf{x}$ , where  $\mathbf{x} \in \mathfrak{S}(\mathcal{H}, \mathfrak{s})$  and  $B \in \pi_2(\mathbf{x}_0, \mathbf{x})$ . Let  $\underline{X}(\mathcal{H}, \mathfrak{s})$  be the  $\mathbb{F}_2$  vector space spanned by  $\underline{\mathfrak{S}}(\mathcal{H}, \mathfrak{s})$ . There is an action of  $\mathcal{I}(\mathcal{Z})$  on  $\underline{X}(\mathcal{H}, \mathfrak{s})$ , where

$$I(s) \cdot e^B \mathbf{x} := \begin{cases} \mathbf{x} & I(s) = I_D(\mathbf{x}) \\ 0 & \text{otherwise} \end{cases}$$

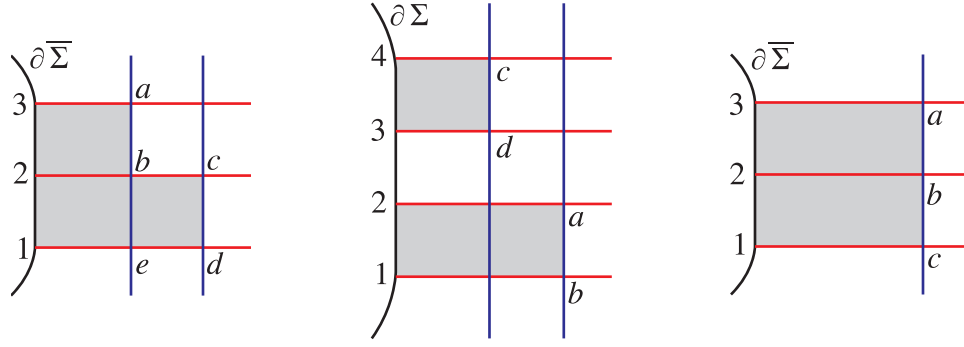


FIGURE 29. **Local illustrations of possible contributions to  $\partial^2$  in the type D module.** Left: a family of curves with one end a two-story building and the other end a join curve end. Center: a family of curves with a two-story building at one end and a collision of two levels (not resulting in any split curves) at the other. Right: a family of curves with a two-story building at one end and a collision of two levels, degenerating a split curve, at the other.

as in Equation (6.1). As a module over  $\mathcal{A}$ , define the twisted chain complex  $\widehat{CFD}(\mathcal{H}, \mathfrak{s}; H_2(Y, \partial Y))$  (or just  $\widehat{CFD}(\mathcal{H}, \mathfrak{s})$ ) to be  $\mathcal{A} \otimes_{\mathcal{I}} \underline{X}(\mathcal{H}, \mathfrak{s})$ . There is a natural action of  $H_2(Y, \partial Y)$  on  $\widehat{CFD}(\mathcal{H}, \mathfrak{s})$ , by composition with the corresponding periodic domain: For  $\alpha \in H_2(Y, \partial Y)$  corresponding to  $B \in \pi_2(\mathbf{x}, \mathbf{x})$ , define

$$\alpha * e^{B_0 \mathbf{x}} := e^{B_0 * B \mathbf{x}}.$$

As in the untwisted case, the action of  $\mathcal{A}$  on  $\widehat{CFD}(\mathcal{H}, \mathfrak{s})$  is given by left multiplication:

$$a \cdot (b \otimes e^B \mathbf{x}) := (ab) \otimes e^B \mathbf{x}.$$

For the boundary operator, we now distinguish the different homotopy classes:

$$\partial(e^{B_0 \mathbf{x}}) := \sum_{\mathbf{y} \in \mathfrak{S}(\mathcal{H}, \mathfrak{s})} \sum_{B \in \pi_2(\mathbf{x}, \mathbf{y})} a_{\mathbf{x}, \mathbf{y}}^B e^{B_0 * B \mathbf{y}}.$$

The proof of Proposition 6.6 carries through, as does Theorem 6.23 below.

**6.3. Gradings.** We now turn to defining a grading on  $\widehat{CFD}(\mathcal{H})$ . As described in Definition 2.26, this grading  $\text{gr}'$  takes values in a set  $G'_D(\mathcal{H}, \mathfrak{s})$  (depending on a  $\text{spin}^c$  structure  $\mathfrak{s}$ ) with a left action of the grading group  $G'(4k)$ . We will also define a refinement  $\text{gr}$  of the grading taking values in a set with an action of the smaller group  $G(\mathcal{Z})$ .

As with the algebra  $\mathcal{A}(\mathcal{Z})$ , the group  $G'(4k)$  is to be interpreted as the group associated to  $\mathcal{Z}$ , which is  $-\partial \mathcal{H}$ . In particular, in the definition of the product on  $G'(4k)$ ,

$$(k_1, \alpha_1) \cdot (k_2, \alpha_2) := (k_1 + k_2 + L(\alpha_1, \alpha_2), \alpha_1 + \alpha_2)$$

(from Equation (3.14)), the correction term  $L(\cdot, \cdot)$  is the negative of what it would be without this orientation reversal. To distinguish between the two possibilities, we

write  $L_{\mathcal{Z}}(\cdot, \cdot)$  for the correction term from the orientation for  $\mathcal{Z}$ , and  $L_{\partial\mathcal{H}}(\cdot, \cdot)$  for the correction term with respect to the orientation induced by  $\mathcal{H}$ .

We start by defining a grading on domains.

**Definition 6.13.** For  $\mathbf{x}, \mathbf{y} \in \mathfrak{S}(\mathcal{H})$  and  $B \in \pi_2(\mathbf{x}, \mathbf{y})$ , define  $g'(B) \in G'(4k)$  by

$$(6.14) \quad g'(B) := (-e(B) - n_{\mathbf{x}}(B) - n_{\mathbf{y}}(B), \partial^{\partial}(B)).$$

**Lemma 6.15.** For  $B_1 \in \pi_2(\mathbf{x}, \mathbf{y})$  and  $B_2 \in \pi_2(\mathbf{y}, \mathbf{z})$ , we have

$$g'(B_1)g'(B_2) = g'(B_1 * B_2).$$

*Proof.* This is obvious for the homological component. For the Maslov component, as in the proof of Proposition 5.53, the Euler number  $e(B)$  is additive, and by Lemma 5.52 the failure of additivity of  $-n_{\mathbf{x}}$  and  $-n_{\mathbf{y}}$  matches the correction term  $L_{\mathcal{Z}}(\partial^{\partial}(B_1), \partial^{\partial}(B_2))$  in the group structure.  $\square$

For  $\vec{\rho}$  a sequence of Reeb chords, let  $\text{gr}'(\vec{\rho})$  be the product  $\prod_{i=1}^n \text{gr}'(a(\rho_i))$ . (If  $a(\vec{\rho}) \neq 0$ , then  $\text{gr}'(\vec{\rho}) = \text{gr}'(a(\vec{\rho}))$ ).

**Lemma 6.16.** For a compatible pair  $(B, \vec{\rho})$ , we have  $g'(B) \text{gr}'(-\vec{\rho}) = (-\text{ind}(B, \vec{\rho}), 0)$ .

*Proof.* The homological component of  $g'(B)$  is the negative of the homological component of  $\text{gr}'(-\vec{\rho})$ , since all the boundary intervals are reversed in the latter. Thus the homological component of the result is 0, and the correction term to the Maslov component of the product vanishes. The Maslov component of  $g'(B)$  is  $-e(B) - n_{\mathbf{x}}(B) - n_{\mathbf{y}}(B)$  by definition. On the other hand, if  $\vec{\rho} = (\rho_1, \dots, \rho_n)$  then

$$\begin{aligned} \text{gr}'(-\vec{\rho}) &= (-\tfrac{1}{2}, -[\rho_1]) \cdots (-\tfrac{1}{2}, -[\rho_n]) \\ &= \left( -\tfrac{n}{2} + \sum_{i < j} L_{\mathcal{Z}}(-[\rho_i], -[\rho_j]), -\sum_i [\rho_i] \right) \\ &= \left( -\tfrac{n}{2} - \sum_{i < j} L_{\partial\mathcal{H}}([\rho_i], [\rho_j]), -\sum_i [\rho_i] \right) \\ &= \left( -n - \iota(\vec{\rho}), -\sum_i [\rho_i] \right), \end{aligned}$$

where we use Lemma 3.17, bilinearity of  $L$ , and the definition of  $\iota$  (Equation (5.44)). Thus, by Definition 5.46, the Maslov component of the product is  $-\text{ind}(B, \vec{\rho})$ .  $\square$

This grading on domains is related to the grading on the modules as follows.

**Proposition 6.17.** For each  $\text{spin}^c$  structure  $\mathfrak{s}$  on  $Y$  there is a map

$$\text{gr}': \underline{\mathfrak{S}}(\mathcal{H}, \mathfrak{s}) \longrightarrow G'(4k),$$

unique up to left translation by an element of  $G'(4k)$ , so that, for  $\mathbf{x}, \mathbf{y} \in \mathfrak{S}(\mathcal{H}, \mathfrak{s})$ ,  $B \in \pi_2(\mathbf{x}, \mathbf{y})$ , and  $B_0 \in \pi_2(\mathbf{x}_0, \mathbf{x})$ ,

$$(6.18) \quad g'(B) \text{gr}'(e^{B_0} \mathbf{x}) = \text{gr}'(e^{B+B_0} \mathbf{y}).$$

Similarly, there is a set  $G'_D(\mathcal{H}, \mathfrak{s})$  with a left action of  $G'(4k)$  and a map

$$\text{gr}': \mathfrak{S}(\mathcal{H}, \mathfrak{s}) \longrightarrow G'_D(\mathcal{H}, \mathfrak{s})$$

so that, for  $\mathbf{x}, \mathbf{y} \in \mathfrak{S}(\mathcal{H}, \mathfrak{s})$  and  $B \in \pi_2(\mathbf{x}, \mathbf{y})$ ,

$$(6.19) \quad g'(B) \text{gr}'(\mathbf{x}) = \text{gr}'(\mathbf{y}).$$

Furthermore,  $G'_D(\mathcal{H}, \mathfrak{s})$  and  $\text{gr}'$  are canonical up to translation by  $G'(4k)$ .

Before proving this proposition, we check that it suffices.

**Proposition 6.20.** *If  $\text{gr}'$  is a grading on  $\widehat{CFD}$  (respectively, on  $\underline{\widehat{CFD}}$ ) so that Proposition 6.17 is satisfied, then  $\widehat{CFD}$  (respectively,  $\underline{\widehat{CFD}}$ ) is a graded differential module in the sense of Definition 2.26.*

*Proof.* Suppose  $(B, \vec{\rho})$  is a compatible pair with  $\text{ind}(B, \vec{\rho}) = 1$ , so that  $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; \vec{\rho})$  has expected dimension 1 and there may be a term  $a(-\vec{\rho})\mathbf{y}$  in  $\partial\mathbf{x}$ . Then

$$\begin{aligned} \text{gr}'(a(-\vec{\rho})\mathbf{y}) &= \text{gr}'(-\vec{\rho}) \text{gr}'(\mathbf{y}) \\ &= g'(B)^{-1} \lambda^{-\text{ind}(B, \vec{\rho})} \text{gr}'(\mathbf{y}) \\ &= \lambda^{-1} \text{gr}'(\mathbf{x}), \end{aligned}$$

as desired. (Here, the second equality uses Lemma 6.16 and the third equality uses Proposition 6.17.) The proof for  $\underline{\widehat{CFD}}$  is similar.  $\square$

*Proof of Proposition 6.17.* For  $\widehat{CFD}(\mathcal{H}, \mathfrak{s})$ , we simply define

$$\text{gr}'(e^B \mathbf{x}) := g'(B).$$

Equation (6.18) is a restatement of Lemma 6.15. Uniqueness is immediate.

For the untwisted theory, the presence of periodic domains means that we can not just define the relative grading of  $\mathbf{x}$  and  $\mathbf{y}$  to be  $g'(B)$  for an arbitrary  $B \in \pi_2(\mathbf{x}, \mathbf{y})$ , as the different possible domains  $B$  have different gradings. Instead, we define a grading with values in the left cosets of an appropriate subgroup of  $G'(4k)$ .

For a generator  $\mathbf{x} \in \mathfrak{S}(\mathcal{H})$ , set  $P(\mathbf{x})$  to be the subgroup of  $G'(4k)$  generated by  $g'(B)$  for  $B \in \pi_2(\mathbf{x}, \mathbf{x})$ . Now pick a base generator  $\mathbf{x}_0 \in \mathfrak{S}(\mathcal{H}, \mathfrak{s})$  and define

$$(6.21) \quad G'_D(\mathcal{H}, \mathfrak{s}) := G'(4k)/P(\mathbf{x}_0).$$

(If  $\mathfrak{s}$  has no generator in the diagram, we have to change the Heegaard diagram in order to find an appropriate representative  $\mathbf{x}_0$ ; however, in this case the Heegaard-Floer homology in this  $\text{spin}^c$  structure is manifestly 0, so the precise definition of the grading set is not very relevant.) For an arbitrary  $\mathbf{x}$  in  $\mathfrak{S}(\mathcal{H}, \mathfrak{s})$ , let  $B$  be an arbitrary element of  $\pi_2(\mathbf{x}, \mathbf{x}_0)$  and set

$$\text{gr}'(\mathbf{x}) := g'(B) \cdot P(\mathbf{x}_0).$$

It is immediate from Lemma 6.15 that this definition is independent of the choice of  $B$  and satisfies Equation (6.19).

Finally, if we pick a different base generator  $\mathbf{x}'_0 \in \mathfrak{S}(\mathcal{H}, \mathfrak{s})$ , we see that  $P(\mathbf{x}'_0) = g'(B_0)^{-1} P(\mathbf{x}_0) g'(B_0)$  for any  $B_0 \in \pi_2(\mathbf{x}_0, \mathbf{x}'_0)$ , and that the grading of an element  $\mathbf{x}$  with respect to  $\mathbf{x}'_0$  is  $\text{gr}'(\mathbf{x}) g'(B_0)$ .  $\square$

Note that Equation (6.21) generalizes the fact that for closed 3-manifolds the gradings in a non-torsion  $\text{spin}^c$  structure take values in a quotient of  $\mathbb{Z}$ .

Finally, we turn to the refinement  $\text{gr}$  of  $\text{gr}'$ . Recall from Section 3.3.2 that we have chosen a base idempotent  $I_0 \in \mathcal{A}(\mathcal{Z})$  and for every other minimal idempotent  $I(s)$  we have chosen an element  $g(s, s_0) \in G'(4k)$  so that  $M_*([g(s, s_0)]) = s - s_0$ . Assume for convenience that  $I(\mathbf{x}_0) = I_0$ . Then, for  $B \in \pi_2(\mathbf{x}, \mathbf{y})$  define

$$(6.22) \quad g(B) = g(I(\mathbf{x}), s_0) g'(B) g(I(\mathbf{y}), s_0)^{-1}.$$

Note that  $g(B_1 * B_2) = g(B_1) * g(B_2)$ . Define a grading on  $\widehat{CFD}$  with respect to the grading  $G(\mathcal{Z})$  on  $\mathcal{A}(\mathcal{Z})$  by

$$\text{gr}(e^B \mathbf{x}) = g(B).$$

Define a grading on  $\widehat{CFD}(\mathcal{H})$  with respect to  $G(\mathcal{Z})$  by

$$\begin{aligned} G_D(\mathcal{H}, \mathfrak{s}) &= G/P(\mathbf{x}_0) \\ \text{gr}(\mathbf{x}) &= g(B)P(\mathbf{x}_0) \end{aligned}$$

where  $B \in \pi_2(\mathbf{x}, \mathbf{x}_0)$  is any domain. It is easy to deduce from the properties of  $\text{gr}'$  that  $\text{gr}$  is, in fact, a grading.

#### 6.4. Invariance.

**Theorem 6.23.** *Up to homotopy equivalence, the differential module  $\widehat{CFD}(\mathcal{H})$  depends only on the bordered three-manifold  $Y$  specified by the provincially admissible pointed bordered Heegaard diagram  $\mathcal{H}$ . That is, if  $\mathcal{H}'$  is another provincially admissible pointed bordered Heegaard diagram for  $Y$ , then  $\widehat{CFD}(\mathcal{H})$  and  $\widehat{CFD}(\mathcal{H}')$  are homotopy equivalent  $\mathcal{A}(\mathcal{Z})$ -modules.*

For the most part the proof of Theorem 6.23 is a simple adaptation of the techniques for closed 3-manifolds [33] (as modified for the cylindrical case in [20], say). In particular, the proof proceeds by showing that each of the three kinds of Heegaard moves from Proposition 4.4 induces a homotopy equivalence. (We also use Proposition 4.16 to ensure that all of the intermediate diagrams are provincially admissible.)

There are essentially two new issues. The first, and easier, is that we must keep track of the algebra elements which occur as coefficients in the maps. We will illustrate how this is done by defining the chain map induced by deformation of the almost complex structure. The second, more serious, issue is that the traditional proof of handleslide invariance uses “triangle maps,” the general definitions of which are subtle for bordered Heegaard diagrams. We will explain how to circumvent this difficulty by proving invariance when handlesliding an  $\alpha$ -arc over an  $\alpha$ -circle (the hardest case). These two issues having been explained, the reader should have no difficulty adapting the rest of the invariance proof to our setting. For instance, the proof of isotopy invariance is similar to the proof of invariance under change of complex structure, but notationally slightly more complicated. The reader may also consult [21, Chapters 6 and 7] for a few more details in a similar setting.

**6.4.1. The chain map for change of complex structure.** Recall that the moduli spaces  $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P})$  from Section 5 depended on a choice of a generic, admissible almost complex structure  $J$  on  $\Sigma \times [0, 1] \times \mathbb{R}$ , and consequently the differential module  $\widehat{CFD}(\mathcal{H})$  also depends on this choice; to make this choice explicit, we will write  $\widehat{CFD}(\mathcal{H}; J)$ . As is usual in Floer homology theories, one proves independence of the almost complex structure by constructing “continuation maps,” and proving that they are chain homotopy equivalences, see for example [8]. More specifically, let  $J_0$  and  $J_1$  be two generic, admissible almost complex structures on  $\Sigma \times [0, 1] \times \mathbb{R}$ , and let  $\partial_0$  and  $\partial_1$  denote the boundary operators on  $\widehat{CFD}(\mathcal{H}; J_0)$  and  $\widehat{CFD}(\mathcal{H}; J_1)$ , respectively. Call a (smooth) path  $\{J_r \mid r \in [0, 1]\}$  of almost complex structures from  $J_0$  to  $J_1$

*admissible* if each  $J_r$  is admissible. To a generic (in a sense to be made precise presently) admissible path  $J_r$  between  $J_0$  and  $J_1$  we will associate a map

$$F^{J_r} : \widehat{CFD}(\mathcal{H}; J_0) \rightarrow \widehat{CFD}(\mathcal{H}; J_1).$$

To the path  $J_r$  we can associate a single almost complex structure  $J$  on  $\Sigma \times [0, 1] \times \mathbb{R}$  by

$$(6.24) \quad J|_{(x,s,t)} = \begin{cases} J_1|_{(x,s,t)} & \text{if } t \geq 1 \\ J_t|_{(x,s,t)} & \text{if } 0 \leq t \leq 1 \\ J_0|_{(x,s,t)} & \text{if } t \leq 0. \end{cases}$$

(Here  $J|_{(x,s,t)}$  means the map  $J$  on the tangent space at  $x \in \Sigma$ ,  $s \in [0, 1]$ , and  $t \in \mathbb{R}$ .) Note that  $J$  is obviously not  $\mathbb{R}$ -translation invariant.

For  $B \in \pi_2(\mathbf{x}, \mathbf{y})$ , let  $\widetilde{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P; J)$  denote the moduli spaces of holomorphic curves defined in Definition 5.8 with respect to the almost complex structure  $J$ . For generic  $J$ , these moduli spaces are all transversely cut out, as in Proposition 5.5. For  $\vec{\rho}$  a sequence of sets of Reeb chords so that  $(\mathbf{x}, \vec{\rho})$  is strongly boundary monotonic and  $B$  is compatible with  $\vec{\rho}$ , define the embedded moduli space by

$$(6.25) \quad \mathcal{M}^B(\mathbf{x}, \mathbf{y}; \vec{\rho}; J) := \bigcup_{\chi(S^\triangleright) = \chi_{\text{emb}}(B, \vec{\rho})} \widetilde{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P}; J).$$

Since we do not divide by  $\mathbb{R}$ -translation, so we will primarily be interested in index 0 moduli spaces, not the index 1 ones.

Then define the map  $F^{J_r}$  by

$$F^{J_r}(\mathbf{x}) = \sum_{\mathbf{y} \in \mathfrak{S}(\mathcal{H})} \sum_{B \in \pi_2(\mathbf{x}, \mathbf{y})} \sum_{\{\vec{\rho} \mid \text{ind}(B, \vec{\rho}) = 0\}} \#(\mathcal{M}^B(\mathbf{x}, \mathbf{y}; \vec{\rho}; J)) a(-\vec{\rho})\mathbf{y},$$

and extend  $F^{J_r}$  to all of  $\widehat{CFD}(\mathcal{H}; J_0)$  by  $F^{J_r}(a\mathbf{x}) = aF^{J_r}(\mathbf{x})$ . Compactness and provincial admissibility imply that this sum is finite, as in Lemma 6.5.

The next task is to show that  $F^{J_r}$  is a chain map. We will use the following analogue Theorem 5.41 for almost complex structures which are not translation invariant. The statement is more general than we need for present purposes, as we will also use it to prove invariance of the type  $A$  module in Section 7.4.

**Proposition 6.26.** *Suppose that  $(\mathbf{x}, \vec{\rho})$  satisfies the strong boundary monotonicity condition. Fix  $\mathbf{y}$ ,  $B \in \pi_2(\mathbf{x}, \mathbf{y})$ ,  $S^\triangleright$ , and  $\vec{P}$  such that  $[\vec{P}] = \vec{\rho}$  and  $\text{ind}(B, S^\triangleright, P) = 1$ . Then the total number of all*

- (1) *two-story ends, with either*
  - (a) *a  $J_0$ -holomorphic curve followed by a  $J$ -holomorphic curve or*
  - (b) *a  $J$ -holomorphic curve followed by a  $J_1$ -holomorphic curve;*
- (2) *join curve ends;*
- (3) *odd shuffle curve ends; and*
- (4) *collision of two levels  $P_i$  and  $P_{i+1}$  from  $\vec{P}$ , where  $P_i$  and  $P_{i+1}$  are weakly composable*

*is even.*

(Recall the definition of two-story ends, join curve ends, odd or even shuffle curve ends, and collisions of two levels from Definition 5.39. As indicated, two-story ends come in two flavors: elements of  $\mathcal{M}_1^B(\mathbf{x}, \mathbf{w}; S_1^\triangleright; \vec{P}_1; J_0) \times \mathcal{M}^{B_2}(\mathbf{w}, \mathbf{y}; S_2^\triangleright; J)$  or elements of  $\mathcal{M}^{B_1}(\mathbf{x}, \mathbf{w}; S_1^\triangleright; \vec{P}_1; J) \times \mathcal{M}^{B_2}(\mathbf{w}, \mathbf{y}; S_2^\triangleright; J_1)$ , where  $B = B_1 * B_2$  and  $S^\triangleright = S_1^\triangleright \uplus S_2^\triangleright$  as before.)

*Proof.* The proof is almost exactly the same as the proof of Theorem 5.41.  $\square$

**Proposition 6.27.** *The map  $F^{J_r}$  is a degree 0 chain map, i.e.,  $\partial_1 \circ F^{J_r} + F^{J_r} \circ \partial_0 = 0$ .*

*Proof.* The proof is almost exactly the same as the proof of Proposition 6.6. First, observe that

$$\partial_1 \circ F^{J_r}(a\mathbf{x}) + F^{J_r} \circ \partial_0(a\mathbf{x}) = (\partial a)F^{J_r}(\mathbf{x}) + a\partial_1 \circ F^{J_r}(\mathbf{x}) + aF^{J_r} \circ \partial_0(\mathbf{x}) + (\partial a)F^{J_r}(\mathbf{x}).$$

So, it suffices to prove that  $\partial_1 \circ F^{J_r}(\mathbf{x}) + F^{J_r} \circ \partial_0(\mathbf{x}) = 0$  for any generator  $\mathbf{x} \in \mathfrak{S}(\mathcal{H})$ .

Now, as in the proof of Proposition 6.6, for given  $\mathbf{x}, \mathbf{y}, B$  and  $\vec{\rho} = (\rho_1, \dots, \rho_n)$ , Proposition 6.26 implies that the sum of the following terms is zero:

- (1) The number of two-story buildings of the form

$$\mathcal{M}^{B_1}(\mathbf{x}, \mathbf{w}; \vec{\rho}_1; J_0) \times \mathcal{M}^{B_2}(\mathbf{w}, \mathbf{y}; \vec{\rho}_2; J)$$

where  $B = B_1 * B_2$  and  $\vec{\rho} = (\vec{\rho}_1, \vec{\rho}_2)$ .

- (2) The number of two-story buildings of the form

$$\mathcal{M}^{B_1}(\mathbf{x}, \mathbf{w}; \vec{\rho}_1; J) \times \mathcal{M}^{B_2}(\mathbf{w}, \mathbf{y}; \vec{\rho}_2; J_1)$$

where  $B = B_1 * B_2$  and  $\vec{\rho} = (\vec{\rho}_1, \vec{\rho}_2)$ .

- (3) The number of join curve ends, i.e., the number of elements of

$$\mathcal{M}^B(\mathbf{x}, \mathbf{y}; (\rho_1, \dots, \rho_{i-1}, \{\rho_j, \rho_k\}, \rho_{i+1}, \dots, \rho_n); J)$$

where  $\rho_i = \rho_j \uplus \rho_k$ .

- (4) The number of split curve ends, i.e., the number of elements of

$$\mathcal{M}^B(\mathbf{x}, \mathbf{y}; (\rho_1, \dots, \rho_{i-1}, \rho_i \uplus \rho_{i+1}, \rho_{i+2}, \dots, \rho_n); J)$$

where  $\rho_i^+ = \rho_{i+1}^-$ .

- (5) The number of other collisions of levels, i.e., the number of elements of

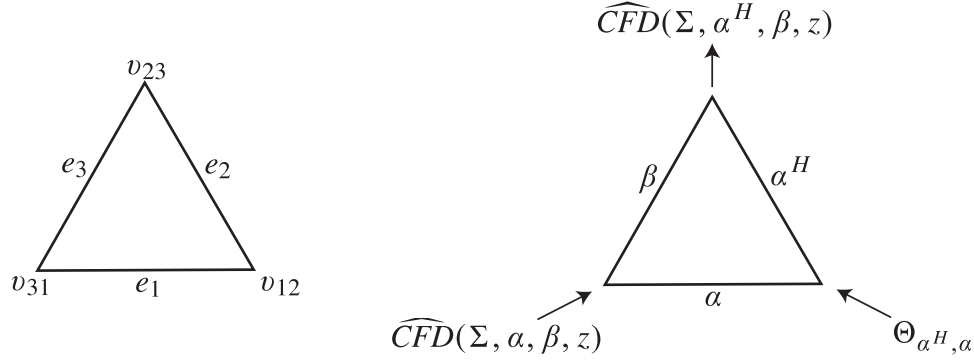
$$\mathcal{M}^B(\mathbf{x}, \mathbf{y}; (\rho_1, \dots, \rho_{i-1}, \{\rho_i, \rho_{i+1}\}, \rho_{i+2}, \dots, \rho_n); J)$$

where  $\rho_i^+ \neq \rho_{i+1}^-$ .

The first term corresponds to  $F^{J_t} \circ \partial_0(\mathbf{x})$ . The second corresponds to those terms in  $\partial_1 \circ F^{J_t}(\mathbf{x})$  coming from applying  $\partial_1$  to the resulting generators  $\mathbf{w} \in \mathfrak{S}(\mathcal{H})$ . The fourth term corresponds to those terms in  $\partial_1 \circ F^{J_t}(\mathbf{x})$  coming from the differential on the algebra  $\mathcal{A}$ . The third and fifth terms cancel in pairs, exactly as in the proof of Proposition 6.6.

The fact that we count curves with  $\text{ind}(B, \vec{\rho}) = 0$  implies that  $F^{J_r}$  is a degree 0 map, following the arguments of Proposition 6.20.  $\square$

One can similarly use a path of almost complex structures from  $J_1$  to  $J_0$  to define a chain map  $\widehat{CFD}(\mathcal{H}; J_1) \rightarrow \widehat{CFD}(\mathcal{H}; J_0)$ . The proof that these maps are mutually inverse chain homotopy equivalences is obtained by adapting the standard arguments in exactly analogous manners to the above, and we leave this to the interested reader. (Some more details are given in the type  $A$  case in Section 7.4.1.)

FIGURE 30. The “triangle”  $\Delta$  (left), and the labeling inducing  $F_{\alpha, \alpha^H, \beta}$  (right).

6.4.2. *Handlesliding an  $\alpha$ -arc over an  $\alpha$ -circle.* Let  $(\Sigma, \alpha, \beta, z)$  denote an pointed bordered Heegaard diagram, which we assume to be admissible for convenience. Write  $\alpha = \{\alpha_1^a, \dots, \alpha_{2k}^a, \alpha_1^c, \dots, \alpha_{g-k}^c\}$ . Provisionally, let  $\alpha_1^{a,H}$  denote the result of performing a handleslide of  $\alpha_1^a$  over  $\alpha_1^c$ , and  $\alpha^H = \{\alpha_1^{a,H}, \alpha_2^a, \dots, \alpha_{g-k}^c\}$ . Assume that the Heegaard diagram  $(\Sigma, \alpha^H, \beta, z)$  is also admissible. We want to show that  $\widehat{CFD}(\Sigma, \alpha, \beta, z)$  and  $\widehat{CFD}(\Sigma, \alpha^H, \beta, z)$  are chain homotopy equivalent.

As is traditional, we will produce a chain map  $F_{\alpha, \alpha^H, \beta}: \widehat{CFD}(\Sigma, \alpha^H, \beta, z) \rightarrow \widehat{CFD}(\Sigma, \alpha, \beta, z)$  by counting holomorphic triangles in  $(\Sigma, \alpha, \alpha^H, \beta, z)$ . That is, let  $\Delta$  denote a disk with three boundary punctures. Label the edges of  $\Delta$  by  $e_1, e_2$  and  $e_3$ , counterclockwise, and let  $v_{ij}$  denote the puncture between  $e_i$  and  $e_j$ ; see Figure 30. To define  $F_{\alpha, \alpha^H, \beta}$  we will count holomorphic maps

$$u: (T, \partial T) \rightarrow (\Sigma \times \Delta, (\alpha \times e_1) \cup (\alpha^H \times e_2) \cup (\beta \times e_3))$$

which do not cover  $z$ .

The  $\alpha^H$  we use are actually a slight modification of the above, as shown in Figure 31. Obtain  $\alpha_i^{c,H}$  from  $\alpha_i^c$  by performing a small Hamiltonian perturbation, so that  $\alpha_i^{c,H}$  intersects  $\alpha_i^c$  transversely in two points, is disjoint from all other  $\alpha$ -curves, and intersects  $\beta_j$  transversally close to  $\beta_j \cap \alpha_i^c$ . For  $i = 2, \dots, 2k$ ,  $\alpha_i^{a,H}$  is an isotopic translate of  $\alpha_i^a$ , intersecting  $\alpha_i^a$  transversely in a single point, and such that there are two short Reeb chords in  $\partial \Sigma$  running from  $\alpha_i^{a,H}$  to  $\alpha_i^a$ . The isotopy is chosen small enough that  $\alpha_i^{a,H}$  is disjoint from all other  $\alpha$ -curves and intersects  $\beta_j$  transversally close to  $\beta_j \cap \alpha_i^a$ . Obtain  $\alpha_1^{a,H}$  by handlesliding  $\alpha_1^a$  over  $\alpha_1^c$  and then performing an isotopy so that  $\alpha_1^{a,H}$  intersects  $\alpha_1^a$  transversely in a single point, and such that there are two short Reeb chords in  $\partial \Sigma$  running from  $\alpha_1^{a,H}$  to  $\alpha_1^a$ . We also arrange that  $\alpha_1^{a,H}$  is disjoint from all other  $\alpha$ -curves and intersects the  $\beta_j$  transversally at points close to  $\beta_j \cap (\alpha_1^a \cup \alpha_1^c)$ . Also define  $\mathbf{a}^H$  to be the set  $\alpha^H \cap \partial \Sigma$ . There is a natural bijection between  $\mathbf{a}$  and  $\mathbf{a}^H$ .

Most of the notions from Section 5 generalize in a completely straightforward manner to the setting of triangles. Following Definition 5.1, we fix an almost complex structure  $J$  on  $\Sigma \times \Delta$  such that

- (1) The projection map  $\pi_\Delta: \Sigma \times \Delta \rightarrow \Delta$  is  $(J, j_\Delta)$ -holomorphic.

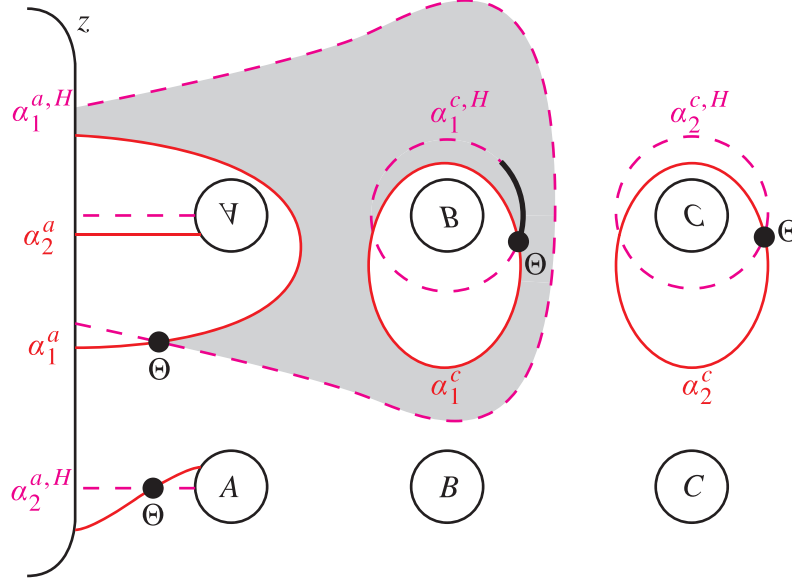


FIGURE 31. The curves  $\alpha$  and  $\alpha^H$  in  $\Sigma$ . One canonical generator  $\Theta$  is marked. The annulus  $R_{1,1}^a$  is shaded. The branch cut corresponding to a possible holomorphic curve is also indicated by a thick arc.

- (2) The fibers of the projection map  $\pi_\Sigma: \Sigma \times \Delta \rightarrow \Sigma$  are  $J$ -holomorphic.
- (3) The complex structure is split near  $p \times \Delta$ . (Recall that  $p$  is the puncture of  $\Sigma$ ).
- (4) For each puncture  $v_{i,j}$  of  $\Delta$ , choose a neighborhood  $U_{i,j}$  which is biholomorphic to  $[0, 1] \times [0, \infty)$ , and such a biholomorphism. Then, there is an action of  $\mathbb{R}_+$  on  $\Sigma \times U_{i,j}$ . We require that these actions be  $J$ -holomorphic.

We call such a  $J$  *admissible*. Notice that the restrictions of an admissible complex structure  $J$  on  $\Sigma \times \Delta$  to  $\Sigma \times U_{i,j}$  induce three admissible structures  $J_{i,j}$  on  $\Sigma \times [0, 1] \times \mathbb{R}$ .

Similarly, we can define decorated sources for maps to  $\Sigma \times \Delta$ , which we denote  $T^\Delta$ : a decorated source  $T^\Delta$  is an oriented surfaces with boundary and punctures, together with a labeling of each puncture by an element of  $\{v_{1,2}, v_{2,3}, v_{3,1}, e\}$  and a further labeling of each  $e$  puncture either by a Reeb chord with endpoints in  $\mathbf{a}$  or by a Reeb chord with endpoints in  $\mathbf{a}^H$ . Notice that we do *not* allow punctures to be labeled by Reeb chords between one point in  $\mathbf{a}$  and one point in  $\mathbf{a}^H$  or vice versa.

Given a decorated source  $T^\Delta$  we consider maps

$$u: (T, \partial T) \rightarrow (\Sigma \times \Delta, (\alpha \times e_1) \cup (\alpha^H \times e_2) \cup (\beta \times e_3))$$

satisfying the obvious analogues of Conditions (1)–(11) from Section 5.1. Such holomorphic curves are asymptotic to a generator  $\mathbf{x}$  for  $(\Sigma, \alpha, \beta, z)$  at  $v_{3,1}$  and a generator  $\mathbf{y}$  for  $(\Sigma, \alpha^H, \beta, z)$  at  $v_{2,3}$ . The asymptotics at  $v_{1,2}$ , however, are more subtle. At each puncture of  $T^\Delta$  mapped to  $v_{1,2}$ ,  $u$  is either asymptotic to a point in  $\alpha \cap \alpha^H$  (in the interior of  $\Sigma$ ), or is asymptotic to a Reeb chord in  $\partial \Sigma$  running from  $\mathbf{a}$  to  $\mathbf{a}^H$ . In defining the map  $F_{\alpha, \alpha^H, \beta}$  we will use curves with only the former kind of asymptotics, but in proving that  $F_{\alpha, \alpha^H, \beta}$  is a chain map we will need to consider curves with the latter asymptotics. Such asymptotics present certain technical issues, as we will discuss below.

For each  $k$ -element subset  $o$  of  $\{1, \dots, 2k\}$  there is a distinguished  $g$ -element subset of  $\alpha \cap \alpha^H$ , denoted  $\Theta_o$ . The set  $\Theta_o$  contains the unique intersection point between  $\alpha_i^a$  and  $\alpha_i^{a,H}$  for  $i \in o$ , as well as the higher graded intersection point<sup>2</sup> between  $\alpha_i^c$  and  $\alpha_i^{c,H}$  for each  $i = 1, \dots, g - k$ . Like curves in  $\Sigma \times [0, 1] \times \mathbb{R}$ , curves in  $\Sigma \times \Delta$  carry (relative) homology classes, in a manner analogous to Definition 4.8. Let  $\pi_2(\mathbf{x}, \mathbf{y}, \Theta_o)$  denote the set of homology classes connecting  $\mathbf{x}$ ,  $\mathbf{y}$ , and  $\Theta_o$ . For a homology class  $B$  in  $\pi_2(\mathbf{x}, \mathbf{y}, \Theta_o)$  we can consider its *domain*, the multiplicities with which  $\pi_\Sigma(B)$  covers each region in the complement of  $\alpha \cup \alpha^H \cup \beta$ . The map taking a homology class to its domain is again injective. Let  $\mathcal{M}^B(\mathbf{x}, \mathbf{y}, \Theta_o; T^\Delta)$  denote the moduli space of holomorphic curves asymptotic to  $\mathbf{x}$  at  $v_{3,1}$ ,  $\mathbf{y}$  at  $v_{2,3}$  and  $\Theta_o$  at  $v_{1,2}$ , with decorated source  $T^\Delta$  and in the homology class  $B$ .

Along  $e_1$  (respectively  $e_2$ ) a curve  $u \in \mathcal{M}^B(\mathbf{x}, \mathbf{y}, \Theta_o; T^\Delta)$  is asymptotic to a collection of Reeb chords in  $(Z, \mathbf{a})$  (respectively  $(Z, \mathbf{a}^H)$ ). The orientation on  $e_1$  (respectively  $e_2$ ) induces a partial order of these collections of Reeb chords. Thus, given ordered partitions  $\vec{P}$  and  $\vec{P}^H$  of Reeb chords in  $(Z, \mathbf{a})$  and  $(Z, \mathbf{a}^H)$  respectively, we may consider the subspace

$$\mathcal{M}^B(\mathbf{x}, \mathbf{y}, \Theta_o; T^\Delta; \vec{P}, \vec{P}^H) \subset \mathcal{M}^B(\mathbf{x}, \mathbf{y}, \Theta_o; T^\Delta)$$

consisting of those holomorphic curves so that the induced ordered partitions of the Reeb chords along  $e_1$  and  $e_2$  are given by  $\vec{P}$  and  $\vec{P}^H$  respectively.

The expected dimension of  $\mathcal{M}^B(\mathbf{x}, \mathbf{y}, \Theta_o; T^\Delta; \vec{P}, \vec{P}^H)$  is

$$(6.28) \quad \text{ind}(B, T^\Delta, P, P^H) = \frac{g}{2} - \chi(T) + 2e(B) + |P| + |P^H|.$$

(The proof is similar to the proof of Proposition 5.6. See [20, p. 1018] for the closed case.) Like Formula (5.7), Formula (6.28) depends on the topology of  $T^\Delta$ , and not just on the homology class and asymptotics. However, as was the case for embedded curves in  $\Sigma \times [0, 1] \times \mathbb{R}$ , the Euler characteristic of an *embedded* curve in  $\Sigma \times \Delta$  is determined by the homology class and asymptotics. To state the formula in a convenient form, first note that for any generator  $\mathbf{x} \in \mathfrak{S}(\Sigma, \alpha, \beta)$ , there is a nearby generator  $\mathbf{x}' \in \mathfrak{S}(\Sigma, \alpha^H, \beta)$  (but not vice versa), as well as a canonical homology class  $T_{\mathbf{x}}$  in  $\pi_2(\mathbf{x}, \mathbf{x}', \Theta_{o(\mathbf{x})})$ , the union of a number of small triangles (supported in the isotopy region) and possibly an annulus with boundary on  $\alpha_1^a$ ,  $\alpha_1^{a,H}$ , and  $\alpha_1^c$ .

**Lemma 6.29.** *Any homology class  $B \in \pi_2(\mathbf{x}, \mathbf{y}, \Theta_o)$  can be written uniquely as*

$$(T_{\mathbf{x}} *_{v_{12}} B_{\alpha, \alpha^H}) *_{v_{23}} B_{\alpha^H, \beta}$$

for some  $B_{\alpha, \alpha^H} \in \pi_2(\Theta_{o(\mathbf{x})}, \Theta_o)$  and  $B_{\alpha^H, \beta} \in \pi_2(\mathbf{x}', \mathbf{y})$ .

Here  $*_{v_{12}}$  is the natural operation of attaching a homology class in  $(\Sigma, \alpha, \alpha^H, z)$  to a homology class in  $(\Sigma, \alpha, \alpha^H, \beta, z)$  at the corner  $v_{12}$  of  $\Delta$ , and likewise  $*_{v_{23}}$  is the operation of attaching a homology class in  $(\Sigma, \alpha^H, \beta, z)$  at the corner  $v_{23}$  of  $\Delta$ .

*Proof.* This is an elementary computation in homology. We use the fact that the space of triply-periodic domains in  $(\Sigma, \alpha, \alpha^H, \beta, z)$  (and therefore  $\pi_2(\mathbf{x}, \mathbf{y}, \Theta_o)$ ) is isomorphic to the second homology of  $Y \times [0, 1]$  minus  $g - k$  disjoint copies of  $S^1$ . In particular, it is isomorphic to  $\pi_2(\mathbf{x}', \mathbf{y}) \oplus \pi_2(\Theta_{o(\mathbf{x})}, \Theta_o)$ .  $\square$

<sup>2</sup>That is, the intersection point such that there are two holomorphic maps  $([0, 1] \times \mathbb{R}, \{1\} \times \mathbb{R}, \{0\} \times \mathbb{R}) \rightarrow (\Sigma, \alpha, \alpha^H)$  mapping  $-\infty$  to  $\Theta$ .

For  $B \in \pi_2(\mathbf{x}, \mathbf{y}, \Theta_o)$ , let  $R(B) \in \pi_2(\mathbf{x}', \mathbf{y})$  be the domain  $B_{\alpha^H, \beta}$  whose existence is guaranteed by Lemma 6.29.

**Lemma 6.30.** *If  $\mathcal{M}^B(\mathbf{x}, \mathbf{y}, \Theta_o; T^\Delta; \vec{P}, \vec{P}^H)$  contains an embedded curve  $u$  then*

$$\chi(T) = \chi_{\text{emb}}(\phi(B), \vec{\rho}),$$

where  $\vec{\rho} = ([\vec{P}], [\vec{P}^H])$ , where we identify Reeb chords in  $(Z, \mathbf{a})$  and  $(Z, \mathbf{a}^H)$ .

*Proof.* Applying the Riemann-Hurwitz formula to  $\pi_\Delta$  shows that the Euler characteristic of  $T$  is  $g - \text{br}(\pi_\Delta \circ u)$ , where  $\text{br}$  denotes the total branching multiplicity. The curve  $u$  tautologically corresponds (in the sense of I. Smith [41]) to a triangle  $\phi: \Delta \rightarrow \text{Sym}^g(\Sigma)$ ; see, for instance, [20, Section 13]. Then,  $\text{br}(\pi_\Delta \circ u) = \phi \cdot D$ , the intersection number of  $\phi$  and the diagonal  $D$  in  $\text{Sym}^g(\Sigma)$ .

This intersection number clearly depends only on the relative homotopy class of the map  $\phi$ , which in turn is determined by  $B$ ,  $[\vec{P}]$ , and  $[\vec{P}^H]$ . This already proves that  $\chi(T)$  is determined by  $(B, [\vec{P}], [\vec{P}^H])$ . Further, one can deform  $\phi$  without changing  $\phi \cdot D$  to a (non-holomorphic) curve  $\phi'$  of the form

$$\phi' = \phi_{\mathbf{x}} * \phi_{\alpha, \alpha^H} * \phi_{\alpha^H, \beta},$$

where  $*$  denotes concatenation and the homology classes of  $\phi_{\mathbf{x}}$ ,  $\phi_1$  and  $\phi_2$  are the homology classes  $T_{\mathbf{x}}$ ,  $B_{\alpha, \alpha^H}$  and  $B_{\alpha^H, \beta}$  from Lemma 6.29.

All of the intersections of  $\phi'$  and  $D$  correspond to intersections of  $\phi_{\alpha^H, \beta}$  and  $D$ . Arrange that  $\phi_{\alpha^H, \beta}$  is holomorphic near its boundary and near  $\phi_{\alpha^H, \beta} \cap D$ . The curve  $\phi_{\alpha^H, \beta}$  tautologically corresponds to a smooth curve

$$\tilde{u}: (\tilde{S}, \partial \tilde{S} \rightarrow (\Sigma \times [0, 1] \times \mathbb{R}, (\alpha^H \times \{1\} \times \mathbb{R}) \cup (\beta \times \{0\} \times \mathbb{R})),$$

holomorphic near the boundary. The proof of Proposition 5.47 then applies to compute  $\chi(S)$  as  $\chi_{\text{emb}}(B_{\alpha^H, \beta}, \rho)$ . It follows that  $\chi(T) = \chi_{\text{emb}}(B_{\alpha^H, \beta}, \rho)$  as well.  $\square$

See also Sarkar [38].

Now, for  $\mathbf{x} \in \mathfrak{S}(\Sigma, \alpha, \beta)$ ,  $\mathbf{y} \in \mathfrak{S}(\Sigma, \alpha^H, \beta)$ ,  $B \in \pi_2(\mathbf{x}', \mathbf{y})$ , and  $\vec{\rho}$  a sequence of Reeb chords so that  $(B, \vec{\rho})$  is compatible, define

$$\mathcal{M}^B(\mathbf{x}, \mathbf{y}, \Theta; \vec{\rho}) := \bigcup_{\substack{\vec{\rho}=(\vec{\rho}_1, \vec{\rho}_2) \\ R(B')=B}} \bigcup_{\substack{[\vec{P}]=\vec{\rho}_1, [\vec{P}^H]=\vec{\rho}_2 \\ \chi(T^\Delta)=\chi_{\text{emb}}(B, \vec{\rho})}} \mathcal{M}^{B'}(\mathbf{x}, \mathbf{y}, \Theta_{o(\mathbf{x}, \vec{\rho}_1)}; T^\Delta; \vec{P}_1, \vec{P}_2).$$

Finally, we set

$$F_{\alpha, \alpha^H, \beta}(\mathbf{x}) := \sum_{\mathbf{y}} \sum_{B \in \pi_2(\mathbf{x}', \mathbf{y})} \sum_{\{\vec{\rho} \mid \text{ind}(B, \vec{\rho})=0\}} \#(\mathcal{M}^B(\mathbf{x}, \mathbf{y}, \Theta; \vec{\rho})) a(-\vec{\rho}) \mathbf{y}.$$

As before, if both  $(\Sigma, \alpha, \beta, z)$  and  $(\Sigma, \alpha^H, \beta)$  is provincially admissible this sum is finite, as the 0-dimensional moduli spaces are compact.

**Proposition 6.31.**  *$F_{\alpha, \alpha^H, \beta}$  is a chain map, i.e., an  $\mathcal{A}$ -module homomorphism.*

*Proof.* The arguments in Proposition 5.20—essentially, considering the curves  $\pi_\Sigma \circ u$  and  $\pi_\Delta \circ u$  separately—show that the moduli spaces  $\mathcal{M}^B(\mathbf{x}, \mathbf{y}, \Theta_o; T^\Delta; \vec{P}_1, \vec{P}_2)$  admit natural compactifications. Other than degenerations at  $v_{12}$ , which we will discuss presently, the limit curves are natural analogues of the holomorphic combs introduced in Section 5.3.

In particular, in one-dimensional families, holomorphic triangles can degenerate in the following ways:

- (1) two-story buildings;
- (2) join curve ends;
- (3) split curve ends;
- (4) other collisions of two levels;
- (5) degenerations at  $v_{12}$ .

The first four kinds of degenerations are, by now, routine. The first corresponds to  $F_{\alpha, \alpha^H, \beta} \circ \partial(\mathbf{x})$  and part of  $\partial \circ F_{\alpha, \alpha^H, \beta}(\mathbf{x})$ . The third kind corresponds to the rest of  $\partial \circ F_{\alpha, \alpha^H, \beta}(\mathbf{x})$ . The fourth kind cancels with itself and the second kind, in pairs. What remains is to study the fifth kind of degeneration—degenerations at  $v_{12}$ .

Degenerations at  $v_{12}$  result in a pair of curves  $(u_T, u_S)$ , where  $u_T$  is a holomorphic triangle as before (with possibly more general asymptotics), and  $u_S$  is a curve

$$u_S: (S, \partial S) \longrightarrow (\Sigma \times [0, 1] \times \mathbb{R}, \alpha^H \times \{1\} \times \mathbb{R} \cup \alpha \times \{0\} \times \mathbb{R}),$$

asymptotic to  $\Theta_o$  at  $-\infty$ . By inspecting the diagram, we see that  $u_S$  can have the following pieces:

- (1) Bigons contained entirely in the interior of  $\Sigma$ .
- (2) Bigons touching  $\partial\bar{\Sigma}$ .
- (3) Annuli, with one boundary component on  $\alpha_1^{c,H}$  and  $\alpha_1^c$ , and the other boundary component on  $\alpha_1^{a,H}$  and  $\alpha_1^a$  (and touching  $\partial\bar{\Sigma}$ ).
- (4) Trivial strips.

Further, generically all but one of the components will be trivial strips.

It is standard that the degenerations of type (1) cancel in pairs: for each such bigon there is a matching bigon with the same endpoints. Whenever one bigon degenerates, one can glue the matching bigon to  $u_T$ , giving another end of the moduli space. We would like to see that the other contributions behave similarly.

On the other hand, the bigons touching  $\partial\bar{\Sigma}$  go out “diagonally” to infinity, which at first sight causes problems in proving the appropriate gluing. That is, the bigons are holomorphic maps from a disk with two punctures on the boundary. One of the punctures maps via  $\pi_{\mathbb{D}}$  to  $+\infty$  in  $[0, 1] \times \mathbb{R}$  and via  $\pi_{\Sigma}$  to the puncture of  $\Sigma$ . (The other puncture maps to a point in  $\Theta_o \times \{-\infty\}$ , one of our standard kinds of asymptotics.) If we look at the image in  $\mathbb{R} \times \mathbb{R}$ , where one  $\mathbb{R}$  is the  $t$  coordinate and the other is the coordinate on the cylindrical end of  $\Sigma$ , the curve is indeed heading out on a diagonal. In general, gluing at such a diagonal infinity would require a new—and rather novel—gluing theorem.

For these particular curves, however, a general result turns out not to be needed. First let us see how these degenerations can occur. For  $i \in [2k]$ , let  $R_{i,\ell}$  ( $\ell = 1, 2$ ) be the two regions that can degenerate that touch  $\partial\bar{\Sigma}$ ,  $\alpha_i^a$ , and  $\alpha_i^{a,H}$ . (These are all bigons except for, say,  $R_{1,1}$ , which is an annulus that we will treat presently.) When  $R_{i,\ell}$  appears as a degeneration, the original curve necessarily has a Reeb chord, either from  $\alpha_j^a$  to  $\alpha_i^a$  or from  $\alpha_j^{a,H}$  to  $\alpha_i^{a,H}$  for some  $j \in [2k]$ . The remaining triangle map after the degeneration then has a puncture running from  $\alpha_j^a$  to  $\alpha_i^{a,H}$  or from  $\alpha_i^a$  to  $\alpha_j^{a,H}$ . We wish to show that we can glue on an appropriate  $u'_S$ , with domain  $R_{j,\ell'}$ , which we suppose to be a bigon. See Figure 32.

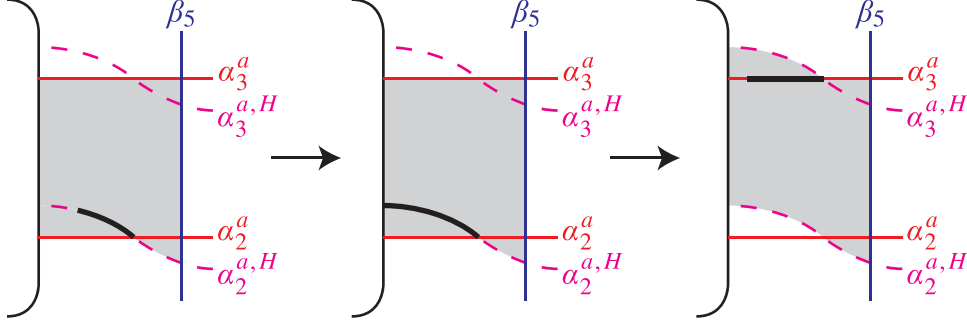


FIGURE 32. An illustration of a boundary bigon degenerating, and a different being glued on the other side.

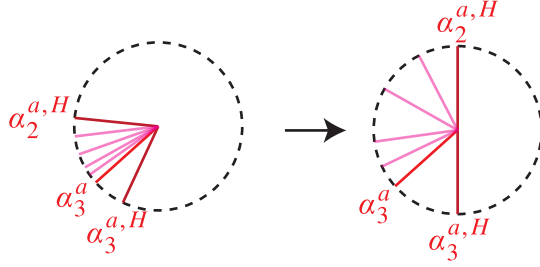


FIGURE 33. An illustration of straightening-out  $\alpha_2^a, H$ , corresponding to the gluing in Figure 32.

Consider the result of filling in the puncture so that  $\Sigma$  becomes a closed Riemann surface. Call the result  $\Sigma_{\bar{e}}$ . The various  $\bar{\alpha}_i^a$  and  $\bar{\alpha}_i^{a,H}$  meet at the puncture  $p$ , in general with a corner.

Let  $U_p$  be a small disk in  $\Sigma_{\bar{e}}$  around  $p$ , and let  $U'_S$  and  $U_T$  be the components of  $(\pi_{\Sigma} \circ u'_S)^{-1}(U_p)$  and  $(\pi_{\Sigma} \circ u_T)^{-1}(U_p)$  containing the punctures to be glued. The image of  $U'_S$  and  $U_T$  in  $U_p$  is a circular sector, which can be straightened by applying a power map as illustrated in Figure 33. By the local nature of gluing arguments, this reduces the gluing problem above to the usual gluing problem from (closed) Heegaard Floer homology: the Reeb chords from  $U'_S$  and  $U_T$  are now just an intersection point.

Thus, for every triangle map with a single Reeb chord running from  $\alpha_i^a$  to  $\alpha_j^{a,H}$ , we can glue on either  $R_{i,\ell}$  or  $R_{j,\ell'}$ , so these two ends of the moduli space cancel (assuming neither is the annular region). Note that this kind of degeneration corresponds to an  $\alpha$ - $\alpha$ -Reeb chord mapped to  $e_1$  turning into an  $\alpha$ - $\alpha^H$ -Reeb chord mapped to  $v_{12}$ , which in turn becomes an  $\alpha^H$ - $\alpha^H$ -Reeb chord after we glue (or vice-versa). That is, an endpoint of  $\mathcal{M}^B(\mathbf{x}, \mathbf{y}, \Theta_o; T^{\Delta}; \vec{P}_{[1,i]}, \vec{P}_{[i+1,n]})$  is being canceled against an endpoint of  $\mathcal{M}^{B'}(\mathbf{x}, \mathbf{y}, \Theta_{o'}; T^{\Delta}; \vec{P}_{[1,i-1]}, \vec{P}_{[i,n]})$ .

We now turn to the case when one of the regions involved is the annular region  $R_{1,1}$ . By the curve-straightening argument used in the case of mirror bigons, these degenerating holomorphic annuli correspond to index 1 holomorphic annuli in a certain Heegaard diagram. These annuli are completely studied in [33, Lemma 9.4], where it

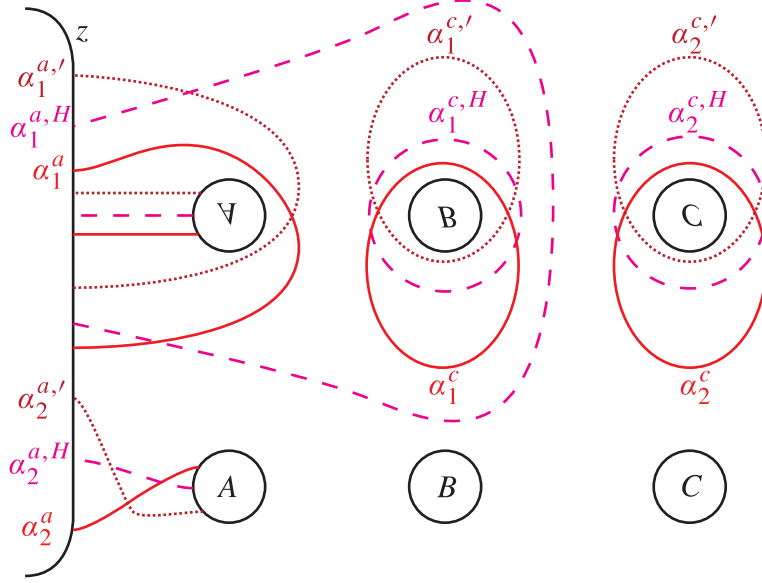


FIGURE 34. The curves  $\alpha$ ,  $\alpha^H$  and  $\alpha'$  in  $\Sigma$ . The curves in  $\alpha$  are solid, those in  $\alpha^H$  are dashed, and those in  $\alpha'$  are dotted.

is proved that the algebraic count of such annuli is 1. It follows that this annulus can be glued on, just as with the bigons. This completes the proof.

Finally, note that for each  $\text{spin}^c$  structure  $\mathfrak{s}$ , we can identify  $G'_D(\Sigma, \alpha, \beta, \mathfrak{s})$  and  $G'_D(\Sigma, \alpha^H, \beta, \mathfrak{s})$  by picking a base generator  $\mathbf{x}_0 \in \mathfrak{S}(\Sigma, \alpha, \beta, \mathfrak{s})$  and the corresponding base generator  $\mathbf{x}'_0 \in \mathfrak{S}(\Sigma, \alpha^H, \beta, \mathfrak{s})$ . Then the fact that we sum over curves with  $\text{ind}(B, \bar{\rho}) = 0$  implies that we preserve the grading, as before.  $\square$

To show that  $F_{\alpha, \alpha^H, \beta}$  is an isomorphism, we wish to construct a chain map the other way and a chain homotopy to the identity. A count of triangles in  $(\Sigma, \alpha^H, \alpha, \beta)$  does not work as desired, as the Reeb chords on the boundary go the wrong way. Instead we define a third set of circles, which we denote  $\alpha'$ . Obtain  $\alpha_i^{c,'}$  from  $\alpha_i^c$  by performing a small Hamiltonian perturbation, so that  $\alpha_i^c$  intersects each of  $\alpha_i^{c,H}$  and  $\alpha_i^{c,'}$  transversely in two points and is disjoint from all other curves in  $\alpha \cup \alpha^H$ . Let  $\alpha_i^{a,'}$  be an isotopic translate of  $\alpha_i^a$ , intersecting each of  $\alpha_i^a$  and  $\alpha_i^{a,H}$  in a single point, and such that there are two short Reeb chords in  $\partial\bar{\Sigma}$  running from  $\alpha_i^{a,'}$  to  $\alpha_i^{a,H}$  (and hence two slightly longer Reeb chords running from  $\alpha_i^{a,'}$  to  $\alpha_i^a$ ). The isotopy is chosen small enough that  $\alpha_i^{a,'}$  is disjoint from all other curves in  $\alpha \cup \alpha^H$ . See Figure 34. Now there is a chain map  $F_{\alpha^H, \alpha', \beta}$  defined using  $(\Sigma, \alpha^H, \alpha', \beta, z)$ , in exactly the same way that  $F_{\alpha, \alpha^H, \beta}$  was defined using  $(\Sigma, \alpha, \alpha^H, \beta, z)$ .

We next consider the map  $F_{\alpha, \alpha', \beta}$  defined in the same way as  $F_{\alpha, \alpha^H, \beta}$  but with  $\alpha'$  in place of  $\alpha^H$ .

**Proposition 6.32.** *The map  $F_{\alpha, \alpha', \beta}$  is an  $\mathcal{A}$ -module isomorphism.*

*Proof.* The proof that  $F_{\alpha, \alpha', \beta}$  is a chain map is parallel to, but marginally easier than, the proof of Proposition 6.31, the only difference being that the discussion of annuli is irrelevant here.

Next we argue that  $F_{\alpha, \alpha', \beta}$  is in fact an isomorphism. The proof, which makes use of the energy filtration, is a straightforward analogue of the proof for closed 3-manifolds [33, Proposition 9.8]. We will use the following standard lemma.

**Lemma 6.33.** *Let  $F: A \rightarrow B$  be a map of filtered groups which is decomposed as a sum  $F = F_0 + \ell$  where  $F_0$  is a filtration-preserving isomorphism and  $\ell$  has strictly lower order than  $F_0$ . Suppose that the filtration on  $B$  is bounded below. Then  $F$  is an isomorphism of groups.*

Choose an area form  $\text{Area}$  on  $\Sigma$  such that if  $D$  is a periodic domain in  $(\Sigma, \alpha, \beta)$  then  $\text{Area}(D) = 0$ . (The existence of  $\text{Area}$  is guaranteed by Lemma 4.17.) Arrange also that periodic domains in  $(\Sigma, \alpha, \alpha')$  have area 0, and that the boundary bigons all have the same area. It follows that for  $D$  a periodic domain in  $(\Sigma, \alpha', \beta)$ ,  $\text{Area}(D) = 0$  as well.

For each  $\text{spin}^c$  structure  $\mathfrak{s}$  on  $Y$ , we define a map  $\mathcal{F}$  from  $\mathfrak{S}(\Sigma, \alpha, \beta, \mathfrak{s}) = \{\mathbf{x} \in \mathfrak{S}(\Sigma, \alpha, \beta) \mid \mathfrak{s}_z(\mathbf{x}) = \mathfrak{s}\}$  to  $\mathbb{R}$ . Choose one generator  $\mathbf{x}_0 \in \mathfrak{S}(\Sigma, \alpha, \beta, \mathfrak{s})$  and declare  $\mathcal{F}(\mathbf{x}_0) = 0$ . Then, for any other  $\mathbf{x} \in \mathfrak{S}(\Sigma, \alpha, \beta, \mathfrak{s})$  pick  $A_{\mathbf{x}_0, \mathbf{x}} \in \pi_2(\mathbf{x}_0, \mathbf{x})$  and define  $\mathcal{F}(\mathbf{x}) = -\text{Area}(A_{\mathbf{x}_0, \mathbf{x}})$ . Since periodic domains have area 0,  $\mathcal{F}(\mathbf{x})$  is independent of the choice of  $A_{\mathbf{x}_0, \mathbf{x}}$ . More generally, for  $a \in \mathcal{A}$  such that  $a\mathbf{x} \neq 0$ , define  $\mathcal{F}(a\mathbf{x}) = \mathcal{F}(\mathbf{x})$ . Since holomorphic curves have positive domains, the map  $\mathcal{F}$  induces a filtration on  $\widehat{\text{CFD}}(\Sigma, \alpha, \beta)$ .

There is an obvious identification between  $\mathfrak{S}(\Sigma, \alpha, \beta, \mathfrak{s})$  and  $\mathfrak{S}(\Sigma, \alpha', \beta, \mathfrak{s})$ . As before, let  $\mathbf{x}'$  denote the generator corresponding to  $\mathbf{x}$ , and let  $T_{\mathbf{x}} \in \pi_2(\mathbf{x}, \mathbf{x}', \Theta_{o(\mathbf{x})})$  be the canonical small triangle. Define a map  $\mathcal{F}'_0: \mathfrak{S}(\Sigma, \alpha', \beta, \mathfrak{s}) \rightarrow \mathbb{R}$  in the same way as  $\mathcal{F}$ , still using the area form  $\text{Area}$  but using  $\mathbf{x}'_0$  in place of  $\mathbf{x}_0$ . Define  $\mathcal{F}': \mathfrak{S}(\Sigma, \alpha', \beta, \mathfrak{s}) \rightarrow \mathbb{R}$  by  $\mathcal{F}' = \mathcal{F} - \text{Area}(T_{\mathbf{x}_0})$ . By extending  $\mathcal{F}'$  by  $\mathcal{F}'(a\mathbf{y}) = \mathcal{F}'(\mathbf{y})$ , we get a filtration  $\mathcal{F}'$  on  $\widehat{\text{CFD}}(\Sigma, \alpha', \beta)$ .

To see that  $F_{\alpha, \alpha', \beta}$  respects the filtrations  $\mathcal{F}$  and  $\mathcal{F}'$ , suppose there is a term  $a\mathbf{y}$  in  $F_{\alpha, \alpha', \beta}(\mathbf{x})$ , and thus a positive domain  $B \in \pi_2(\mathbf{x}, \mathbf{y}', \Theta_o)$ , and pick a domain  $A_{\mathbf{x}_0, \mathbf{x}} \in \pi_2(\mathbf{x}_0, \mathbf{x})$ . Then, for an appropriate  $o' = o(\mathbf{x}_0)$  and domain  $A_{o, o'} \in \pi_2(\Theta_o, \Theta_{o'})$ , the domain  $A_{\mathbf{x}_0, \mathbf{x}} + B - T_{\mathbf{x}_0} - A_{o, o'}$  connects  $\mathbf{x}'_0$  and  $\mathbf{y}'$ . Our assumptions guarantee that  $\text{Area}(A_{o, o'}) = 0$ , so

$$\mathcal{F}'(\mathbf{y}') = -\text{Area}(A_{\mathbf{x}_0, \mathbf{x}}) - \text{Area}(B) < \mathcal{F}(\mathbf{x})$$

as desired.

Now, if we choose  $\text{Area}$  so that for all  $\mathbf{x}$  the triangle  $T_{\mathbf{x}}$  is the unique triangle of minimal area connecting  $\mathbf{x}$ ,  $\mathbf{y}'$  and  $\Theta_o$  for any  $\mathbf{y}'$  and  $o$  (this is easily accomplished), then the top order part of  $F_{\alpha, \alpha', \beta}$  with respect to  $\mathcal{F}$  and  $\mathcal{F}'$  is simply the map  $\mathbf{x} \mapsto \mathbf{x}'$ . In particular, the top order part of  $F_{\alpha, \alpha', \beta}$  is a group isomorphism.

Consequently, it follows from Lemma 6.33 that the  $\mathcal{A}$ -module homomorphism  $F_{\alpha, \alpha', \beta}$  is an  $\mathbb{F}_2$  vector space isomorphism  $\widehat{\text{CFD}}(\Sigma, \alpha, \beta, z) \rightarrow \widehat{\text{CFD}}(\Sigma, \alpha', \beta, z)$ , and hence also a  $\mathcal{A}$ -module isomorphism.  $\square$

*Remark 6.34.* We could weaken our assumptions and prove handleslide invariance for Heegaard diagrams  $\mathcal{H}$  that are provincially admissible (rather than admissible) by being slightly more clever in the choice of filtration  $\mathcal{F}$ , as follows. Pick a map  $f: H_1(Z', \mathbf{a}) \rightarrow \mathbb{R}$  so that for any periodic domain  $B$ ,  $f(\partial^\partial B) + \text{Area}(B) = 0$ . (This is possible for any provincially admissible diagram.) Then define  $\mathcal{F}(\mathbf{x}) = \text{Area}(A_{\mathbf{x}_0, \mathbf{x}}) +$

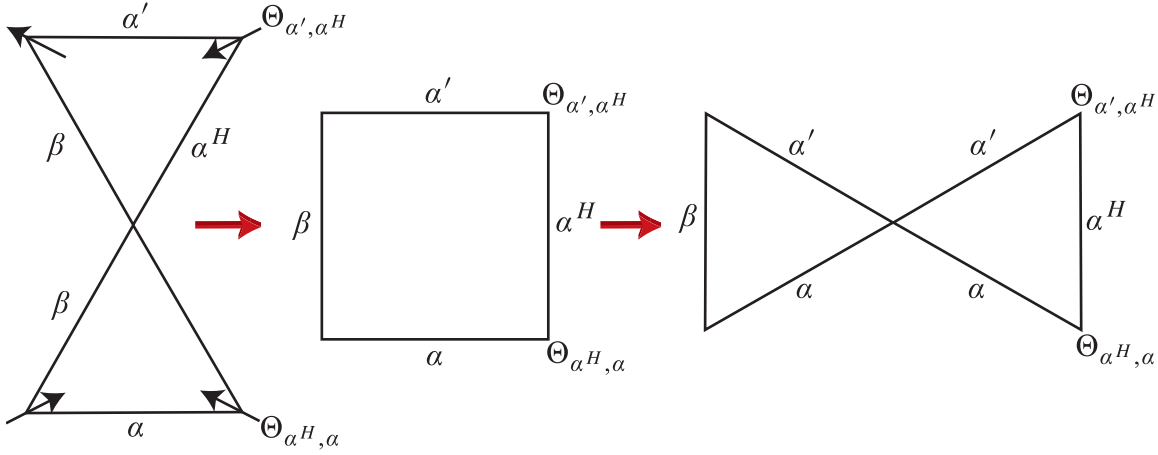


FIGURE 35. Two compositions of triangles, as degenerations of a rectangle.

$f(\partial^\partial A_{\mathbf{x}_0, \mathbf{x}})$ , and make similar other adjustments. Alternatively, we can do an isotopy to make  $\mathcal{H}$  admissible, do the handleslide, and then do an inverse isotopy.

**Proposition 6.35.** *The map  $F_{\alpha, \alpha', \beta}$  is chain homotopic to  $F_{\alpha^H, \alpha', \beta} \circ F_{\alpha, \alpha^H, \beta}$ .*

*Proof.* The proof is in two steps. First, the composition  $F_{\alpha^H, \alpha', \beta} \circ F_{\alpha, \alpha^H, \beta}$  is a count of pairs of triangles, in  $(\Sigma, \alpha, \alpha^H, \beta, z)$  and  $(\Sigma, \alpha^H, \alpha', \beta, z)$ . This map is chain homotopic to counting pairs of triangles in  $(\Sigma, \alpha, \alpha', \beta, z)$  and  $(\Sigma, \alpha, \alpha^H, \alpha', z)$ , via a chain homotopy counting (index  $-1$ ) rectangles. (See Figure 35 for a schematic illustration.) The reader unfamiliar with such arguments is directed to any of [33, Theorem 8.16], [20, Proposition 10.29] or [21, Lemma 7.2.5]. The only new types of degeneration are curves in either  $(\Sigma, \alpha, \alpha^H, z)$  or  $(\Sigma, \alpha^H, \alpha', z)$  touching  $\partial \Sigma$ , either small bigons or annuli. As in Proposition 6.31, these degenerations cancel against themselves.

Second, a direct computation shows that the only rigid triangles in  $(\Sigma, \alpha, \alpha^H, \alpha', z)$  asymptotic to  $\Theta_{o, \alpha^H, \alpha}$  and  $\Theta_{o, \alpha', \alpha^H}$  at two of the corners are asymptotic to  $\Theta_{o, \alpha, \alpha'}$  at the third corner—and that there is algebraically a single such triangle. Again, the reader is directed to [33, Lemma 9.7] for details—the proof there applies without change in our case.  $\square$

**Corollary 6.36.** *The  $\mathcal{A}$ -modules  $\widehat{CFD}(\Sigma, \alpha, \beta, z)$  and  $\widehat{CFD}(\Sigma, \alpha^H, \beta, z)$  are homotopy equivalent (over  $\mathcal{A}$ ).*

*Proof.* If we arrange that  $\alpha'$  and  $\alpha$  are close enough then the obvious identification of generators of  $\widehat{CFD}(\Sigma, \alpha, \beta, z)$  and  $\widehat{CFD}(\Sigma, \alpha', \beta, z)$  induces an isomorphism of  $\mathcal{A}$ -modules (because all of the moduli spaces counted in the definitions of  $\partial$  are the same). Hence, by Propositions 6.32 and 6.35, the composition  $F_{\alpha^H, \alpha', \beta} \circ F_{\alpha, \alpha^H, \beta}$  is chain homotopic to an automorphism of  $\widehat{CFD}(\Sigma, \alpha, \beta, z)$ .

Choose a fourth set of curves  $\alpha^{H'}$  isotopic to  $\alpha^H$  and define another map  $F_{\alpha', \alpha^{H'}, \beta} : \widehat{CFD}(\Sigma, \alpha', \beta, z) \rightarrow \widehat{CFD}(\Sigma, \alpha^{H'}, \beta, z)$ . The same argument shows that  $F_{\alpha', \alpha^{H'}, \beta} \circ F_{\alpha^H, \alpha'}$  is chain homotopic to an automorphism of  $\widehat{CFD}(\Sigma, \alpha^H, \beta, z)$ . The result follows.  $\square$

## 7. TYPE A MODULES

We now turn to defining the type  $A$  module  $\widehat{CFA}(\mathcal{H})$  associated to a bordered Heegaard diagram  $\mathcal{H}$ . This uses more of the moduli spaces from Section 5 than the type  $D$  module does; in particular non-trivial partitions appear.

**7.1. Definition.** Fix a pointed bordered Heegaard diagram  $\mathcal{H} = (\Sigma, \alpha, \beta, z)$ , satisfying the provincial admissibility criterion of Definition 4.14. The module  $\widehat{CFA}(\mathcal{H})$  is a right  $(\mathcal{A}_\infty)$ - $\mathcal{A}(\mathcal{Z})$ -module, where  $\mathcal{Z}$  is the pointed matched circle  $\partial\mathcal{H}$ . Unlike  $\widehat{CFD}$ , which is essentially free over  $\mathcal{A}$ , much of the data of the Heegaard diagram is encoded in the multiplication on  $\widehat{CFA}(\mathcal{H})$ , as well as in certain higher products.

The module  $\widehat{CFA}(\mathcal{H})$ , also denoted simply  $\widehat{CFA}$ , is generated over  $\mathbb{F}_2$  by the set of generators  $\mathfrak{S}(\mathcal{H})$ . Given  $\mathbf{x} \in \mathfrak{S}(\mathcal{H})$ , define  $I_A(\mathbf{x}) := I(o(\mathbf{x}))$ . (Recall that  $o(\mathbf{x}) \subset [2k]$  is the set of  $\alpha$ -arcs occupied by  $\mathbf{x}$ .) We define a right action of  $\mathcal{I}$  on  $\widehat{CFA}(\mathcal{H})$  by

$$\mathbf{x} \cdot I(s) = \begin{cases} \mathbf{x} & I_A(\mathbf{x}) = I(s) \\ 0 & \text{otherwise.} \end{cases}$$

Note that, as promised in Section 3.2, this implies that the summands  $\mathcal{A}(\mathcal{Z}, i)$  of  $\mathcal{A}(\mathcal{Z})$  act trivially on  $\widehat{CFA}(\mathcal{H})$  for  $i \neq 0$ .

Recall from Section 3.1.3 that to a set  $\rho$  of Reeb chords we associate an algebra element  $a(\rho)$ .

**Lemma 7.1.** *For  $\mathbf{x} \in \mathfrak{S}(\mathcal{H})$  and  $\vec{\rho}$  a sequence of sets of Reeb chords,  $(\mathbf{x}, \vec{\rho})$  is strongly boundary monotonic (Definition 5.34) if and only if the tensor product*

$$o(\mathbf{x}) \otimes_{\mathcal{I}} a(\rho_1) \otimes_{\mathcal{I}} \cdots \otimes_{\mathcal{I}} a(\rho_n)$$

*is not zero.*

*Proof.* Both statements are equivalent to saying that there is a sequence of subsets  $o_i \subset [2k]$  with  $o_0 = o(\mathbf{x})$  and  $I(o_{i-1})a(\rho_i)I(o_i) \neq 0$ .  $\square$

As explained in Definition 2.3, an  $\mathcal{A}_\infty$ -module structure on  $\widehat{CFA}$  is a compatible family of maps

$$m_{n+1} : \widehat{CFA} \otimes_{\mathcal{I}} \mathcal{A} \otimes_{\mathcal{I}} \cdots \otimes_{\mathcal{I}} \mathcal{A} \rightarrow \widehat{CFA}.$$

In particular, in view of Lemma 7.1, since the tensor products are over  $\mathcal{I}$ , it suffices to define

$$m_{n+1}(\mathbf{x}, a(\rho_1), \dots, a(\rho_n))$$

when  $(\mathbf{x}, \rho_1, \dots, \rho_n)$  is strongly boundary monotonic. (In the following sections we will use commas to separate the tensor factors in the argument to  $m_{n+1}$  and similar maps.)

**Definition 7.2.** For  $(\mathbf{x}, \vec{\rho})$  strongly boundary monotonic, define

$$\begin{aligned} m_{n+1}(\mathbf{x}, a(\rho_1), \dots, a(\rho_n)) &:= \sum_{\mathbf{y} \in \mathfrak{S}(\mathcal{H})} \sum_{\substack{B \in \pi_2(\mathbf{x}, \mathbf{y}) \\ \text{ind}(B, \vec{\rho})=1}} \#(\mathcal{M}^B(\mathbf{x}, \mathbf{y}; \rho_1, \dots, \rho_n)) \mathbf{y} \\ m_2(\mathbf{x}, 1) &:= \mathbf{x} \\ m_{n+1}(\mathbf{x}, \dots, 1, \dots) &:= 0, \quad n > 1. \end{aligned}$$

Extend this action multilinearly (over  $\mathcal{I}$ ) to all of  $\mathcal{A}$ .

*Remark 7.3.* The value of  $m_{n+1}$  when one of the arguments is 1 can be made more natural by allowing the corresponding  $\rho_i$  to be empty (since  $a(\emptyset) = 1$ ). The corresponding moduli spaces should be interpreted as having a choice of height  $t_i$ , and no other conditions at  $t_i$ ; such moduli spaces are never 0-dimensional (and so do not contribute to  $m_{n+1}$ ) unless the original curve was not stable.

**Lemma 7.4.** *Suppose that  $\mathcal{H}$  is provincially admissible. Fix generators  $\mathbf{x}$  and  $\mathbf{y}$  and a consistent sequence  $\vec{\rho}$  of sets of Reeb chords. Then there are at most finitely many  $B \in \pi_2(\mathbf{x}, \mathbf{y})$  such that  $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; \vec{\rho})$  is nonempty.*

*Proof.* As in Lemma 6.5, this is immediate from Proposition 4.19 and the fact that if a domain admits a holomorphic representative then all of its coefficients must be non-negative.  $\square$

*Remark 7.5.* Notice that the operation  $m_1: \widehat{CFA} \rightarrow \widehat{CFA}$  is the differential obtained by counting only provincial holomorphic curves, i.e., curves which do not approach  $\partial\bar{\Sigma}$ .

*Remark 7.6.* In Section 8 we will show that for special kinds of diagrams (the obvious analogues of “nice diagrams” from [39]) the  $m_i$  vanish for  $i > 2$ , so  $\widehat{CFA}$  is an ordinary (not  $\mathcal{A}_\infty$ ) differential module. However, it is not clear how to prove invariance if one restricts only to nice diagrams. Further, in spite of the added analytic and algebraic complications, it is often easier to compute with a diagram which is not nice, as general diagrams often have a much smaller set of generators  $\mathfrak{S}(\mathcal{H})$ .

We will sometimes use the alternate notation  $m_1(\mathbf{x}) = \partial\mathbf{x}$  and  $m_2(\mathbf{x}, a) = \mathbf{x} \cdot a$ .

**7.2. Compatibility with algebra.** In this section we will prove that  $\widehat{CFA}(\mathcal{H})$  is an  $\mathcal{A}_\infty$ -module over  $\mathcal{A}$ . Recall from Definition 2.3 that the basic relation for an  $\mathcal{A}_\infty$ -module (over an  $\mathcal{A}_\infty$ -algebra over  $\mathbb{F}_2$ ) is

$$\begin{aligned} 0 &= \sum_{i+j=n+1} m_i(m_j(\mathbf{x}, a_1, \dots, a_{j-1}), \dots, a_{n-1}) \\ &\quad + \sum_{i+j=n+1} \sum_{\ell=1}^{n-j} m_i(\mathbf{x}, a_1, \dots, a_{\ell-1}, \mu_j(a_\ell, \dots, a_{\ell+j-1}), \dots, a_{n-1}). \end{aligned}$$

For the algebra  $\mathcal{A}$ , the  $\mu_i$  vanish for  $i > 2$ . Therefore, we must check that

$$\begin{aligned}
 0 = & \sum_{i+j=n+1} m_i(m_j(\mathbf{x}, a_1, \dots, a_{j-1}), \dots, a_{n-1}) \\
 (7.7) \quad & + \sum_{\ell=1}^{n-1} m_n(\mathbf{x}, a_1, \dots, \partial a_\ell, \dots, a_{n-1}) \\
 & + \sum_{\ell=1}^{n-2} m_{n-1}(\mathbf{x}, a_1, \dots, a_\ell a_{\ell+1}, \dots, a_{n-1}).
 \end{aligned}$$

**Proposition 7.8.** *The data  $(\widehat{CFA}(\mathcal{H}), \{m_i\}_{i=1}^\infty)$  forms an  $\mathcal{A}_\infty$ -module over  $\mathcal{A}$ .*

We give the proof shortly. In fact, the most interesting points in the proof of Proposition 7.8 can be seen in the following examples. (The reader may wish to compare Examples 7.9–7.11 with Examples 6.10–6.12.)

*Example 7.9.* In the left of Figure 36 is a piece of a diagram with four generators. The nontrivial multiplications in the corresponding module  $\widehat{CFA}$  are

$$\begin{aligned}
 \partial\{b, d\} &= \{c, e\} \\
 \{a, d\} \cdot \begin{bmatrix} 1 & 3 \\ 2 & \end{bmatrix} &= \{a, c\} \\
 \{e, c\} \cdot \begin{bmatrix} 1 & 2 \\ 3 & \end{bmatrix} &= \{a, c\} \\
 \{b, d\} \cdot \begin{bmatrix} 2 & 1 \\ 3 & \end{bmatrix} &= \{a, d\} \\
 \{b, d\} \cdot \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} &= \{a, c\}
 \end{aligned}$$

There is an  $\mathcal{A}_\infty$  compatibility equation

$$m_2(\partial\{b, d\} \cdot \begin{bmatrix} 1 & 2 \\ 3 & \end{bmatrix}) + m_2(\{b, d\} \cdot \partial\begin{bmatrix} 1 & 2 \\ 3 & \end{bmatrix}) = 0,$$

bearing in mind that  $\partial\begin{bmatrix} 1 & 2 \\ 3 & \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ . Geometrically, this corresponds to considering the two-dimensional moduli space of curves from  $\{b, d\}$  to  $\{a, c\}$  where the Reeb chord from 2 to 3 is encountered before the Reeb chord from 1 to 2. This moduli space has two ends, one of which is a two-story building, and the other of which is a join curve end.

*Example 7.10.* In the center of Figure 36 is a piece of a diagram with four generators. The nontrivial products are

$$\begin{aligned}
 \{b, d\} \cdot \begin{bmatrix} 1 & 3 \\ 2 & \end{bmatrix} &= \{a, d\} \\
 \{b, c\} \cdot \begin{bmatrix} 1 & 4 \\ 2 & \end{bmatrix} &= \{a, c\} \\
 \{a, d\} \cdot \begin{bmatrix} 3 & 2 \\ 4 & \end{bmatrix} &= \{a, c\} \\
 \{b, d\} \cdot \begin{bmatrix} 3 & 1 \\ 4 & \end{bmatrix} &= \{b, c\} \\
 \{b, d\} \cdot \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} &= \{a, c\}.
 \end{aligned}$$

Here, associativity follows from the fact that

$$\begin{bmatrix} 1 & 3 \\ 2 & \end{bmatrix} \cdot \begin{bmatrix} 3 & 2 \\ 4 & \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 4 & \end{bmatrix} \cdot \begin{bmatrix} 1 & 4 \\ 2 & \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}.$$

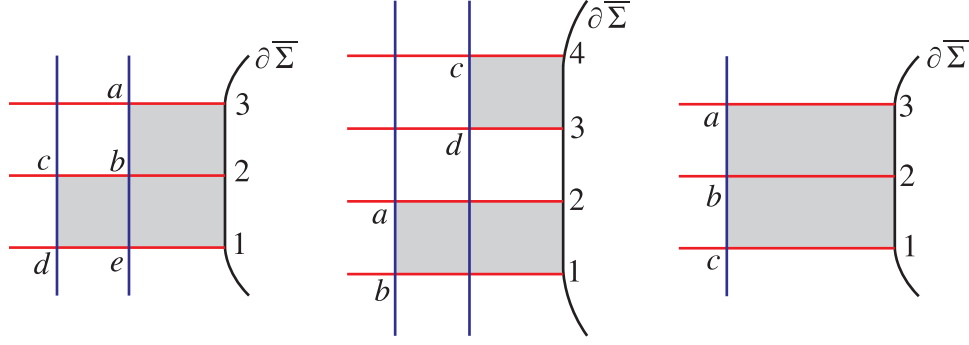


FIGURE 36. **Local illustrations of terms occurring the  $\mathcal{A}_\infty$ -compatibility relation for  $\widehat{CFA}$ .** The moduli spaces are the same as the moduli spaces in Figure 29 in Section 6.1, but the algebraic interpretations of the phenomena are different for  $\widehat{CFA}$  than for  $\widehat{CFD}$ .

This cancellation can be seen geometrically as follows. Consider the two-dimensional moduli space connecting  $\{b, d\}$  to  $\{a, c\}$  where the Reeb chord from 3 to 4 is encountered before the one from 1 to 2, then this moduli space has two ends, one of which is a two story building, while at the other end the heights of two Reeb chords are the same.

*Example 7.11.* On the right in Figure 36 is a piece of a diagram with three generators. The nontrivial products are

$$\begin{aligned} \{c\} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} &= \{b\} \\ \{c\} \cdot \begin{bmatrix} 1 \\ 3 \end{bmatrix} &= \{a\} \\ \{b\} \cdot \begin{bmatrix} 2 \\ 3 \end{bmatrix} &= \{a\}. \end{aligned}$$

Associativity follows from the fact that

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

Geometrically, at one end of the moduli spaces section is a two story building. At the other end, two levels collide and a split curve degenerates.

*Example 7.12.* Figure 37 shows an example in which the higher product  $m_3$  is needed for  $\mathcal{A}_\infty$ -associativity. The diagram has four generators,  $\{a, c\}$ ,  $\{a, d\}$ ,  $\{b, c\}$  and  $\{b, d\}$ . The differential is given by  $\partial\{a, c\} = \{a, d\}$  and  $\partial\{b, c\} = \{b, d\}$ . For an appropriate choice of complex structure  $J$ , there is a nontrivial product ( $m_2$ ) given by  $\{b, c\} \cdot \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \{a, c\}$ ; this corresponds to the domain and branch cuts indicated in the right of the figure. One can factor  $\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ . Obviously  $\{b, c\} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 0$ , so ordinary associativity fails. There is, however, for the same choice of complex structure  $J$  as above, a higher product

$$m_3(\{b, d\}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}) = \{a, c\}.$$

Thus, the  $\mathcal{A}_\infty$ -associativity condition is satisfied:

$$m_3((\partial\{b, c\}), \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}) + (\{b, c\} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix}) \cdot \begin{bmatrix} 2 \\ 3 \end{bmatrix} + \{b, c\} \cdot \begin{bmatrix} 1 \\ 3 \end{bmatrix} = 0.$$

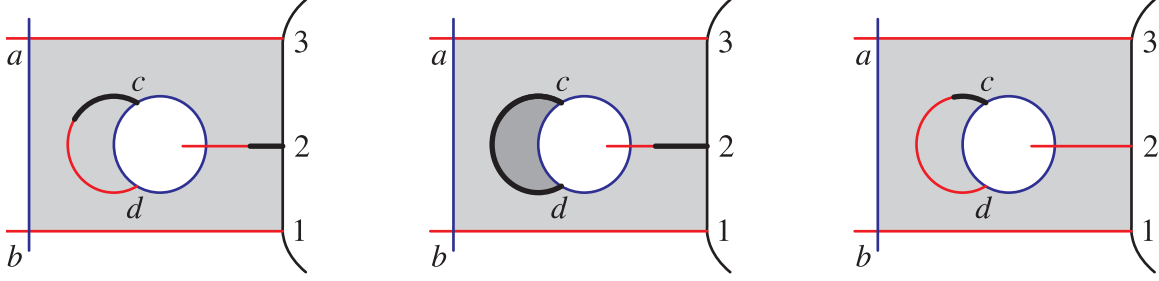


FIGURE 37. **A domain where  $m_3$  is necessary for associativity.**

The domain is shaded; the dark arcs denote branch cuts. The one parameter family is indicated on the left. In the center is a two-story end of the moduli space. At the right is the other end, corresponding to degenerating a split curve.

Geometrically, this corresponds to the one-parameter moduli space of curves asymptotic to two Reeb chords (at different heights), shown at the left of Figure 37. One end (pictured in the center) is a two-story building where one of the stories contains both Reeb chords. At the other end (pictured at the right), the two Reeb chords collide and a split curve degenerates.

*Proof of Proposition 7.8.* Morally, the proof proceeds by considering the ends of the index 2 moduli spaces. As in the type  $D$  module case, however, we will instead appeal to Theorem 5.41.

Fix generators  $\mathbf{x}$  and  $\mathbf{y}$ ,  $B \in \pi_2(\mathbf{x}, \mathbf{y})$ , and a sequence of sets of Reeb chords  $\vec{\rho} = (\rho_1, \dots, \rho_n)$  so that  $(x, \vec{\rho})$  is strongly boundary monotonic,  $B$  is compatible with  $\vec{\rho}$ , and  $\text{ind}(B, \vec{\rho}) = 2$ . Set  $\rho_i = \{\rho_{i,j}\}$ . Theorem 5.41 applied to the data  $(\mathbf{x}, \mathbf{y}, B, \vec{\rho})$  (for all sources  $S^\triangleright$  with  $\chi(S^\triangleright) = \chi_{\text{emb}}(B, \vec{\rho})$ ) shows that the sum of the following numbers is zero:

- (1) The number of two-story ends, i.e., the number of elements of

$$\mathcal{M}^{B_1}(\mathbf{x}, \mathbf{w}; \vec{\rho}_1) \times \mathcal{M}^{B_2}(\mathbf{w}, \mathbf{y}; \vec{\rho}_2)$$

where  $B = B_1 * B_2$  and  $\vec{\rho} = (\vec{\rho}_1, \vec{\rho}_2)$ .

- (2) The number of join curve ends, i.e., the number of elements of

$$\mathcal{M}^B(\mathbf{x}, \mathbf{y}; (\rho_1, \dots, \rho_i^{a,b}, \dots, \rho_n))$$

where  $\rho_{i,j} = \rho_a \uplus \rho_b$  and  $\rho_i^{a,b}$  obtained from  $\rho_i$  by replacing  $\rho_{i,j}$  with  $\{\rho_a, \rho_b\}$ .

- (3) The number of odd shuffle curve ends, i.e., the number of elements of

$$\mathcal{M}^B(\mathbf{x}, \mathbf{y}; \vec{\rho}')$$

where  $\vec{\rho}'$  is obtained from  $\vec{\rho}$  by performing a weak shuffle on one  $\rho_i$ .

- (4) The number of collisions of levels, i.e., the number of elements of

$$\mathcal{M}^B(\mathbf{x}, \mathbf{y}; \rho_1, \dots, \rho_i \uplus \rho_{i+1}, \dots, \rho_n)$$

where  $\rho_i$  and  $\rho_{i+1}$  are weakly composable.

We are also using Corollary 5.55 to ensure that the limiting curves are embedded.

The two-story ends correspond to the first term in Formula (7.7) by definition.

Next we discuss the second term in Formula (7.7). From Lemma 3.8 we see that, for a consistent set  $\boldsymbol{\rho}$  of Reeb chords,

$$\partial a(\boldsymbol{\rho}) = \sum_{\substack{\boldsymbol{\rho}' \text{ a splitting} \\ \text{of } \boldsymbol{\rho}}} a(\boldsymbol{\rho}') + \sum_{\substack{\boldsymbol{\rho}' \text{ a shuffle} \\ \text{of } \boldsymbol{\rho}}} a(\boldsymbol{\rho}').$$

On the other hand, sums (2) and (3) above correspond, respectively, to the sum over all weak splittings and weak shuffles of  $\boldsymbol{\rho}_i$ , one of the sets of Reeb chords in  $\vec{\boldsymbol{\rho}}$ . By Lemma 5.56, only proper (not weak) splittings or shuffles contribute, and therefore the second term in Formula (7.7) corresponds to sums (2) and (3) above.

Finally, sum (4) corresponds to the third term in Formula (7.7), as follows. Since, by Lemma 3.6,  $a(\boldsymbol{\rho}_i)a(\boldsymbol{\rho}_{i+1})$  is  $a(\boldsymbol{\rho}_i \uplus \boldsymbol{\rho}_{i+1})$  if  $\boldsymbol{\rho}_i$  and  $\boldsymbol{\rho}_{i+1}$  are composable, and is zero otherwise, the coefficient of  $\mathbf{y}$  in  $m_{n-1}(\mathbf{x}, a(\boldsymbol{\rho}_1), \dots, a(\boldsymbol{\rho}_i)a(\boldsymbol{\rho}_{i+1}), \dots, a(\boldsymbol{\rho}_n))$  is equal to sum (4) if  $\boldsymbol{\rho}_i$  and  $\boldsymbol{\rho}_{i+1}$  are composable, and is 0 otherwise. But by Lemma 5.56, sum (4) is also 0 if  $\boldsymbol{\rho}_i$  and  $\boldsymbol{\rho}_{i+1}$  are not composable, and otherwise gives the third term of Formula (7.7).  $\square$

We can also define a twisted theory  $\widehat{CFA}(\mathcal{H}, \mathfrak{s}; H_2(Y, \partial Y))$ , for  $\mathfrak{s}$  a  $\text{spin}^c$  structure on  $Y$ , in an entirely analogous way to Section 6.2.

**7.3. Gradings.** The gradings on the type  $A$  modules are quite similar to the gradings on the type  $D$  modules from Section 6.3, except that since  $\widehat{CFA}(\mathcal{H})$  is a right module, the gradings take values in a *right*  $G'(4k)$  set. As before, pick a base generator  $\mathbf{x}_0$  for each occupied  $\text{spin}^c$  structure  $\mathfrak{s}$ , and let

- $P(\mathbf{x}_0)$  be the group generated by  $g'(B)$  for  $B \in \pi_2(\mathbf{x}_0, \mathbf{x}_0)$ ,
- $G'_A(\mathcal{H}, \mathfrak{s})$  be the set of right cosets of  $P(\mathbf{x}_0)$ , and
- $\text{gr}'(\mathbf{x})$  be  $P(\mathbf{x}_0) \backslash g'(B)$  for an arbitrary  $B \in \pi_2(\mathbf{x}_0, \mathbf{x})$ .

It is clear that this definition satisfies an analogue of Proposition 6.17:

$$(7.13) \quad \text{gr}'(\mathbf{x})g'(B) = \text{gr}'(\mathbf{y})$$

where  $g'(B)$  is defined in Definition 6.13.

**Proposition 7.14.** *With this grading,  $\widehat{CFA}(\mathcal{H}, \mathfrak{s})$  is a graded right  $\mathcal{A}_\infty$ -module.*

*Proof.* From Definition 2.3 we see that we want to show that if  $a_1, \dots, a_\ell$  are homogeneous, then  $m_{\ell+1}(\mathbf{x}, a_1, \dots, a_\ell)$  is homogeneous of degree  $\lambda^{\ell-1} \text{gr}'(x) \text{gr}'(a_1) \cdots \text{gr}'(a_\ell)$ . Let  $B \in \pi_2(\mathbf{x}, \mathbf{y})$  and let  $\vec{\boldsymbol{\rho}} = (\boldsymbol{\rho}_1, \dots, \boldsymbol{\rho}_\ell)$  be a sequence of sets of Reeb chords compatible with  $B$ . Suppose that  $\text{ind}(B, \vec{\boldsymbol{\rho}}) = 1$  so that  $\mathbf{y}$  may appear as a term in  $m_{\ell+1}(\mathbf{x}, a(\boldsymbol{\rho}_1), \dots, a(\boldsymbol{\rho}_\ell))$ . Then by Definition 5.46,

$$(7.15) \quad 1 = \text{ind}(B, \vec{\boldsymbol{\rho}}) = e + n_{\mathbf{x}} + n_{\mathbf{y}} + \ell + \iota(\vec{\boldsymbol{\rho}}).$$

We therefore have

$$\begin{aligned} \prod_{i=1}^{\ell} \text{gr}'(a(\boldsymbol{\rho}_i)) &= (\iota(\vec{\boldsymbol{\rho}}), \partial^{\partial} B) \\ &= (1 - \ell - e - n_{\mathbf{x}} - n_{\mathbf{y}}, \partial^{\partial} B) \\ &= \lambda^{1-\ell} g'(B) \end{aligned}$$

where we apply, in turn, Lemma 5.45, Equation (7.15), and the definition of  $g'(B)$ . From this we see that

$$\mathrm{gr}'(\mathbf{y}) = \mathrm{gr}'(\mathbf{x})g'(B) = \lambda^{\ell-1} \mathrm{gr}'(\mathbf{x}) \mathrm{gr}'(a_1) \cdots \mathrm{gr}'(a_\ell). \quad \square$$

We can also define a grading on  $\widehat{CFD}(\mathcal{H}, \mathfrak{s})$  with values in  $G'(4k)$  by  $\mathrm{gr}'(e^B \mathbf{x}) := g'(B)$ .

As in Section 6.3, it is easy to define variants  $g$  of the gradings  $g'$  on  $\widehat{CFA}$  and  $\widehat{CFA}$ , which are gradings with respect to the smaller group  $G(\mathcal{Z})$ , taking values the  $G(\mathcal{Z})$ -set  $G_A = P(\mathbf{x}_0) \backslash G(\mathcal{Z})$ .

#### 7.4. Invariance.

**Theorem 7.16.** *Up to  $\mathcal{A}_\infty$ -homotopy equivalence, the  $\mathcal{A}_\infty$ -module  $\widehat{CFA}(\mathcal{H})$  depends only on the bordered three-manifold  $Y$  specified by the provincially admissible pointed bordered Heegaard diagram  $\mathcal{H}$ . That is, if  $\mathcal{H}'$  is another provincially admissible pointed bordered Heegaard diagram for  $Y$ , then  $\widehat{CFA}(\mathcal{H})$  and  $\widehat{CFA}(\mathcal{H}')$  are  $\mathcal{A}_\infty$ -homotopy equivalent ( $\mathcal{A}_\infty$ )  $\mathcal{A}(\mathcal{Z})$ -modules.*

As for invariance of  $\widehat{CFD}$ , the proof of Theorem 7.16 differs from the standard arguments in essentially two ways: the additional algebraic complications in dealing with  $\mathcal{A}_\infty$ -maps and homotopies, and the additional analytic complications in the case of handleslide invariance because of the presence of east  $\infty$ . Both of these issues are slightly more involved for  $\widehat{CFA}$  than for  $\widehat{CFD}$ . We will take a similar approach as for  $\widehat{CFD}$ : first we will explain succinctly the proof of invariance under change in almost complex structure, and then we will discuss the (mild) new complications in the triangle maps used to prove handleslide invariance.

**7.4.1. Change of complex structure.** We first address invariance of  $\widehat{CFA}$  under change in almost complex structure. The main purpose of this section is to relate the algebra of  $\mathcal{A}_\infty$ -maps and  $\mathcal{A}_\infty$ -homotopies to holomorphic curve counts; because this is somewhat more involved than the algebra of differential graded algebras required in Section 6.4.1, we will explain more steps in the proof.

Fix a Heegaard diagram  $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$  and generic, admissible almost complex structures  $J_0$  and  $J_1$ . The complex structures  $J_0$  and  $J_1$  lead to potentially different moduli spaces, and hence to two potentially different  $\mathcal{A}_\infty$ -modules  $\widehat{CFA}(\mathcal{H}; J_0)$  and  $\widehat{CFA}(\mathcal{H}; J_1)$ . We will show that these two  $\mathcal{A}_\infty$ -modules are  $\mathcal{A}_\infty$ -homotopy equivalent.

As in Section 6.4.1, call a (smooth) path  $\{J_r \mid r \in [0, 1]\}$  of almost complex structures from  $J_0$  to  $J_1$  *admissible* if each  $J_r$  is admissible. To the path  $J_r$  we can associate a single almost complex structure  $J$  on  $\Sigma \times [0, 1] \times \mathbb{R}$  by Formula (6.24), and define embedded moduli spaces  $\mathcal{M}(\mathbf{x}, \mathbf{y}; \vec{\rho}; J)$  by Formula (6.25).

Recall from Definition 2.4 that an  $\mathcal{A}_\infty$ -morphism between  $\mathcal{A}_\infty$ -modules  $M_1$  and  $M_2$  is a compatible family of maps

$$f_n: M_1 \otimes \overbrace{\mathcal{A} \otimes \cdots \otimes \mathcal{A}}^{n-1} \rightarrow M_2,$$

for  $n \in \mathbb{N}$ . So define an  $\mathcal{A}_\infty$ -map  $f^{J_r}$  from  $\widehat{CFA}(\mathcal{H}; J_0)$  to  $\widehat{CFA}(\mathcal{H}; J_1)$  by setting

$$f_n^{J_r}(\mathbf{x}, a(\boldsymbol{\rho}_1), \dots, a(\boldsymbol{\rho}_{n-1})) := \sum_{\mathbf{y} \in \mathfrak{S}(\mathcal{H})} \sum_{\substack{B \in \pi_2(\mathbf{x}, \mathbf{y}) \\ \text{ind}(B, \bar{\rho})=0}} \#(\mathcal{M}^B(\mathbf{x}, \mathbf{y}; \boldsymbol{\rho}_1, \dots, \boldsymbol{\rho}_{n-1}; J)) \mathbf{y}$$

$$f_n^{J_r}(\mathbf{x}, \dots, 1, \dots) := 0$$

and extending  $f^{J_r}$  to all of  $\widehat{CFA}(\mathcal{H}; J_0)$  multilinearly over  $\mathcal{I}$ .

Our first task is to show that  $f^{J_r}$  is an  $\mathcal{A}_\infty$ -morphism. We again use Proposition 6.26.

**Lemma 7.17.** *The map  $f^{J_r} = (f_n^{J_r})_{n \in \mathbb{N}}: \widehat{CFA}(\mathcal{H}; J_0) \rightarrow \widehat{CFA}(\mathcal{H}; J_1)$  is a strictly unital morphism of  $\mathcal{A}_\infty$ -modules, as defined in Definition 2.4.*

*Proof.*  $f^{J_r}$  is strictly unital by definition. So we need to check the  $\mathcal{A}_\infty$ -relation for morphisms, which we recall from Definition 2.4 (specialized for  $\mu_i = 0$  for  $i > 2$ ):

$$\begin{aligned} 0 = & \sum_{i+j=n+1} m_i^{J_1}(f_j(\mathbf{x}, a_1, \dots, a_{j-1}), \dots, a_{n-1}) \\ & + \sum_{i+j=n+1} f_i(m_j^{J_0}(\mathbf{x}, a_1, \dots, a_{j-1}), \dots, a_{n-1}) \\ & + \sum_{\ell=1}^{n-1} f_n(\mathbf{x}, a_1, \dots, a_{\ell-1}, \partial a_\ell, a_{\ell+1}, \dots, a_{n-1}) \\ & + \sum_{\ell=1}^{n-2} f_{n-1}(\mathbf{x}, a_1, \dots, a_{\ell-1}, a_\ell a_{\ell+1}, a_{\ell+2}, \dots, a_{n-1}). \end{aligned}$$

This follows in a similar manner to Proposition 7.8. The first sum corresponds to the two-story ends with a  $J$ -holomorphic curve followed by a  $J_1$ -holomorphic curve. The second sum corresponds to two-story ends with a  $J_0$ -holomorphic curve followed by a  $J$ -holomorphic curve. The third sum corresponds to join curve ends and odd shuffle curve ends. The fourth sum corresponds to collisions of two levels.

It follows from the index formula that  $f^{J_r}$  preserves the grading, as in Proposition 7.14.  $\square$

Similarly, fix a generic path  $J'_r$  of admissible almost complex structures with  $J'_0 = J_1$  and  $J'_1 = J_0$ . Let  $J'$  denote the associated complex structure on  $\Sigma \times [0, 1] \times \mathbb{R}$ . Then  $J'_r$  defines an  $\mathcal{A}_\infty$ -morphism  $f^{J'_r}: \widehat{CFA}(\mathcal{H}; J_1) \rightarrow \widehat{CFA}(\mathcal{H}; J_0)$ . We will show that  $f^{J_r}$  and  $f^{J'_r}$  are homotopy inverses to each other. This proof is again essentially standard, but as the algebra is unfamiliar we will give a few details.

For  $R \gg 0$ , let  $J \natural_R J'$  denote the almost complex structure on  $\Sigma \times [0, 1] \times \mathbb{R}$  given by

$$J \natural_R J'|_{(x,s,t)} = \begin{cases} J_0|_{(x,s,t)} & \text{if } R+1 \leq t \\ J'_{t-R}|_{(x,s,t-R)} & \text{if } R \leq t \leq R+1 \\ J_1|_{(x,s,t)} & \text{if } 1 \leq t \leq R \\ J_t|_{(x,s,t)} & \text{if } 0 \leq t \leq 1 \\ J_0|_{(x,s,t)} & \text{if } t \leq 0. \end{cases}$$

Associated to  $J \natural_R J'$  is an  $\mathcal{A}_\infty$ -morphism  $f^{J \natural_R J'} : \widehat{CFA}(\mathcal{H}; J_0) \rightarrow \widehat{CFA}(\mathcal{H}; J_0)$ .

**Lemma 7.18.** *For  $R$  sufficiently large the map  $f^{J \natural_R J'}$  is the  $\mathcal{A}_\infty$ -composite  $f^{J'_r} \circ f^{J_r}$ .*

*Proof.* Recall from Section 2.1 that  $f^{J'_r} \circ f^{J_r}$  has

$$(f^{J'_r} \circ f^{J_r})_n(\mathbf{x}, a_1, \dots, a_{n-1}) = \sum_{i+j=n+1} f_j^{J'_r} (f_i^{J_r}(\mathbf{x}, a_1, \dots, a_{i-1}), \dots, a_{n-1}).$$

The coefficient of  $\mathbf{y}$  in

$$f_j^{J'_r} (f_i^{J_r}(\mathbf{x}, a(\boldsymbol{\rho}_1), \dots, a(\boldsymbol{\rho}_{i-1})), \dots, a(\boldsymbol{\rho}_{n-1}))$$

counts holomorphic curves in  $\mathcal{M}^{B_1}(\mathbf{x}, \mathbf{w}; \vec{\rho}_{[1, i-1]}; J) \times \mathcal{M}^{B_2}(\mathbf{x}, \mathbf{w}; \vec{\rho}_{[i, n-1]}; J')$  where  $\mathbf{w} \in \mathfrak{S}(\mathcal{H})$ ,  $B_1 \in \pi_2(\mathbf{x}, \mathbf{w})$ ,  $B_2 \in \pi_2(\mathbf{w}, \mathbf{x})$ ,  $\text{ind}(B_1, \vec{\rho}_{[1, i-1]}) = 0$ , and  $\text{ind}(B_2, \vec{\rho}_{[i, n-1]}) = 0$ . It follows from obvious non- $\mathbb{R}$ -invariant analogues of Propositions 5.20 and 5.21 (compactness and gluing) that for  $R$  large enough such curves correspond to curves in

$$\bigcup_{B \in \pi_2(\mathbf{x}, \mathbf{y})} \mathcal{M}^B(\mathbf{x}, \mathbf{y}; \boldsymbol{\rho}_1, \dots, \boldsymbol{\rho}_{n-1}; J \natural_R J').$$

The result follows.  $\square$

We next define an  $\mathcal{A}_\infty$ -homotopy  $h$  between  $f^{J \natural_R J'}$  and the identity map. Fix a path of almost complex structures  $\{J^a\}$  on  $\Sigma \times [0, 1] \times \mathbb{R}$  with  $J^0 = J \natural_R J'$  and  $J^1 \equiv J_0$ , and such that each  $J^a$  satisfies conditions (1), (2) and (4) of Definition 5.1—i.e.,  $J^a$  is admissible except for not being  $\mathbb{R}$ -invariant. For a generic path  $J^a$  some  $B, S^\triangleright, P$  with  $\text{ind}(B, S^\triangleright, P) = -1$ , there are finitely many  $a \in (0, 1)$  for which  $\widetilde{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P; J^a)$  is nonempty. So set

$$\begin{aligned} \mathcal{M}^B(\mathbf{x}, \mathbf{y}; \vec{\rho}; \{J^a\}) &:= \bigcup_{a \in [0, 1]} \widetilde{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; \vec{\rho}; J^a) \\ h_n(\mathbf{x}, a(\boldsymbol{\rho}_1), \dots, a(\boldsymbol{\rho}_{n-1})) &:= \sum_{\mathbf{y} \in \mathfrak{S}(\mathcal{H})} \sum_{\substack{B \in \pi_2(\mathbf{x}, \mathbf{y}) \\ \text{ind}(B, \vec{\rho}) = -1}} \#(\mathcal{M}^B(\mathbf{x}, \mathbf{y}; \vec{\rho}; \{J^a\})) \mathbf{y}. \end{aligned}$$

In this context, the analogue of Theorem 5.41 is:

**Proposition 7.19.** *Suppose that  $(\mathbf{x}, \vec{\rho})$  satisfies the strong boundary monotonicity condition. Fix  $\mathbf{y}$ ,  $B \in \pi_2(\mathbf{x}, \mathbf{y})$ ,  $S^\triangleright$ , and  $\vec{P}$  such that  $[\vec{P}] = \vec{\rho}$  and  $\text{ind}(B, S^\triangleright, \vec{P}) = 0$ . Then, in the moduli space  $\bigcup_{a \in [0, 1]} \widetilde{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P}; J^a)$ , the total number of all*

- (1) elements of  $\widetilde{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P}; J^0)$ ;
- (2) elements of  $\widetilde{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P}; J^1)$ ;
- (3) two-story ends, with either
  - (a) a  $J_0$ -holomorphic curve followed by a  $J^a$ -holomorphic curve or
  - (b) a  $J^a$ -holomorphic curve followed by a  $J_0$ -holomorphic curve;
- (4) join curve ends;
- (5) odd shuffle curve ends; and
- (6) collisions of two levels  $P_i$  and  $P_{i+1}$  from  $\vec{P}$ , where  $P_i$  and  $P_{i+1}$  are composable

is even.

Using this, we prove:

**Lemma 7.20.** *The map  $h$  is an  $\mathcal{A}_\infty$ -homotopy between  $f^{J_{\mathbb{R}}J'}$  and the identity map.*

*Proof.* We must check that

$$(7.21) \quad f_1^{J_{\mathbb{R}}J'}(\mathbf{x}) - \mathbf{x} = h_1(m_1^{J_0}(\mathbf{x})) - m_1^{J_0}(h_1(\mathbf{x}))$$

and that for  $n > 1$ ,

$$(7.22) \quad \begin{aligned} f_n^{J_{\mathbb{R}}J'}(\mathbf{x}, a_1, \dots, a_{n-1}) = & \sum_{i+j=n+1} h_i(m_j^{J_0}(\mathbf{x}, a_1, \dots, a_{j-1}), \dots, a_{n-1}) \\ & + \sum_{i+j=n+1} m_i^{J_0}(h_j(\mathbf{x}, a_1, \dots, a_{j-1}), \dots, a_{n-1}) \\ & + \sum_{\ell=1}^{n-1} h_n(\mathbf{x}, a_1, \dots, \partial(a_\ell), \dots, a_{n-1}) \\ & + \sum_{\ell=1}^{n-2} h_{n-1}(\mathbf{x}, a_1, \dots, a_\ell a_{\ell+1}, \dots, a_{n-1}). \end{aligned}$$

As expected, this follows from Proposition 7.19. The  $f^{J_{\mathbb{R}}J'}$  terms in Formulas (7.21) and (7.22) correspond to item (1) of Proposition 7.19. The  $-\mathbf{x}$  term of Formula (7.21) corresponds to item (2), since  $J^1 \equiv J_0$  and for an  $\mathbb{R}$ -invariant complex structure the only index 0 holomorphic curves are trivial strips. As usual, two-story ends correspond to the remaining two terms in Formula (7.21) as well as the first two sums in Formula (7.22). The join curve shuffle curve ends account for the third sum in Formula (7.22). Collisions of two levels account for the final sum in Formula (7.22).  $\square$

Together with the obvious analogues for the composition  $f^{J_r} \circ f^{J'_r}$ , Lemmas 7.18 and 7.20 imply that the maps  $f^{J_r}$  and  $f^{J'_r}$  are homotopy inverses to each other, and hence that  $\widehat{CFA}(\mathcal{H}; J_0)$  and  $\widehat{CFA}(\mathcal{H}; J_1)$  are  $\mathcal{A}_\infty$ -homotopy equivalent.

**7.4.2. Handlesliding an  $\alpha$ -arc over an  $\alpha$ -circle.** The subtle cases of handleslide invariance are handlesliding an  $\alpha$ -circle over an  $\alpha$ -circle or an  $\alpha$ -arc over an  $\alpha$ -circle. The goal of this section is to explain briefly that no new analytic issues arise for  $\widehat{CFA}$ , as compared with  $\widehat{CFD}$ .

The main analytic issue in Section 6.4.2 relates to gluing bigons to triangles with corners at “ $+e$ -infinity” or “ $-e$ -infinity,” cf. the proof of Propositions 6.31 and 6.35. More precisely, one encounters triple Heegaard diagrams like  $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\alpha}^H, \boldsymbol{\beta})$  where  $\boldsymbol{\alpha}$  and  $\boldsymbol{\alpha}^H$  are as shown in Figure 34. One is considering holomorphic maps, say, of the form

$$u: (T, \partial T) \rightarrow (\Sigma \times \Delta, (\boldsymbol{\alpha} \times e_1) \cup (\boldsymbol{\alpha}^H \times e_2) \cup (\boldsymbol{\beta} \times e_3)).$$

where  $\Delta$  is a triangle with edges  $e_1$ ,  $e_2$  and  $e_3$ . Such maps can degenerate off a curve in

$$(\Sigma \times [0, 1] \times \mathbb{R}, (\boldsymbol{\alpha} \times \{0\} \times \mathbb{R}) \cup (\boldsymbol{\alpha}^H \times \{1\} \times \mathbb{R}))$$

asymptotic to a generator  $\theta_i$  at  $-\infty$  and a  $\alpha$ - $\alpha^H$  Reeb chord at  $+\infty$ . Gluing these curves to curves in  $\Sigma \times \Delta$  requires some kind of gluing statement at “ $+e$ -infinity.”

In general, such gluing is subtle. We got around this issue by filling in the puncture  $p$  in  $\Sigma$ , and observing that for the curves under consideration the gluing problem then

corresponds to the gluing of an ordinary 2-story holomorphic curve. This works for  $\widehat{CFA}$  as well:

**Lemma 7.23.** *Let  $\vec{\rho} = (\rho_1, \dots, \rho_n)$  be a consistent sequence of sets of Reeb chords,  $\mathbf{x}$  and  $\mathbf{y}$  generators of  $\widehat{CFA}(\Sigma, \alpha, \beta)$  and  $\widehat{CFA}(\Sigma, \alpha^H, \beta)$  respectively. Then the ends of the moduli space*

$$\mathcal{M}^B(\mathbf{x}, \mathbf{y}, \Theta_o; (\rho_1, \dots, \rho_i), (\rho_{i+1}, \dots, \rho_n))$$

*where  $\rho_i$  approaches the puncture are in bijection with the ends of*

$$\mathcal{M}^B(\mathbf{x}, \mathbf{y}, \Theta_o; (\rho_1, \dots, \rho_{i-1}), (\rho_i, \dots, \rho_n))$$

*where  $\rho_i$  approaches the puncture.*

*Proof.* The case when  $\rho_i$  has only one Reeb chord is covered in Section 6.4.2. We consider the case that  $\rho_i$  contains exactly two Reeb chords,  $\rho_1$  and  $\rho_2$ . When  $\rho_i$  approaches the puncture, the curve degenerates a curve  $u_S$  in  $(\Sigma, \alpha, \alpha^H, z)$  with two non-trivial components, each of which is either one of the small bigons or the annulus discussed in Section 6.4.2. There is also a leftover component  $u_T$  in  $\Sigma \times \Delta$ . For each of the components of  $u_S$  there is a matching bigon (or annulus). Together these give a family of curves  $u'_S$  in  $\Sigma \times [0, 1] \times \mathbb{R}$ —one can slide the two non-trivial components past each other.

Since gluing is local in the source, the trick of filling in  $p$  (as in Section 6.4.2) still applies to show that one can glue any member of the family of  $u'_S$  to  $u_T$  to obtain a new curve  $u$  in  $\Sigma \times \Delta$ . (Gluing different curves in the family is equivalent to gluing the two different components with two different gluing parameters.) We want to show that, algebraically, there is one sliding which yields a curve satisfying the height constraint imposed by  $\rho_i$ .

Let  $u_S^1$  (respectively  $u_S^2$ ) be the component of  $u'_S$  containing  $\rho_1$  (respectively  $\rho_2$ ). If we glue  $u_S^1$  first and then  $u_S^2$  we obtain a curve with  $\rho_1$  higher than  $\rho_2$  (in the obvious sense). If we glue  $u_S^2$  first and then  $u_S^1$  we obtain a curve with  $\rho_2$  higher than  $\rho_1$ . It follows that there is algebraically one curve with  $\rho_1$  and  $\rho_2$  at equal height.

The case that  $\rho_i$  has more than two Reeb chords is similar to the above, using an argument analogous to the one in Proposition 5.29.  $\square$

Using this lemma, the rest of the handleslide invariance proof follows as before.

## 8. PAIRING THEOREM VIA NICE DIAGRAMS

We now give a proof of the pairing theorem (Theorem 1.3), adapting methods from Sarkar and Wang [39].

Given a bordered Heegaard diagram  $\mathcal{H} = (\Sigma, \alpha, \beta, z)$ , by a *region* we mean a connected component of  $\bar{\Sigma} \setminus (\bar{\alpha} \cup \bar{\beta})$ . Let  $D_z$  denote the region containing  $z$ .

**Definition 8.1.** A bordered Heegaard diagram  $(\Sigma, \alpha, \beta, z)$  is called *nice* if every region in  $\bar{\Sigma}$  except for  $D_z$  is a (topological) disk with at most 4 corners. That is, each region in the interior of  $\Sigma$  is either a bigon or a quadrilateral and any region at the boundary of  $\Sigma$  except for  $D_z$  has two  $\alpha$ -arcs, one  $\beta$ -arc and one arc of  $\partial\bar{\Sigma}$  in its boundary.

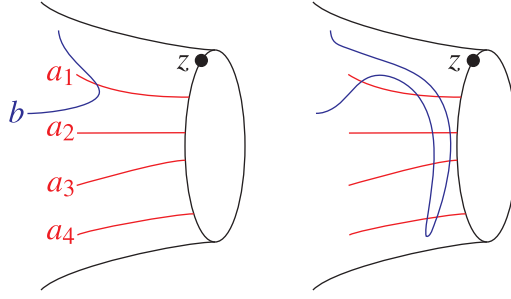


FIGURE 38. First step in making a Heegaard diagram with boundary nice.

**Proposition 8.2.** *Any Heegaard diagram with boundary  $(\Sigma, \alpha, \beta, z)$  can be turned into a nice Heegaard diagram with boundary  $(\Sigma, \alpha, \beta', z)$  via*

- isotopies of the  $\beta_i$  not crossing  $\partial\Sigma$  and
- handleslides among the  $\beta_i$ .

*Proof.* We start by doing a finger move as in Figure 38. That is, let  $D_z$  denote the region in  $\Sigma$  containing  $z$ . Let  $a_1, \dots, a_{4k}$  denote the ends of the  $\alpha$ -arcs intersecting  $\partial\Sigma$ , enumerated (say) clockwise around  $\partial\Sigma$ , with  $a_1$  and  $a_{4k}$  forming part of  $\partial D_z$ . Let  $b$  be the  $\beta$ -arc on the boundary of  $D_z$  which intersects  $a_1$ . Do a finger move pushing  $b$  consecutively through  $a_1, \dots, a_{4k}$ . In the resulting diagram, the regions  $R_1, \dots, R_{4k-1}$  adjacent to  $\partial\bar{\Sigma}$  other than  $D_z$  are all rectangles.

Now apply the Sarkar-Wang algorithm [39], which we briefly recall, to the resulting diagram. Assign to each region  $R$  its *distance*, which is the minimum number of  $\beta$ -arcs which a path in  $\Sigma \setminus \alpha$  connecting  $R$  and  $D_z$  must cross. A region is called *bad* if it is not a bigon or rectangle. The algorithm is inductive. At each step, choose a bad region  $R$  of maximal distance and with minimal badness among such regions. Let  $b_R$  be a  $\beta$ -arc separating  $R$  from a region of smaller distance. Do a finger move, pushing  $b_R$  through one of the  $\alpha$ -edges of  $R$ , and continue the finger move until reaching either a lower distance region, a bigon, another bad region, or the region  $R$  again. In the last case, the algorithm tells us to either choose a different edge through which to push  $b_R$  or perform an appropriate handleslide; otherwise, one repeats the process. The crucial point for us is that the finger move will never reach any of the regions  $R_1, \dots, R_{2n-1}$ . The proof in the closed case thus implies that the process eventually yields a nice Heegaard diagram with boundary.  $\square$

Nice diagrams are automatically admissible:

**Lemma 8.3.** *If  $\mathcal{H} = (\Sigma, \alpha, \beta, z)$  is nice then  $\mathcal{H}$  is admissible.*

*Proof.* The proof is exactly the same as in the closed case ([22, Corollary 3.2]). One shows that the region containing  $z$  occurs both to the left and to the right of each  $\beta$ -circle: if all the regions to the left (say) of some  $\beta_i$  are bigons or rectangles then one sees that the  $\beta$ -circles are homologically linearly dependent.  $\square$

Sarkar and Wang apply the index formula to show that for nice Heegaard diagrams (in the closed case), the differential counts only bigons and rectangles. This fact has the following analogue for nice Heegaard diagrams with boundary.

**Proposition 8.4.** *Let  $\mathcal{H}$  be a nice Heegaard diagram with boundary. Suppose that  $\text{ind}(B, \vec{\rho}) = 1$ . Then any holomorphic curve  $u \in \mathcal{M}^B(\mathbf{x}, \mathbf{y}; \vec{\rho})$  has one of the following three forms:*

- (1) *The source  $S$  of  $u$  consists of  $g$  bigons. Each bigon is either a trivial strip or has a unique puncture mapped to east  $\infty$ . Further,  $\vec{\rho}$  has only one part.*
- (2) *The source  $S$  consists of  $g$  bigons. All but one of the bigons are trivial strips, and the other is mapped by  $\pi_\Sigma \circ u$  to the interior of  $\Sigma$ .*
- (3) *The source  $S$  consists of  $g - 1$  bigons and one quadrilateral. All of the bigons are trivial strips, and the quadrilateral is mapped by  $\pi_\Sigma \circ u$  to the interior of  $\Sigma$ .*

*Conversely, suppose the homology class  $B$  and asymptotic data  $\vec{\rho}$  admit a topological map satisfying the condition of Case (1), (2), or (3), and  $\text{ind}(B, \vec{\rho}) = 1$ . Then there is a holomorphic map in  $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; \vec{\rho})$ , unique up to translation.*

*Proof.* By Formula (5.7), for a rigid holomorphic curve we have

$$1 = g - \chi(S) + 2e(D(u)) + |P|.$$

Since  $S$  has at most  $g$  connected components,  $\chi(S) \leq g$ . Since the Heegaard diagram is nice,  $e(D(u)) \geq 0$ . Consequently, one of the following three cases holds:

- $g = \chi(S)$ ,  $e(D(u)) = 0$  and  $|P| = 1$ . This corresponds to Case 1, above.
- $g = \chi(S)$ ,  $e(D(u)) = 1/2$  and  $|P| = 0$ . This corresponds to Case 2, above.
- $g - 1 = \chi(S)$ ,  $e(D(u)) = 0$  and  $|P| = 0$ . This corresponds to Case 3, above.

It is clear that in each case the holomorphic representative is unique. The converse, that such domains admit holomorphic representatives, is standard.  $\square$

**Corollary 8.5.** *Let  $\mathcal{H}$  be a nice Heegaard diagram with boundary, and  $\widehat{CFA} = \widehat{CFA}(\mathcal{H})$  the associated type A module. Then the higher  $\mathcal{A}_\infty$  operations  $m_i$  ( $i \geq 3$ ) vanish on  $\widehat{CFA}$ .*

*Proof.* The higher multiplications  $m_i$  ( $i \geq 3$ ) count rigid holomorphic curves in moduli spaces  $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; \vec{\rho})$  with  $|\vec{\rho}| > 1$ . But by Proposition 8.4, in a nice diagram there are no such curves.  $\square$

As a consequence of these remarks, we have a quick proof of Theorem 1.3.

*Proof of Theorem 1.3.* Let  $\mathcal{H}_1 = (\Sigma_1, \alpha_1, \beta_1, z)$  and  $\mathcal{H}_2 = (\Sigma_2, \alpha_2, \beta_2, z)$  be nice Heegaard diagrams with boundary for  $Y_1$  and  $Y_2$  respectively. Their union  $\mathcal{H} = (\bar{\Sigma}_1 \cup_{\partial} \bar{\Sigma}_2, \bar{\alpha}_1 \cup_{\partial} \bar{\alpha}_2, \beta_1 \cup \beta_2, z)$  is a Heegaard diagram for  $Y = Y_1 \cup_F Y_2$ . We want to show that  $\widehat{CF}(\mathcal{H}) \cong \widehat{CFA}(\mathcal{H}_1) \tilde{\otimes} \widehat{CFD}(\mathcal{H}_2)$ .

It is easy to see that  $\mathcal{H}$  is a nice diagram, since  $\mathcal{H}_1$  and  $\mathcal{H}_2$  are. Sarkar and Wang showed ([39]) that the differential for  $\widehat{CF}(\mathcal{H})$  counts embedded rectangles and bigons. (This is also a special case of Proposition 8.4.) We compare this complex with  $\widehat{CFA}(\mathcal{H}_1) \boxtimes \widehat{CFD}(\mathcal{H}_2)$  from Section 2.3, which by Proposition 2.20 serves as a model for the derived tensor product  $\tilde{\otimes}$ . (Note, in particular, that since  $\mathcal{H}_2$  is admissible,  $\widehat{CFD}(\mathcal{H}_2)$  is bounded.)

There is an obvious identification of generators of  $\widehat{CFA}(\mathcal{H}_1) \boxtimes \widehat{CFD}(\mathcal{H}_2)$  and  $\widehat{CF}(\mathcal{H})$ . Let  $\mathbf{x}_1 \otimes \mathbf{x}_2 \in \widehat{CFA}(\mathcal{H}_1) \boxtimes \widehat{CFD}(\mathcal{H}_2)$ .

By Proposition 8.4, the type  $D$  differential  $\partial(\mathbf{x}_2)$  can be written as  $\sum_{\mathbf{y}_2} a_{\mathbf{x}_2, \mathbf{y}_2} \otimes \mathbf{y}_2$ , where each term in  $a_{\mathbf{x}_2, \mathbf{y}_2}$  is either 1—coming from a provincial rectangle or bigon—or an algebra element in which a single strand moves. Indeed, applying Corollary 8.5, we see that the differential in  $\widehat{CFA}(\mathcal{H}_1) \boxtimes \widehat{CFD}(\mathcal{H}_2)$  is given by

$$\partial^{\boxtimes}(\mathbf{x}_1 \otimes \mathbf{x}_2) = (m_1(\mathbf{x}_1)) \otimes \mathbf{x}_2 + \sum_{\mathbf{y}_2} m_2(\mathbf{x}_1, a_{\mathbf{x}_2, \mathbf{y}_2}) \otimes \mathbf{y}_2.$$

The first term in the sum counts holomorphic bigons and rectangles contained entirely in  $(\Sigma_1, \boldsymbol{\alpha}_1, \boldsymbol{\beta}_1, z)$ . The second sum counts both rectangles and bigons contained entirely in  $(\Sigma_2, \boldsymbol{\alpha}_2, \boldsymbol{\beta}_2, z)$  (corresponding to terms of 1 in  $a_{\mathbf{x}_2, \mathbf{y}_2}$ ) and rectangles in  $(\bar{\Sigma}_1 \cup_{\partial} \bar{\Sigma}_2, \boldsymbol{\alpha}_1 \cup \boldsymbol{\alpha}_2, \boldsymbol{\beta}_1 \cup \boldsymbol{\beta}_2, z)$  crossing  $\partial \bar{\Sigma}_1$  (corresponding to terms with a single moving strand in  $a_{\mathbf{x}_2, \mathbf{y}_2}$ ). But, since  $\mathcal{H}$  is a nice diagram, this is exactly the differential on  $\widehat{CF}(\mathcal{H})$ .  $\square$

*Remark 8.6.* The idea to use the Sarkar-Wang algorithm to prove a gluing result is partly inspired by work of Juhász [15].

*Remark 8.7.* Since for  $\mathcal{H}_1$  a nice diagram  $\widehat{CFA}(\mathcal{H}_1)$  is an ordinary  $\mathcal{A}$ -module, we can also consider the ordinary tensor product  $\widehat{CFA}(\mathcal{H}_1) \otimes \widehat{CFD}(\mathcal{H}_2)$ . It is easy to see that this agrees, on the one hand, with  $\widehat{CFA}(\mathcal{H}_1) \boxtimes \widehat{CFD}(\mathcal{H}_2)$ . On the other hand, since  $\mathcal{H}_2$  is admissible it is straightforward to check that  $\widehat{CFD}(\mathcal{H}_2)$  is a projective  $\mathcal{A}$ -module, and thus  $\widehat{CFA}(\mathcal{H}_1) \otimes \widehat{CFD}(\mathcal{H}_2) \cong \widehat{CFA}(\mathcal{H}_1) \tilde{\otimes} \widehat{CFD}(\mathcal{H}_2)$ . This gives an alternative to the use of Proposition 2.20.

In fact, these methods prove a graded version of the pairing theorem. This will be spelled out in Section 9.4.

## 9. PAIRING THEOREM VIA TIME DILATION

Let  $F$  be a parameterized surface, and let  $Y_1$  and  $Y_2$  be two three-manifolds with compatible parameterized boundary:  $\partial Y_1 = F$  and  $\partial Y_2 = -F$ ; let  $Y = Y_1 \cup_F Y_2$ . In this section we give a second proof of Theorem 1.3, reconstructing  $\widehat{HF}(Y)$  from  $\widehat{CFA}(Y_1)$  and  $\widehat{CFD}(Y_2)$ . The idea here is to start from suitably admissible diagrams  $\mathcal{H}_1$  for  $Y_1$  and  $\mathcal{H}_2$  for  $Y_2$  which are glued together to form a Heegaard diagram  $\mathcal{H}$  for  $Y$ . The differential for  $Y$  is then deformed to the differential on the derived tensor product by *time dilation*.

More precisely, we construct a family of chain complexes  $\widehat{CF}(T; \mathcal{H}_1, \mathcal{H}_2)$  from fibered products of the moduli spaces for  $\mathcal{H}_1$  and  $\mathcal{H}_2$ , indexed by a real parameter  $T$  which, informally, speeds up the time parameter  $t$  by a factor of  $T$  on the type  $D$  side (i.e.,  $\mathcal{H}_2$ ), as compared to time on the type  $A$  side. Gluing theory identifies  $\widehat{CF}(1; \mathcal{H}_1, \mathcal{H}_2)$  with  $\widehat{CF}(\mathcal{H}_1 \cup_{\partial} \mathcal{H}_2)$ . A continuation argument shows that the chain homotopy type of the complex  $\widehat{CF}(T; \mathcal{H}_1, \mathcal{H}_2)$  is independent of  $T$ . A compactness argument shows that for all sufficiently large  $T$ , the behavior of the complex stabilizes. In the large  $T$  limit, the complex is then identified with the  $\boxtimes$ -product, in the sense of Definition 2.17, of the chain complexes  $\widehat{CFA}(\mathcal{H}_1)$  and  $\widehat{CFD}(\mathcal{H}_2)$ . Since

this in turn is identified with the derived tensor product (see Proposition 2.20), we deduce Theorem 1.3.

**9.1. Moduli of matched pairs.** Let  $\mathcal{H}_1 = (\Sigma_1, \alpha_1, \beta_1, z)$  and  $\mathcal{H}_2 = (\Sigma_2, \alpha_2, \beta_2, z)$  be provincially admissible Heegaard diagrams for  $Y_1$  and  $Y_2$ , respectively, inducing compatible markings of  $F$ ; i.e.,  $\partial\mathcal{H}_1 = -\partial\mathcal{H}_2$ . Let  $\Sigma = \bar{\Sigma}_1 \cup_{\partial} \bar{\Sigma}_2$ ,  $\alpha = \bar{\alpha}_1 \cup_{\partial} \bar{\alpha}_2$ , and  $\beta = \beta_1 \cup \beta_2$ . Also assume that one of  $\mathcal{H}_1$  or  $\mathcal{H}_2$  is admissible. Then  $(\Sigma, \alpha, \beta, z)$  is an admissible Heegaard diagram (denoted  $\mathcal{H}$ ) for  $Y$ . Recall that generators  $\mathfrak{S}(\mathcal{H})$  of  $\widehat{CF}(\mathcal{H})$  correspond to certain elements  $(\mathbf{x}_1, \mathbf{x}_2)$  of  $\mathfrak{S}(\mathcal{H}_1) \times \mathfrak{S}(\mathcal{H}_2)$ . Moreover,  $\pi_2(\mathbf{x}, \mathbf{y})$  is naturally a subset of  $\pi_2(\mathbf{x}_1, \mathbf{y}_1) \times \pi_2(\mathbf{x}_2, \mathbf{y}_2)$  by Lemma 4.21.

**Definition 9.1.** By a *compatible pair* of decorated sources we mean decorated sources  $S_1^\triangleright$  and  $S_2^\triangleright$  together with a bijection  $\varphi$  between  $E(S_1^\triangleright)$  and  $E(S_2^\triangleright)$ , such that for each puncture  $q_i$  of  $S_1^\triangleright$ , the Reeb chord labeling  $\varphi(q_i)$  is the orientation reverse of the Reeb chord labeling  $q_i$ . A compatible pair of sources  $S_1^\triangleright$  and  $S_2^\triangleright$  can be *preglued* to form a surface  $S_1^\triangleright \natural S_2^\triangleright$ .

**Definition 9.2.** Fix generators  $\mathbf{x}_1, \mathbf{y}_1 \in \mathfrak{S}(\mathcal{H}_1)$  and  $\mathbf{x}_2, \mathbf{y}_2 \in \mathfrak{S}(\mathcal{H}_2)$ , such that  $\mathbf{x} = \mathbf{x}_1 \cup \mathbf{x}_2$  and  $\mathbf{y} = \mathbf{y}_1 \cup \mathbf{y}_2$  are generators in  $\mathfrak{S}(\mathcal{H})$ , and homology classes  $B_1 \in \pi_2(\mathbf{x}_1, \mathbf{y}_1)$  and  $B_2 \in \pi_2(\mathbf{x}_2, \mathbf{y}_2)$  inducing a homology class  $B \in \pi_2(\mathbf{x}, \mathbf{y})$ . Fix compatible sources  $S_1^\triangleright$  and  $S_2^\triangleright$  connecting  $\mathbf{x}_1$  to  $\mathbf{y}_1$  and  $\mathbf{x}_2$  to  $\mathbf{y}_2$  respectively. Then define the *weak moduli space of matched pairs* to be the fibered product

$$\widetilde{\mathcal{MM}}_0^B(\mathbf{x}_1, \mathbf{y}_1, S_1^\triangleright; \mathbf{x}_2, \mathbf{y}_2, S_2^\triangleright) := \widetilde{\mathcal{M}}^{B_1}(\mathbf{x}_1, \mathbf{y}_1; S_1^\triangleright) \times_{\text{ev}_1=\text{ev}_2} \widetilde{\mathcal{M}}^{B_2}(\mathbf{x}_2, \mathbf{y}_2; S_2^\triangleright);$$

that is, pairs  $(u_1, u_2)$  with  $u_i \in \widetilde{\mathcal{M}}^{B_i}(\mathbf{x}_i, \mathbf{y}_i; S_i^\triangleright)$  such that  $\text{ev}(u_1) = \text{ev}(u_2)$ , under the correspondence induced by  $\varphi$ .

Define the index of a matched pair by

$$(9.3) \quad \begin{aligned} \text{ind}(B_1, S_1^\triangleright; B_2, S_2^\triangleright) &:= \text{ind}(B_1, S_1^\triangleright, P) + \text{ind}(B_2, S_2^\triangleright, P) - m \\ &= g_1 + g_2 + 2e(B_1) + 2e(B_2) - \chi(S_1) - \chi(S_2) + m, \end{aligned}$$

where  $P$  is the discrete partition on  $E(S_1^\triangleright)$  (or equivalently  $E(S_2^\triangleright)$ ) and  $m = |P| = |E(S_1^\triangleright)|$ . This is the expected dimension of the moduli space coming from its description as a fibered product of two moduli spaces over  $\mathbb{R}^m$ .

**Lemma 9.4.** *For generic, admissible almost complex structures on  $\Sigma_i \times [0, 1] \times \mathbb{R}$ , the weak moduli spaces of matched pairs  $\widetilde{\mathcal{MM}}_0^{B_1 \natural B_2}(\mathbf{x}_1, \mathbf{y}_1, S_1^\triangleright; \mathbf{x}_2, \mathbf{y}_2, S_2^\triangleright; \vec{P})$  are transversely cut out by the  $\bar{\partial}$ -equation and the evaluation map. Thus, the moduli space is a manifold whose dimension is given by  $\text{ind}(B_1, S_1^\triangleright; B_2, S_2^\triangleright)$ .*

*Proof.* This is an immediate application of Proposition 5.5 and the index formula, Formula (5.7), in the case when the partition is discrete.  $\square$

We wish to consider only strongly boundary monotonic moduli spaces.

**Definition 9.5.** Define the (strongly boundary monotonic) *moduli space of matched pairs*  $\widetilde{\mathcal{MM}}^B(\mathbf{x}_1, \mathbf{y}_1, S_1^\triangleright; \mathbf{x}_2, \mathbf{y}_2, S_2^\triangleright)$  to be the subset of  $\widetilde{\mathcal{MM}}_0^B(\mathbf{x}_1, \mathbf{y}_1, S_1^\triangleright; \mathbf{x}_2, \mathbf{y}_2, S_2^\triangleright)$  of pairs of curves  $(u_1, u_2)$  which are both strongly boundary monotonic.

**Lemma 9.6.** *Let  $\vec{\rho}$  be a sequence of sets of Reeb chords and let  $o_1, o_2 \subset [2k]$  be disjoint,  $k$ -element sets of occupied  $\alpha$ -arcs. If  $(o_1, \vec{\rho})$  and  $(o_2, -\vec{\rho})$  are both strongly boundary monotonic, then for each  $\rho_i \in \vec{\rho}$ ,  $M(\rho_i^-)$  and  $M(\rho_i^+)$  are disjoint.*

*Proof.* This follows the same argument as Lemma 6.7.  $\square$

**Lemma 9.7.** *For every  $\mathbf{x}_i, \mathbf{y}_i$ , and  $S_i^\triangleright$ , the strongly boundary monotonic moduli space of matched pairs  $\widetilde{\mathcal{MM}}^B(\mathbf{x}_1, \mathbf{y}_1, S_1^\triangleright; \mathbf{x}_2, \mathbf{y}_2, S_2^\triangleright)$  is a closed and open subset of the corresponding weak moduli space  $\widetilde{\mathcal{MM}}_0^B(\mathbf{x}_1, \mathbf{y}_1, S_1^\triangleright; \mathbf{x}_2, \mathbf{y}_2, S_2^\triangleright)$ .*

*Proof.* Let the two moduli spaces be  $\widetilde{\mathcal{M}}$  and  $\widetilde{\mathcal{M}}_0$ . Let  $(u_1, u_2)$  be a curve in  $\widetilde{\mathcal{M}}$ , and let  $\vec{P}$  be the induced ordered partition on  $E(S_1^\triangleright)$ . Since both  $(\mathbf{x}, [\vec{P}])$  and  $(\mathbf{y}, -[\vec{P}])$  are strongly boundary monotonic, it follows from Lemma 9.6 that any resolution of  $\vec{P}$  into a discrete ordered partition remains strongly boundary monotonic on both sides. Thus a neighborhood of  $(u_1, u_2)$  inside  $\widetilde{\mathcal{M}}_0$  consists entirely of curves which are strongly boundary monotonic, and so  $\widetilde{\mathcal{M}}$  is open. On the other hand, in general the limit of strongly boundary monotonic curves is strongly boundary monotonic, so  $\widetilde{\mathcal{M}}$  is closed.  $\square$

Translation induces an  $\mathbb{R}$ -action on  $\widetilde{\mathcal{MM}}^B(\mathbf{x}_1, \mathbf{y}_1, S_1^\triangleright; \mathbf{x}_2, \mathbf{y}_2, S_2^\triangleright)$ . This action is free except where both sides of the matching are trivial strips. When the action is free we say the moduli space is *stable* and denote the quotient  $\widetilde{\mathcal{MM}}^B(\mathbf{x}_1, \mathbf{y}_1, S_1^\triangleright; \mathbf{x}_2, \mathbf{y}_2, S_2^\triangleright)/\mathbb{R}$  by  $\mathcal{MM}^B(\mathbf{x}_1, \mathbf{y}_1, S_1^\triangleright; \mathbf{x}_2, \mathbf{y}_2, S_2^\triangleright)$ .

Given a compatible pair  $S_1^\triangleright$  and  $S_2^\triangleright$ , preglue them to form  $S^\diamond = S_1^\triangleright \natural S_2^\triangleright$ . Consider the corresponding moduli space for the diagram  $\mathcal{H}$  (representing the closed manifold  $Y$ ),  $\mathcal{M}^B(\mathbf{x}, \mathbf{y}, S^\diamond)$ . This moduli space has an expected dimension given by the index

$$\text{ind}(B, S^\diamond) = g - \chi(S) + 2e(B).$$

(This is [20, Formula (6)], which is a special case of Equation (5.7).) If  $|E(S_1^\triangleright)| = m$ , we have  $\chi(S_1 \natural S_2) = \chi(S_1) + \chi(S_2) - m$ . It follows from Formula (9.3) that the index of the moduli space of curves from  $\mathbf{x}$  to  $\mathbf{y}$  represented by the pre-glued source  $S^\diamond$  coincides with the index of the moduli space of matched pairs, i.e.,

$$\text{ind}(B_1, S_1^\triangleright; B_2, S_2^\triangleright) = \text{ind}(B_1 \natural B_2, S_1^\triangleright \natural S_2^\triangleright).$$

A stronger identification occurs at the level of moduli spaces. Specifically, given  $\mathbf{x}, \mathbf{y} \in \mathfrak{S}(Y)$ ,  $B \in \pi_2(\mathbf{x}, \mathbf{y})$ , and  $S^\diamond$  an appropriately decorated surface, let  $\widetilde{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}, S^\diamond)$  denote the moduli space of holomorphic curves with source  $S^\diamond$  in  $\Sigma \times [0, 1] \times \mathbb{R}$  connecting  $\mathbf{x}$  to  $\mathbf{y}$  (satisfying boundary monotonicity).

**Proposition 9.8.** *There are generic admissible complex structures on  $\Sigma \times [0, 1] \times \mathbb{R}$  and on  $\Sigma_i \times [0, 1] \times \mathbb{R}$  for  $i = 1, 2$  with the property that for all  $\mathbf{x}, \mathbf{y} \in \mathfrak{S}(\mathcal{H})$ ,  $B \in \pi_2(\mathbf{x}, \mathbf{y})$ , and source  $S^\diamond$  with  $\text{ind}(B, S^\diamond) = 1$ , the number of elements in  $\mathcal{M}^B(\mathbf{x}, \mathbf{y}, S^\diamond)$  is equal (modulo 2) to the number of elements in*

$$\bigcup_{S^\diamond = S_1^\triangleright \natural S_2^\triangleright} \mathcal{MM}^B(\mathbf{x}_1, \mathbf{y}_1, S_1^\triangleright; \mathbf{x}_2, \mathbf{y}_2, S_2^\triangleright),$$

where  $\mathbf{x} = \mathbf{x}_1 \cup \mathbf{x}_2$  and  $\mathbf{y} = \mathbf{y}_1 \cup \mathbf{y}_2$ .

*Proof.* The result follows from standard compactness and gluing techniques, in a completely analogous way to Propositions 5.20 and 5.22.  $\square$

As always, the curves of primary importance to us are the embedded ones. For a Heegaard diagram  $\mathcal{H}$  for a closed manifold with genus  $g$ ,  $\mathbf{x}, \mathbf{y} \in \mathfrak{S}(\mathcal{H})$ , and  $B \in \pi_2(\mathbf{x}, \mathbf{y})$ , define

$$\begin{aligned}\chi_{\text{emb}}(B) &:= g + e(B) - n_{\mathbf{x}}(B) - n_{\mathbf{y}}(B) \\ \text{ind}(B) &:= e(B) + n_{\mathbf{x}}(B) + n_{\mathbf{y}}(B).\end{aligned}$$

(This is a special case of Definition 5.46.)

**Definition 9.9.** The *embedded matched moduli space*  $\widetilde{\mathcal{MM}}^B(\mathbf{x}_1, \mathbf{y}_1; \mathbf{x}_2, \mathbf{y}_2)$  is the union of  $\widetilde{\mathcal{MM}}^B(\mathbf{x}_1, \mathbf{y}_1, S_1^\triangleright; \mathbf{x}_2, \mathbf{y}_2, S_2^\triangleright)$  over all pairs  $S_1^\triangleright, S_2^\triangleright$  with  $\chi(S_1^\triangleright \natural S_2^\triangleright) = \chi_{\text{emb}}(B)$ . (Note there are only finitely many terms in this union.) Also define  $\mathcal{MM}^B(\mathbf{x}_1, \mathbf{y}_1; \mathbf{x}_2, \mathbf{y}_2)$  to be the quotient of  $\widetilde{\mathcal{MM}}^B(\mathbf{x}_1, \mathbf{y}_1; \mathbf{x}_2, \mathbf{y}_2)$  by the  $\mathbb{R}$ -action.

It follows from Lemma 9.4 that  $\mathcal{MM}^B(\mathbf{x}_1, \mathbf{y}_1; \mathbf{x}_2, \mathbf{y}_2)$  is a manifold of dimension  $\text{ind}(B) - 1$ .

**Lemma 9.10.** *A matched pair of curves  $(u_1, u_2)$  is in the corresponding embedded matched moduli space  $\widetilde{\mathcal{MM}}^B(\mathbf{x}_1, \mathbf{y}_1; \mathbf{x}_2, \mathbf{y}_2)$  if and only if both  $u_i$  are embedded and every pair of Reeb chords appearing at the same height in  $u_1$  are either nested or disjoint.*

*Proof.* Let  $(u_1, u_2)$  be any matched pair of curves. Let  $S_i^\triangleright$  be the source of  $u_i$ , and let  $\vec{\rho}$  be sequence of sets of Reeb chords on  $u_1$ . Let  $m = |E(S_1^\triangleright)|$  and  $B_1 \natural B_2 = B$ . By Lemma 5.54,  $\chi(S_1) \geq \chi_{\text{emb}}(B_1, \vec{\rho})$  and  $\chi(S_2) \geq \chi_{\text{emb}}(B_2, -\vec{\rho})$ .

We claim that

$$\iota(\vec{\rho}) + \iota(-\vec{\rho}) \leq -m,$$

as follows. The  $L(\rho_i, \rho_j)$  terms in the definition of  $\iota$  (Formula (5.44)) contribute oppositely to  $\iota(\vec{\rho})$  and  $\iota(-\vec{\rho})$ , as the orientation on the circle is reversed, so the only contributions to  $\iota(\vec{\rho}) + \iota(-\vec{\rho})$  are from the  $\iota(\rho_i)$  terms. From Lemma 5.43 we see that each chord contributes  $-1$ , and each pair of chords  $\rho_1, \rho_2$  has a further negative contribution of  $-2|L([\rho_1], [\rho_2])|$ , from which the claim follows.

Thus

$$\begin{aligned}\chi(S_1^\triangleright \natural S_2^\triangleright) &= \chi(S_1) + \chi(S_2) - m \\ &\geq \chi_{\text{emb}}(B_1, \vec{\rho}) + \chi_{\text{emb}}(B_2, \vec{\rho}) - m \\ &= g + e(B) - n_{\mathbf{x}}(B) - n_{\mathbf{y}}(B) - \iota(\vec{\rho}) - \iota(-\vec{\rho}) - m \\ &\geq \chi_{\text{emb}}(B).\end{aligned}$$

The pair  $(u_1, u_2)$  is in the embedded moduli space if and only if we have equality, which is equivalent to both  $u_1$  and  $u_2$  being embedded and all  $|L([\rho_1], [\rho_2])|$  terms vanishing. These last terms vanish if and only if for all  $\rho_1$  and  $\rho_2$  on the same level,  $\rho_1$  and  $\rho_2$  are nested or disjoint, as desired.  $\square$

**Definition 9.11.** Let  $\widehat{CF}(\mathcal{H}_1, \mathcal{H}_2)$  be the  $\mathbb{F}_2$ -vector space with basis those  $\mathbf{x}_1 \times \mathbf{x}_2 \in \mathfrak{S}(\mathcal{H}_1) \times \mathfrak{S}(\mathcal{H}_2)$  so that  $\mathbf{x}_1 \cup \mathbf{x}_2$  is in  $\mathfrak{S}(\mathcal{H}_1 \cup_{\partial} \mathcal{H}_2)$ . Define a boundary operator  $\partial_1$  on  $\widehat{CF}(\mathcal{H}_1, \mathcal{H}_2)$  by

$$\partial_1(\mathbf{x}_1 \times \mathbf{x}_2) := \sum_{\mathbf{y}_1, \mathbf{y}_2} \sum_{\{B | \text{ind}(B)=1\}} \#(\mathcal{MM}^B(\mathbf{x}_1, \mathbf{y}_1; \mathbf{x}_2, \mathbf{y}_2)) \cdot (\mathbf{y}_1 \times \mathbf{y}_2).$$

Since  $\mathcal{H}_1 \cup_{\partial} \mathcal{H}_2$  is admissible by Lemma 4.22, the argument from Lemma 6.5 shows that this sum is finite.

**Theorem 9.12.**  $\widehat{CF}(\mathcal{H}_1, \mathcal{H}_2)$  is a chain complex isomorphic to the complex  $\widehat{CF}(\mathcal{H}_1 \cup_{\partial} \mathcal{H}_2)$ , for suitable choices of almost-complex structure.

*Proof.* According to [20, Theorem 2],  $\widehat{CF}(\mathcal{H}_1 \cup_{\partial} \mathcal{H}_2)$  can be calculated by the chain complex whose differential counts points in  $\mathcal{M}^B(\mathbf{x}, \mathbf{y}, S^{\triangleright})$ , where we take the union over all choices of  $B \in \pi_2(\mathbf{x}, \mathbf{y})$  and compatible  $S^{\triangleright}$  so that  $\text{ind}(S^{\triangleright}, B) = \text{ind}(B) = 1$ . According to Proposition 9.8, these counts agree with  $\partial_1$ .  $\square$

**9.2. Dilating time.** We can generalize the notion of matched pairs by inserting a real parameter  $T$ , as follows.

**Definition 9.13.** Fix a real number  $T > 0$ . Fix generators  $\mathbf{x}_1, \mathbf{y}_1 \in \mathfrak{S}(Y_1)$  and  $\mathbf{x}_2, \mathbf{y}_2 \in \mathfrak{S}(Y_2)$ . Fix compatible sources  $S_1^{\triangleright}$  and  $S_2^{\triangleright}$  connecting  $\mathbf{x}_1$  to  $\mathbf{y}_1$  and  $\mathbf{x}_2$  to  $\mathbf{y}_2$  respectively. The *weak moduli space of  $T$ -matched pairs*  $\widetilde{\mathcal{MM}}_0^B(T; \mathbf{x}_1, \mathbf{y}_1, S_1^{\triangleright}; \mathbf{x}_2, \mathbf{y}_2, S_2^{\triangleright})$  is defined to be the fibered product

$$\widetilde{\mathcal{M}}^{B_1}(\mathbf{x}_1, \mathbf{y}_1; S_1^{\triangleright}) \times_{T \cdot \text{ev}_1 = \text{ev}_2} \widetilde{\mathcal{M}}^{B_2}(\mathbf{x}_2, \mathbf{y}_2; S_2^{\triangleright});$$

that is, it consists of pairs  $(u_1, u_2)$  with  $u_i \in \widetilde{\mathcal{M}}^{B_i}(\mathbf{x}_i, \mathbf{y}_i; S_i^{\triangleright})$  and  $T \cdot \text{ev}(u_1) = \text{ev}(u_2)$ . Also define the *moduli space of  $T$ -matched pairs*  $\mathcal{MM}^B(T; \mathbf{x}_1, \mathbf{y}_1, S_1^{\triangleright}; \mathbf{x}_2, \mathbf{y}_2, S_2^{\triangleright})$  following Definition 9.5 and the *moduli space of embedded  $T$ -matched pairs*  $\mathcal{MM}^B(T; \mathbf{x}_1, \mathbf{y}_1; \mathbf{x}_2, \mathbf{y}_2)$  following Definition 9.9, except using  $T$ -matched pairs rather than matched pairs (which are the case  $T = 1$ ).

**Definition 9.14.** The moduli space of  $T$ -matched pairs has an  $\mathbb{R}$ -action defined by  $\tau_t(u_1, u_2) = (\tau_{T \cdot t}(u_1), \tau_t(u_2))$ . We let  $\mathcal{MM}^B(T; \mathbf{x}_1, \mathbf{y}_1, S_1^{\triangleright}; \mathbf{x}_2, \mathbf{y}_2, S_2^{\triangleright})$  denote the quotient of  $\widetilde{\mathcal{MM}}^B(T; \mathbf{x}_1, \mathbf{y}_1, S_1^{\triangleright}; \mathbf{x}_2, \mathbf{y}_2, S_2^{\triangleright})$  by this action, and similarly for  $\mathcal{MM}^B(T; \mathbf{x}_1, \mathbf{y}_1; \mathbf{x}_2, \mathbf{y}_2)$ .

Of course, the index of a  $T$ -matched pair does not depend on the  $T$ -parameter; it is still given by  $\text{ind}(B_1, S_1^{\triangleright}; B_2, S_2^{\triangleright})$  as defined in Equation (9.3).

**Lemma 9.15.** For generic admissible almost complex structures  $J_i$  on  $\Sigma_i \times [0, 1] \times \mathbb{R}$ , for all choices of  $\mathbf{x}_i, \mathbf{y}_i \in \mathfrak{S}(\mathcal{H}_i)$  and domain  $B \in \pi_2(\mathbf{x}_1 \cup \mathbf{x}_2, \mathbf{y}_1 \cup \mathbf{y}_2)$  so that  $B \neq 0$ , we have that

- for generic values of  $T$ ,  $\mathcal{MM}^B(T; \mathbf{x}_1, \mathbf{y}_1; \mathbf{x}_2, \mathbf{y}_2)$  is a manifold of dimension  $\text{ind}(B) - 1$ , and
- $\bigcup_{T>0} \mathcal{MM}^B(T; \mathbf{x}_1, \mathbf{y}_1; \mathbf{x}_2, \mathbf{y}_2)$  is a manifold of dimension  $\text{ind}(B)$ .

*Proof.* As in Lemma 9.4, the first statement follows from Proposition 5.5 and Equation (5.7), as well as Lemma 9.7. The proof of the second statement is standard; see, for instance, [28, Section 3.4] for a nice explanation of this type of argument.  $\square$

**Definition 9.16.** For  $T \in (0, \infty)$ , let  $\widehat{CF}(T; \mathcal{H}_1, \mathcal{H}_2)$  be the vector space with the same basis as  $\widehat{CF}(\mathcal{H}_1, \mathcal{H}_2)$  from Definition 9.11 and with boundary operator  $\partial_T: \widehat{CF}(T, \mathcal{H}_1, \mathcal{H}_2) \longrightarrow \widehat{CF}(T, \mathcal{H}_1, \mathcal{H}_2)$  defined by

$$\partial_T(\mathbf{x}_1 \times \mathbf{x}_2) := \sum_{\mathbf{y}_1, \mathbf{y}_2} \sum_{\{B | \text{ind}(B)=1\}} \#(\mathcal{MM}^B(T; \mathbf{x}_1, \mathbf{y}_1; \mathbf{x}_2, \mathbf{y}_2)) \cdot (\mathbf{y}_1 \times \mathbf{y}_2).$$

The next goal is to show that  $(\partial_T)^2 = 0$ . As usual, this requires considering the ends of  $\mathcal{MM}^B(T; \mathbf{x}_1, \mathbf{y}_1; \mathbf{x}_2, \mathbf{y}_2)$  when  $\text{ind}(B) = 2$ . To this end, we consider a compactification of  $\mathcal{MM}^B(T; \mathbf{x}_1, \mathbf{y}_1, S_1^\triangleright; \mathbf{x}_2, \mathbf{y}_2, S_2^\triangleright)$  by holomorphic combs, following Section 5.3.

**Definition 9.17.** Given  $\mathbf{x}_i, \mathbf{y}_i \in \mathfrak{S}(\mathcal{H}_i)$  for  $i = 1, 2$ , a  $T$ -matched story from  $(\mathbf{x}_1, \mathbf{y}_1)$  to  $(\mathbf{x}_2, \mathbf{y}_2)$  is a sequence  $(u_1, v_1, \dots, v_k, u_2)$  where  $u_i \in \mathcal{M}^B(\mathbf{x}_i, \mathbf{y}_i; S_i^\triangleright)$  and  $v_j \in \mathcal{N}(T_j^\diamond)$  for some sources  $S_i^\triangleright$  and  $T_j^\diamond$ , together with one-to-one correspondences between  $E(S_1^\triangleright)$  and  $W(T_1^\diamond)$ ,  $E(T_i^\diamond)$  and  $W(T_{i+1}^\diamond)$ , and  $E(T_k^\diamond)$  and  $E(S_2^\triangleright)$ , so that the correspondences preserve the labelings by Reeb chords, with orientation reversal between  $E(T_k^\diamond)$  and  $E(S_2^\triangleright)$ , and

$$\begin{aligned} \text{ev}(u_1) &= \text{ev}_w(v_1), \\ \text{ev}_e(v_i) &= \text{ev}_w(v_{i+1}), \text{ and} \\ T \cdot \text{ev}_w(v_k) &= \text{ev}(u_2). \end{aligned}$$

We call  $u_1$  the *west-most level* of the  $T$ -matched story and  $u_2$  the *east-most level* of the  $T$ -matched story. A  $T$ -matched story is *stable* if either  $u_1$  or  $u_2$  is stable. (In particular, if  $|E(S_1^\triangleright)| > 0$ , the comb is automatically stable.)

A  $T$ -matched comb of height  $N$  is a sequence of stable  $T$ -matched stories running from  $(\mathbf{x}_j, \mathbf{y}_j)$  to  $(\mathbf{x}_{j+1}, \mathbf{y}_{j+1})$  for some sequences of generators  $\mathbf{x}_j$  and  $\mathbf{y}_j$  (for  $j = 1, \dots, N+1$ ).

As in Section 5.3, there is a natural compactification  $\overline{\mathcal{MM}}^B(\mathbf{x}_1, \mathbf{y}_1, S_1^\triangleright; \mathbf{x}_2, \mathbf{y}_2, S_2^\triangleright)$  of  $\mathcal{MM}^B(\mathbf{x}_1, \mathbf{y}_1, S_1^\triangleright; \mathbf{x}_2, \mathbf{y}_2, S_2^\triangleright)$  inside the space of stable  $T$ -matched holomorphic combs. The following is the analogue of Proposition 5.32, and the proof is similar.

**Proposition 9.18.** *Suppose that  $\text{ind}(B_1, S_1^\triangleright; B_2, S_2^\triangleright) = 2$ . Then for generic  $J$ , every  $T$ -matched comb in  $\partial \overline{\mathcal{MM}}^B(T; \mathbf{x}_1, \mathbf{y}_1, S_1^\triangleright; \mathbf{x}_2, \mathbf{y}_2, S_2^\triangleright)$  has one of the following forms:*

- (1) *a two-story  $T$ -matched comb  $(u'_1, u'_2) * (u''_1, u''_2)$ ;*
- (2) *a  $T$ -matched comb  $(u_1, v, u_2)$  where the only non-trivial component of  $v$  is a join component; or*
- (3) *a  $T$ -matched comb  $(u_1, v, u_2)$  where the only non-trivial component of  $v$  is a split component.*

*Proof.* First suppose that we have a one-story  $T$ -matched comb  $(u_1, v_1, \dots, v_\ell, u_2)$  in the boundary of the moduli space, with source  $(S_1^\triangleright, T_1^\diamond, \dots, T_\ell^\diamond, S_2^\triangleright)$ . Let  $T^\diamond = T_1^\diamond \natural \dots \natural T_\ell^\diamond$ . By the index hypothesis,

$$2 = g - \chi(S_1^\triangleright) - \chi(S_2^\triangleright) + 2e(B) + m_0,$$

where  $m_0 = |E(S_1^\triangleright)| = |E(S_2^\triangleright)|$ . We also have

$$S_1^\triangleright \natural S_2^\triangleright \cong S_1^{\triangleright'} \natural T^\diamond \natural S_2^{\triangleright'}$$

and so

$$\chi(S_1^\triangleright) + \chi(S_2^\triangleright) - m_0 = \chi(S_1^{\triangleright'}) + \chi(T^\diamond) + \chi(S_2^{\triangleright'}) - m_1 - m_2$$

where  $m_1 = |E(S_1^{\triangleright'})|$  and  $m_2 = |E(S_2^{\triangleright'})|$ . Let  $k$  be the number of components of  $T^\diamond$ . Let  $P_i$  be the partition of  $E(S_i^{\triangleright'})$  according to the components of  $T^\diamond$ , so that  $|P_i| = k$ . Then an upper bound for the dimension of the space of limit curves for generic  $J$  is

$$\begin{aligned} \text{ind}(B_1, S_1^{\triangleright'}, P_1) + \text{ind}(B_2, S_2^{\triangleright'}, P_2) - k - 1 \\ &= g - \chi(S_1^{\triangleright'}) - \chi(S_2^{\triangleright'}) + 2e(B) + k - 1 \\ &= \chi(S_1^\triangleright) - \chi(S_1^{\triangleright'}) + \chi(S_2^\triangleright) - \chi(S_2^{\triangleright'}) - m_0 + k + 1 \\ &= (\chi(T^\diamond) - k) + (k - m_1) + (k - m_2) + 1. \end{aligned}$$

All three terms in parentheses are non-positive, so if the space of limit curves is non-empty, at most one can be negative. If all are equal to 0,  $T^\diamond$  consists of trivial strips and there is no degeneration. If  $\chi(T^\diamond) < k$ , one component of  $T^\diamond$  is an annulus, which is ruled out as in Proposition 5.32. If  $m_1 = k + 1$  or  $m_2 = k + 1$ , then  $T^\diamond$  is a join or split curve, respectively.

The arguments above also show that the index of a story in the degenerate curve is at least 1, and that if the index is 1 there are no components at east infinity. Thus the only other possibility for a degeneration is a two-story matched curve.  $\square$

In our setting, the matched combs with components at east infinity cancel in pairs, as boundary branch points move between  $\Sigma_1$  and  $\Sigma_2$ . Roughly speaking, the next proposition states that rigid  $T$ -matched combs  $(u_1, v, u_2)$  appear with multiplicity one in the boundaries of their corresponding one-dimensional moduli spaces.

**Proposition 9.19.** *Let  $(u_1, v, u_2)$  be a  $T$ -matched comb of index two with source  $(S_1^\triangleright, T^\diamond, S_2^\triangleright)$ , where  $T^\diamond$  has one non-trivial component, which is either a join or split component. Then there are arbitrarily small open neighborhoods  $U_1$  of  $(u_1, v, u_2)$  in*

$$\overline{\mathcal{M}}^B(\mathbf{x}_1, \mathbf{y}_1, S_1^\triangleright \natural T^\diamond) \times \overline{\mathcal{M}}^B(\mathbf{x}_2, \mathbf{y}_2, S_2^\triangleright)$$

*with the property that  $\partial \overline{U}_1$  meets  $\overline{\mathcal{M}}^B(T; \mathbf{x}_1, \mathbf{y}_1, S_1^\triangleright \natural T^\diamond; \mathbf{x}_2, \mathbf{y}_2, S_2^\triangleright)$  in an odd number of points. Similarly, there are arbitrarily small open neighborhoods  $U_2$  of  $(u_1, v, u_2)$  in*

$$\overline{\mathcal{M}}^B(\mathbf{x}_1, \mathbf{y}_1, S_1^\triangleright) \times \overline{\mathcal{M}}^B(\mathbf{x}_2, \mathbf{y}_2, T^\diamond \natural S_2^\triangleright)$$

*with the property that  $\partial \overline{U}_2$  meets  $\overline{\mathcal{M}}^B(T; \mathbf{x}_1, \mathbf{y}_1, S_1^\triangleright; \mathbf{x}_2, \mathbf{y}_2, T^\diamond \natural S_2^\triangleright)$  in an odd number of points.*

*Proof.* Suppose that  $v$  is a split curve. We label the west punctures of  $S_1^\triangleright \natural T^\diamond$  by  $\{w_i\}_{i=0}^\ell$  so that  $w_0$  and  $w_1$  are the ones which are in the same component in the split curve. Let  $\{e_i\}_{i=0}^\ell$  be the corresponding punctures of  $S_2^\triangleright$ . There are open neighborhoods  $N_1 \subset \overline{\mathcal{M}}^{B_1}(\mathbf{x}_1, \mathbf{y}_1, S_1^\triangleright \natural T^\diamond)$  of  $(u_1, v)$  and  $N_2 \subset \overline{\mathcal{M}}^{B_2}(\mathbf{x}_2, \mathbf{y}_2, S_2^\triangleright)$  of  $u_2$  which admit evaluation maps

$$\begin{aligned} \text{ev}^1: N_1 &\longrightarrow \mathbb{R}^\ell \\ \text{ev}^2: N_2 &\longrightarrow \mathbb{R}^\ell, \end{aligned}$$

where the  $i^{\text{th}}$  coordinate of  $\text{ev}^2$  is  $\text{ev}_{e_i, e_0}$ , and the  $i^{\text{th}}$  coordinate of  $\text{ev}^1$  is  $T \cdot \text{ev}_{w_i, w_0}$ . Let

$$f_i: N_i \longrightarrow \mathbb{R}$$

be the first component of  $\text{ev}^i$ , i.e., using the difference  $\text{ev}_{e_1} - \text{ev}_{e_0}$  in the case of  $f_2$ .

By the assumptions on the index,  $\dim N_1 + \dim N_2 = \ell + 1$ , so if we knew that the maps  $\text{ev}^1$  and  $\text{ev}^2$  were smooth it would follow that the images intersect in a 1-manifold, which would prove the result. The difficulty is that we do not know that  $\text{ev}^1$  is smooth on the boundary. Instead, we will use linking number considerations.

Let  $U$  be a neighborhood of  $u_1$  inside  $\overline{\mathcal{M}}^{B_1}(\mathbf{x}_1, \mathbf{y}_1, S_1^\circ)$ . Gluing (Proposition 5.22) gives a map

$$\gamma: U \times [0, \epsilon] \longrightarrow N_1 \subset \overline{\mathcal{M}}^{B_1}(\mathbf{x}_1, \mathbf{y}_1, S_1^\circ \natural T^\diamond)$$

which is a homeomorphism onto a neighborhood of  $(u_1, v)$ . We choose  $U$  so that

- $U$  is a ball, and
- $U$  is sufficiently small that  $\text{ev}^1(\overline{U} \times v) \cap \text{ev}^2(N_2)$  consists of a single point.

The second point is possible by transversality (Proposition 5.5).

In particular,  $\text{ev}^1(\partial \overline{U} \times v) \cap \text{ev}^2(N_2)$  is empty and hence, for sufficiently small  $\epsilon$ , the intersection  $\text{ev}^1(\gamma(\partial \overline{U} \times [0, \epsilon])) \cap \text{ev}^2(N_2)$  is also empty. Since  $\gamma(U \times \epsilon)$  stays away from the degenerate strata,  $f_1$  is positive on  $\gamma(U \times \epsilon)$ . Therefore, for sufficiently small  $\delta > 0$ ,  $f_2^{-1}((-\infty, \delta])$  is disjoint from  $\gamma(U \times \epsilon)$ . Consider now the open ball  $B_1 = \overline{U} \times [0, \epsilon]$  and choose an open ball  $B_2^\circ \subset N_2$  which is a neighborhood of  $u_2$  and contained in  $f_2^{-1}((-\infty, \delta])$ . Let  $B_2$  be the closure of  $B_2^\circ$ . Then  $\text{ev}^1(\partial B_1)$  intersects  $\text{ev}^2(B_2)$  only in the single point  $\text{ev}^1(u_1, v)$ , which is in  $\text{ev}^2(B_2^\circ)$ . Therefore linking number considerations show that  $\text{ev}^1(B_1) \cap \text{ev}^2(\partial B_2)$  consists of an odd number of points. We can therefore take  $U_1$  to be  $\gamma(U \times [0, \epsilon)) \times B_2^\circ$ .

A similar analysis holds for the other moduli spaces. □

**Proposition 9.20.** *The map  $\partial_T$  is a differential.*

*Proof.* Fix  $\mathbf{x}_i, \mathbf{y}_i \in \mathfrak{S}(\mathcal{H}_i)$  and a homology class  $B \in \pi_2(\mathbf{x}_1 \cup \mathbf{x}_2, \mathbf{y}_1 \cup \mathbf{y}_2)$  with  $\text{ind}(B) = 2$ . Consider the ends of the moduli space  $\overline{\mathcal{M}}^B(T; \mathbf{x}_1, \mathbf{x}_2; \mathbf{y}_1, \mathbf{y}_2)$  of embedded  $T$ -matched holomorphic curves representing  $B$ . According to Proposition 9.18, these ends correspond either to simple matched combs, which cancel in pairs according to Proposition 9.19, or to two-story holomorphic buildings, with each story an index one  $T$ -matched curve. Indeed, each possible two-story building appears as an end an odd number of times, according to an extension of Proposition 5.21 along the lines of Proposition 9.19. Hence, the number of such two-story buildings must be even. But the number of such terms (over all homology classes  $B$  with index two) is the  $\mathbf{y}_1 \times \mathbf{y}_2$  coefficient of  $(\partial_T)^2(\mathbf{x}_1 \times \mathbf{x}_2)$ . □

The chain complex  $\widehat{CF}(1; \mathcal{H}_1, \mathcal{H}_2)$  is  $\widehat{CF}(\mathcal{H}_1 \cup_\partial \mathcal{H}_2)$  by Theorem 9.12. We will next show that the homotopy type of  $\widehat{CF}(T; Y_1, Y_2)$  is in fact independent of the choice of  $T$ .

**Definition 9.21.** Let  $T_1$  and  $T_2$  be two positive real numbers, and fix a smooth function  $\psi: \mathbb{R} \longrightarrow \mathbb{R}$  with positive derivative so that

$$\psi(t) = \begin{cases} T_1 \cdot t & t \leq -1 \\ T_2 \cdot t & t \geq 1. \end{cases}$$

There are induced maps  $\psi^m: \mathbb{R}^m \longrightarrow \mathbb{R}^m$  defined by  $\psi^m(t_1, \dots, t_m) = (\psi(t_1), \dots, \psi(t_m))$ . Fix generators  $\mathbf{x}_1, \mathbf{y}_1 \in \mathfrak{S}(Y_1)$  and  $\mathbf{x}_2, \mathbf{y}_2 \in \mathfrak{S}(Y_2)$ . Fix compatible sources  $S_1^\triangleright$  and  $S_2^\triangleright$  connecting  $\mathbf{x}_1$  to  $\mathbf{y}_1$  and  $\mathbf{x}_2$  to  $\mathbf{y}_2$  respectively. The *moduli space of  $\psi$ -matched pairs*, denoted

$$\mathcal{M}^{B_1 \natural B_2}(\psi; \mathbf{x}_1, \mathbf{y}_1, S_1^\triangleright; \mathbf{x}_2, \mathbf{y}_2, S_2^\triangleright),$$

is defined as the fibered product

$$\mathcal{M}^{B_1}(\mathbf{x}_1, \mathbf{y}_1; S_1^\triangleright) \times_{\psi^m \circ \text{ev}_1 = \text{ev}_2} \mathcal{M}^{B_2}(\mathbf{x}_2, \mathbf{y}_2; S_2^\triangleright),$$

where  $m = |E(S_1^\triangleright)|$ . That is, the moduli space consists of pairs  $(u_1, u_2)$  with  $u_i \in \mathcal{M}^{B_i}(\mathbf{x}_i, \mathbf{y}_i; S_i^\triangleright)$  such that  $\psi^m \circ \text{ev}(u_1) = \text{ev}(u_2)$ . The space of  $\psi$ -matched pairs has a natural compactification by  *$\psi$ -matched combs*, defined analogously to Definition 9.17, except using the map induced by  $\psi$  rather than simply rescaling on the left side.

The moduli spaces of  $\psi$ -matched pairs do not in general admit a natural action by  $\mathbb{R}$ .

For  $\psi$  as above, define

$$F_{T_1, T_2}: \widehat{CF}(T_1; \mathcal{H}_1, \mathcal{H}_2) \longrightarrow \widehat{CF}(T_2; \mathcal{H}_1, \mathcal{H}_2)$$

by

$$F_{T_1, T_2}(\mathbf{x}_1 \times \mathbf{x}_2) = \sum_{\mathbf{y}_1, \mathbf{y}_2} \sum_{\{B | \text{ind}(B)=0\}} \#(\mathcal{M}^B(\psi; \mathbf{x}_1, \mathbf{y}_1; \mathbf{x}_2, \mathbf{y}_2)) \cdot (\mathbf{y}_1 \times \mathbf{y}_2).$$

We sketch now familiar arguments (generalizing the above proof of Proposition 9.20) which show that  $F_{T_1, T_2}$  is a chain map.

**Lemma 9.22.** *Suppose that  $\mathcal{M}^B(\psi; \mathbf{x}_1, \mathbf{y}_1, S_1^\triangleright; \mathbf{x}_2, \mathbf{y}_2, S_2^\triangleright)$  is 1-dimensional. Then for generic  $J$ , every  $\psi$ -matched comb in  $\partial \mathcal{M}^B(\psi; \mathbf{x}_1, \mathbf{y}_1, S_1^\triangleright; \mathbf{x}_2, \mathbf{y}_2, S_2^\triangleright)$  has one of the following forms:*

- (1) a two-story matched curve  $(u'_1, u'_2) * (u''_1, u''_2)$ , where  $(u'_1, u'_2)$  is a  $\psi$ -matched curve and  $(u''_1, u''_2)$  is a  $T_2$ -matched curve;
- (2) a two-story matched curve  $(u'_1, u'_2) * (u''_1, u''_2)$ , where  $(u'_1, u'_2)$  is a  $T_1$ -matched curve and  $(u''_1, u''_2)$  is a  $\psi$ -matched curve;
- (3) a  $\psi$ -matched comb  $(u_1, v, u_2)$  where the only non-trivial component of  $v$  is a join component; or
- (4) a  $\psi$ -matched comb  $(u_1, v, u_2)$  where the only non-trivial component of  $v$  is a split component.

*Proof.* This follows along the same lines of Proposition 9.18 above.  $\square$

**Proposition 9.23.**  *$F_{T_1, T_2}$  induces a chain map from  $\widehat{CF}(T_1; \mathcal{H}_1, \mathcal{H}_2)$  to  $\widehat{CF}(T_2; \mathcal{H}_1, \mathcal{H}_2)$ . Indeed,  $F_{T_1, T_2}$  is a chain homotopy equivalence.*

*Proof.* For fixed  $\mathbf{x}_i, \mathbf{y}_i \in \mathfrak{S}(\mathcal{H}_i)$  and homology class  $B$  with  $\text{ind}(B) = 1$ , consider the moduli space  $\mathcal{M}^B(\psi; \mathbf{x}_1, \mathbf{y}_1; \mathbf{x}_2, \mathbf{y}_2)$ . Lemma 9.22 accounts for the ends of these moduli spaces. A straightforward adaptation of Proposition 9.19 shows that the ends enumerated in Lemma 9.22 which do not *a priori* cancel in pairs are the two-story  $\psi$ -matched curves. Gluing as in the proof of Proposition 9.20 shows that those ends are in one-to-one correspondence with the types of two-story buildings enumerated

in Lemma 9.22. But these counts, which must add up to zero, are precisely the coefficients of  $\mathbf{y}_1 \times \mathbf{y}_2$  in  $\partial_{T_2} \circ F_{T_1, T_2}(\mathbf{x}_1 \times \mathbf{x}_2) + F_{T_1, T_2} \circ \partial_{T_1}(\mathbf{x}_1 \times \mathbf{x}_2)$ .

Showing that  $F_{T_1, T_2}$  is a chain homotopy equivalence also follows along familiar lines in Floer homology. The homotopy inverse is provided by  $F_{T_2, T_1}$  defined with a suitable function  $\psi_{21}$ , and the chain homotopy to the identity is provided by counting curves matched by a suitable family of functions, depending on a parameter  $c$ , interpolating between multiplication by  $T_1$  and a function which agrees with  $\psi(t)$  for  $t$  sufficiently small, and  $\psi_{21}(-t + c)$  for  $t$  sufficiently large.  $\square$

**9.3. Dilating to infinity.** Now that we have established suitable independence of  $\widehat{CF}(T; \mathcal{H}_1, \mathcal{H}_2)$  from  $T$ , we turn to the large  $T$  behavior of this complex. As  $T$  goes to infinity, we see ideal objects as follows.

**Definition 9.24.** Let  $\overline{\mathbb{R}} := [-\infty, +\infty]$  be the standard compactification of  $\mathbb{R}$ . For  $U$  a holomorphic comb of height  $N$ , define its *time-parameter space*  $T(U)$  to be the union of  $N$  copies of  $\overline{\mathbb{R}}$  modulo identifying  $+\infty$  in the  $i^{\text{th}}$  copy with  $-\infty$  in the  $(i+1)^{\text{st}}$  copy:

$$T(U) := \left( \bigcup_{i=1}^N \overline{\mathbb{R}}_i \right) / ((+\infty)_i \sim (-\infty)_{i+1}).$$

When  $U$  is the trivial comb ( $N = 0$ ), let  $T(U)$  be a single point. Then an *ideal matched holomorphic comb* is

- a pair  $(U_1, U_2)$  of stable holomorphic combs,
- an order-preserving map  $\Phi$  from the set of stories of  $U_2$  to  $T(U_1)$ , and
- a correspondence  $\varphi$  from  $E(U_2)$  to  $E(U_1)$ ,

so that, for every east puncture  $p$  of  $U_2$ ,

- if  $p$  is labeled by the Reeb chord  $\rho$ , then  $\varphi(p)$  is labeled by  $-\rho$  and
- if  $p$  is a puncture on the story  $C$  of  $U_2$ , the  $t$ -coordinate of  $\varphi(p)$  is  $\Phi(C)$ .

As usual, we will often suppress  $\varphi$  from the notation. Let the *total west source*  $S_1^\triangleright$  and *total east source*  $S_2^\triangleright$  of  $(U_1, U_2)$  be the result of pregluing all the sources of  $U_1$  and  $U_2$ , respectively. We view such an ideal matched comb as a point in

$$\overline{\mathcal{M}}^{B_1}(\mathbf{x}_1, \mathbf{y}_1; S_1^\triangleright; \vec{P}_1) \times \overline{\mathcal{M}}^{B_2}(\mathbf{x}_2, \mathbf{y}_2; S_2^\triangleright; P_2)$$

where  $\mathbf{x}_i, \mathbf{y}_i$ , and  $B_i$  are as usual,  $\vec{P}_1$  is the induced partition on  $E(S_1^\triangleright)$ , and  $P_2$  is the discrete partition on  $E(S_2^\triangleright)$ .

In the definition,  $U_1$  or  $U_2$  may be of height 0, in which case the other is an ordinary provincial comb.

**Definition 9.25.** Let  $\{U^j\}_{j=1}^\infty$  be a sequence of  $T_j$ -matched combs, where  $T_j \rightarrow \infty$ . Let  $(U_1, U_2)$  be an ideal matched holomorphic comb. From each  $T_j$ -matched comb  $U^j$  extract a pair of combs  $u_1^j$  and  $u_2^j$  by first taking  $u_2^j$  to consist of the east-most levels of each story and  $u_1^j$  to be the rest of each story and then throwing out any stories consisting of trivial strips. We say that the sequence  $\{U^j\}$  *converges* to  $(U_1, U_2)$  if the following conditions are satisfied:

- The sequences  $\{u_1^j\}$  and  $\{u_2^j\}$  converge to  $U_1$  and  $U_2$ , respectively.

- For sufficiently large  $j$  the identification between the east-most punctures of  $u_1^j$  and the east punctures of  $u_2^j$  corresponds to the identification between the east-most punctures of  $U_1$  and the east-most punctures of  $U_2$ .
- For each story  $C$  of  $U_2$ ,  $\Phi(C)$  is  $r \in T(U_1)$  determined as follows. Place marked points  $p_j$  on the source of  $u_2^j$  so that the sequence  $\{p_j\}$  converges to a point on the source of  $C$ . Choose marked points  $q_j$  on the source of  $u_1^j$  so that

$$T_j \cdot t(u_1^j(q_j)) = t(u_2^j(p_j)).$$

Then  $r$  is the limit of  $t(u_1^j(q_j))$  (which may be  $\pm\infty$ ).

**Proposition 9.26.** *An sequence  $\{(u_1^j, u_2^j)\}_{j=1}^\infty$  of  $T_j$ -matched holomorphic combs with  $T_j \rightarrow \infty$  has a subsequence which converges to an ideal matched holomorphic comb.*

*Proof.* This is clear from the definitions and the compactness theorem for holomorphic combs, Proposition 5.20.  $\square$

*Remark 9.27.* Observe that the limit  $(U_1, U_2)$  is not necessarily unique. To make it unique, we should identify ideal matched combs  $(U_1, U_2)$  and  $(U'_1, U'_2)$  where  $U'_2$  is gotten by deleting some collection of eastern levels of  $U_2$  and  $U'_1$  is gotten by annexing those eastern levels to  $U_1$ .

**Definition 9.28.** An *ideal matched curve* is an ideal matched comb  $(U_1, U_2)$  in which

- $U_1$  has at most one story,
- $U_2$  has no components at east infinity, and
- every component of  $U_1$  at east infinity is a disk with a single west puncture.

In the above definition, the east infinity components of  $U_1$  are like split components, except that there may be any number of east punctures. Note that the comb  $U_2$  can have many stories.

**Lemma 9.29.** *Let  $\mathbf{x}_i, \mathbf{y}_i \in \mathfrak{S}(\mathcal{H}_i)$ ,  $B_i \in \pi_2(\mathbf{x}_i, \mathbf{y}_i)$ , and  $S_i^\triangleright$  be an appropriate source for  $i = 1, 2$ . Let  $\mathcal{M}$  be the space of stable ideal matched combs from  $\mathbf{x}_1 \times \mathbf{x}_2$  to  $\mathbf{y}_1 \times \mathbf{y}_2$  with total east source  $S_1^\triangleright$  and total west source  $S_2^\triangleright$ . If  $\text{ind}(B_1, S_1^\triangleright; B_2, S_2^\triangleright) < 1$ , then  $\mathcal{M}$  is empty. If  $\text{ind}(B_1, S_1^\triangleright; B_2, S_2^\triangleright) = 1$ , then  $\mathcal{M}$  consists of ideal matched curves  $(U_1, U_2)$ , and exactly one of the two following conditions holds.*

- (1)  $E(S_1^\triangleright) = E(S_2^\triangleright) = \emptyset$  and one of  $U_1$  or  $U_2$  is the trivial comb of height 0, the other having index one.
- (2) Each story of  $U_2$  has index one (with the discrete partition), and  $U_1$  has index one (with the induced partition).

In particular, in the second case  $U_2$  has no components at east infinity, although  $U_1$  may. Define a *simple ideal matched curve* to be an ideal matched curve satisfying one of the two conditions above.

*Proof.* This is similar to the proofs of Propositions 5.32 and 9.18. The case with no east punctures is immediate. Otherwise, let  $(U_1, U_2) \in \mathcal{M}$ . Suppose first that  $U_1$  has a single story, and for notational simplicity assume that  $U_2$  also has a single story, so that all east punctures of  $U_1$  appear at a single height. Let  $S_1^{\triangleright'}$  and  $S_2^{\triangleright'}$  be the sources of the main components of  $U_1$  and  $U_2$ , respectively, and similarly let  $T_1^\diamond$  and  $T_2^\diamond$  be the sources of the east components of  $U_1$  and  $U_2$ . Let  $k_2$  be the number of

components of  $T_2^\diamond$ ,  $m_0 = |E(S_1^\triangleright)| = |E(S_2^\triangleright)|$ , and  $m_i = |E(S_i^{\triangleright'})|$  for  $i = 1, 2$ . For generic  $J$ , an upper bound for the dimension of  $\mathcal{M}$  is then

$$\begin{aligned} & \text{ind}(B_1, S_1^{\triangleright'}, P_1') + \text{ind}(B_2, S_2^{\triangleright'}, P_2') - 2 \\ &= g + 2e(B) - \chi(S_1^{\triangleright'}) - \chi(S_2^{\triangleright'}) + k_2 - 1 \\ &= (\chi(T_1^\diamond) - m_1) + (\chi(T_2^\diamond) - m_0) + (k_2 - m_2) + \text{ind}(B_1, S_1^\triangleright; B_2, S_2^\triangleright) - 1. \end{aligned}$$

When  $\text{ind}(B_1, S_1^\triangleright; B_2, S_2^\triangleright) < 1$ , the sum is negative and so there are no ideal matched combs. When  $\text{ind}(B_1, S_1^\triangleright; B_2, S_2^\triangleright) = 1$ , all three terms in parentheses must be 0, which implies that  $(U_1, U_2)$  is an ideal matched curve as stated.

The case when  $U_2$  has multiple stories is similar. If  $U_1$  has more than one story, each story must have index at least one by the above analysis, contradicting the index assumption.  $\square$

Now define a moduli space

$$\mathcal{MM}^B(\geq T_0; \mathbf{x}_1, \mathbf{y}_1, S_1^\triangleright; \mathbf{x}_2, \mathbf{y}_2, S_2^\triangleright) := \bigcup_{T \geq T_0} (T, \mathcal{MM}^B(T, \mathbf{x}_1, \mathbf{y}_1, S_1^\triangleright; \mathbf{x}_2, \mathbf{y}_2, S_2^\triangleright)).$$

**Proposition 9.30.** *For all  $\mathbf{x}_i, \mathbf{y}_i, B_i$ , and  $S_i^\triangleright$  with  $\text{ind}(B_1, S_1^\triangleright; B_2, S_2^\triangleright) = 0$  and the sources  $S_i^\triangleright$  not both trivial strips, there is a  $T_0$  so that  $\mathcal{MM}^{B_1 \natural B_2}(\geq T_0; \mathbf{x}_1, \mathbf{y}_1, S_1^\triangleright; \mathbf{x}_2, \mathbf{y}_2, S_2^\triangleright)$  is empty.*

*Proof.* Suppose we have a sequence of  $T_i$ -matched curves  $U_i \in \mathcal{MM}^B(T_i; \mathbf{x}_1, \mathbf{y}_1, S_1^\triangleright; \mathbf{x}_2, \mathbf{y}_2, S_2^\triangleright)$  with  $T_i$  approaching  $\infty$ . By Proposition 9.26, there is be a subsequence converging to an ideal matched building, but by Lemma 9.29 this is impossible for stable curves in index 0.  $\square$

**Proposition 9.31.** *For generic  $T$  sufficiently large,  $\mathbf{x}_i, \mathbf{y}_i \in \mathfrak{S}(\mathcal{H}_i)$  and  $B_1, B_2$  so that  $\text{ind}(B_1, B_2) = 1$ , the embedded moduli space  $\mathcal{MM}^{B_1 \natural B_2}(T; \mathbf{x}_1, \mathbf{y}_1; \mathbf{x}_2, \mathbf{y}_2)$  is a zero-manifold whose point count has the same parity as the number of stable ideal matched curves from  $\mathbf{x}_1 \times \mathbf{x}_2$  to  $\mathbf{y}_1 \times \mathbf{y}_2$  in the same homology class.*

*Proof.* Let  $T_0$  be the maximum value from Proposition 9.30 for all relevant sources  $S_i^\triangleright$ . Let

$$\pi_T: \mathcal{MM}^B(\geq T_0; \mathbf{x}_1, \mathbf{y}_1; \mathbf{x}_2, \mathbf{y}_2) \longrightarrow [T_0, \infty)$$

be the natural projection to the  $T$  coordinate. We claim that  $\pi_T$  is proper. To this end, observe that if  $\{u_i\}_{i=1}^\infty \in \mathcal{MM}^{B_1, B_2}(T_i; \mathbf{x}_1, \mathbf{y}_1; \mathbf{x}_2, \mathbf{y}_2)$  is a sequence of matched curves with  $T_i \geq T_0$  but  $|T_i|$  bounded above, then the matched curves have a subsequence which converges to a  $T$ -matched comb  $U$  for some  $T$ . Suppose this comb has  $k$  stories. Additivity of the expected dimension, together with Lemma 9.15, ensures that  $k \leq 2$ . The case  $k = 2$  is excluded by Proposition 9.30 (since one of the stories must have index 0). In the case where  $k = 1$ , by Proposition 9.18, either  $U$  is a  $T$ -matched curve (rather than a comb), or  $U$  has a join or split component at east-west infinity. Suppose that the latter happens for arbitrarily large  $T$ . Then we can take the limit of a subsequence to get a ideal matched comb  $(u_1, u_2)$  with either a join or split component at east infinity in  $u_1$ . If we have a join component, we violate Lemma 9.29; if we have a split component, then  $u_2$  has two punctures at the same level, violating the condition that  $u_2$  have index 1.

It follows that for any two sufficiently large, generic  $T_1$  and  $T_2$ , the moduli spaces

$$\mathcal{MM}^B(T_i; \mathbf{x}_1, \mathbf{y}_1; \mathbf{x}_2, \mathbf{y}_2)$$

are compactly cobordant.

Next, we study the ends of the space as the projection to  $T$  goes to infinity. Specifically, let  $U_i \in \mathcal{M}^B(T_i, \mathbf{x}_1 \times \mathbf{x}_2, \mathbf{y}_1 \times \mathbf{y}_2)$  be a sequence of matched curves where  $T_i \rightarrow \infty$ . Such a sequence converges to an ideal matched comb, which by Lemma 9.29 is an ideal matched curve.

It remains to verify that each end appears: each simple ideal, matched curve has a neighborhood with the property that the number of  $T$ -matched curves in this neighborhood (for sufficiently large, generic  $T$ ) is odd. We first fix some notation. Fix a moduli space  $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright)$  in which  $S^\triangleright$  has  $\ell$  east punctures, an ordered partition  $\vec{P} = (P_1, \dots, P_m)$  of  $E(S^\triangleright)$ , with none of the  $P_i$  empty. Let the punctures in  $P_i$  be  $p_i^1$  through  $p_i^{n_m}$ . We can then regroup the evaluation maps into two functions as follows:

$$\begin{aligned} I_P: \mathcal{M}(\mathbf{x}, \mathbf{y}; S^\triangleright) &\longrightarrow \mathbb{R}^{m-1} \\ I_P(u_2) &= (\text{ev}(p_2^1, p_1^1; u_2), \dots, \text{ev}(p_m^1, p_{m-1}^1; u_2)) \\ J_P: \mathcal{M}(\mathbf{x}, \mathbf{y}; S^\triangleright) &\longrightarrow \mathbb{R}^{\ell-m} \\ J_P(u_2) &= (\text{ev}(p_1^2, p_1^1; u_2), \dots, \text{ev}(p_1^{n_1}, p_1^{n_1-1}; u_2), \\ &\quad \text{ev}(p_2^2, p_2^1; u_2), \dots, \text{ev}(p_2^{n_2}, p_2^{n_2-1}; u_2), \\ &\quad \vdots \\ &\quad \text{ev}(p_m^2, p_m^1; u_2), \dots, \text{ev}(p_m^{n_m}, p_m^{n_m-1}; u_2)). \end{aligned}$$

Here we use  $\text{ev}(p, q; u)$  for what we previously denoted  $\text{ev}_{p,q}(u)$  or  $\text{ev}_p(u) - \text{ev}_q(u)$ .

Now, fix an ideal matched curve  $(u_1, u_2)$ , with  $u_2 = (f_1, \dots, f_m)$ . (Recall that  $u_1$  consists of a one-story holomorphic building with generalized split curves at east infinity.) Let  $P_d$  be the discrete partition on the east punctures of  $u_2$ . Then Proposition 5.21 tells us that a neighborhood  $\mathcal{U}_2$  of  $(f_1, \dots, f_m)$ , thought of as an  $m$ -story building in  $\overline{\mathcal{M}}^{B_2}(\mathbf{x}_2, \mathbf{y}_2, S_2^\triangleright; P_d)$ , is homeomorphic to  $\prod_{i=1}^{m-1} (t, \infty]$ . Partition the punctures on  $S_2^\triangleright$  into  $m$  groups according to the stories in the limit and order the punctures within each group according to their height in  $u_2$ . We can then regroup the evaluation maps as above to give the two maps  $I_P$  and  $J_P$  as above. Then the map

$$I_P: \mathcal{U}_2 \longrightarrow \overline{\mathbb{R}}_+^{m-1}$$

maps degree one onto all points in  $\mathbb{R}_+^{m-1}$  with sufficiently large norm, while  $J_P$  maps  $\mathcal{U}_2$  into an  $\epsilon$ -neighborhood of some vector  $c_2$  in the interior of  $\mathbb{R}_+^{\ell-m}$ .

We now turn to the  $U_1$  side. For a small neighborhood  $\mathcal{U}_1$  of  $u_1$  in  $\overline{\mathcal{M}}(\mathbf{x}_1, \mathbf{y}_1, S_1^\triangleright; P_d)$ , the function

$$J_P: \mathcal{U}_1 \longrightarrow \mathbb{R}^{\ell-m}$$

maps to a neighborhood of 0. The precise image depends on which east punctures of  $S_1^\triangleright$  are on the same generalized split curve, but in any case transversality and the argument of Proposition 5.29 guarantee that  $J_P$  maps  $\mathcal{U}_1$  with degree one (mod 2) onto all points with positive coordinates and sufficiently small norm. In addition,  $I_P$  maps  $\mathcal{U}_1$  into an  $\epsilon$ -neighborhood of some vector  $c_1$  in the interior of  $\mathbb{R}_+^{m-1}$ .

See Figure 39 for an illustration of the map  $\text{ev}$  on  $\mathcal{U}_1$  and  $\mathcal{U}_2$ .

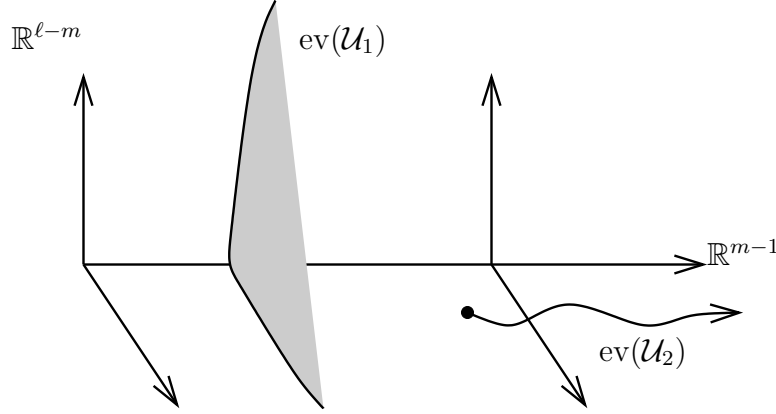


FIGURE 39. Schematic illustration of the proof of Proposition 9.31. The curve to the right represents the image of  $\mathcal{U}_2$  under evaluation, while the piece of surface represents the image of  $\mathcal{U}_1$  under evaluation. After rescaling the image of  $\mathcal{U}_2$  by a factor of  $1/T$ , it intersects the image of  $\mathcal{U}_1$  an odd number of times.

We now show that for all sufficiently large  $T$ , the fibered product

$$X_T := \mathcal{U}_1 \times_{T \cdot \text{ev}_1 = \text{ev}_2} \mathcal{U}_2 = \mathcal{U}_1 \times_{\text{ev}_1 = 1/T \cdot \text{ev}_2} \mathcal{U}_2$$

consists of an odd number of points. Specifically, the map

$$\frac{1}{T} \cdot (I_P, J_P): \mathcal{U}_2 \longrightarrow \overline{\mathbb{R}}_+^{m-1} \times \mathbb{R}_+^{\ell-m}$$

can be homotoped smoothly by a homotopy  $H$  to the map

$$\frac{1}{T} \cdot (I_P, c): \mathcal{U}_2 \longrightarrow \overline{\mathbb{R}}_+^{m-1} \times \mathbb{R}_+^{\ell-m}$$

for some generically chosen vector  $c \in \mathbb{R}_+^{\ell-m}$ . For sufficiently large  $T$ , the image of  $\partial(\mathcal{U}_2)$  under  $H$  is disjoint from the image of  $\mathcal{U}_1$  under the evaluation maps. (Recall that the image of  $\mathcal{U}_1$  is contained in some  $\epsilon$  neighborhood of a vector  $c_1$  in  $\mathbb{R}_+^{m-1}$  with no vanishing coordinate, whereas for  $T$  sufficiently large,  $\partial(\mathcal{U}_2)$  is mapped under  $\frac{1}{T}(I_P, J_P)$  outside of this neighborhood). It follows now that the zero-manifold  $X_T$  is compactly cobordant to

$$\mathcal{U}_1 \times_{(I_P, J_P) = \frac{1}{T} \cdot (I_P, c)} \mathcal{U}_2,$$

In turn, since  $I_P: \mathcal{U}_2 \longrightarrow \overline{\mathbb{R}}_+^{m-1}$  has degree one onto all points with sufficiently large norm, it follows that  $X_T$  is also compactly cobordant to

$$\mathcal{U} \times_{(I_P, J_P) = \frac{1}{T} \cdot (\iota, c)} \prod_{i=1}^{m-1} [t, \infty],$$

where  $\iota: \prod_{i=1}^{m-1} [t, \infty] \longrightarrow \overline{\mathbb{R}}_+^m$  is simply inclusion; i.e.,  $X_T$  is compactly cobordant to  $J_P^{-1}(c/T)$  in  $\mathcal{U}$ . Since  $J_P: \mathcal{U} \longrightarrow \mathbb{R}_+^{\ell-m}$  has degree one onto all points with positive coordinates and sufficiently small norm,  $X_T$  is compactly cobordant to a single point, as desired.  $\square$

*Proof of Theorem 1.3.* Write  $\mathcal{H} = \mathcal{H}_1 \cup_{\partial} \mathcal{H}_2$ . According to Theorem 9.12,  $\widehat{CF}(\mathcal{H}_1 \cup_{\partial} \mathcal{H}_2)$  is identified with  $\widehat{CF}(T; \mathcal{H}_1, \mathcal{H}_2)$  with  $T = 1$ . By Proposition 9.23, the chain

homotopy type of  $\widehat{CF}(T; \mathcal{H}_1, \mathcal{H}_2)$  is independent of  $T$ . According to Proposition 9.31, if  $T$  is sufficiently large, the differential count in  $\widehat{CF}(T; \mathcal{H}_1, \mathcal{H}_2)$  is counting ideal matched curves with formal dimension one. More precisely, we conclude that for large  $T$ ,  $\widehat{CF}(T; \mathcal{H}_1, \mathcal{H}_2)$  is generated by  $\mathbf{x} \in \mathfrak{S}(\mathcal{H}_1)$  and  $\mathbf{y} \in \mathfrak{S}(\mathcal{H}_2)$ , with differential defined as follows. Let  $a_i$  be a basis for the algebra  $\mathcal{A}(F)$  over  $\mathbb{F}_2$ , so that we can write

$$\delta(\mathbf{y}) = \sum_i a_i \otimes D_i(\mathbf{y})$$

for  $\mathbf{y} \in \widehat{CFD}(\mathcal{H}_2)$ . Then, for large  $T$ , we have that

$$\partial_T(\mathbf{x} \otimes \mathbf{y}) = \sum m_{k+1}(\mathbf{x}, a_{i_1}, \dots, a_{i_k})(D_{i_k} \circ \dots \circ D_{i_1})(\mathbf{y}),$$

where here the sum is taken over all  $k$ -element sequences  $i_1, \dots, i_k$  of elements in  $\{1, \dots, n\}$  (including the empty sequence with  $k = 0$ ). (The possibility, in an ideal matched curve  $(u_1, u_2)$ , for  $u_1$  to have generalized split components at east infinity accounts for the different ways to write an algebra element as a product of Reeb chords.) Comparison with Equation (2.22) shows that the differential we are considering here coincides with the differential in  $\widehat{CFA}(\mathcal{H}_1) \boxtimes \widehat{CFD}(\mathcal{H}_2)$ , in the sense of Definition 2.17. According to Proposition 2.20, this in turn is identified with the derived tensor product of  $\widehat{CFA}(\mathcal{H}_1)$  with  $\widehat{CFD}(\mathcal{H}_2)$ . (Here, we are using the fact that we can find an admissible diagram isotopic to  $\mathcal{H}_2$ , so  $\widehat{CFD}(\mathcal{H}_2)$  is homotopy equivalent to a bounded type  $D$  structure.)  $\square$

**9.4. Gradings.** We now turn our attention to a graded version of the pairing theorem. We continue to assume that  $Y_1$  and  $Y_2$  are two bordered three-manifolds which agree along their boundary, represented by Heegaard diagrams  $\mathcal{H}_1$  and  $\mathcal{H}_2$ . We let  $Y = Y_1 \cup_{\partial} Y_2$  be represented by the Heegaard diagram  $\mathcal{H} = \mathcal{H}_1 \cup_{\partial} \mathcal{H}_2$ .

Recall that for each  $\text{spin}^c$  structure  $\mathfrak{s}_1$  over  $Y_1$ ,  $\widehat{CFA}(\mathcal{H}_1, \mathfrak{s}_1)$  is graded by the set  $P_1(\mathbf{x}_1) \backslash G(\mathcal{Z})$ , where here  $\mathbf{x}_1 \in \mathfrak{S}(\mathcal{H}_1)$  is a generator representing  $\mathfrak{s}_1$ , and  $P_1(\mathbf{x}_1)$  denotes the image of the group of periodic domains in the grading group. Similarly, for each  $\text{spin}^c$  structure  $\mathfrak{s}_2$  over  $Y_2$ ,  $\widehat{CFD}(\mathcal{H}_2, \mathfrak{s}_2)$  is graded by the set  $G(\mathcal{Z})/P_2(\mathbf{x}_2)$ , where now  $\mathbf{x}_2 \in \mathfrak{S}(\mathcal{H}_2)$  is a generator representing  $\mathfrak{s}_2$ , and  $P_2(\mathbf{x}_2)$  denotes the image of the corresponding space of periodic domains in the grading group. Correspondingly, the tensor product  $\widehat{CFA}(\mathcal{H}_1, \mathfrak{s}_1) \boxtimes \widehat{CFD}(\mathcal{H}_2, \mathfrak{s}_2)$  inherits a grading  $\text{gr}^{\boxtimes}$  with values in the double coset space

$$P_1(\mathbf{x}_1) \backslash G(\mathcal{Z}) / P_2(\mathbf{x}_2).$$

We now assume that  $\mathbf{x}_1$  and  $\mathbf{x}_2$  occupy complementary  $\alpha$ -arcs, i.e., that  $\mathbf{x} := \mathbf{x}_1 \times \mathbf{x}_2 \in \mathfrak{S}(\mathcal{H})$ . In addition, we will use  $I_A(\mathbf{x}_1) = I_D(\mathbf{x}_2) = I(s_0)$  as the reference idempotent for defining the grading  $\text{gr}$  as in Section 3.3.2.

Note that the above double coset space still inherits an action by  $\mathbb{Z}$  (acting on the Maslov component). The quotient by this action is naturally identified with the quotient of  $H_1(F)$  by the image of  $H_2(Y_1, F) \oplus H_2(Y_2, F)$  under the boundary homomorphism, or, equivalently, by the image of  $H_1(F)$  in  $H_1(Y)$ . Indeed, we can encode this as the following right exact sequence:

$$\mathbb{Z} \longrightarrow P_1(\mathbf{x}_1) \backslash G / P_2(\mathbf{x}_2) \longrightarrow \text{Im}(H_1(F) \rightarrow H_1(Y)) \longrightarrow 0.$$

More invariantly, we have

$$\mathbb{Z} \longrightarrow P_1(\mathbf{x}_1) \backslash G / P_2(\mathbf{x}_2) \xrightarrow{\Pi} \{\mathfrak{s} \in \text{spin}^c(Y) \mid \mathfrak{s}|_{Y_i} \cong \mathfrak{s}_i, i = 1, 2\} \longrightarrow 0,$$

where the map  $\Pi$  is defined by

$$\Pi(\gamma) = \mathfrak{s}(\mathbf{x}) + [\gamma].$$

Here  $\gamma \mapsto [\gamma]$  is the map from the double-coset space to  $H_1(Y)$ , and we are implicitly using Poincaré duality to think of  $\text{spin}^c(Y)$  as an affine space over  $H_1(Y)$  (rather than the usual  $H^2(Y)$ ).

**Lemma 9.32.** *The  $\mathbb{Z}$  orbit of a double-coset in  $P_1(\mathbf{x}_1) \backslash G / P_2(\mathbf{x}_2)$  mapping to  $\mathfrak{s}' \in \text{spin}^c(Y)$  under the map  $\Pi$  is identified with*

$$\mathbb{Z} / \text{div}(\mathfrak{s}'),$$

where here  $\text{div}(\mathfrak{s}')$  denotes the divisibility of the first Chern class of  $\mathfrak{s}'$ , i.e.,

$$H^1(Y; \mathbb{Z}) \cup c_1(\mathfrak{s}') = \text{div}(\mathfrak{s}') \cdot H^3(Y; \mathbb{Z}).$$

*Proof.* Suppose that  $\gamma$  represents a double-coset with the property that  $\gamma$  and  $\lambda^t \cdot \gamma$  represent the same double coset. This means that there are  $B_i \in \pi_2(\mathbf{x}_i, \mathbf{x}_i)$  with the property that

$$\lambda^t \cdot g(B_1) \cdot \gamma \cdot g(B_2) = \gamma$$

where  $g(B_i)$  is as defined in Formula (6.14). (The difference between  $g'$  and  $g$  from Formula (6.22) makes no difference here.) This implies that  $\partial^\partial B_1 + \partial^\partial B_2 = 0$ , so that we can form the periodic domain  $B = B_1 \natural B_2$ . Using the formula  $e(B) + 2n_{\mathbf{x}}(B) = \langle c_1(\mathfrak{s}(\mathbf{x})), [B] \rangle$ , (see [32, Proposition 7.4]), we can interpret the above equation as

$$\begin{aligned} 1 &= \lambda^t \cdot \lambda^{-e(B) - 2n_{\mathbf{x}}(B)} (0, \partial^\partial B_1) \cdot \gamma \cdot (0, -\partial^\partial B_1) \cdot \gamma^{-1} \\ &= \lambda^{t - \langle c_1(\mathfrak{s}(\mathbf{x})), [B] \rangle - 2[\gamma] \cap [\partial^\partial B_1]} \\ &= \lambda^{t - \langle c_1(\mathfrak{s}(\mathbf{x})), [B] \rangle - 2[\gamma] \cap [B]} \\ &= \lambda^{t - \langle c_1(\Pi(\gamma)), [B] \rangle}, \end{aligned}$$

where  $[\gamma] \cap [\partial^\partial B_1]$  denotes the intersection number of two elements of  $H_1(F; \mathbb{Z})$ , while  $[\gamma] \cap [B]$  denotes the intersection number of  $[\gamma] \in H_1(Y; \mathbb{Z})$  with  $[B] \in H_2(Y; \mathbb{Z})$ . The second equality uses Equation (3.22) and the last uses  $c_1(\Pi(\gamma)) = c_1(\mathfrak{s}(\mathbf{x})) + 2[\gamma]$ . From this (and the fact that any  $B$  can be decomposed into  $B_1$  and  $B_2$ ) the desired result follows.  $\square$

**Theorem 9.33.** *Fix  $\mathfrak{s}_i \in \text{spin}^c(Y_i)$ . There is a quasi-isomorphism*

$$\Phi: \widehat{CFA}(\mathcal{H}_1, \mathfrak{s}_1) \boxtimes \widehat{CFD}(\mathcal{H}_2, \mathfrak{s}_2) \longrightarrow \bigoplus_{\substack{\mathfrak{s} \in \text{spin}^c(Y) \\ \mathfrak{s}|_{Y_i} = \mathfrak{s}_i, i=1,2}} \widehat{CF}(\mathcal{H}, \mathfrak{s})$$

which respects the identification between grading sets in the following sense:

- (1) *If  $m_1 \boxtimes m_2$  is a homogeneous element with grading  $\text{gr}^\boxtimes(m_1 \boxtimes m_2) \in P(\mathbf{x}_1) \backslash G / P(\mathbf{x}_2)$ , then  $\Phi(m_1 \boxtimes m_2)$  lies in the  $\Pi(\text{gr}^\boxtimes(m_1 \boxtimes m_2)) \in \text{spin}^c(Y)$  summand of  $\widehat{CF}(Y)$ .*

- (2) If  $m_1 \boxtimes m_2$  and  $n_1 \boxtimes n_2$  are two homogeneous elements whose gradings are related by

$$\text{gr}^{\boxtimes}(m_1 \boxtimes m_2) = \lambda^t \text{gr}^{\boxtimes}(n_1 \boxtimes n_2),$$

so that  $\Pi(\text{gr}^{\boxtimes}(m_1 \boxtimes m_2)) = \Pi(\text{gr}^{\boxtimes}(n_1 \boxtimes n_2)) = \mathfrak{s}' \in \text{spin}^c(Y)$ , then  $\Phi(m_1 \boxtimes m_2)$  and  $\Phi(n_1 \boxtimes n_2)$  are elements whose relative Maslov grading is given by  $t \pmod{c_1(\text{div}(\mathfrak{s}'))}$ .

*Proof.* The map  $\Phi$  is the quasi-isomorphism from Theorem 1.3. We must prove the map  $\Phi$  respects the grading sets.

To define the identification of grading sets we chose generators  $\mathbf{x}_1$  and  $\mathbf{x}_2$  with  $\mathfrak{s}(\mathbf{x}_i) = \mathfrak{s}_i$ . We will use the idempotent  $I_A(\mathbf{x}_1) = I_D(\mathbf{x}_2) = I(s_0)$  to pin down the gradings in the algebra as in Section 3.3.2, and we also use  $\mathbf{x}_i$  to pin down the gradings on  $\widehat{CFA}(\mathcal{H}_1, \mathfrak{s}_1)$  and  $\widehat{CFD}(\mathcal{H}_2, \mathfrak{s}_2)$ , as in Sections 7.3 and 6.3.

Let  $\mathbf{y}_1, \mathbf{y}'_1 \in \mathfrak{S}(\mathcal{H}_1)$  and  $\mathbf{y}_2, \mathbf{y}'_2 \in \mathfrak{S}(\mathcal{H}_2)$  be generators with  $\mathfrak{s}(\mathbf{y}_i) = \mathfrak{s}(\mathbf{y}'_i) = \mathfrak{s}_i$  and such that  $\mathbf{y} = \mathbf{y}_1 \times \mathbf{y}_2$  and  $\mathbf{y}' = \mathbf{y}'_1 \times \mathbf{y}'_2$  are generators for  $\mathcal{H}$ . Let  $B_i \in \pi_2(\mathbf{x}_i, \mathbf{y}_i)$  and  $B'_i \in \pi_2(\mathbf{x}_i, \mathbf{y}'_i)$ . We have

$$\begin{aligned} \text{gr}(\mathbf{y}_1) &= P(\mathbf{x}_1)g(B_1) \in G_A = P(\mathbf{x}_1) \backslash G \\ \text{gr}(\mathbf{y}_2) &= g(B_2)P(\mathbf{x}_2) \in G_D = G/P(\mathbf{x}_2). \end{aligned}$$

Moreover, recall that

$$\begin{aligned} \text{gr}'(\mathbf{y}_i) &= (-e(B_i) - n_{\mathbf{x}_i}(B_i) - n_{\mathbf{y}_i}(B_i), \partial^\partial(B_i)) \\ \text{gr}(\mathbf{y}_1) &= \text{gr}'(\mathbf{y}_1) \cdot g(I_A(\mathbf{y}_1), I_A(\mathbf{x}_1))^{-1} \\ \text{gr}(\mathbf{y}_2) &= g(I_D(\mathbf{y}_2), I_D(\mathbf{x}_2)) \cdot \text{gr}'(\mathbf{y}_2) \end{aligned}$$

with similar equations involving  $\mathbf{y}'_i$  and  $B'_i$ . Consequently,

$$\begin{aligned} \text{gr}(\mathbf{y}_1 \times \mathbf{y}_2) &= \text{gr}'(\mathbf{y}_1) \cdot g(I_A(\mathbf{y}_1), I_A(\mathbf{x}_1))^{-1} \cdot g(I_D(\mathbf{y}_2), I_D(\mathbf{x}_2)) \cdot \text{gr}'(\mathbf{y}_2) \\ &= \text{gr}'(\mathbf{y}_1) \cdot \text{gr}'(\mathbf{y}_2), \end{aligned}$$

where we have used the fact that  $I_A(\mathbf{x}_1) = I_D(\mathbf{x}_2)$  and  $I_A(\mathbf{y}_1) = I_D(\mathbf{y}_2)$ . We wish to use this to calculate the difference element between the two  $\text{spin}^c$  structures associated to  $\mathbf{x} = \mathbf{x}_1 \times \mathbf{x}_2$  and  $\mathbf{y} = \mathbf{y}_1 \times \mathbf{y}_2$ . The homological component of  $\text{gr}'(\mathbf{y}_1) \cdot \text{gr}'(\mathbf{y}_2)$  is  $\partial^\partial B_1 + \partial^\partial B_2$ . This represents a homology class  $\gamma$  in  $H_1(F)$ . Moreover, the image of  $\gamma$  in  $H_1(Y)$  agrees with the difference element  $\epsilon(\mathbf{x}_1 \times \mathbf{x}_2, \mathbf{y}_1 \times \mathbf{y}_2)$  defined in [33, Section 2.6], whose Poincaré dual is  $\mathfrak{s}_z(\mathbf{x}_1 \times \mathbf{x}_2) - \mathfrak{s}_z(\mathbf{y}_1 \times \mathbf{y}_2)$ . It follows that  $\Pi(\text{gr}'(\mathbf{y}_1) \cdot \text{gr}'(\mathbf{y}_2))$  represents the same  $\text{spin}^c$  structure as  $\mathbf{y} = \mathbf{y}_1 \times \mathbf{y}_2$ . This verifies Part 1.

For Part 2, fix some  $B'' \in \pi_2(\mathbf{y}, \mathbf{y}')$ , where  $B'' = B'_1 \natural B'_2$ . Set  $B'_i = B_i * B''_i \in \pi_2(\mathbf{x}_i, \mathbf{y}'_i)$ . Now, the grading of  $\mathbf{y}'$  is given by

$$\begin{aligned} \text{gr}(\mathbf{y}') &= g'(B_1 * B''_1) \cdot g'(B_2 * B''_2)^{-1} \\ &= g'(B_1) \cdot g'(B''_1) \cdot g'(B''_2)^{-1} \cdot g'(B_2)^{-1} \\ &= g'(B_1) \cdot \lambda^{-e(B'') - n_{\mathbf{y}}(B'') - n_{\mathbf{y}'}(B'')} \cdot g'(B_2)^{-1} \\ &= \lambda^{-e(B'') - n_{\mathbf{y}}(B'') - n_{\mathbf{y}'}(B'')} \cdot \text{gr}(\mathbf{y}). \end{aligned}$$

This proves Part 2. □

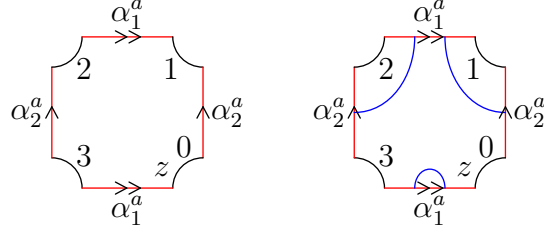


FIGURE 40. Bordered Heegaard diagrams for torus boundary. Left: the necessary arrangement of the  $\alpha$ -arcs. Opposite sides are identified as indicated. There may be additional handles attached with corresponding  $\alpha$ -circles. Right: a  $\beta$ -circle to create an admissible diagram for a solid torus.

## 10. BORDERED MANIFOLDS WITH TORUS BOUNDARY

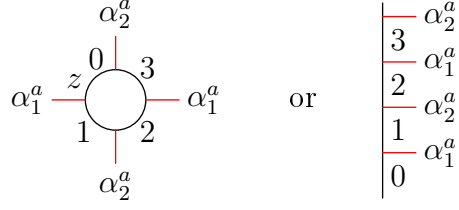
In this section, we specialize to the case of bordered manifolds with torus boundary. As a warm-up, we use the pairing theorem to give a quick proof of the surgery exact triangle. We then turn to the relationship between bordered Floer homology for manifolds with torus boundary, and knot Floer homology. In Section 10.3 we recall the conventions on knot Floer homology. The easier task of extracting knot Floer homology from bordered Floer homology is addressed in Section 10.4. This follows quickly from a suitable adaptation of the pairing theorem, and a simple model calculation for the solid torus.

Following this, Section 10.5 explains how to extract the type  $D$  module for a knot complement in  $S^3$  from the knot Floer homology. After further developing the holomorphic curve machinery in Section 10.6, this result is proved in Sections 10.7 and 10.8. Finally, in Section 10.9, we use bordered Floer homology to study satellite knots.

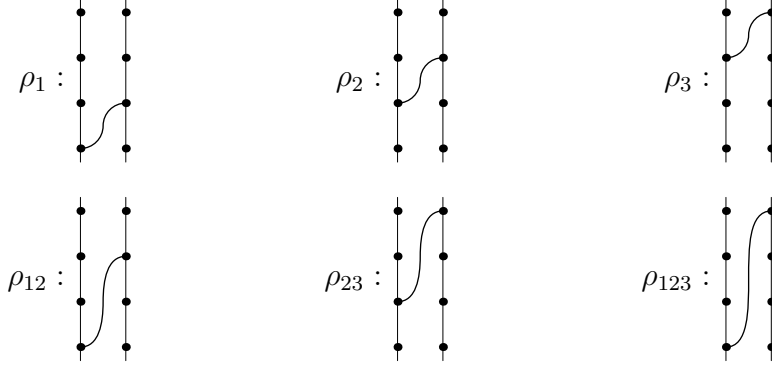
**10.1. Torus algebra.** A bordered Heegaard diagram  $\mathcal{H}$  for a 3-manifold  $Y$  with torus boundary necessarily has two  $\alpha$ -arcs,  $g - 1$   $\alpha$ -circles, and  $g$   $\beta$ -circles. The  $\alpha$ -arcs are necessarily arranged as in Figure 40.

As a result, the parametrization of  $\partial Y$  is specified by a pair of a meridian and a longitude, intersecting once, and the matching on  $\partial \mathcal{H}$  is always the same. Let  $\mathcal{A}(\mathbb{T}) = \mathcal{A}(\mathbb{T}, 0)$  denote the  $i = 0$  part of the corresponding algebra. We will think of the Heegaard diagram as for a type  $D$  module. Accordingly, we label the regions between the arcs in the opposite order to the induced orientation on the boundary, as in Figure 41. The point  $z$  is in the region labeled 0. Further, label the  $\alpha$ -arcs so that regions 0 and 1 are separated from regions 2 and 3 by  $\alpha_2^a$ . (From the type  $A$  side, the regions would be labeled in the opposite order around the puncture.)

The algebra  $\mathcal{A}(\mathbb{T})$  has two idempotents  $\iota_0$  and  $\iota_1$ , corresponding respectively to  $\alpha_1^a$  and  $\alpha_2^a$  being occupied on the type  $A$  side, and 6 Reeb elements, denoted graphically

FIGURE 41. Labeling of the regions around  $\partial\overline{\Sigma}$ .

as follows:



The differential is zero, and the non-zero products are

$$\rho_1\rho_2 = \rho_{12} \quad \rho_2\rho_3 = \rho_{23} \quad \rho_1\rho_{23} = \rho_{123} \quad \rho_{12}\rho_3 = \rho_{123}.$$

(All other products of two Reeb elements vanish identically.) There are also compatibility conditions with the idempotents:

$$\begin{aligned} \rho_1 &= \iota_0\rho_1\iota_1 & \rho_2 &= \iota_1\rho_2\iota_0 & \rho_3 &= \iota_0\rho_3\iota_1 \\ \rho_{12} &= \iota_0\rho_{12}\iota_0 & \rho_{23} &= \iota_1\rho_{23}\iota_1 & \rho_{123} &= \iota_0\rho_{123}\iota_1. \end{aligned}$$

One grading takes values in the group  $G'$  generated by quadruples  $(m; a, b, c)$ , with group law

$$(m_1; a_1, b_1, c_1) \cdot (m_2; a_2, b_2, c_2) = \left( m_1 + m_2 + \frac{1}{2} \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} + \frac{1}{2} \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}; \right. \\ \left. a_1 + a_2, b_1 + b_2, c_1 + c_2 \right).$$

We have gradings on the algebra:

$$\begin{aligned} \text{gr}'(\rho_1) &= \left(-\frac{1}{2}; 1, 0, 0\right) \\ \text{gr}'(\rho_2) &= \left(-\frac{1}{2}; 0, 1, 0\right) \\ \text{gr}'(\rho_3) &= \left(-\frac{1}{2}; 0, 0, 1\right). \end{aligned}$$

There is also a refined grading, as in Section 3.3.2, with values in the group  $G$ , given explicitly by triples  $(m; p, q)$  with group law

$$(m_1; p_1, q_1) \cdot (m_2; p_2, q_2) = \left( m_1 + m_2 + \begin{vmatrix} p_1 & q_1 \\ p_2 & q_2 \end{vmatrix}; p_1 + p_2, q_1 + q_2 \right).$$

The inclusion from  $G$  to  $G'$  sends  $(m; p, q)$  to  $(m; p, p + q, q)$ . To define the refined grading, choose  $\iota_0$  as the base idempotent and set  $g(\iota_0, \iota_1) := (0; \frac{1}{2}, 0, \frac{1}{2})$ . From Equation (3.23) we then find

$$(10.1) \quad \begin{aligned} \text{gr}(\rho_1) &= \left(-\frac{1}{2}; \frac{1}{2}, -\frac{1}{2}\right) \\ \text{gr}(\rho_2) &= \left(-\frac{1}{2}; \frac{1}{2}, \frac{1}{2}\right) \\ \text{gr}(\rho_3) &= \left(-\frac{1}{2}; -\frac{1}{2}, \frac{1}{2}\right). \end{aligned}$$

Note that for any algebra element  $a$ , the gradings  $\text{gr}'(a)$  and  $\text{gr}(a)$  are related by the homomorphism from  $G'$  to  $G$  (after extending scalars) defined by

$$(10.2) \quad (m; a, b, c) \mapsto \left(m; \frac{a+b-c}{2}, \frac{-a+b+c}{2}\right).$$

Since the torus algebra is so simple, we can understand the structure of type  $D$  modules over it more explicitly. (This will be helpful in Section 10.7, where we understand how to construct the type  $D$  module of a knot complement from the knot Floer complex.) To this end, we introduce the notion of a collection of “coefficient maps.”

**Definition 10.3.** Let  $V = V^0 \oplus V^1$  be a  $\mathbb{Z}/2\mathbb{Z}$ -graded vector space. A collection of (torus) *coefficient maps* is a collection of maps

$$D_I: V^{[i_0-1]} \longrightarrow V^{[i_n]}$$

indexed by increasing sequences of consecutive integers  $I = \{i_0, \dots, i_n\} \subset \{1, 2, 3\}$ , where  $[i]$  denotes the reduction of  $i \pmod{2}$ , satisfying the compatibility equation stated below. The empty sequence also has a corresponding map  $D_\emptyset$  or just  $D$ , which respects the splitting of  $V$  (i.e., it maps  $V^i$  to  $V^i$  for  $i = 0, 1$ ). Thus, there are 7 coefficient maps. The coefficient maps are required to satisfy the following compatibility equations, also indexed by increasing sequences of consecutive integers  $I \subset \{1, 2, 3\}$ :

$$(10.4) \quad \sum_{\{I=J \cup K \mid J < K\}} D_K \circ D_J = 0$$

where here  $J < K$  means that each element of  $J$  is smaller than each element of  $K$ .

For example, Equation (10.4) in the case where  $I = \emptyset$  says that  $D$  is a differential; similarly,  $I = \{1\}$  says that  $D_1$  is a chain map;  $I = \{1, 2\}$  says that  $D_{12}$  is a null-homotopy of the composite  $D_2 \circ D_1$ ;  $I = \{123\}$  gives the following:

$$D \circ D_{123} + D_3 \circ D_{12} + D_{23} \circ D_1 + D_{123} \circ D = 0.$$

**Lemma 10.5.** A type  $D$  structure in the sense of Definition 2.12 over  $\mathcal{A}(\mathbb{T})$  with base ring  $\mathcal{I}(\mathbb{T})$  corresponds to a pair of  $\mathbb{Z}/2\mathbb{Z}$ -graded vector spaces equipped with coefficient maps in the sense of Definition 10.3.

*Proof.* Let  $V$  be a space with a type  $D$  structure over  $\mathcal{A}(\mathbb{T})$ . The projection  $V \longrightarrow V^0$  is induced by multiplication by the idempotent  $\iota_0$ , while the projection  $V \longrightarrow V^1$  is

induced by multiplication by the idempotent  $\iota_1$ . The coefficient maps can be extracted from the map  $\delta_1$  by the formula

$$\delta_1(x) = \mathbb{I} \otimes Dx + \sum_i \rho_i \otimes D_i + \sum_{\{i,j|j=i+1\}} \rho_{ij} \otimes D_{ij} + \rho_{123} \otimes D_{123}.$$

Equations (10.4) are the components of the compatibility condition for  $\delta_1$ .  $\square$

*Remark 10.6.* In particular, for a module  $\widehat{CFD}(Y)$ , choosing a subspace  $V \subset \widehat{CFD}(Y)$  which is closed under multiplication by  $\mathcal{I}(\mathbb{T})$  and so that  $\mathcal{A}(\mathbb{T}) \otimes V$  maps isomorphically to  $\widehat{CFD}(Y)$  gives a type  $D$  structure and so defines coefficient maps. The coefficient maps in general will depend on the subspace  $V$ .

**10.2. Surgery exact triangle.** Recall that Heegaard Floer homology admits a surgery exact triangle [32]. Specifically, for a pair  $(M, K)$  of a 3-manifold  $M$  and a framed knot  $K$  in  $M$ , there is an exact triangle

$$(10.7) \quad \begin{array}{ccc} \widehat{HF}(M_{-1}) & \longrightarrow & \widehat{HF}(M_0) \\ & \nwarrow \quad \nearrow & \\ & \widehat{HF}(M_\infty) & \end{array}$$

where  $M_{-1}$ ,  $M_0$ , and  $M_\infty$  are respectively  $-1$ ,  $0$ , and infinity surgery on  $K$ . As a simple application of the present material, we prove a version of this result. A similar computation was carried out in [21, Section 5.3]. We do not, however, verify that the maps in our present triangle agree with those in the original version.

We prove this exact triangle by constructing three provincially admissible bordered Heegaard diagrams  $\mathcal{H}_{-1}$ ,  $\mathcal{H}_0$ , and  $\mathcal{H}_\infty$  for the three solid tori filled at the corresponding slopes, and exhibiting a short exact sequence relating the chain complexes  $\widehat{CFD}(\mathcal{H}_\bullet)$ . For any knot complement with (admissible) bordered Heegaard diagram  $\mathcal{H}'$ , we can then take the derived tensor product of  $\widehat{CFA}(\mathcal{H}')$  with this exact triangle. Since derived tensor product is an exact functor, we get the triangle in (10.7).

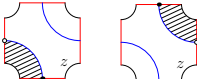
The three diagrams we use are

$$(10.8) \quad \mathcal{H}_\infty : \begin{array}{c} r \\ \begin{array}{|c|c|} \hline 2 & 1 \\ \hline 3 & z0 \\ \hline \end{array} \\ r \end{array} \quad \mathcal{H}_{-1} : \begin{array}{c} b \\ \begin{array}{|c|c|} \hline 2 & 1 \\ \hline 3 & z0 \\ \hline \end{array} \\ b \end{array} \quad \mathcal{H}_0 : \begin{array}{c} \begin{array}{|c|c|} \hline 2 & 1 \\ \hline 3 & z0 \\ \hline \end{array} \\ n \end{array}$$

Each generator for the chain complexes consists of a single intersection point between the  $\beta$ -circle and an  $\alpha$ -arc. These intersections are as labeled above.

The boundary operator on the  $\widehat{CFD}(\mathcal{H}_\bullet)$  (and the support of the relevant domains) is given by

$$\begin{array}{llll} \partial r = \rho_{23}r & \partial a = \rho_3b + \rho_1b & \partial b = 0 & \partial n = \rho_{12}n \end{array}$$


It is easy to write down a short exact sequence

$$0 \longrightarrow \widehat{CFD}(\mathcal{H}_\infty) \xrightarrow{\varphi} \widehat{CFD}(\mathcal{H}_{-1}) \xrightarrow{\psi} \widehat{CFD}(\mathcal{H}_0) \longrightarrow 0$$

by

$$\begin{aligned} \varphi(r) &= \begin{array}{c} b \\ \text{[diagram: rectangle with blue vertical line and green diagonal line]} \end{array} + \begin{array}{c} \rho_2 a \\ \text{[diagram: rectangle with blue vertical line and green diagonal line, shaded]} \end{array} & \psi(a) &= \begin{array}{c} n \\ \text{[diagram: rectangle with blue vertical line and green diagonal line]} \end{array} \\ & & \psi(b) &= \begin{array}{c} \rho_2 n \\ \text{[diagram: rectangle with blue vertical line and green diagonal line, shaded]} \end{array} . \end{aligned}$$

Indeed, one can guess the maps  $\phi$  and  $\psi$  by considering holomorphic triangles, as indicated above. (This is presumably part of a general theory of holomorphic triangles for bordered manifolds.) It is straightforward to verify that  $\psi \circ \varphi = 0$ , that  $\varphi$  is injective, that  $\psi$  is surjective, and that  $\ker \psi = \text{im } \varphi$ .

**10.3. Preliminaries on knot Floer homology.** We focus attention now on the relationship between Heegaard Floer invariants of three-manifolds with torus boundary and knot Floer homology. Before delving into the details, we recall some of the basics of knot Floer homology. More detailed accounts may be found in the original papers [31, 37].

For simplicity of notation, we restrict attention to the case where the ambient three-manifold is  $S^3$ .

The knot Floer homology invariant of a knot is the filtered chain homotopy type of a filtered chain complex over  $\mathbb{F}_2[U]$ . Specifically, a knot in  $S^3$  can be specified by a suitable Heegaard diagram  $(\Sigma, \alpha, \beta, w, z)$ , where  $w$  and  $z$  are a pair of basepoints in the complement of the  $\alpha$ - and  $\beta$ -circles. The knot Floer complex is defined as a chain complex  $CFK^-(S^3)$  generated by  $\mathfrak{S}_K$ , which are the usual generators of Heegaard Floer homology with respect to the Heegaard diagram, endowed with the differential

$$\partial^-(\mathbf{x}) := \sum_{\mathbf{y} \in \mathfrak{S}_K} \sum_{\substack{B \in \pi_2(\mathbf{x}, \mathbf{y}) \\ \text{ind}(\phi)=1}} \#(\mathcal{M}^B(\mathbf{x}, \mathbf{y})) U^{n_w(B)} \cdot \mathbf{y}.$$

The complex is endowed with an Alexander filtration. The Alexander depth  $A$  (filtration degree) of the generators is characterized, up to an overall translation, by

$$\begin{aligned} A(\mathbf{x}) - A(\mathbf{y}) &= n_w(B) - n_z(B) \\ A(U \cdot \mathbf{x}) &= A(\mathbf{x}) - 1 \end{aligned}$$

for any  $\mathbf{x}, \mathbf{y} \in \mathfrak{S}_K$  and homology class  $B \in \pi_2(\mathbf{x}, \mathbf{y})$ . Its homological grading is the Maslov grading  $M$ , defined up to overall translation by

$$\begin{aligned} M(\mathbf{x}) - M(\mathbf{y}) &= \text{ind}(B) - 2n_w(B) \\ M(U \cdot \mathbf{x}) &= M(\mathbf{x}) - 2. \end{aligned}$$

(Note that the knot enters the picture only through the induced Alexander filtration.)

The homology groups of the associated graded object (with respect to the Alexander filtration) are the *knot Floer homology groups* of  $K$ ,

$$HFK^-(K) = \bigoplus_r HFK^-(K, r).$$

The grading  $r$  here is the Alexander grading (induced from the filtration). There is also a Maslov grading induced from the Maslov grading of the knot Floer complex.

The above constructions can be specialized to  $U = 0$ , where we have a different version of knot Floer homology. The homology of the associated graded object (with respect to the Alexander grading) of the  $U = 0$  theory is denoted

$$\widehat{HFK}(K) = \bigoplus_r \widehat{HFK}(K, r).$$

Recall also that knot Floer homology naturally gives rise to an integral valued concordance invariant  $\tau$  for knots [29]. One construction of  $\tau$  comes from considering the filtered chain homotopy type of  $\widehat{CFK}(S^3) = CFK^-(S^3)/(U = 0)$  with its induced knot filtration. From this point of view,  $\tau(K)$  is the minimal  $s$  for which the generator of  $H_*(\widehat{CF}(S^3))$  can be represented as a sum of generators in Alexander grading less than or equal to  $s$ . Alternatively,  $\tau$  has an interpretation from the associated graded object  $HFK^-(K)$ , namely,

$$(10.9) \quad \tau(K) = -\max\{s \mid \forall d \geq 0, U^d \cdot HFK^-(K, s) \neq 0\}.$$

(see, for example, [36, Lemma A.2]).

**10.4. From  $\widehat{CFD}$  to  $HFK^-$ .** Let  $K$  be a null-homologous knot in a 3-manifold  $Y$ . Recall from Section 4.2 that we can construct a bordered Heegaard diagram for  $Y \setminus \text{nbid}(K)$  as follows. Let  $(\Sigma_g, \alpha_1^c, \dots, \alpha_{g-1}^c, \beta_1, \dots, \beta_g)$  be a Heegaard diagram for  $Y \setminus \text{nbid}(K)$ , in the classical sense. Let  $\mu \subset \Sigma$  and  $\lambda \subset \Sigma$  denote a meridian and longitude for  $K$ , respectively. Arrange that  $\lambda$  and  $\mu$  intersect in a single point  $p$ , and that they are disjoint from all  $\alpha_i^c$ . Set  $\alpha_1^a := \lambda \setminus \{p\}$  and  $\alpha_2^a := \mu \setminus \{p\}$ . Then

$$(\Sigma \setminus \{p\}, \alpha_1^a, \alpha_2^a, \alpha_1^c, \dots, \alpha_{g-1}^c, \beta_1, \dots, \beta_g)$$

is a bordered Heegaard diagram for  $Y \setminus \text{nbid}(K)$ .

Number the four regions in  $(\Sigma, \alpha, \beta)$  adjacent to  $p$ , in cyclic order. Let  $z$  and  $w$  be basepoints in regions 0 and 2, respectively. Then

$$(\Sigma, \mu, \alpha_1^c, \dots, \alpha_{g-1}^c, \beta_1, \dots, \beta_g)$$

is a doubly-pointed Heegaard diagram for  $K$  in  $Y$ . So, with notation as in Section 10.1, for the Heegaard diagram  $(\Sigma, \alpha, \beta, z)$  it is immediate that  $(V^0, D)$  is the knot Floer chain complex  $\widehat{CFK}(Y, K)$ . (The chain complex  $(V^1, D)$  is the longitude Floer complex described by Eftekhary [4].)

In fact, it follows that for any bordered Heegaard diagram  $(\Sigma, \alpha, \beta, z)$  for  $Y \setminus \text{nbid}(K)$ , with framing such that  $\alpha_2^a$  is a meridian of  $K$ , the complex  $(V^0, D)$  is homotopy equivalent to  $\widehat{CFK}(Y, K)$ . Indeed, define a  $\mathcal{A}$ -module  $W_0$ , one-dimensional over  $\mathbb{F}_2$ , by letting  $\iota_0$  act as the identity and all other elements of  $\mathcal{A}$  act as 0. Then the tensor product  $W_0 \boxtimes \widehat{CFD}(\Sigma, \alpha, \beta, z)$  is exactly  $(V^0, D)$ . By Lemma 2.19,

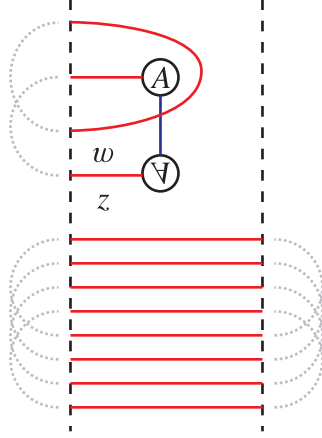


FIGURE 42. **Turning a bordered diagram for  $Y \setminus (K \cup A)$  into a doubly-pointed bordered diagram for  $(Y, K)$ .** This diagram can be thought of as a bordered Heegaard diagram with two basepoints for the standard 1-handle attachment cobordism  $M$  from a surface of genus  $g$  to a surface of genus  $g + 1$ . Gluing this diagram, or an obvious analogue, to a bordered Heegaard diagram for  $(Y', \partial Y' = T^2 \# \Sigma_g)$  gives a doubly-pointed Heegaard diagram for  $(Y' \cup_{\partial} M, K)$ , where  $K$  is the core of the 1-handle in  $M$ .

the homotopy type of this tensor product depends only on the homotopy type of  $\widehat{CFD}(\Sigma, \alpha, \beta, z)$ . Thus, we have the following.

**Proposition 10.10.** *Let  $(\mathcal{H}, z)$  be any bordered Heegaard diagram for  $Y \setminus \text{nbdd}(K)$ , with framing such that  $\alpha_2^g$  is a meridian of  $K$ . Then, with notation as in Section 10.1,  $(V^0, D)$  is homotopy equivalent to  $\widehat{CFK}(Y, K)$ . In particular,  $\widehat{CFK}(Y, K)$  is determined by  $\widehat{CFD}(\mathcal{H}, z)$ .*

The relative Maslov grading within each  $\text{spin}^c$  structure is induced in an obvious way from  $\widehat{CFD}(\mathcal{H}, z)$ . Alexander gradings and a somewhat finer Maslov grading can also be determined; we return to this point later.

There is a simple geometric interpretation of  $W_0$ , which we explain after a more general discussion.

A *doubly-pointed bordered Heegaard diagram* is a bordered Heegaard diagram  $\mathcal{H}$  or  $(\Sigma, \alpha, \beta)$  for  $(Y, \partial Y)$ , together with basepoints  $z$  and  $w$  in  $\bar{\Sigma} \setminus (\bar{\alpha} \cup \bar{\beta})$ . We further assume  $z \in \partial \bar{\Sigma}$ . A doubly-pointed bordered Heegaard diagram specifies a knot  $K$  in  $Y$ : let  $\gamma_{\alpha}$  (respectively  $\gamma_{\beta}$ ) denote a path in  $\Sigma \setminus \alpha$  (respectively  $\Sigma \setminus \beta$ ) from  $z$  to  $w$ . Let  $\tilde{\gamma}_{\alpha}$  denote the result of pushing the interior of  $\gamma_{\alpha}$  into the  $\alpha$ -handlebody and  $\tilde{\gamma}_{\beta}$  the result of pushing the interior of  $\gamma_{\beta}$  into the  $\beta$ -handlebody. Then  $K = \tilde{\gamma}_{\alpha} \cup \tilde{\gamma}_{\beta}$ . We orient  $K$  as  $\tilde{\gamma}_{\alpha} - \tilde{\gamma}_{\beta}$ .

Every knot in a bordered three-manifold can be realized by a doubly-pointed bordered Heegaard diagram. Specifically, let  $Y$  be a bordered three-manifold with genus  $g$  boundary, and let  $K \subset Y$  be a knot. By connecting  $K$  to the boundary by an arc  $A$  and deleting a regular neighborhood of  $K \cup A$ , we can write  $Y$  as the union of a three-manifold  $Y'$  with genus  $g + 1$  boundary and a standard cobordism  $M$  (a

two-handle attachment) from the surface of genus  $g + 1$  to a surface with genus  $g$ . Gluing the (partial) diagram shown in Figure 42 to a bordered Heegaard diagram for  $Y'$  gives a doubly-pointed bordered Heegaard diagram for  $K$  in  $Y$ .

To a doubly-pointed bordered Heegaard diagram  $(\mathcal{H}, z, w)$ , we can associate type  $D$  and type  $A$  modules as before, now working over a ground ring of  $\mathbb{F}_2[U]$ , with the understanding that holomorphic disks crossing  $w$  with multiplicity  $n$  contribute a factor of  $U^n$ . We denote the resulting modules by  $CFA^-(\mathcal{H}, z, w)$  and  $CFD^-(\mathcal{H}, z, w)$ . Setting  $U = 1$  recaptures  $\widehat{CFA}(\mathcal{H}, z)$  and  $\widehat{CFD}(\mathcal{H}, z)$ , while setting  $U = 0$  gives pair of modules denoted  $\widehat{CFA}(\mathcal{H}, z, w)$  and  $\widehat{CFD}(\mathcal{H}, z, w)$  where we count only those holomorphic disks with multiplicity zero at  $w$ .

*Remark 10.11.* We do not claim that these new modules  $CFD^-(\mathcal{H}, z, w)$ , etc., are invariants of the pair  $(Y, K)$ . Because the proofs of invariance in Sections 6.4 and 7.4 require one basepoint,  $z$ , to remain on the boundary in the course of the handleslides and isotopies, the module  $CFD^-(\mathcal{H}, z, w)$  is a priori an invariant of the graph in  $(Y, \partial Y)$  obtained by attaching a point in  $K$  to  $\partial Y$ .

For a doubly-pointed bordered Heegaard diagram, we can enhance the grading set to include an Alexander grading. Specifically, the enhanced grading group has the form  $\tilde{G} = G \times \mathbb{Z}$ . (This is a direct product.) We call the new  $\mathbb{Z}$  summand the *Alexander factor*. Given  $\mathbf{x}, \mathbf{y} \in \mathfrak{S}(\mathcal{H})$  and  $B \in \pi_2(\mathbf{x}, \mathbf{y})$ , define

$$\tilde{g}(B) := (g(B), n_w(B) - n_z(B)),$$

where  $g(B)$  is the grading from Equation (6.22). We can use this enhanced grading group to define an enhanced grading on  $CFD^-(\mathcal{H}, z, w)$ . Following the discussion from Section 6.3, this gives rise to a grading on the type  $D$  module with values in the coset space

$$\tilde{G}_D(\mathcal{H}, \mathfrak{s}) := (\tilde{G}/\tilde{P}(\mathbf{x}_0)),$$

where  $\mathbf{x}_0$  is any intersection point representing  $\mathfrak{s}$ , and  $\tilde{P}(\mathbf{x}_0)$  denotes the image of the space of periodic domains in  $\tilde{G}$  under the homomorphism  $\tilde{g}$ . Namely, we define  $\tilde{g}(\mathbf{x}) := g(B) \cdot P(\mathbf{x}_0)$ , where  $B \in \pi_2(\mathbf{x}, \mathbf{x}_0)$ . Note that when we extend this grading to  $CFD^-(\mathcal{H}, z, w)$ , we take the convention that multiplication by  $U$  drops the Alexander grading by one and the Maslov grading by two. That is,  $\tilde{g}(Ux) = (\lambda^{-2}, -1) \cdot \tilde{g}(x)$ .

Suppose next that  $(\mathcal{H}_1, z)$  is a pointed bordered Heegaard diagram for  $(Y_1, \partial Y_1 = F)$  and  $(\mathcal{H}_2, z, w)$  is a doubly-pointed bordered Heegaard diagram for  $(Y_2, \partial Y_2 = -F)$ , equipped with the knot  $K$ . Then, the grading set of the tensor product  $CFD(\mathcal{H}_1, z) \boxtimes CFA^-(\mathcal{H}_2, z, w)$  is naturally the double-coset space  $\tilde{P}(\mathbf{x}_1) \backslash \tilde{G} / \tilde{P}(\mathbf{x}_2)$ . (In the definition of  $\tilde{P}(\mathbf{x}_1)$ , we take the convention that  $n_w(B) = 0$  since, of course,  $w$  is thought of as located on the  $Y_2$  side of the Heegaard diagram.)

Projecting onto the Alexander factor in the double-coset space induces a map from the grading set to a cyclic group. Indeed, the cyclic group is identified with  $\mathbb{Z}/n_w(\pi_2(\mathbf{x}, \mathbf{x}))$  for any generator  $\mathbf{x}$  of  $\mathcal{H}_1 \cup_{\mathcal{Z}} \mathcal{H}_2$ . In particular, if  $Y$  is an integer homology sphere, the projection induces a map from the double-coset-space to  $\mathbb{Z}$ .

We now have a pairing theorem for knot Floer homology.

**Theorem 10.12.** *Let  $(\mathcal{H}_1, z) = (\Sigma_1, \boldsymbol{\alpha}_1, \boldsymbol{\beta}_1, z)$  be a pointed bordered Heegaard diagram for  $(Y_1, \partial Y_1 = F)$  and  $(\mathcal{H}_2, z, w) = (\Sigma_2, \boldsymbol{\alpha}_2, \boldsymbol{\beta}_2, z, w)$  a doubly pointed bordered*

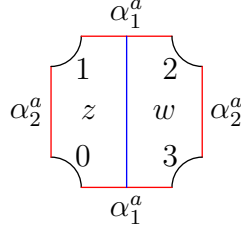


FIGURE 43. **A doubly pointed Heegaard diagram for the longitude in the solid torus.** This is a genus 1 Heegaard diagram with boundary, with  $\alpha$ -arcs  $\alpha_1^a$  and  $\alpha_2^a$ , and a  $\beta$ -circle  $\beta_1$  intersecting  $\alpha_1^a$  in a single point and disjoint from  $\alpha_2^a$ . Note that the boundary markings follow the conventions of a type  $A$  module.

Heegaard diagram for  $(Y_2, \partial Y_2 = -F, K)$ . Then there are quasi-isomorphisms

$$CFK^-(Y_1 \cup_Z Y_2, K) \simeq \widehat{CFD}(\mathcal{H}_1, z) \boxtimes CFA^-(\mathcal{H}_2, z, w)$$

$$CFK^-(Y_2 \cup_Z Y_1, K) \simeq CFD^-(\mathcal{H}_2, z, w) \boxtimes \widehat{CFA}(\mathcal{H}_1, z).$$

In particular, by setting  $U = 0$  we obtain the following quasi-isomorphisms:

$$\widehat{CFK}(Y_1 \cup_Z Y_2, K) \simeq \widehat{CFD}(\mathcal{H}_1, z) \boxtimes \widehat{CFA}(\mathcal{H}_2, z, w)$$

$$\widehat{CFK}(Y_2 \cup_Z Y_1, K) \simeq \widehat{CFD}(\mathcal{H}_2, z, w) \boxtimes \widehat{CFA}(\mathcal{H}_1, z).$$

Under all these identifications, the Alexander grading on the closed manifold is induced from the projection onto the Alexander factor in the double-coset space grading the tensor complexes.

*Proof.* The identification of complexes follows from either of the two proofs of Theorem 1.3 to the doubly-pointed context. The statement about Alexander gradings now follows from Lemma 4.21.  $\square$

As in Section 9.4, we can refine Theorem 10.12 to calculate relative Maslov gradings.

Proposition 10.10 can be seen as a special case of Theorem 10.12, as follows. Consider the doubly pointed Heegaard diagram with boundary for the solid torus shown in Figure 43. If we restrict to holomorphic curves which cross neither  $z$  nor  $w$ , the resulting type  $A$ -module has a single generator over  $\mathbb{F}_2$ , no differential, and indeed is isomorphic to  $W_0$  as described before the statement of Proposition 10.10. Thus, Proposition 10.10 follows from the  $U = 0$  version of Theorem 10.12 (with the solid torus on the type  $A$  side).

To reconstruct  $CFK^-$  of a knot from  $\widehat{CFD}$ , we must describe the type  $A$  module for the solid torus. But this is easily done.

**Lemma 10.13.** *The type  $A$  module of the doubly-pointed solid torus, equipped with a longitudinal unknot, has a single generator  $\mathbf{x}_0$  and higher products given by*

$$m_{3+i}(\mathbf{x}_0, \rho_3, \overbrace{\rho_{23}, \dots, \rho_{23}}^i, \rho_2) = U^{i+1} \cdot \mathbf{x}_0$$

for all  $i \geq 0$ .

*Proof.* This is obvious from the genus one Heegaard diagram in Figure 43.  $\square$

Thus, according to Theorem 10.12,  $HFK^-(Y, K)$  can be calculated as a derived tensor product of  $\widehat{CFD}(Y \setminus K)$  with the type  $A$  module from Lemma 10.13, generalizing Proposition 10.10.

In a different direction, Theorem 10.12 gives a technique for studying the knot Floer homology of satellite knots. We return to this point in Section 10.9 (where we give also some examples), contenting ourselves for now with the following corollary.

**Corollary 10.14.** *Suppose that  $K_1$  and  $K_2$  are knots in  $S^3$  such that  $\widehat{CFD}(S^3 \setminus K_1) \simeq \widehat{CFD}(S^3 \setminus K_2)$  (with respect to the 0-framing on  $S^3 \setminus K_1$  and  $S^3 \setminus K_2$ , say). Let  $K_1^C$  (respectively  $K_2^C$ ) denote the satellite of  $K_1$  (respectively  $K_2$ ) with companion  $C$ . Then  $HFK^-(S^3, K_1^C) \cong HFK^-(S^3, K_2^C)$ .*

*Proof.* According to Theorem 10.12, the knot Floer homology of the satellite knot is obtained as the derived tensor product of the type  $D$  module of the companion knot ( $K_1$  or  $K_2$ ) with the type  $A$  module of a doubly-pointed Heegaard diagram for the pattern knot in the solid torus. The result then follows immediately.  $\square$

**10.5. From  $CFK^-$  to  $\widehat{CFD}$ : Statement of results.** We now turn to the opposite direction, computing  $\widehat{CFD}(S^3 \setminus K)$  from  $CFK^-(K)$ . Recall that  $CFK^-(K)$  is a finitely generated, free,  $\mathbb{Z}$ -filtered, graded chain complex over  $\mathbb{F}_2[U]$ . We first look at such complexes in general. Let  $C$  be a finitely-generated free  $\mathbb{F}_2[U]$  module equipped with

- an integer-valued grading, which we refer to as the *Maslov grading*  $M$ ,
- an integer-valued filtration, the *Alexander filtration*  $A$ , and
- a differential, denoted  $\partial$ .

These data satisfy the following compatibility conditions:

- The differential  $\partial$  drops the Maslov grading by one: for homogeneous  $\xi \in C$ ,  $M(\partial\xi) = M(\xi) - 1$ .
- The differential  $\partial$  respects the Alexander filtration:  $A(\partial\xi) \leq A(\xi)$ . (Here  $A(\xi)$  is the lowest degree of the filtration containing  $\xi$ .)
- Multiplication by  $U$  is compatible with the Maslov grading, in the sense that  $M(U \cdot \xi) = M(\xi) - 2$ .
- Multiplication by  $U$  respects the Alexander filtration, in the sense that  $A(U \cdot \xi) \leq A(\xi) - 1$ .

Given such a  $C$ , there are two naturally associated chain complexes. One complex, which we call the *vertical complex*  $C^{\text{vert}} := C/(U \cdot C)$ , is a filtered complex which inherits the Alexander filtration. Its homology is called the *vertical homology*, denoted  $H^{\text{vert}}(C)$ . Next define  $C_0 := C \otimes_{\mathbb{F}_2[U]} \mathbb{F}_2[U, U^{-1}]$ . This has two filtrations, namely the Alexander filtration and the filtration by the powers of  $U$ . Then the *horizontal complex*  $C^{\text{horz}}$  is defined to be the degree 0 part of the associated graded space to  $C_0$  with respect to the Alexander filtration. (Note that  $C^{\text{vert}}$  is the degree 0 part of the associated graded space to  $C_0$  with respect to the filtration by  $U$  powers.) The complex  $C^{\text{horz}}$  still has the filtration by powers of  $U$  and inherits a differential, whose homology is called the *horizontal homology*, denoted  $H^{\text{horz}}(C)$ .

We illustrate  $C$  graphically by plotting a generator over  $\mathbb{F}_2$  of  $C \otimes \mathbb{F}_2[U, U^{-1}]$  of the form  $U^x \cdot \xi_i$  with Alexander depth  $y$  on the plane at the position  $(-x, y)$ . Then the

differential of a generator at  $(x, y)$  can be graphically represented by arrows connecting the point at  $(x, y)$  with the coordinates of other generators. These arrows necessarily point (non-strictly) to the left and down. The complex  $C^{\text{vert}}$  is obtained by restricting to one vertical column, and  $C^{\text{horz}}$  is obtained by restricting to one horizontal row.

**Definition 10.15.** We call  $C$  *reduced* if  $\partial$  strictly drops either the Alexander or  $U$  power filtration; i.e., if  $\partial \mathbf{x} = \mathbf{y}_1 + U \cdot \mathbf{y}_2$  where  $A(\mathbf{y}_1) < A(\mathbf{x})$ .

We call a basis  $\{\xi_i\}$  over  $\mathbb{F}_2[U]$  for  $C$  *vertically simplified* if for each basis vector  $\xi_i$ , either  $\partial \xi_i \in U \cdot C$  or  $\partial \xi_i \equiv \xi_{i+1} \pmod{U \cdot C}$ . In the latter case we say that there is a *vertical arrow* from  $\xi_i$  to  $\xi_{i+1}$ . The *length* of the vertical arrow is the difference  $A(\xi_i) - A(\xi_{i+1})$ .

Similarly, we call a basis  $\{\eta_i\}$  over  $\mathbb{F}_2[U]$  for  $C$  *horizontally simplified* if for each basis vector  $\eta_i$ , either  $A(\partial \eta_i) < A(\eta_i)$  or there is an  $m$  so that  $\partial \eta_i = U^m \cdot \eta_{i+1} + \epsilon$  where  $A(\eta_i) = A(U^m \cdot \eta_{i+1})$  and  $A(\epsilon) < A(\eta_i)$ . In the latter case we say that there is a *horizontal arrow* from  $\eta_i$  to  $\eta_{i+1}$ . The *length* of the horizontal arrow is the integer  $m$ .

*Remark 10.16.* These notions could alternatively be formulated without reference to a basis: vertical arrows correspond to certain non-vanishing differentials in the spectral sequence belonging to the filtered complex  $C/U \cdot C$ , while horizontal arrows correspond to arrows in an analogous spectral sequence for  $C/[(U-1) \cdot C]$ .

Every finitely generated chain complex  $C$  as above is quasi-isomorphic to one which is reduced; moreover, it can also be vertically simplified or horizontally simplified. See Lemma 10.37.

All of these constructions can be carried out for knot Floer homology  $CFK^-(K)$ . Indeed,  $C^{\text{vert}}(CFK^-(K))$  and  $C^{\text{horz}}(CFK^-(K))$  are both canonically isomorphic to  $\widehat{CF}(S^3)$  and so if  $K$  is a knot in  $S^3$ , then  $H^{\text{vert}}(CFK^-(K)) \cong H^{\text{horz}}(CFK^-(K)) \cong \widehat{HF}(S^3) \cong \mathbb{F}_2$ . As such, in a vertically (resp. horizontally) simplified basis  $\xi_i$  (resp.  $\eta_i$ ) for  $CFK^-(K)$ , there is a preferred element which (after reordering) we label  $\xi_0$  (resp.  $\eta_0$ ), characterized by the property that it has no in-coming or out-going vertical (resp. horizontal) arrows.

**Theorem 10.17.** *Let  $CFK^-(K)$  be a model for a chain complex for a knot in  $S^3$  which is reduced. Let  $Y$  be the bordered three-manifold  $S^3 - \text{nb}d K$ , given any sufficiently large integer  $n$ , we consider  $Y$  with framing  $-n$ . The associated type D module  $\widehat{CFD}(Y)$  can be extracted from  $CFK^-(K)$  by the following procedure.*

*In the idempotent  $\iota_0$ ,  $\widehat{CFD}(Y)$  is isomorphic as an  $\mathbb{F}_2$ -module to  $CFK^-(K)$ .<sup>3</sup> Let  $\{\xi_i\}$  be a vertically simplified basis for  $CFK^-(K)$ . To each vertical arrow of length  $\ell$  from  $\xi_i$  to  $\xi_{i+1}$ , we introduce a string of basis elements  $\kappa_1^i, \dots, \kappa_\ell^i$  for  $\iota_1 \widehat{CFD}(Y)$  and differentials*

$$\xi_i \xrightarrow{D_1} \kappa_1^i \xleftarrow{D_{23}} \dots \xleftarrow{D_{23}} \kappa_k^i \xleftarrow{D_{23}} \kappa_{k+1}^i \xleftarrow{D_{23}} \dots \xleftarrow{D_{23}} \kappa_\ell^i \xleftarrow{D_{123}} \xi_{i+1}$$

*where  $\xi_i$  and  $\xi_{i+1}$  are viewed as elements of  $\widehat{CFD}(Y)$ . (The reader is cautioned to note the directions of the above arrows.)*

*Similarly, endow  $CFK^-(K)$  with a horizontally simplified basis  $\{\eta_i\}$ . To each horizontal arrow from  $\eta_i$  to  $\eta_{i+1}$  of length  $\ell$ , we introduce a string of basis elements*

<sup>3</sup>That is, the submodule  $\iota_0 \widehat{CFD}(Y)$  is identified with  $CFK^-(K)$  as an  $\mathbb{F}_2$ -module.

$\lambda_1^i, \dots, \lambda_\ell^i$  connected by a string of differentials

$$\eta_i \xrightarrow{D_3} \lambda_1^i \xrightarrow{D_{23}} \dots \xrightarrow{D_{23}} \lambda_k^i \xrightarrow{D_{23}} \lambda_{k+1}^i \xrightarrow{D_{23}} \dots \xrightarrow{D_{23}} \lambda_\ell^i \xrightarrow{D_2} \eta_{i+1}.$$

(As before,  $\eta_i$  and  $\eta_{i+1}$  are viewed as elements of  $\widehat{CFD}(Y)$ , via the identification of  $\iota_0 \widehat{CFD}(Y)$  with  $CFK^-(K)$ .)

There is a final string, the unstable chain, of generators  $\mu_1, \dots, \mu_m$  connecting  $\xi_0$  and  $\eta_0$ , connected by a string of differentials,

$$\xi_0 \xrightarrow{D_1} \mu_1 \xleftarrow{D_{23}} \mu_2 \xleftarrow{D_{23}} \dots \xleftarrow{D_{23}} \mu_m \xleftarrow{D_3} \eta_0.$$

The length  $m$  of this string is  $2\tau(K) + n + 1$ , where  $n$  is the framing parameter.

*Remark 10.18.* As stated in the theorem above, we prove here the relationship between knot Floer homology and  $\widehat{CFD}$  for all sufficiently negative framing parameters. In [23], we give bimodules which allow us to deduce the type  $D$  module for arbitrary framings, which we summarize in Section 11. In fact, it turns out that most of the above statement goes through for arbitrary integral framings; the only part which depends on the framing is the unstable chain, connecting  $\xi_0$  and  $\eta_0$ , which can be explicitly describe. In particular, for framing  $2\tau(K)$ , this entire chain can be replaced by a single map

$$(10.19) \quad \xi_0 \xrightarrow{D_{12}} \eta_0.$$

See Theorem 11.7 for the complete statement.

It is interesting to note that the above construction of  $\widehat{CFD}$  uses only the vertical and horizontal arrows, while  $CFK^-$  in general can contain diagonal arrows. In concrete terms, the vertical arrows correspond to counting holomorphic disks  $\phi$  with  $n_w(\phi) = 0$ , while horizontal ones correspond to counts of holomorphic disks with  $n_z(\phi) = 0$ . In general,  $CFK^-(K)$  might also contain “diagonal” arrows. (An example is the knot  $9_{41}$  [31, Proposition 6.6].) These arrows may affect the filtered chain homotopy type of  $CFK^-$ , but they do not affect the associated type  $D$  module.

We have illustrated the procedure of Theorem 10.17 in Figure 44.

We will deduce Theorem 10.17 from a more complicated-looking statement (Theorem 10.26, below) which does not make reference to a choice of basis, and which comes more directly from a Heegaard diagram. Before proceeding to the statement and proof, we consider counts of some further holomorphic curves which will be used in the proof of Theorem 10.26. Theorem 10.17 is proved in Section 10.8.

**10.6. Generalized coefficient maps and boundary degenerations.** So far, the holomorphic curves we count in this paper do not cover the basepoint, and in particular do not involve the Reeb chord  $\rho_0$  that contains the basepoint. In this section, we explain certain further structure on the type  $D$  module coming from counting certain holomorphic curves which contain  $\rho_0$  in their boundary as well. These additional maps will be used in Section 10.7 to construct  $\widehat{CFD}$  from  $CFK^-$ . This further structure exists for arbitrary (connected) boundary, but for notational simplicity we will restrict our attention to the torus boundary case.

Fix a bordered Heegaard diagram  $(\mathcal{H}, z)$  for  $(Y^3, T^2)$ , and label the regions around  $\partial\overline{\Sigma}$  as in Section 10.1. Recall that for  $I = \{i_0, \dots, i_n\}$  an interval in  $\{1, 2, 3\}$  there is

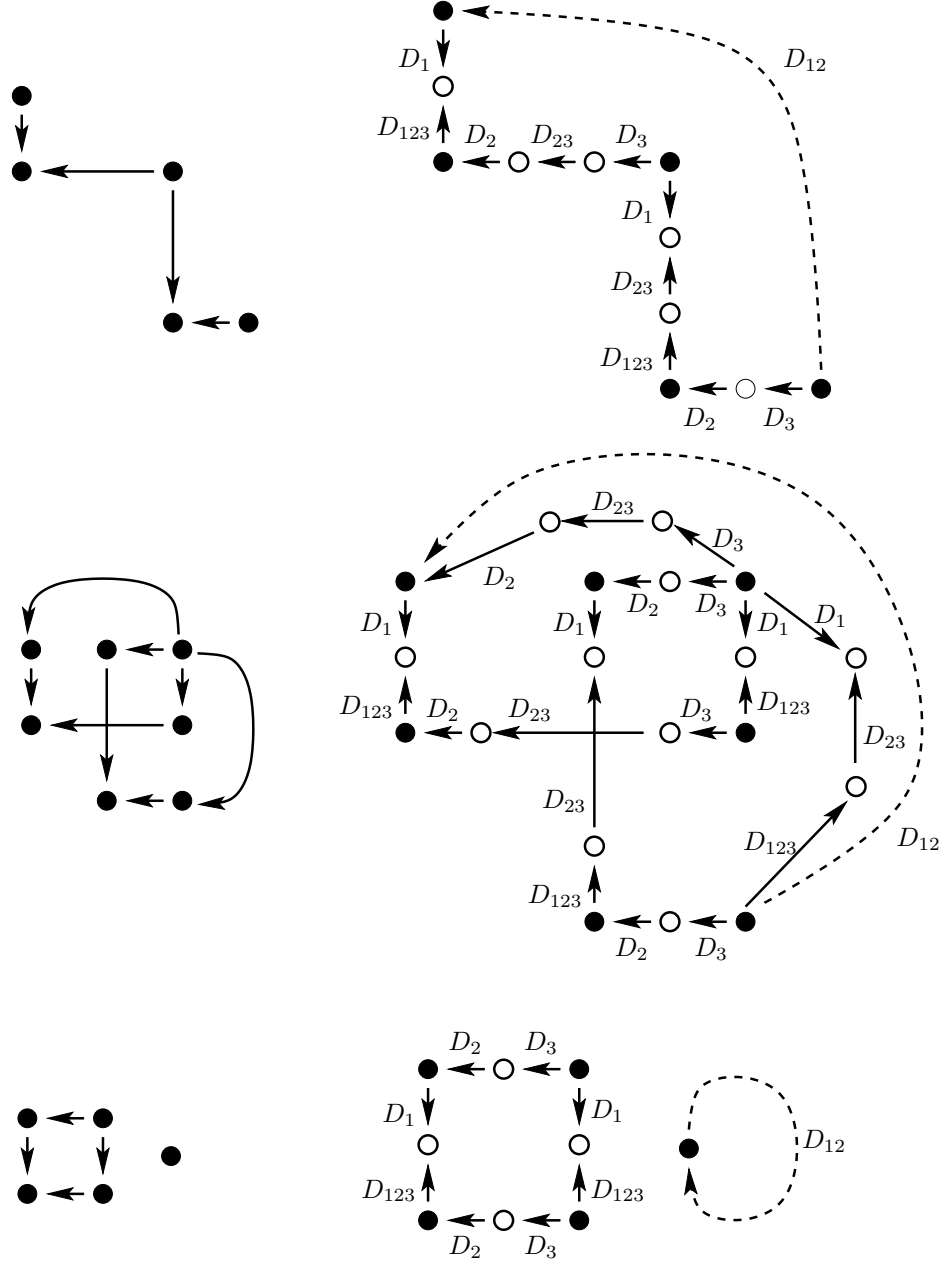


FIGURE 44. **Reconstructing the type  $D$  module.** A table with data from knot Floer homology on the left, where vertical differentials correspond to  $\partial_z$  and horizontal ones corresponding to  $\partial_w$ ; the length of the arrow represents the change in Alexander grading. Top: the torus knot  $T_{3,4}$ . Middle: the  $(2,1)$  cable of  $T_{2,3}$ . Bottom: the figure eight knot. The corresponding type  $D$  modules are on the right, with black dots for generators in the idempotent  $\iota_0$  and white ones for generators in the idempotent  $\iota_1$ . The unstable chain is denoted by the dotted arrow. Here we take framing parameter  $2\tau(K)$  as in Equation (10.19). Explicitly, these are framings 6, 4, and 0, respectively.

an associated map  $D_I: V^{[i_0-1]} \rightarrow V^{[i_n]}$ . This is induced by the differential on  $\widehat{CFD}$ , which in turn is defined by counting holomorphic curves in  $\Sigma \times [0, 1] \times \mathbb{R}$ , cf. Section 6. By moving the basepoint  $z$  from the region labeled 0 to one of the regions labeled 1, 2 or 3, we obtain maps  $D_I: V \rightarrow V$  for intervals  $I$  in  $\{0, 1, 2, 3\}$  (with respect to the cyclic ordering), of length at most 3. (For instance, putting  $z$  in region 1 the new maps we obtain are  $D_0$ ,  $D_{30}$  and  $D_{230}$ .) We will call these new maps *generalized coefficient maps*.

We define four more generalized coefficient maps,  $D_{0123}$ ,  $D_{1230}$ ,  $D_{2301}$  and  $D_{3012}$  as follows. By dropping Condition (9) on holomorphic maps from Section 5.1, we can consider moduli spaces of holomorphic curves asymptotic to Reeb chords  $\rho_0$ ,  $\rho_{01}$  and so on. Collecting these into moduli spaces as in Section 6.1, for  $\mathbf{x}$  a generator of  $\widehat{CFD}(\mathcal{H}, z)$  in idempotent  $\iota_0$ , we set

$$D_{0123}(\mathbf{x}) := \sum_{\substack{\mathbf{y} \in \mathfrak{S}(\mathcal{H}) \\ B \in \pi_2(\mathbf{x}, \mathbf{y}) \\ \vec{\rho} | \text{ind}(B, \vec{\rho}) = 1}} \#(\mathcal{M}^B(\mathbf{x}, \mathbf{y}; \vec{\rho})) \mathbf{y}$$

where the sum over  $\vec{\rho}$  is over  $\{(\rho_0, \rho_1, \rho_2, \rho_3), (\rho_{012}, \rho_3), (\rho_0, \rho_{123})\}$ . (These are the three sequences of realizable Reeb chords for which  $\prod a(-\rho_\alpha) = a(-\rho_{0123})$  in a suitable generalization of  $\mathcal{A}$ .) The maps  $D_{1230}$ ,  $D_{2301}$  and  $D_{3012}$  are defined similarly.

By moving the basepoint  $z$ , it follows from the discussion in Section 10.1 that these maps satisfy the compatibility conditions

$$\sum_{\{I=J \cup K \mid J < K\}} D_K \circ D_J = 0$$

where  $<$  now denotes the cyclic ordering on  $\{0, 1, 2, 3\}$  and  $I$  is any *proper* cyclic interval in  $\{0, 1, 2, 3\}$ . The compatibility equations involving all four Reeb chords are, perhaps, more surprising, as there is now a right hand side.

**Proposition 10.20.** *The maps  $D_{0123}$  and  $D_{2301}$  satisfy*

$$(10.21) \quad D \circ D_{0123} + D_3 \circ D_{012} + D_{23} \circ D_{01} + D_{123} \circ D_0 + D_{0123} \circ D = \mathbb{I}: V^1 \rightarrow V^1,$$

and

$$(10.22) \quad D \circ D_{2301} + D_1 \circ D_{230} + D_{01} \circ D_{23} + D_{301} \circ D_2 + D_{2301} \circ D = \mathbb{I}: V^1 \rightarrow V^1.$$

*The maps  $D_{1230}$  and  $D_{3012}$  satisfy similar formulas.*

(Here,  $\mathbb{I}$  denotes the identity map.)

*Remark 10.23.* The grading shifts of the new coefficient maps may be computed directly. Alternatively, they are uniquely specified by the grading shifts for the original grading shift maps, together with the above formulas, which are homogeneous.

The reason that the right hand sides of Equations (10.21) and (10.22) are the identity map rather than zero is the existence of certain boundary degenerations. We can see these boundary degenerations explicitly in simple cases. For example, let  $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$  be a genus 1 Heegaard diagram for a solid torus—for instance, the diagram in Figure 45. Then  $\Sigma \setminus \boldsymbol{\alpha}$  is a rectangle (disk with four boundary punctures). There is a holomorphic map  $u: S \rightarrow \Sigma \times [0, 1] \times \mathbb{R}$ , where  $S$  is the union of a trivial

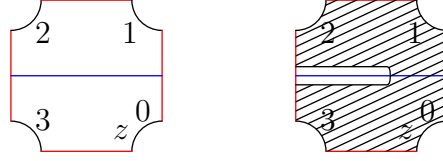


FIGURE 45. Left: a genus 1 Heegaard diagram  $(\Sigma, \alpha, \beta)$  for the solid torus. Here,  $\beta_1$  is disjoint from  $\alpha_1^a$  and intersects  $\alpha_2^a$  in a single point. Right: one point in a one-parameter family of holomorphic curves in  $\Sigma \times [0, 1] \times \mathbb{R}$ , with one end a two-story building and the other end a boundary degeneration.

strip and a rectangle, and the rectangle is mapped in the obvious way to  $\Sigma \setminus \alpha$ , and by a constant map to  $[0, 1] \times \mathbb{R}$ .

Such curves look quite strange in the cylindrical setting. (In particular, it is not clear when they are transversely cut out.) We therefore invoke, for the first and only time in this paper, the *tautological correspondence* between curves in  $\Sigma \times [0, 1] \times \mathbb{R}$  and disks in  $\text{Sym}^g(\Sigma)$ . That is, recall that there is a bijective correspondence

|                                                                                                                                                                                                                   |                       |                                                                                                                                                                                    |
|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-----------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Holomorphic maps<br>$\phi: [0, 1] \times \mathbb{R} \rightarrow \text{Sym}^g(\Sigma)$<br>with $\phi(\{0\} \times \mathbb{R}) \subset \prod \beta_i$<br>and $\phi(\{1\} \times \mathbb{R}) \subset \prod \alpha_i$ | $\longleftrightarrow$ | Holomorphic curves<br>$u$ in $\Sigma \times [0, 1] \times \mathbb{R}$<br>with boundary in $\beta \times \{0\} \times \mathbb{R}$<br>and in $\alpha \times \{1\} \times \mathbb{R}$ |
| so that $\pi_{\mathbb{D}} \circ u$ is a $g$ -fold branched cover.                                                                                                                                                 |                       |                                                                                                                                                                                    |

See, for instance, [20, Section 13] for a detailed explanation. In the bordered setting,  $\text{Sym}^g(\Sigma)$  is a symplectic manifold with cylindrical ends. When the map  $u$  is asymptotic to a Reeb chord between  $\alpha_1^a$  and  $\alpha_2^a$ , the corresponding disk  $\phi$  is asymptotic to a chord between  $\alpha_1^a \times \alpha_1^c \times \cdots \times \alpha_{g-1}^c$  and  $\alpha_2^a \times \alpha_1^c \times \cdots \times \alpha_{g-1}^c$ . The ordering of the Reeb chords induced by the  $t$ -coordinate in the cylindrical setting corresponds to their ordering along  $\{0\} \times \mathbb{R}$  on the boundary of  $\phi$ .

With respect to this correspondence, the boundary degeneration mentioned above comes from a disk in  $\text{Sym}^g(\Sigma)$  with boundary entirely in the  $\alpha$ -tori. There is nothing strange about this disk. In particular, for generic  $J$  on the symmetric product  $\text{Sym}^g(\Sigma)$ , the moduli spaces of these disks will be transversely cut out.

We collect the boundary degenerations in  $\text{Sym}^g(\Sigma)$  into moduli spaces:

**Definition 10.24.** Given  $i = 1$  or  $2$ ,  $\mathbf{x} \in \alpha_i^a \times \alpha_1^c \times \cdots \times \alpha_{g-1}^c$ , a sequence  $\vec{\rho}$  of Reeb chords, and an almost complex structure  $J$  on  $\text{Sym}^g(\Sigma)$ , let  $\mathcal{M}^{[\Sigma]}(\mathbf{x}; \vec{\rho}; J)$  denote the moduli space of  $J$ -holomorphic maps  $\phi: \mathbb{D}^2 \rightarrow \text{Sym}^g(\Sigma)$  such that the homology class of  $\phi$  is  $[\Sigma]$ ,  $\phi$  is asymptotic to  $\vec{\rho}$  at east  $\infty$ , and  $\phi$  passes through  $\mathbf{x}$ .

**Proposition 10.25.** *There is a nonempty open subset of nearly-symmetric almost complex structures  $J$  on  $\text{Sym}^g(\Sigma)$ , as in [33, Section 3], such that the moduli spaces  $\mathcal{M}^{[\Sigma]}(\mathbf{x}; \vec{\rho}; J)$ , where  $\vec{\rho}$  is a cyclic permutation of  $(\rho_0, \rho_1, \rho_2, \rho_3)$  and  $\mathbf{x} \in \mathfrak{S}(\mathcal{H})$ , are transversely cut out and have an odd number of points. Further, for these almost complex structures, all  $\mathcal{M}^{[\Sigma]}(\mathbf{x}; \vec{\rho}; J)$  for other  $\vec{\rho}$  are empty.*

*Proof.* As already discussed, it is standard that  $\mathcal{M}^{[\Sigma]}(\mathbf{x}; \vec{\rho}; J)$  is transversely cut out for generic  $J$ . To prove the moduli space has an odd number of points, we start

by studying a model case. Consider the Heegaard diagram  $(\Sigma_1, \alpha_1, \beta_1)$  shown in Figure 45. In this diagram, there is a single generator  $\mathbf{x}$ . With respect to any almost complex structure on  $\text{Sym}^g(\Sigma_1)$  of the form  $\text{Sym}^g(j_\Sigma)$  there is a one-parameter family of holomorphic disks in  $\text{Sym}^g(\Sigma)$ , in the homology class  $[\Sigma]$ , with asymptotics  $(\rho_0, \rho_1, \rho_2, \rho_3)$ . One end of this moduli space is a broken holomorphic curve and the other end is the boundary degeneration under discussion. It is clear that these curves, including the boundary degeneration, are transversely cut out. So, for an almost complex structure of the form  $\text{Sym}^g(j_\Sigma)$ , the count of boundary degenerations is 1. It follows that the same holds for almost complex structures near  $\text{Sym}^g(j_\Sigma)$ .

Now, for the general case, let  $(\Sigma, \alpha, \beta)$  be any bordered Heegaard diagram with torus boundary. Then  $(\Sigma, \alpha)$  can be decomposed as a connect sum  $(\Sigma_1, \alpha_1) \# (\Sigma_2, \alpha_2)$  where  $(\Sigma_1, \alpha_1)$  is the diagram discussed above and  $\alpha_2$  consists of closed curves. A simple adaptation of the arguments showing stabilization invariance of  $HF$  of closed 3-manifolds [33, Section 10] shows that for almost complex structures with a long neck between  $(\Sigma_1, \alpha_1)$  and  $(\Sigma_2, \alpha_2)$ , there are an odd number of holomorphic curves in  $\mathcal{M}^{[\Sigma]}(\mathbf{x}; (\rho_0, \rho_1, \rho_2, \rho_3); J)$ . Similar arguments apply for other cyclic orderings of  $(\rho_0, \rho_1, \rho_2, \rho_3)$ . This proves the first claim in the proposition.

Curves with other asymptotics (e.g.,  $(\rho_0, \rho_{123})$ ) cannot occur in  $(\Sigma_1, \alpha_1, \beta_1)$  by inspection. Hence, if there is a long neck between  $(\Sigma_1, \alpha_1)$  and  $(\Sigma_2, \alpha_2)$ , they do not occur for  $(\Sigma, \alpha)$ . This proves the second claim in the proposition.  $\square$

*Proof of Proposition 10.20.* We will prove Relation (10.21); the other relations are symmetric. Consider the moduli space

$$\bigcup_{\text{ind}(B, (\rho_0, \rho_1, \rho_2, \rho_3))=2} \mathcal{M}^B(\mathbf{x}, \mathbf{y}; (\rho_0, \rho_1, \rho_2, \rho_3)).$$

For  $\mathbf{x} \neq \mathbf{y}$ , the ends of this moduli space all correspond to terms on the left side of Formula (10.21). By contrast, for  $\mathbf{x} = \mathbf{y}$  it follows from Proposition 10.25, together with a standard gluing argument (compare [34]), and the tautological correspondence that an odd number of ends of the moduli space correspond to boundary degenerations. The result is immediate.  $\square$

**10.7. From  $CFK^-$  to  $\widehat{CFD}$ : Basis-free version.** We can now explain how to extract the type  $D$  module from information on the knot Floer homology. Specifically, we state a variant of Theorem 10.17. The version we prove first is given a more basis-free but perhaps more complicated statement. The translation to the version in Theorem 10.17 is in the next section.

Before stating it, we introduce a little more notation for knot Floer homology.

Fix a doubly pointed Heegaard diagram  $\mathcal{H}$  for  $K$ , with basepoints  $w$  and  $z$ . We write  $C$  for the associated graded space to  $\widehat{CFK}(\mathcal{H})$  and  $C(r)$  for  $\widehat{CFK}(\mathcal{H}, r)$ , the part of  $C$  in Alexander grading  $r$ . There are maps  $\partial^i: C(r) \rightarrow C(r+i)$  defined by counting holomorphic disks  $\phi$  with  $n_z(\phi) = -i$  and  $n_w(\phi) = 0$  if  $i \leq 0$ , and  $n_z(\phi) = 0$  and  $n_w(\phi) = i$  if  $i \geq 0$ . Set  $\partial_w := \sum_{i \geq 0} \partial^i$  and  $\partial_z := \sum_{i \leq 0} \partial^i$ . The maps  $\partial_w$  and  $\partial_z$  each give differentials on  $C$ . Furthermore,  $C$  has two filtrations,

$$C(\geq s) := \bigoplus_{r \geq s} C(r) \quad \text{and} \quad C(\leq s) := \bigoplus_{r \leq s} C(r),$$

preserved by  $\partial_w$  and  $\partial_z$ , respectively. We will principally use not the complex  $(C(\geq s), \partial_w)$ , but the complex  $(C(\leq s), \partial_w)$ , which can be thought of a quotient of  $(C, \partial_w)$  by the subcomplex  $(C(\geq s+1), \partial_w)$ . We likewise need  $(C(\geq s), \partial_z)$ , with a similar definition.

The above constructions can be interpreted using the notions from Section 10.5. The complex  $CFK^-(K)$  is a finitely generated, free,  $\mathbb{Z}$ -filtered,  $\mathbb{Z}$ -graded chain complex over  $\mathbb{F}_2[U]$  satisfying compatibility conditions spelled out in that subsection. The complex  $(C, \partial_w)$  is the horizontal complex associated to  $CFK^-(K)$ , while  $(C, \partial_z)$  is the vertical complex. In particular,  $(C(\geq s), \partial_w)$  is identified with the  $s^{\text{th}}$  summand of the associated graded complex to  $CFK^-(\mathcal{H}, s)$ , while  $(C(\leq s), \partial_z)$  is identified with the  $s^{\text{th}}$  filtered complex of  $\widehat{CFK}(\mathcal{H})$ . Thus  $(C(\geq s), \partial_w)$  and  $(C(\geq s), \partial_z)$  for  $s$  sufficiently small both have homology isomorphic to  $\mathbb{F}_2$ ; likewise for  $(C(\leq s), \partial_z)$  and  $(C(\leq s), \partial_w)$  for  $s$  sufficiently small.

**Theorem 10.26.** *Let  $K \subset S^3$  be a knot with meridian  $\mu$  and 0-framed longitude  $\lambda$ , and let  $C$ ,  $\partial_w$ , and  $\partial_z$  be the data associated as above. Fix a sufficiently large positive integer  $n$ . Then  $\widehat{CFD}(Y \setminus \text{nbid}(K))$ , with framing  $\mu$  and  $-n \cdot \mu + \lambda$ , is homotopy equivalent to the type D module associated to the following data. Let*

$$V^0 := \bigoplus_{s \in \mathbb{Z}} V_s^0 \quad \text{and} \quad V^1 := \bigoplus_{s \in \mathbb{Z} + \frac{n+1}{2}} V_s^1$$

where

$$V_s^0 := C(s)$$

$$V_s^1 := \begin{cases} C(\leq s + \frac{n-1}{2}) & s \leq -\frac{n}{4} \\ \mathbb{F}_2 & |s| < \frac{n}{4} \\ C(\geq s - \frac{n-1}{2}) & s \geq \frac{n}{4} \end{cases}$$

Let  $V := V^0 \oplus V^1$ . Endow  $V$  with coefficient maps as follows:

- The differential  $D$  restricted to  $V_s^0 = C(s)$  is the differential  $\partial^0$  on the knot complex; for  $s \leq -\frac{n}{4}$ ,  $D$  restricted to  $V_s^1 = C(\leq s + \frac{n-1}{2})$  is the differential  $\partial_w$ ; for  $s \geq \frac{n}{4}$ ,  $D$  restricted to  $V_s^1 = C(\geq s - \frac{n-1}{2})$  is the differential  $\partial_z$ ; for  $|s| < \frac{n}{4}$ ,  $D$  restricted to  $V_s^1 = \mathbb{F}_2$  is identically zero.
- The map

$$D_1: V_s^0 = C(s) \longrightarrow V_{s+\frac{n-1}{2}}^1 = C(\geq s)$$

is the obvious inclusion of the subcomplex.

- The map  $D_2$  is nonzero only on  $V_s^1$  for  $s \leq -\frac{n}{4}$ , in which case

$$D_2: V_s^1 = C(\leq s + \frac{n-1}{2}) \longrightarrow V_{s+\frac{n+1}{2}}^0 = C(s + \frac{n+1}{2})$$

is given by

$$\pi \circ \partial_w: C(\leq s + \frac{n-1}{2}) \longrightarrow C(s + \frac{n+1}{2})$$

where  $\pi: C \rightarrow C(s + \frac{n+1}{2})$  is the projection.

- The map

$$D_3: V_s^0 = C(s) \longrightarrow V_{s-\frac{n-1}{2}}^1 = C(\leq s)$$

is the obvious inclusion of the subcomplex.

- The map  $D_{12}$  is identically zero.

- For  $s < -\frac{n}{4}$ , the map

$$D_{23}: V_s^1 = C \left( \leq s + \frac{n-1}{2} \right) \longrightarrow V_{s+1}^1 = C \left( \leq s + \frac{n+1}{2} \right)$$

is the obvious inclusion map; for the  $s$  with  $s \leq -\frac{n}{4} < s+1$ ,

$$D_{23}: V_s^1 = (C, \partial_w) \longrightarrow \mathbb{F}_2$$

is a chain map inducing an isomorphism in homology; for  $-\frac{n+2}{4} < s < \frac{n-2}{4}$ ,  $D_{23}: \mathbb{F}_2 \longrightarrow \mathbb{F}_2$  is the isomorphism; for the  $s$  with  $s < \frac{n}{4} \leq s+1$ ,

$$D_{23}: V_s^1 = \mathbb{F}_2 \longrightarrow (C, \partial_z)$$

is a chain map inducing an isomorphism on homology; for  $s > \frac{n}{4}$ ,

$$D_{23}: V_s^1 = C \left( \geq s - \frac{n-1}{2} \right) \longrightarrow V_{s+1}^1 = C \left( \geq s - \frac{n+1}{2} \right)$$

is the obvious projection map.

- The map

$$D_{123}: V_s^0 \longrightarrow V_{s+\frac{n+1}{2}}^1$$

is the composite of one component of  $\partial_w$ ,

$$\partial^1: C(s) \longrightarrow C(s+1),$$

with the inclusion of  $C(s+1)$  in  $(C(\geq s+1), \partial_z) = V_{s+\frac{n+1}{2}}^1$ .

Then for  $n$  large enough,  $(V, \{D_I\})$  satisfies the compatibility condition (10.4). Moreover, the associated type  $D$  module is homotopy equivalent to  $\widehat{CFD}(S^3 \setminus \text{nbld}(K))$ , marked by the curves  $\mu$  and  $-\mu + \lambda$ . The  $s$  grading on  $V^0 \oplus V^1$  is identified with the homological grading on  $\widehat{CFD}$ , induced from a projection of  $G$  to  $\frac{1}{2}\mathbb{Z}$  which descends to  $G_D(K)$ .

See Figure 46 for a schematic illustration of some of the maps from Theorem 10.26.

In the theorem, we have associated a type  $D$  module to the complex  $C$ , equipped with boundary operators  $\partial_w$  and  $\partial_z$  and filtrations, which in turn can be thought of as associated to  $CFK^-(K)$ , a  $\mathbb{Z}$ -filtered,  $\mathbb{Z}$ -graded complex over  $\mathbb{F}_2[U]$ . It is easy to see that the quasi-isomorphism type of the resulting type  $D$  module  $(V, \{D_I\})$  depends only on the filtered homotopy type of  $CFK^-(K)$ ; i.e., a filtered homotopy equivalence between two  $\mathbb{Z}$ -filtered,  $\mathbb{Z}$ -graded complexes over  $\mathbb{F}_2[U]$   $C$  and  $C'$  induces a homotopy equivalence between their associated type  $D$  modules  $(V, \{D_I\}) \longrightarrow (V', \{D'_I\})$ .

We prove Theorem 10.26 by considering a bordered Heegaard diagram  $\mathcal{H}_K$  for the knot complement with sufficiently large framing  $n$ , in a way analogous to the surgery formula for Heegaard Floer homology [31]. For an analogous choice of bordered Heegaard diagram  $\mathcal{H}$ , the corresponding type  $D$  module  $\widehat{CFD}(\mathcal{H}) = W^0 \oplus W^1$ , has generators which correspond to the generators of  $V = V^0 \oplus V^1$  as stated in the theorem, except for  $W_s^1$  with  $|s| < \frac{n}{4}$ . Moreover, for  $|s| \geq \frac{n}{4}$ , the coefficient maps are as stated in the theorem. We then construct a quasi-isomorphism which fixes the cases with  $|s| < \frac{n}{4}$ . We give the proof after some preliminary lemmas.

To describe our bordered Heegaard diagram we start with a doubly-pointed Heegaard diagram  $(\Sigma_0, \alpha, \beta, z, w)$  for the knot  $K$ . We stabilize this diagram to obtain a new diagram  $\mathcal{H}_K := (\Sigma, \alpha, \beta, z, w)$  so that there is a meridian (which we label  $\alpha_g$ ) which meets a corresponding circle  $\beta_1$  in a single point.

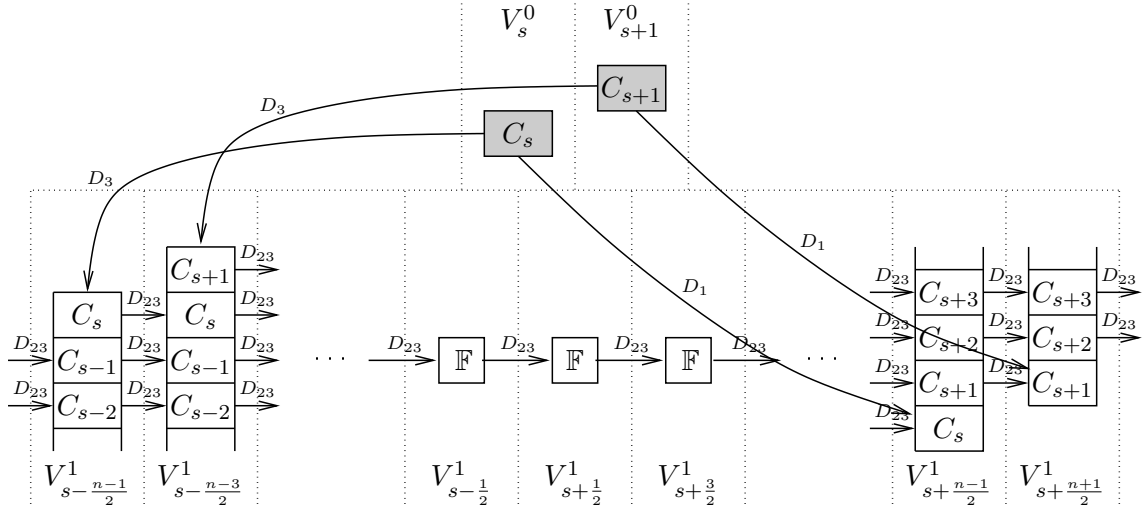


FIGURE 46. A schematic illustration of some of the maps from Theorem 10.26. The horizontal coordinate represents the grading by relative spin<sup>c</sup> structures. The shaded squares correspond to  $V^0$ , while the unshaded ones represent  $V^1$ .

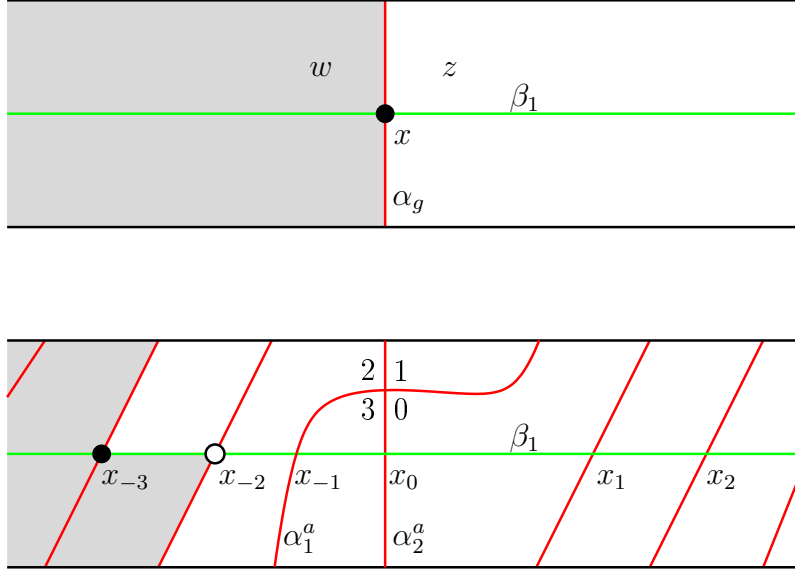


FIGURE 47. **Twisting.** At the top, we have pictured a portion of a doubly-pointed Heegaard diagram for a knot. Below it, we have a bordered diagram for the knot complement, with suitably twisted framing. The gray shading represents a domain from  $\mathbf{a} \times x$  to  $\mathbf{b} \times x$  (for some  $\mathbf{a}$  and  $\mathbf{b}$ ) which, in the top diagram, crosses  $w$  with multiplicity one and  $z$  with multiplicity zero. The corresponding (provincial) domain below connects  $\mathbf{a} \times x_i$  to  $\mathbf{b} \times x_{i+1}$ , provided that  $i \leq -2$ .

Identify a neighborhood  $\mathcal{W}$  of  $\alpha_g$  in  $\Sigma$  with  $[-1, 1] \times S^1$  so that  $\alpha_g$  is identified with  $\{0\} \times S^1$  and  $\beta_1 \cap \mathcal{W}$  is identified with  $[-1, 1] \times \{e^{-\pi i/2}\}$ . Let  $\lambda$  denote a 0-framed longitude of  $K$  in  $\Sigma$ , disjoint from the  $\alpha_i$  for  $i \neq g$ , and with  $\lambda \cap \mathcal{W} = [-1, 1] \times \{1\}$ . Let

$$\bar{\alpha}_1^a = (\lambda \cap (\Sigma \setminus \mathcal{W})) \cup \{(t, e^{\pi i n(t+1)}) \in \mathcal{W} \mid t \in [-1, 1]\}.$$

That is,  $\bar{\alpha}_1^a$  is obtained from the 0-framed longitude  $\lambda$  by winding  $n$  times around the meridian  $\alpha_g$  inside  $\mathcal{W}$ . We refer to  $\mathcal{W}$  as the *winding region*. Note that  $\bar{\alpha}_1^a$  intersects  $\beta_1$  in  $n$  points in  $\mathcal{W}$ , and intersects  $\alpha_g$  in a single point  $p$ . Define  $\alpha_1^a := \bar{\alpha}_1^a \setminus \{p\}$  and  $\alpha_2^a := \alpha_g \setminus \{p\}$ . Then, as in Section 4.2,

$$\mathcal{H}(n) := (\Sigma, \alpha_1^a, \alpha_2^a, \alpha_1, \dots, \alpha_{g-1}, \beta)$$

is a bordered Heegaard diagram for  $S^3 \setminus \text{nbd}(K)$ , with framing  $n$ . We will also write  $\mathcal{H}$  for  $\mathcal{H}(n)$  when the framing is clear. We label the four quadrants around  $p$  by  $0, \dots, 3$ , arranged in a counterclockwise order (i.e., the order induced by  $\partial \text{nbd}(p)$ ), so that, if you forget  $\alpha_2^a$ , 0 and 1 in the region which contains  $z$  in  $\mathcal{H}_K$ . As usual, we place the basepoint in region 0. For convenience of notation, we will henceforth assume that  $n$  is divisible by 4. See Figure 47 for an illustration.

We now turn to the grading on  $\widehat{CFD}(\mathcal{H})$ . Following Section 6.3, the grading takes values in the coset space  $G/P(\mathbf{x}_0)$ , where  $P(\mathbf{x}_0)$  is the span of  $g(B)$  over all periodic domains  $B \in \pi(\mathbf{x}_0, \mathbf{x}_0)$ . For a homology solid torus like a knot complement there is (up to scale) only one periodic domain  $B_0$  not covering  $z$ , which for the diagram  $\mathcal{H}$  above has local multiplicities  $(0, 1, 1 - n, -n)$  in the regions  $(0, 1, 2, 3)$ , respectively. Thus  $P(\mathbf{x}_0) \subset G$  is  $\langle (v; 1, -n) \rangle$  for some  $v \in \mathbb{Z}$  (depending on  $\mathbf{x}_0$ ). For present purposes it suffices to consider only the homological component of the grading on  $\widehat{CFD}(\mathcal{H})$ , which we can recover with the homomorphism from  $G$  to  $\mathbb{Z}$  given by

$$(10.27) \quad (m; p, q) \longmapsto -np - q.$$

(Note that  $P(\mathbf{x}_0)$  is in the kernel of this map.)

More generally, let  $S: \mathfrak{S}(\mathcal{H}) \rightarrow \frac{1}{2}\mathbb{Z}$  be the function characterized (up to overall translation) by the formula

$$(10.28) \quad S(\mathbf{x}) - S(\mathbf{y}) = \left(\frac{n+1}{2}\right) \cdot n_0(B) - \left(\frac{n-1}{2}\right) \cdot n_1(B) - \left(\frac{n+1}{2}\right) \cdot n_2(B) + \left(\frac{n-1}{2}\right) \cdot n_3(B),$$

where  $B$  is any homology class in  $\pi_2(\mathbf{x}, \mathbf{y})$  (possibly covering the basepoint  $z$ ).

**Lemma 10.29.** *Let  $Y$  be a homology solid torus, i.e., a three-manifold with torus boundary and  $H_1(Y; \mathbb{Z}) \cong \mathbb{Z}$ . Let  $\mu$  generate its first homology, and  $\lambda$  generate the kernel of the map on  $H_1$  induced by inclusion of the boundary. Let  $\widehat{CFD}(Y) \cong W^0 \oplus W^1$  be the decomposition by idempotents of the type D structure for  $Y$ , with boundary marked by  $\mu$  and  $-n \cdot \mu + \lambda$ . Then the function  $S$  defined above gives a decomposition*

$$W_s^i \cong \bigoplus_{s \in \frac{1}{2}\mathbb{Z}} W_s^i,$$

well-defined up to overall translation. The coefficient maps respect this grading in the following sense:

$$\begin{aligned} D_1: W_s^0 &\longrightarrow W_{s+\frac{n-1}{2}}^1 \\ D_2: W_s^1 &\longrightarrow W_{s+\frac{n+1}{2}}^0 \\ D_3: W_s^0 &\longrightarrow W_{s-\frac{n-1}{2}}^1 \\ D_{12}: W_s^0 &\longrightarrow W_{s+n}^0 \\ D_{23}: W_s^1 &\longrightarrow W_{s+1}^1 \\ D_{123}: W_s^0 &\longrightarrow W_{s+\frac{n+1}{2}}^1. \end{aligned}$$

*Proof.* The function  $S$  is well-defined since it vanishes on all periodic domains (i.e., the domain with multiplicity 1 everywhere on  $\Sigma$  and the domain with local multiplicities  $(0, 1, 1 - n, -n)$  above). We may assume  $n_0(B) = 0$ . For such a domain  $B$ , the function  $S$  can be obtained from the map in Formula (10.2) followed by the map in Formula (10.27) applied to  $g(B)$ . The result then follows by considering the gradings of the algebra elements from Formula (10.1).  $\square$

Let  $\mathfrak{S}_K$  denote the set of generators for the knot Floer complex  $\widehat{CFK}(\mathcal{H}_K)$ , and  $\mathfrak{S}(n)$  (or just  $\mathfrak{S}$ ) denote the generators for  $\widehat{CFD}(\mathcal{H}(n))$ . Order the intersection points of  $(\alpha_1 \cup \alpha_2)$  with  $\beta_1$  in the winding region in the order they are encountered along  $\beta_1$ ,  $\{x_i\}_{i=-\frac{n}{2}}^{\frac{n}{2}}$ , so that  $x_0$  is the intersection point of  $\alpha_2$  with  $\beta_1$ . For each generator  $\mathbf{x} \in \mathfrak{S}_K$  there are  $n + 1$  corresponding generators in  $\mathfrak{S}(n)$ , gotten by adding a point in  $(\alpha_1 \cup \alpha_2) \cap \beta_1$  in the winding region. That is, we let  $\{\mathbf{x}_i\}_{i=-n/2}^{n/2}$  denote the sequence of generators with the property that the  $(\alpha_1 \cup \alpha_2) \cap \beta_1$ -component of  $\mathbf{x}_i$  is  $x_i$  and, outside the winding region,  $\mathbf{x}_i$  agrees with  $\mathbf{x}$ .

**Lemma 10.30.** *For a given  $n$ , we can normalize the function  $S$  on  $\mathfrak{S}(n)$  so that it is related to the Alexander grading  $A: \mathfrak{S}_K \longrightarrow \mathbb{Z}$  of elements in  $\mathfrak{S}_K$  by the formula*

$$(10.31) \quad S(\mathbf{x}_k) = A(\mathbf{x}) - k - \left( \frac{(n+1) \cdot \text{sgn}(k)}{2} \right),$$

where  $\text{sgn}(k) = -1, 0$ , or  $1$  if  $k < 0, k = 0$ , or  $k > 0$  respectively.

*Proof.* From Figure 47 we see that region 3 is a domain connecting  $\mathbf{x}_0$  to  $\mathbf{x}_{-1}$ , and that for  $k < 0$  there is a domain connecting  $\mathbf{x}_k$  to  $\mathbf{x}_{k-1}$  with multiplicities  $(0, 0, 1, 1)$  in regions  $(0, 1, 2, 3)$ . (See the middle of Figure 48.) Similarly region 1 connects  $\mathbf{x}_0$  to  $\mathbf{x}_1$  and, for  $k > 0$ , the generators  $\mathbf{x}_k$  and  $\mathbf{x}_{k+1}$  are connected by a domain with multiplicities  $(1, 1, 0, 0)$ . It follows from Equation (10.28) that

$$S(\mathbf{x}_k) = S(\mathbf{x}_0) - k - \left( \frac{(n+1) \cdot \text{sgn}(k)}{2} \right).$$

Recall that the Alexander grading on knot Floer homology [31] can be characterized (up to translation) by the property that if  $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$  is any domain in  $\mathcal{H}_K$ , then  $A(\mathbf{x}) - A(\mathbf{y}) + n_z(\phi) - n_w(\phi) = 0$ . We can convert  $\phi$  into a domain  $\phi'$  in  $\mathcal{H}(n)$  that

connects  $\mathbf{x}_0$  to  $\mathbf{y}_0$  with  $n_0(\phi) = n_1(\phi) = n_z(\phi)$  and  $n_2(\phi) = n_3(\phi) = n_w(\phi)$  (as is evident from Figure 47). We conclude that

$$S(\mathbf{x}_0) = A(\mathbf{x}) + c.$$

for some constant independent of the intersection point. For the purpose of our lemma, we choose  $c = 0$ .  $\square$

We now have a collection of generators in  $\mathfrak{S}(n)$  which have corresponding elements of  $\mathfrak{S}_K$ . The remaining generators (coming from intersections of  $\alpha_2^a$  with the  $\beta$ -curves outside of the winding region) are called *exterior* generators.

**Lemma 10.32.** *With respect to the normalization on  $S$  specified in Lemma 10.30, there is a constant  $c$  (independent of  $n$ ) with the property that all exterior generators  $\mathbf{y} \in \mathfrak{S}(n)$  have  $|S(\mathbf{y})| \leq c$ .*

*Proof.* Note that the set of exterior generators is independent of the framing parameter  $n$ . Start with some initial framing parameter  $n_0$ , fix some  $\mathbf{x} \in \mathfrak{S}_K$ , and consider  $\mathbf{x}_{-1} \in \mathfrak{S}(n_0)$ . For each exterior generator  $\mathbf{y}$ , we can find a domain  $B_0 \in \pi_2(\mathbf{y}, \mathbf{x}_{-1})$  with  $n_0(B_0) = n_1(B_0) = 0$  and  $n_2(B_0) = n_3(B_0) = c(\mathbf{y})$ . (Start with an arbitrary domain in  $\pi_2(\mathbf{y}, \mathbf{x}_{-1})$  and add periodic domains to make  $n_0 = n_1 = 0$ ; since  $\mathbf{y}$  and  $\mathbf{x}_{-1}$  are in the same idempotent, we will then have  $n_2 = n_3$ .) Now, in  $\mathcal{H}(n)$ , the domain  $B_0$  can be used to construct a new domain  $B_n \in \pi_2(\mathbf{y}, \mathbf{x}_{\frac{n_0-n}{2}-1})$  with  $n_0(B_n) = n_1(B_n) = 0$  and  $n_2(B_n) = n_3(B_n) = c(\mathbf{y})$ . Since there are only finitely many possible exterior generators, we see that  $|S(\mathbf{y}) - S(\mathbf{x}_{-\frac{n}{2}})|$  on exterior generators  $\mathbf{y}$  is bounded independent of  $n$ . It follows from Lemma 10.30 that  $|S(\mathbf{y})|$  is bounded independent of  $n$ , as claimed.  $\square$

**Lemma 10.33.** *Let  $K \subset Y$  be a knot. Fix a sufficiently large surgery coefficient  $n$  and a bordered Heegaard diagram  $\mathcal{H}$  as above. Then there is an almost complex structure  $J$  on  $\mathcal{H}$  with the following property: For each generator  $\mathbf{x} \in \mathfrak{S}_K$ , with corresponding generators  $\{\mathbf{x}_i\}$  in  $\mathfrak{S}$ , we have:*

- $D_{23}(\mathbf{x}_i) = \mathbf{x}_{i-1}$  if  $i < 0$  and  $S(\mathbf{x}_i) < -c - 1$ ;
- $D_{01}(\mathbf{x}_i) = \mathbf{x}_{i+1}$  if  $i > 0$  and  $S(\mathbf{x}_i) > c + 1$ ;
- $D_3(\mathbf{x}_0) = \mathbf{x}_{-1}$ ; and
- $D_1(\mathbf{x}_0) = \mathbf{x}_1$

where  $c$  is the constant from Lemma 10.32. That is, for each  $\mathbf{x} \in \mathfrak{S}_K$ , we have a string of generators connected by coefficient maps as follows:

$$\cdots \xleftarrow{D_{23}} \mathbf{x}_{-3} \xleftarrow{D_{23}} \mathbf{x}_{-2} \xleftarrow{D_{23}} \mathbf{x}_{-1} \xleftarrow{D_3} \mathbf{x}_0 \xrightarrow{D_1} \mathbf{x}_1 \xrightarrow{D_{01}} \mathbf{x}_2 \xrightarrow{D_{01}} \mathbf{x}_3 \xrightarrow{D_{01}} \cdots$$

*Proof.* The domains realizing the stated maps can be found inside the winding region. For example, the domain realizing  $D_3$  carrying  $\mathbf{x}_0$  to  $\mathbf{x}_{-1}$  is realized by the bigon domain consisting of the region marked by 3 in Figure 47. Domains from  $\mathbf{x}_{-k}$  to  $\mathbf{x}_{-k-1}$  are realized as annuli  $\phi_k$  which have a boundary component in  $\alpha_1^a$  as in the middle of Figure 48. By cutting along  $\alpha_2^a$  at  $\mathbf{x}_{-k}$ , we obtain a bigon realizing a non-zero coefficient of  $\mathbf{x}_{-k-1}$  in  $D_{23}(\mathbf{x}_{-k})$ .

We next show that this is the only term in  $D_{23}(\mathbf{x}_{-k})$ . We first consider output terms of the form  $\mathbf{y}_{-k-1}$ . Considering the corners shows that any domain  $\phi \in \pi_2(\mathbf{x}_{-k}, \mathbf{y}_{-k-1})$

is a disjoint union of  $\phi_k$  with a different positive domain  $\phi'$ . As such, the corresponding moduli space naturally has an  $(\mathbb{R} \oplus \mathbb{R})$ -action (translating the two components independently). If this moduli space is generic and one-dimensional, it follows that the other component  $\phi'$  is trivial, as desired.

Output terms which are exterior generators in  $D_{01}$  and  $D_{23}$  are impossible by Lemma 10.29 and our assumption on  $S(\mathbf{x}_k)$ . For  $D_3$  and  $D_1$  they are similarly impossible for sufficiently large  $n$ .

We must now argue that there are no other possible terms in these maps. The arguments are all similar; we consider  $D_{23}(\mathbf{x}_{-k})$ . This follows from a neck-stretching argument. Specifically, the boundary of the winding region  $\mathcal{W}$  consists of two circles parallel to the meridional  $\alpha$ -circle. Let  $\delta$  be the component on the side of the regions 2 and 3. We degenerate  $\Sigma$  along the curve  $\delta$ . Suppose that there is  $\varphi \in \widetilde{\mathcal{M}}(\mathbf{x}_{-k}, \mathbf{y}_{-k+i})$  with multiplicity +1 in regions 2 and 3 and 0 in regions 0 and 1. Then we will show that by positivity considerations it follows that  $i \geq -1$ ; in fact, we will see that the homology class  $B$  can be constructed as a kind of connected sum of a domain  $B_0 \in \pi_2(\mathbf{x}, \mathbf{y})$  with  $n_w(B_0) = i + 1$  and  $n_z(B_0) = 0$  with a homology class  $B_1 \in \pi_2(x_{-k}, x_{-k+i})$ .

More precisely, let  $B \in \pi_2(\mathbf{x}_{-k}, \mathbf{y}_{-k+i})$  be a homology class giving a term in  $D_{23}(\mathbf{x}_{-k})$ . This homology class has multiplicity zero in regions 0 and 1. Since  $B$  has multiplicity zero at the regions 0 and 1 and no corners outside of the winding region on  $\beta_1$  or  $\alpha_1^a$ , after degeneration  $B$  induces a two-chain  $B_0$  on the destabilized surface  $\Sigma_0$ , having local multiplicity zero at  $z$ , and some multiplicity  $j$  at the point  $w$  (which in turn corresponds to  $\delta$ , shrunk to a point).

The domain at the winding region induces another two-chain, supported in the disk. Specifically, as we degenerate our curve  $\delta$  to a point, we obtain a disk  $\Sigma_1$  whose outer boundary is  $\alpha_2^a$ , the meridian of the knot from Figure 47, with a preferred central point  $q$ , the node obtained by degenerating  $\delta$ . See Figure 48. The two-chain  $B_1$  induced from  $B$  must have local multiplicity  $j$  at  $q$ . There are two arcs in  $\Sigma_1$  connecting  $q$  to the boundary. One is a portion of  $\beta_1$ . The other is a portion of  $\alpha_1^a$  which winds around meeting  $\beta_1$  in a sequence of intersection points  $\{x_k\}_{k=-n/2}^0$ . For each  $k$  there is a homology class  $\phi_k$  connecting  $x_k$  to  $x_{k-1}$ , with multiplicity +1 at the boundary regions 2 and 3, and multiplicity zero at  $q$ . There is also a homology class  $\psi_k$  connecting  $x_{k-1}$  to  $x_k$ , which can be thought of as the complement of  $\phi_k$ . It is easy to see that a positive domain in this disk region which connects two intersections  $x_k$  and  $x_{k+i}$  of the  $\alpha$ - and  $\beta$ -arcs and has multiplicity one in regions 2 and 3 necessarily decomposes as a composite  $\phi_k * \psi_k * \psi_{k+1} * \cdots * \psi_{k+i}$ . In particular, for our initial  $B \in \pi_2(\mathbf{x}_{-k}, \mathbf{y}_{-k+i})$ , we get  $B_1 \in \pi_2(\mathbf{x}_{-k}, \mathbf{x}_{-k+i})$ , which we decompose as  $\phi_k * \psi_k * \psi_{k+1} * \cdots * \psi_{k+i}$ . Consequently, the multiplicities of  $B_1$  near the boundary of the winding region are all equal (and indeed equal to  $i + 1$ ). Thus, the homology class  $B_0$  on  $\Sigma_0$  has multiplicity  $j = i + 1$  at  $w$ .

Now, we claim that  $\text{ind}(B_1, (\rho_2, \rho_3)) = 2i + 3$ . (Note that a term contributing to  $D_{23}$  for the type  $D$  module necessarily has  $(\rho_2, \rho_3)$  as its asymptotics.) This is easy to see by the decomposition  $B_1 = \phi_k * \psi_k * \psi_{k+1} * \cdots * \psi_{k+i}$ :  $\phi_k$  contributes 1 to the Maslov index, and each factor of  $\psi_l$  adds two more.

Suppose now that  $i \geq 0$ , but  $M(B)$  is non-empty for a large values of the stretching parameter. By Gromov compactness, we can extract a convergent subsequence which

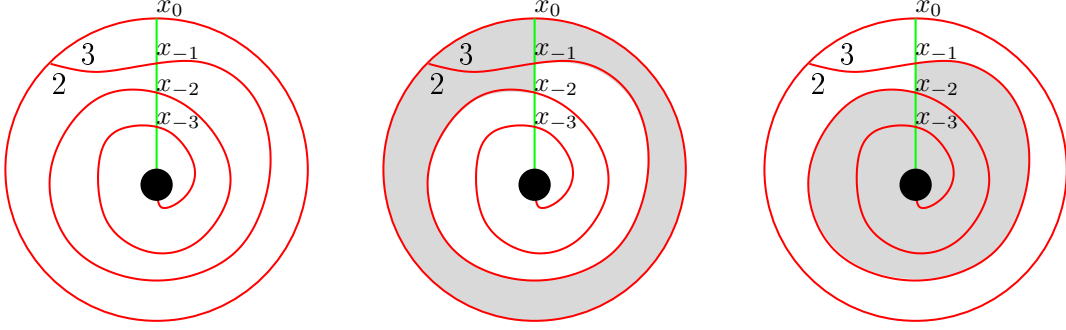


FIGURE 48. The winding region to the left of the meridian, from Figure 47, inverted so that the outer boundary of the winding region corresponds to the dark marked point, and the outer boundary corresponds to the  $\alpha$ -meridional arc. We have illustrated two fundamental homotopy classes:  $\phi_1 \in \pi_2(\mathbf{x}_{-1}, \mathbf{x}_{-2})$  and its complement,  $\psi_1 \in \pi_2(\mathbf{x}_{-2}, \mathbf{x}_{-1})$ .

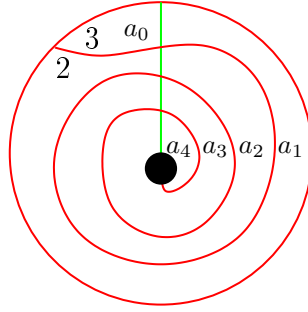


FIGURE 49. Local multiplicities of a region are indicated  $a_0, \dots, a_4$ .

converges to a pair of holomorphic buildings  $u_0$  representing  $B_0$  and  $u_1$  representing  $B_1$ . However, the limiting objects might *a priori* have cuts going through the stretching region. We next show that this does not occur. Suppose that a cut from  $x_{-k}$  or  $x_{-k+i}$  does, in fact, pass through the stretching region. This can happen in one of two ways: either we pass to a limiting holomorphic curve starting at our initial  $\mathbf{x}_{-k}$  at  $-\infty$ , and which contains Reeb chords connecting the  $\alpha$ - and  $\beta$ -arcs going through the region; or it does not, in which case the limit has some Reeb chords which connect some  $\alpha$ -arc or some  $\beta$ -arc to itself.

The former case is excluded by positivity considerations, as follows. For such a configuration, we have  $x_{-k}$  as an initial corner, and the final corner is represented by our Reeb arc connecting  $\alpha_1$  and  $\beta_1$ . Since the local multiplicities around basepoints 2 and 3 are both 1 (for, if it were less, we would have to have split off a component of Maslov index one), it follows that the local multiplicity around the singular point must be negative. More precisely, we number the local multiplicities in the various regions  $\{a_i\}_{i=0}^{n/2+1}$  ordered by distance from the region containing the Reeb chord  $\rho_3$ , as indicated in Figure 49. Note that  $a_1$  coincides with the local multiplicity around  $\rho_2$ . Since  $a_0$  and  $a_1$  both are one, it follows, that  $1 = a_0 = \dots a_1 = \dots = a_k$ . Since  $x_{-k}$  is

an initial corner, it follows that  $a_{k+1} = 0$ , and since there are no other corners it follows that  $a_{k+2} = -1$ . It follows that our curve cannot have holomorphic representatives. (Here we use again the assumption that  $n$  is sufficiently large, to guarantee that  $k + 2 < n/2$  from the assumption on  $S(\mathbf{x}_i)$ .)

In the second case, when there are two cuts connecting either  $\alpha_1^a$  or  $\beta_1$  to itself, we can extract a holomorphic curve  $u_0$  representing  $B_0$ . The source  $S_0$  of this curve has some boundary component which is mapped to, say,  $\alpha_1^a$ . (There is a symmetric argument when the boundary component is mapped to  $\beta_1$  instead). Since  $\mathbf{x}$  and  $\mathbf{y}$  have no components in  $\alpha_1^a \cap \Sigma_0$ , the image of this boundary component in  $u_0$  remains bounded under the projection from  $[0, 1] \times \mathbb{R}$  to  $\mathbb{R}$ . By the maximum principle, we see that in fact this boundary component and therefore the entire curve component must be mapped to a point in  $[0, 1] \times \mathbb{R}$ . In fact, since the multiplicity of this component, which has boundary entirely inside the  $\alpha$  circles, is zero at  $z$ , we conclude that the entire component is mapped to a constant in  $\Sigma_0$ , i.e., it represents the trivial homology class. Thus, after removing the component, we have our desired representative  $u_0$ .

In sum, we have extracted holomorphic representatives  $u_0$  and  $u_1$  for  $B_0$  and  $B_1$ , where cuts do not go through the winding region. Now, it is easy to see that

$$\begin{aligned} \text{ind}(B, (\rho_2, \rho_3)) &= \text{ind}(B_0) + \text{ind}(B_1, (\rho_2, \rho_3)) - 2j \\ &= \text{ind}(B_0) + 1. \end{aligned}$$

Since by hypothesis  $\text{ind}(B) = 1$ , we conclude that  $\text{ind}(B_0) = 0$ . Thus, we have extracted a representative  $u_0$  for the homology class  $B_0 \in \pi_2(\mathbf{x}, \mathbf{y})$  with  $\text{ind}(B_0) = 0$ . This in turn forces the representative of  $B_0$  to be constant, and  $i = -1$ , as needed.

The same considerations apply to the other displayed maps.  $\square$

**Lemma 10.34.** *For all  $n$  sufficiently large, if  $\mathbf{y} \in \mathfrak{S}(n)$  with  $|S(\mathbf{y})| < \frac{n}{4}$  then  $D_{230}(\mathbf{y}) = D_2(\mathbf{y}) = D_0(\mathbf{y}) = D_{012} = 0$ .*

*Proof.* Generators of  $\mathbf{y} \in \mathfrak{S}(n)$  with  $|S(\mathbf{y})| < \frac{n}{4}$  are either exterior generators or of the form  $\mathbf{x}_s$  for some  $\mathbf{x} \in \mathfrak{S}_K$  and  $|s| \geq \frac{n}{4}$ . For any such  $y$ , it is straightforward to see by considering the  $s$  (Alexander) grading that there are no positive domains leaving it which could possibly represent a term in  $D_{230}$ ,  $D_2$ ,  $D_0$ , or  $D_{012}$ .  $\square$

**Lemma 10.35.** *For each positive domain  $B \in \pi_2(\mathbf{x}, \mathbf{y})$  in  $\mathcal{H}_K$  with  $\mathbf{x}, \mathbf{y} \in \mathfrak{S}_K$ ,  $n_z(B) = k \geq 0$ , and  $n_w(B) = 0$ , there is a corresponding sequence of positive domains of the same index  $B_i \in \pi_2(\mathbf{x}_{i+k}, \mathbf{y}_i)$  for  $i \geq 0$ . For  $i > 0$ ,  $B_i$  is provincial; if  $i = 0$  and  $k > 0$ ,  $B_i$  crosses region 0 with multiplicity one. Indeed, if  $B' \in \pi_2(\mathbf{x}_{i+k}, \mathbf{y}_i)$  satisfies*

$$\begin{aligned} n_1(B') &= n_2(B') = n_3(B') = 0 \\ \text{ind}(B', \vec{\rho}(B')) &= 1 \\ \mathcal{M}(B', \vec{\rho}(B')) &\neq \emptyset, \end{aligned}$$

*then  $B' = B_i$  for some  $B \in \pi_2(\mathbf{x}, \mathbf{y})$  and  $\#\mathcal{M}(B', \vec{\rho}(B')) = \#\mathcal{M}(B)$ , where  $\vec{\rho}(B')$  is  $(\rho_0)$  if  $n_0(B') = 1$  and  $\emptyset$  otherwise.*

*Similarly, for each positive domain  $B \in \pi_2(\mathbf{x}, \mathbf{y})$  in  $\mathcal{H}_K$  with  $n_z(B) = 0$  and  $n_w(B) = k \geq 0$ , there is a corresponding sequence of positive domains of the same index  $B_i \in \pi_2(\mathbf{x}_{i-k}, \mathbf{y}_i)$  for  $i \leq 0$ . For  $i < 0$ ,  $B_i$  is provincial; if  $i = 0$  and  $k > 0$ ,  $B_i$*

crosses region 2 with multiplicity one. Indeed, if  $B' \in \pi_2(\mathbf{x}_{i-k}, \mathbf{y}_i)$  satisfies

$$\begin{aligned} n_0(B') &= n_1(B') = n_3(B') = 0 \\ \text{ind}(B', \vec{\rho}(B')) &= 1 \\ \mathcal{M}(B', \vec{\rho}(B')) &\neq \emptyset, \end{aligned}$$

then  $B' = B_i$  for  $B \in \pi_2(\mathbf{x}, \mathbf{y})$  as above, and  $\#\mathcal{M}(B', \vec{\rho}(B')) = \#\mathcal{M}(B)$ , where  $\vec{\rho}(B')$  is  $(\rho_2)$  if  $n_2(B') = 1$  and  $\emptyset$  otherwise.

*Proof.* Given  $B$ , the domain  $B_i$  is constructed using the procedure illustrated by the shaded region in Figure 47.

Suppose next that  $B' \in \pi_2(\mathbf{x}_{-s}, \mathbf{y}_{-s-t})$  for  $s, t \geq 0$  is some Maslov index one homology class which has a holomorphic representative. We claim that  $B' = B_{-s}$  for  $B \in \pi_2(\mathbf{x}, \mathbf{y})$  with  $n_w(B) = k$ ,  $n_z(B) = 0$ , and  $\#\mathcal{M}(B', \vec{\rho}(B')) = \#\mathcal{M}(B)$ .

By stretching normal to the neck region, as in the proof of Lemma 10.33, we see that  $B'$  can be written as a connected sum of a homology class  $B_0$  with a standard chain  $\psi_{s,t} \in \pi_2(x_{-s}, y_{-s-t})$ . We have that  $\text{ind}(\psi_{s,t}, \vec{\rho}(B')) = 2t$ , and  $\text{ind}(\psi) = \text{ind}(\phi) + \text{ind}(\psi_{s,t}) - 2t$ . Thus,  $\text{ind}(B_0) = 1$ .

We claim that moduli space is obtained as a fibered product  $\widetilde{\mathcal{M}}(\varphi_0) \times_{\text{Sym}^t(\mathbb{D})} \widetilde{\mathcal{M}}(\psi_{s,t})$ . This in turn follows from the fact that cuts cannot go through the degenerating point, for if they do, we extract a holomorphic representative  $\phi_0$  for  $B_0$ , which has some limit with the property that the preimage of the degenerating point projects to a collection of points in  $\mathbb{D}$ , some of which lie on the boundary of  $\mathbb{D}$ . As in the proof of Lemma 10.33, any components of this curve which lie on the  $\alpha_g$ -boundary are constant components, and hence they can be discarded. The remaining curve has preimage of the special point projecting to a tuple of points in the interior of  $\mathbb{D}$ . Since the components discarded were constant, it follows that the tuple of points is in fact a  $t$ -tuple.

Having established that our moduli space is identified with the fiber product  $\widetilde{\mathcal{M}}(B_0) \times_{\text{Sym}^t(\mathbb{D})} \widetilde{\mathcal{M}}(\psi_{s,t})$ , we next claim that the map  $\widetilde{\mathcal{M}}(\psi_{s,t}) \rightarrow \text{Sym}^t(\mathbb{D})$ , gotten by projecting the preimage of the connect sum point  $q$ , has degree one (onto  $\text{Sym}^t(\mathbb{D})$ ), so that the moduli space has the same count modulo two as  $\widetilde{\mathcal{M}}(B_0)$ , as desired.

In the case where  $t = 1$ , this can be seen directly. Consider the moduli space  $\psi_{s,1}$ , which is  $\psi_{s-1}$  (in the notation of the proof of Lemma 10.33; see also the right of Figure 48). Our claim is that the map of  $\widetilde{\mathcal{M}}(\psi_s)$  to  $[0, 1] \times \mathbb{R}$  has degree one or, equivalently, after we divide out the space  $\widetilde{\mathcal{M}}(\psi_s)$  by translations, the preimage of  $q$  is mapped with degree one onto the interval  $[0, 1]$ . But the quotient of  $\widetilde{\mathcal{M}}(\psi_s)$  by translations can be parameterized by depth of the cut at  $x_{-s-1}$ . Now, in the limit as the cut goes out to the marked point along the  $\alpha$ -circle, the preimage of  $q$  is mapped (under projection to  $[0, 1]$ ) to  $\{1\}$ ; if the cut goes out along the  $\beta$ -circle, then the preimage of  $q$  is mapped to  $\{0\}$ . Hence the evaluation map has degree one when  $t = 1$  as stated.

For  $t > 1$ , the evaluation map has degree one, as follows. According to the analysis of flowlines,  $\psi_{s,t}$  can be written as a juxtaposition of flowlines with Maslov index two. In fact, any decomposition of  $\psi_{s,t}$  consisting of non-negative and non-constant

homology classes must necessarily be a decomposition into homology classes each of which has non-zero multiplicity at  $q$ . It follows readily that the map  $\widetilde{\mathcal{M}}(\psi_{s,t}) \longrightarrow \text{Sym}^t(\mathbb{D})$  is proper. To calculate its degree, we consider the fibers of points in  $\text{Sym}^t(\mathbb{D})$  where one of the components has  $\mathbb{R}$  coordinate much larger than all the others. Such holomorphic disks are in the image of the gluing map of a moduli space with multiplicity 1 with another moduli space with multiplicity  $t - 1$ . By induction, it follows then that the evaluation map has degree one.  $\square$

We henceforth make the assumption that  $n$  is larger than  $4c$ , where  $c$  is the constant from Lemma 10.32, so that all exterior generators  $\mathbf{x}$  have  $|S(\mathbf{x})| \leq \frac{n}{4}$ . Moreover, we make the additional hypothesis that  $n$  is chosen large enough that Lemmas 10.33 and 10.34 hold.

*Proof of Theorem 10.26.* Let  $W = W^0 \oplus W^1$  be the type  $D$  module for  $\widehat{CFD}(S^3 \setminus \text{nbnd}(K))$ . We see from Lemma 10.30 that  $W_s^0 = C(s)$ , and its differential  $D$  is clearly identified with the differential on knot Floer homology.

Consider next  $\mathbf{x}$  representing elements of  $W^1$ . In the case where  $|S(\mathbf{x})| \geq \frac{n}{4}$ , then, according to Lemmas 10.30 and 10.32, all generators with the same  $S$ -grading are of the form  $\mathbf{x}_t$  for  $\mathbf{x} \in \mathfrak{S}_K$ ; and indeed  $t$  are all either non-positive or they are all positive. In effect, this gives the identification of  $W_s^1$  with  $V_s^1$  for  $|s| \geq \frac{n}{4}$  as stated in the theorem. For instance, for  $s \leq -\frac{n}{4}$ ,  $W_s^1$  contains  $\mathbf{x}_k$  for  $S(\mathbf{x}) = s + \frac{n+1}{2} - k$  and can therefore be identified with  $C(\leq s + \frac{n-1}{2})$ , as desired. Moreover, Lemma 10.35 gives then identifies the differential  $D$  on  $W_s^1$  with the stated differential  $D$  on  $V_s^1$ .

Next, we turn to the other coefficient maps. Any domain representing  $D_1$  must start at a generator of the form  $\mathbf{x}_0$ . By Lemma 10.33, the map  $D_1$  is represented by the bigon from  $\mathbf{x}_0$  to  $\mathbf{x}_1$ . Similarly, any domain representing  $D_3$  must start at a generator of the form  $\mathbf{x}_0$ . We conclude that  $D_1$  and  $D_3$  are the maps from  $\mathbf{x}_0$  to  $\mathbf{x}_1$  and  $\mathbf{x}_{-1}$ , respectively, as desired.

Next, any domain representing  $D_2$  must start at a generator of the form  $\mathbf{x}_s$  with  $s < 0$  and end at a generator of the form  $\mathbf{y}_0$ . By Lemma 10.35, such a domain must be induced by some domain  $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$  with  $n_z(\phi) = 0$  and  $n_w(\phi) = k \geq 1$ .

An analogous argument shows that  $D_{123}$  has the stated form, except that we necessarily have  $n_w(\phi) = 1$ .

There are no domains which could represent a map of the form  $D_{12}$ , for such a domain would need to have a cut going out to the intersection point  $x_0$ .

The fact that  $D_{23}$  has the stated form on  $\mathbf{x}_t$  with  $t < 0$  was established in Lemma 10.33. We claim that

$$D_{23}: W_s^1 \longrightarrow W_{s+1}^1$$

has the stated form for  $s \geq \frac{n}{4}$ , as well.

We see this in two steps. First, it is clear that  $D_{23}(\mathbf{x}_1) = 0$  for all  $\mathbf{x} \in \mathfrak{S}_K$  (as there are no representing domains); i.e.,  $D_{23}$  vanishes on  $C(s - \frac{n-1}{2}) \subset W_s^1$ . Since we know the form of  $D_{01}$  by Lemma 10.33, it then suffices to prove that

$$(10.36) \quad D_{23} \circ D_{01}(\mathbf{x}_t) = \mathbf{x}_t$$

if  $t \geq 1$ . We start by considering Formula (10.21) from Proposition 10.20 with input and output both equal to the generator  $\mathbf{x}_t$ . We assume first that  $t > 1$ . Then, by a neck-stretching argument, contributions from  $D \circ D_{0123}$  in Formula (10.21) mapping

$\mathbf{x}_t$  to  $\mathbf{x}_t$  can be identified with points in  $\mathcal{M}(\xi_1) \times \mathcal{M}(\xi_2)$ , where  $\xi_1 \in \pi_2(\mathbf{x}, \mathbf{y})$  and  $\xi_2 \in \pi_2(\mathbf{y}, \mathbf{x})$  are homology classes in  $\mathcal{H}_K$  whose sum covers the surface with multiplicity one, so that  $\xi_2$  covers  $w$  with multiplicity one. Similarly, contributions from  $D_{0123} \circ D$  can be identified with points in  $\mathcal{M}(\eta_1) \times \mathcal{M}(\eta_2)$ , where  $\eta_1 \in \pi_2(\mathbf{x}, \mathbf{y})$  and  $\eta_2 \in \pi_2(\mathbf{y}, \mathbf{x})$  are homology classes in  $\mathcal{H}_K$  whose sum covers the surface with multiplicity one, so that  $\eta_1$  covers  $w$  with multiplicity one. Thus, we can conclude that these two terms cancel from the fact that  $\partial^2 = 0$  on the knot complex. The term  $D_3 \circ D_{012}(\mathbf{x}_t)$  cannot contribute because there is no generator with Alexander grading of  $D_{012}(\mathbf{x}_t)$ . Finally, the term  $D_{123} \circ D_0(\mathbf{x}_t)$  cannot contribute because  $\mathbf{x}_t$  for  $t > 1$  cannot be in the image of  $D_{123}$ . The remaining terms are as desired, thus proving that the domain  $\Sigma \setminus \phi_t$  has a holomorphic representative.

We must now argue that there are no other domains which can contribute to  $D_{23}$ . To this end, suppose that  $\phi \in \pi_2(\mathbf{x}_t, \mathbf{y})$  is any homology class which has representatives. Since the asymptotics at  $p$  are  $(\rho_2, \rho_3)$ , there is a cut along  $\alpha_1^a$  from  $p$ , through the complement of  $\mathcal{W}$ , to  $x_t$ . After stretching the neck, this cut gives a boundary component which lies entirely on  $\alpha_1^a$ , and so a component mapping to a point in  $[0, 1] \times \mathbb{R}$ . After removing this component, the remainder has expected dimension zero, hence must be constant. It follows that for  $t > 1$ ,  $D_{23} \circ D_{01}(\mathbf{x}_t) = \mathbf{x}_t$ , and so for  $r > 2$ ,  $D_{23}(\mathbf{x}_r) = \mathbf{x}_{r-1}$ .

The case of  $D_{23}(\mathbf{x}_2)$  follows similarly, except that  $D_{123} \circ D_0$  may contribute. But these contributions all correspond to points in  $\mathcal{M}(\zeta_1) \times \mathcal{M}(\zeta_2)$ , where  $\zeta_1 \in \pi_2(\mathbf{x}, \mathbf{y})$  and  $\zeta_2 \in \pi_2(\mathbf{y}, \mathbf{x})$  are domains in  $\mathcal{H}_K$ , and  $\zeta_1$  contains  $z$  (and  $\zeta_2$  contains  $w$ ). These terms cancel with terms in  $\mathcal{M}(\eta_1) \times \mathcal{M}(\eta_2)$  as before.

We have now shown that all the coefficient maps have the right form for  $W_s^1$  with  $|s| \geq \frac{n}{4}$  and for  $W_s^0$ . We claim next that for all  $|s| \leq \frac{n}{4}$ ,  $H_*(W_s^1, D) \cong \mathbb{F}_2$ , and in fact

$$D_{23}: W_s^1 \longrightarrow W_{s+1}^1$$

is a chain map (with respect to  $D$ ) inducing an isomorphism on homology. We investigate the terms in Equation (10.21) and Equation (10.22), bearing in mind Lemma 10.34. Specifically, for  $\mathbf{x}$  with  $|S(\mathbf{x})| \leq \frac{n}{4}$ ,  $D_3 \circ D_{012}(\mathbf{x}) = 0$ ,  $D_1 \circ D_{230}(\mathbf{x}) = 0$ , and  $D_{301} \circ D_2(\mathbf{x}) = 0$  for degree reasons (comparing Lemma 10.29 with the observation that  $W_s^0 = 0$  for all  $|s| \geq \frac{n}{4}$ ). Similarly, since

$$D_0: W_s^1 \longrightarrow W_{s-\frac{n+1}{2}}^0,$$

(by the same considerations as in Lemma 10.29), it follows that  $D_{123} \circ D_0(\mathbf{x}) = 0$  for  $|S(\mathbf{x})| \leq \frac{n}{4}$ . Thus  $D_{01}$  and  $D_{23}$  are homotopy inverses in this range, and so the map induced on homology by  $D_{23}: W_s^1 \longrightarrow W_{s+1}^1$  is an isomorphism. Then since  $W_{-\frac{n+1}{4}}^1$  is  $(C, \partial_w)$ , a chain complex with homology isomorphic to  $\mathbb{F}_2$ , we can conclude that

$$H_*(W_s^1, D) \cong \mathbb{F}_2$$

for all  $|s| \leq \frac{n}{4}$ .

It follows that for  $|s| \leq \frac{n}{4}$ , we have a chain map  $\phi: W_s^1 \longrightarrow \mathbb{F}_2$  inducing an isomorphism on homology. The map

$$\Phi: W \longrightarrow V$$

given by

$$\Phi(\mathbf{x}) = \begin{cases} \phi(\mathbf{x}) & \text{if } \mathbf{x} \in W_s^1 \text{ with } |s| < \frac{n}{4} \\ \mathbf{x} & \text{otherwise} \end{cases}$$

induces a homotopy equivalence between the type  $D$  module determined by  $W$  and the one determined by  $V$ .  $\square$

**10.8. Proof of Theorem 10.17.** We next deduce Theorem 10.17 from Theorem 10.26. But first, we show that  $CFK^-(K)$  can be horizontally or vertically reduced, as in Definition 10.15.

**Lemma 10.37.** *Let  $C$  be a  $\mathbb{Z}$ -filtered,  $\mathbb{Z}$ -graded, finitely generated chain complex over  $\mathbb{F}_2[U]$  which is free as an  $\mathbb{F}_2[U]$ -module. Then  $C$  is  $\mathbb{Z}$ -filtered,  $\mathbb{Z}$ -graded homotopy equivalent to a chain complex  $C'$  which is reduced. Further, one can choose a basis for  $C'$  over  $\mathbb{F}_2[U]$  which is vertically simplified or, if one prefers, a basis which is horizontally simplified instead.*

*Proof.* We show that  $C$  can be reduced by induction on the rank of  $C$ . If  $C$  is not reduced, by definition it admits a non-trivial differential which does not change the Alexander filtration, so that  $\xi \in C$  with  $\partial\xi \neq 0$  and  $A(\xi) = A(\partial\xi)$ . Then  $\xi$  and  $\partial\xi$  generate a subcomplex of  $C$  whose quotient complex  $Q$  is homotopy equivalent to  $C$  with rank two less than the rank of  $C$ .

Next, we argue that  $C$  can be vertically simplified. Consider  $C^{\text{vert}} = C/U$ . Find a basis  $\{\xi_2, \xi_4, \dots, \xi_{2m}\}$  for the image of  $\partial^{\text{vert}}$ , and extend this by vectors  $\{\xi_1, \xi_3, \dots, \xi_{2m-1}\}$  with  $\partial^{\text{vert}}(\xi_i) = \xi_{i+1}$ . (These vectors are linearly independent of the earlier ones, since  $\partial^{\text{vert}}$  is a differential.) Complete this basis to a basis of  $C$  by vectors in the kernel of  $\partial^{\text{vert}}$ . The resulting basis for  $C^{\text{vert}}$  can be thought of as a basis for  $C$ , as well.

A similar argument applies for  $C^{\text{horz}}$ .  $\square$

*Proof of Theorem 10.17.* We will write  $\widehat{CFD}$  for  $\widehat{CFD}(S^3 \setminus \text{nbdd}(K))$  with a suitably large framing  $n$ . Let  $C$  be a reduced complex homotopy equivalent to  $CFK^-(K)$ , and let  $\{\xi_i\}$  and  $\{\eta_i\}$  be bases of  $C$  which are vertically and horizontally simplified, respectively. These bases are related by

$$(10.38) \quad \xi_i = \sum_j \Lambda_i^j \eta_j$$

for a suitable change of basis matrix  $\Lambda$ . As in Theorem 10.26,  $\{\xi_i\}$  and  $\{\eta_i\}$  can also be thought of as two different bases for  $V_s^0$  over  $\mathbb{F}_2$  or, alternatively, two bases for  $\iota_0 \widehat{CFD}$  over  $\mathcal{A}(\mathbb{T})$ . We will modify and extend these to two bases for all of  $\widehat{CFD}$  which

- on  $\iota_0 \widehat{CFD}$ , are still related by Formula (10.38),
- agree on  $\iota_1 \widehat{CFD}$ , and
- split as a direct sum of a contractible complex and a complex with the coefficient maps from Theorem 10.17.

Recall from Remark 10.6 that the coefficient maps depend the subspace of  $\widehat{CFD}$  used to define them, which we take to be the  $\mathbb{F}_2$ -span of the basis.

To give an initial basis for  $\iota_1 \widehat{CFD}$ , recall that all  $\xi_i$  except for  $\xi_0$  come in pairs  $\xi_j$  and  $\xi_{j+1}$ , where  $\partial_z \xi_j = \xi_{j+1}$ . Fix  $i$  so that there is a height  $\ell$  vertical arrow from  $\xi_i$  to  $\xi_{i+1}$ . Then the generators for  $V_s^1$  for  $s > 0$  coming from this pair have coefficient maps like

$$(10.39) \quad \begin{array}{ccccccccccccccc} \xi_i & \xrightarrow{D_1} & \kappa_1^i & \xleftarrow{D_{23}} & \dots & \xleftarrow{D_{23}} & \kappa_k^i & \xleftarrow{D_{23}} & \kappa_{k+1}^i & \xleftarrow{D_{23}} & \dots & \xleftarrow{D_{23}} & \kappa_\ell^i & \xleftarrow{D_{23}} & \alpha^i & \xleftarrow{D_{23}} & \dots \\ & & & & & & & & & & & & & & \downarrow D_\emptyset & & \downarrow D_\emptyset \\ & & & & & & & & & & & & & & \xi_{i+1} & \xrightarrow{D_1} & \beta^i & \xleftarrow{D_{23}} & \dots \end{array}$$

That is, the only time we get a contribution from this pair to the  $D_\emptyset$ -homology of  $V_s^1$  for  $s > 0$  is for  $A(\xi_{i+1}) + (\frac{n-1}{2}) < s \leq A(\xi_i) + (\frac{n-1}{2})$ . We call these generators  $\kappa_k^i$  as in the statement of the theorem, where  $k = A(\xi_i) - s + (\frac{n-1}{2}) + 1$ . For  $s = A(\xi_{i+1}) + (\frac{n-1}{2})$ , we have a canceling pair of generators  $\alpha^i$  (from  $\xi_i$ ) and  $\beta^i$  (from  $\xi_{i+1}$ ). We also have canceling pairs for smaller  $s$ .

Similarly, for a horizontal arrow from  $\eta_i$  to  $\eta_{i+1}$  of width  $\ell$ , we get generators and coefficient maps like

$$(10.40) \quad \begin{array}{ccccccccccccccc} \eta_i & \xrightarrow{D_3} & \lambda_1^i & \xrightarrow{D_{23}} & \dots & \xrightarrow{D_{23}} & \lambda_k^i & \xrightarrow{D_{23}} & \lambda_{k+1}^i & \xrightarrow{D_{23}} & \dots & \xrightarrow{D_{23}} & \lambda_\ell^i & \xrightarrow{D_{23}} & \gamma^i & \xrightarrow{D_{23}} & \dots \\ & & & & & & & & & & & & \downarrow D_2 & \downarrow D_\emptyset & \downarrow D_\emptyset \\ & & & & & & & & & & & & \eta_{i+1} & \xrightarrow{D_3} & \delta^i & \xrightarrow{D_{23}} & \dots \end{array}$$

(The  $\lambda_k^i$  are elements of  $V_s^1$  with  $s < 0$ .) In particular, there is an undesirable term in  $D_3(\eta_{i+1})$ . In addition, we usually have  $D_{123}(\eta_i) = 0$ ; the exception is when the arrow has width equal to one, when

$$(10.41) \quad D_{123}(\eta_i) = D_1(\eta_{i+1}).$$

Consider next  $\xi_0$ , which has no vertical differential either into or out of it. We see that its corresponding element comes from a chain

$$\dots \xrightarrow{D_{23}} \varphi_3 \xrightarrow{D_{23}} \varphi_2 \xrightarrow{D_{23}} \varphi_1 \xleftarrow{D_1} \xi_0,$$

where the chain of elements  $\varphi_i$  extends back to  $V_s^1$  where  $s$  is large in magnitude (and hence  $V_s^1$  is one-dimensional). Similarly, consider the element  $\eta_0$  which has no horizontal differential either into or out of it. (Remember that  $\xi_0$  and  $\eta_0$  can coincide.) Associated to  $\eta_0$  is a similar string

$$\eta_0 \xrightarrow{D_3} \mu_1 \xrightarrow{D_{23}} \mu_2 \xrightarrow{D_{23}} \mu_3 \xrightarrow{D_{23}} \dots$$

where the chain of elements  $\mu_i$  extends forward to  $V_s^1$  where  $s$  is large in magnitude (and hence  $V_s^1$  is one-dimensional). Thus, for some sufficiently large  $a$  and  $b$ , we must have that  $\varphi_a = \mu_b$ ; i.e., we can connect the two chains to a single one of the form

$$\eta_0 \xrightarrow{D_3} \mu_1 \xrightarrow{D_{23}} \mu_2 \xrightarrow{D_{23}} \mu_3 \xrightarrow{D_{23}} \dots \xrightarrow{D_{23}} \mu_m \xleftarrow{D_1} \xi_0$$

We have now constructed bases for all the  $V_s^1$ :

- $\lambda_k^i$  and  $\mu_k$  for  $s \leq -n/4$ ,
- $\phi_k = \mu_k$  for  $-n/4 < s < n/4$ , and
- $\kappa_k^i$  and  $\phi_k$  for  $s \geq n/4$ .

However, we have extra generators for the modules and terms in the coefficient maps. To bring the module to the desired form, we must change basis so as to

- split off the chain of canceling generators in (10.40) by arranging for  $D_3(\eta_{i+1})$  and  $D_{23}(\lambda_\ell^i)$  to vanish;
- eliminate the generators with canceling differentials in (10.39), in particular  $\alpha^j$  and  $\beta^j$ ; and
- bring  $D_{123}$  to the desired form, by eliminating the non-trivial terms belonging to width one arrows (Equation (10.41)).

We do this by successively changing basis (and, at some point, dropping out  $\beta^j$ ).

We start by modifying the base to eliminate the unwanted terms in  $D_3$  (from  $\eta_{i+1}$  when there is a horizontal arrow from  $\eta_i$ ). To this end, we change basis by

$$\eta'_{i+1} := \eta_{i+1} + \rho_3 \cdot \gamma^i.$$

With this substitution, we have that  $D_3(\eta'_{i+1}) = 0$ ,  $D_2(\lambda_\ell^i) = \eta'_{i+1}$ , and  $D_{23}(\lambda_\ell^i) = 0$ . Let  $\{\xi'_j\}$  denote the corresponding change of basis applied to the  $\{\xi_j\}$  so that Formula (10.38) remains true. Note that the other coefficient maps are unchanged; in particular  $D_1(\xi'_j) = D_1(\xi_j)$ , and there are no coefficient maps other than  $D_2$  which enter  $\eta'_{i+1}$ . Also, the chain of canceling generators from this horizontal arrow now splits off as a direct sum, so we can drop them.

We pass now to a submodule to eliminate  $\alpha^i$  and  $\beta^i$ . To do this, for each vertical arrow from  $\xi_i$  to  $\xi_{i+1}$  of height  $\ell$ , (and corresponding new elements  $\xi'_i$  and  $\xi'_{i+1}$ ), we replace  $\xi'_{i+1}$  by

$$\xi''_{i+1} := \xi'_{i+1} + \rho_1 \cdot \alpha^i.$$

Now we have  $D_{123}(\xi''_{i+1}) = \kappa_\ell^i + D_{123}(\xi_{i+1})$ . Again, there is a corresponding modification to the other generators, replacing  $\eta'_i$  by  $\eta''_i$ . Now the submodule without the chain of canceling generators from (10.39) is easily seen to be homotopy equivalent to the original module.

Finally, we wish to bring  $D_{123}$  to the desired form, eliminating the terms from horizontal arrows of length one. This is done as follows. Suppose that  $\xi_i$  is the initial point of some vertical arrow. Replace  $\kappa_1^i$  by

$$\tilde{\kappa}_1^i := \kappa_1^i + \rho_{23} \cdot D_{123}(\xi_i).$$

With this new basis, then,  $D_{123}(\xi''_i) = 0$  (as its contribution is absorbed into  $D_1$ ). Moreover, this change does not affect any of the other coefficient maps.

It remains to make one last change of basis, corresponding to terminal points  $\xi_{i+1}$  of vertical arrows, so that  $D_{123}(\xi_{i+1})$  is the element taking the place of  $\kappa_\ell^i$ . A priori we have  $D_{123}(\xi''_{i+1}) = \kappa_\ell^i + D_{123}(\xi_{i+1})$ . In the case where the length of the arrow from  $\xi_i$  to  $\xi_{i+1}$  is greater than one, we make the change by letting

$$\tilde{\kappa}_\ell^i := \kappa_\ell^i + D_{123}(\xi_{i+1}).$$

Since  $D_{123}(\xi_{i+1})$  is in the kernel of  $D_{23}$  (from the description in Theorem 10.26), all other coefficient maps remain unchanged.

In the case where the length of the arrow from  $\xi_i$  to  $\xi_{i+1}$  is one, proceed as follows. First, we find an element  $\zeta_i$  so that

$$\partial \zeta_i = \rho_{23} \cdot D_{123}(\xi_{i+1}).$$

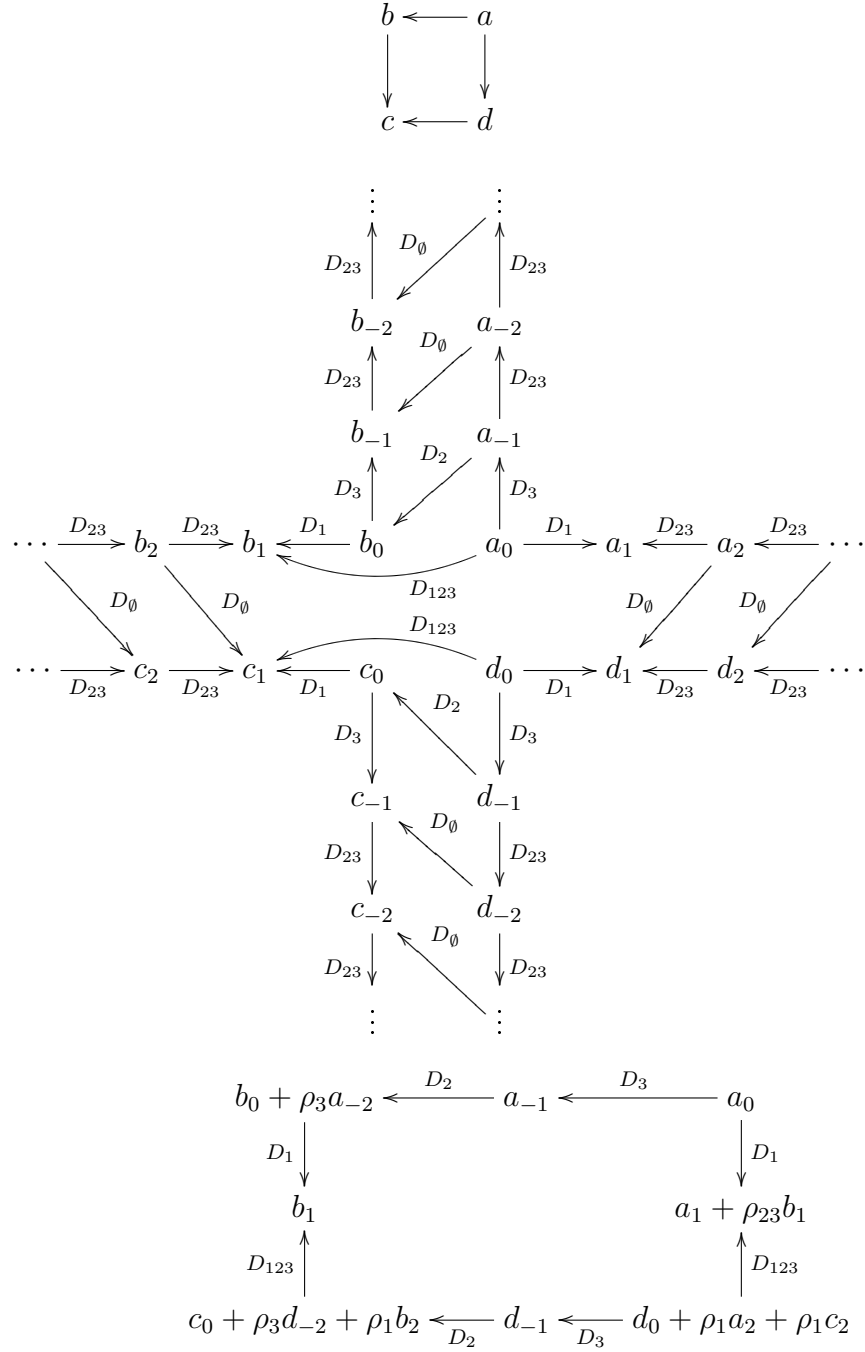


FIGURE 50. **An illustration of the proof of Theorem 10.17.** The top illustrates a summand in the knot complex; the middle illustrates a corresponding piece coming from Theorem 10.26; the bottom is obtained by making the substitutions coming from the proof of Theorem 10.17, and dropping acyclic summands.

In particular,  $D_0(\zeta_i) = 0$  and  $D_{23}(\zeta_i) = D_{123}(\xi_{i+1})$ . This element is found as follows. Recall that  $\xi_i$  and  $\xi_{i+1}$  correspond to elements  $\mathbf{x}_i$  and  $\mathbf{x}_{i+1}$  in  $C(s)$  and  $C(s-1)$ , respectively. Let  $\mathbf{y} := \partial^1(\mathbf{x}_i) \in C(s+1)$ , i.e., the image of  $\mathbf{x}_i$  under the map which counts holomorphic disks which cross  $w$  exactly once. (Recall from Theorem 10.26 that  $D_{123}(\xi_{i+1})$  is  $\partial^1(\mathbf{x}_{i+1})$  considered as an element of  $C(\geq s)$ .) We then let  $\zeta_i = \partial_z(\mathbf{y})$ , where we consider  $\mathbf{y}$  as a chain in  $C(\geq s-1)$ . Then  $\zeta_i$ , thought of as a generator of the type  $D$  module, has the desired properties. Making the substitution

$$\xi_{i+1}''' := \xi_{i+1}'' + \rho_1 \cdot \zeta_i,$$

we see that  $D_1(\xi_{i+1}''') = D_1(\xi_{i+1}'')$  and  $D_{123}(\xi_{i+1}''')$  has the desired form.

This module now has the form promised in Theorem 10.17.

As a final point, we can relate the length  $m$  of the chain of  $\mu_i$  to the framing parameter and the knot invariant  $\tau(K)$ , as follows. We let  $\mathbf{x}$  be some intersection point which appears with non-zero multiplicity in  $\xi_0$ , and  $\mathbf{y}$  be one which appears with non-zero multiplicity in  $\eta_0$ . The number  $2\tau(K)$  is the Alexander difference between  $\mathbf{x}$  and  $\mathbf{y}$ . Concretely, there is some  $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$  with  $n_w(\phi) = -\tau(K)$  and  $n_z(\phi) = \tau(K)$ . By adding copies of  $\Sigma$ , we can turn this into a domain connecting the corresponding intersection points, thought now as generators for  $\widehat{CFD}(K)$ , whose local multiplicity is zero near 0 and 1, and whose local multiplicity at the regions 2 and 3 is  $2\tau(K)$ . Combining this with our chain of domains going through the  $\mu_i$  gives a periodic domain with local multiplicities  $(0, 1, 1-n, -n)$  at the four regions around  $(0, 1, 2, 3)$ , where  $n = m + 2\tau(K)$ .  $\square$

In Figure 50 we have illustrated some of the changes of basis from Theorem 10.17 in the case where the knot complex has a summand which is a square, as illustrated on the top of the figure. (Such a summand appears, e.g., in the knot Floer complex for the figure eight knot, on the bottom of Figure 44.) Initially, Theorem 10.26 gives a corresponding summand as illustrated in the middle of Figure 50. Going through the cancellations prescribed in the proof of Theorem 10.17, we end up with the complex on the bottom of Figure 50.

**10.9. Cables revisited.** In this subsection, we illustrate the power of the above arguments by computing  $HFK^-$  of the  $(2, 1)$ -cable of a knot  $K$  in  $S^3$  in terms of  $CFK^-(S^3, K)$ . The techniques extend to give a formula for  $HFK^-$  of any particular satellite in terms of  $CFK^-$  of the pattern.

Figure 51 is a Heegaard diagram for the  $(2, 1)$  cabling operation. Since this diagram has genus equal to one, the holomorphic curves are straightforward to count. In fact, the type  $A$  module  $CFA^-(C)$  associated to this diagram is given as follows. It has three generators  $a$ ,  $b_1$ , and  $b_2$ , and the following relations:

$$\begin{aligned} m_1(b_1) &= U \cdot b_2 \\ m_2(a, \rho_1) &= b_2 \\ m_{3+i}(a, \rho_3, \overbrace{\rho_{23}, \dots, \rho_{23}}^i, \rho_2) &= U^{2i+2} \cdot a, & i \geq 0 \\ m_{4+i}(a, \rho_3, \overbrace{\rho_{23}, \dots, \rho_{23}}^i, \rho_2, \rho_1) &= U^{2i+1} \cdot b_1, & i \geq 0. \end{aligned}$$

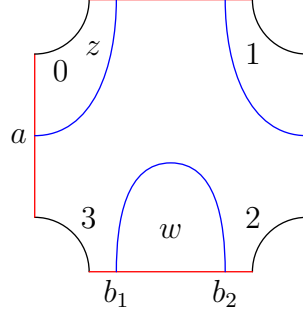


FIGURE 51. **Heegaard diagram type A module for the  $(2,1)$  cable.** This is a doubly-pointed Heegaard diagram for the  $(2,1)$  cable (of the unknot), thought of as a knot in the solid torus. The basepoint  $z$  lies in the region marked with a 0.

Consider next the type  $D$  module for the left-handed trefoil knot  $T$  with framing  $-2$ , as pictured in Figure 52. (Although this framing is not large enough for Theorem 10.17 to apply,  $\widehat{CFD}(T)$  may still be computed; see Theorem 11.7, whose proof uses results from [23].)

The tensor product  $CFA^-(C) \boxtimes \widehat{CFD}(T)$  has generators  $a \boxtimes x_i$  and  $b_k \boxtimes y_j$ . Differentials are easily computed; they are displayed in Figure 52. In fact, that figure also illustrates the Alexander gradings. Note also that the concordance invariant  $\tau$  can be immediately read from this data, bearing in mind the interpretation of  $\tau$  in terms of the Alexander grading on  $HFK^-$ , as in Equation (10.9). Specifically, we find that  $HFK^-$  has a free summand, generated by the element  $U \cdot ax_3 + b_1y_2$ , an element with Alexander grading 3, which in turn is equal to  $-\tau$  by one definition of  $\tau$ . Thus,  $\tau(c_{2,-1}T_{2,-3}) = -3$ .

To calculate the bigraded knot Floer homology complexes, we must first grade the factors in the tensor product.

The (enhanced) grading on the type  $D$  module is given by

$$\begin{aligned} \text{gr}(x_1) &= (1; 0, 2; 0)/\langle h \rangle \\ \text{gr}(x_2) &= (\tfrac{1}{2}; 0, 1; 0)/\langle h \rangle \\ \text{gr}(x_3) &= (0; 0, 0; 0)/\langle h \rangle \\ \text{gr}(y_1) &= (\tfrac{3}{2}; \tfrac{1}{2}, \tfrac{3}{2}; 0)/\langle h \rangle \\ \text{gr}(y_2) &= (-\tfrac{1}{2}; -\tfrac{1}{2}, \tfrac{1}{2}; 0)/\langle h \rangle, \end{aligned}$$

where we use notation for the enhanced grading  $(a; b_1, b_2; c)$  where  $(a; b_1, b_2)$  denotes the component in the ordinary grading group, as in Section 10.1, and  $c$  denotes the Alexander component. The indeterminacy  $h$  will be calculated shortly.

In fact, the displayed gradings are uniquely determined from the gradings on the algebra, the normalization  $\text{gr}(x_3) = (0; 0, 0; 0)$ , and the algebraic structure of the type  $D$  module displayed on the left in Figure 52. For example, given the grading of  $y_2$ , we can calculate the grading of  $x_2$ , since  $\rho_{123} \cdot y_2$  appears in  $\partial x_2$ ; so that

$$\lambda^{-1} \cdot \text{gr}(x_2) = \text{gr}(\rho_{123}) \cdot \text{gr}(y_2).$$

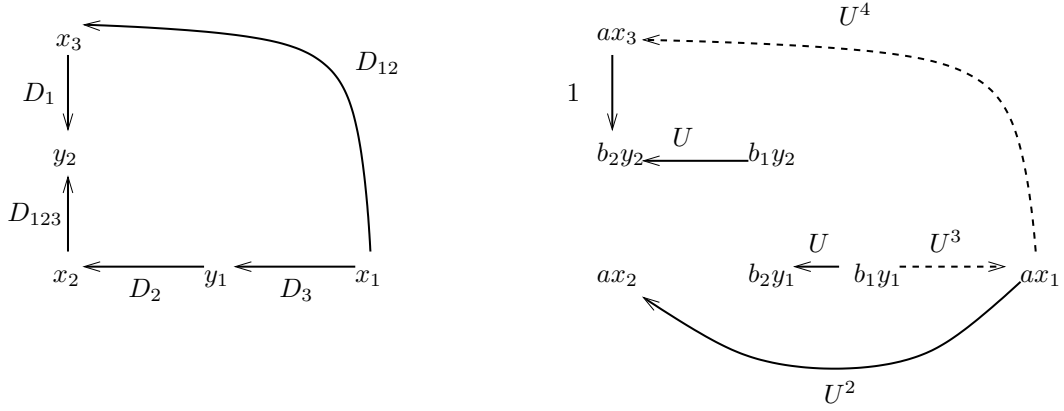


FIGURE 52. **Cable of the trefoil.** On the left, the coefficient maps for the left-handed trefoil. On the right, the result of cabling. Arrows with solid arrows represent differentials; they are labeled by their corresponding coefficients (powers of  $U$ ). Dotted arrows do not represent differentials; they represent domains, and the labels over them represent changes in Alexander gradings.

Continuing in the vein, the indeterminacy  $h$  is determined from the fact that  $\rho_{12}x_3$  appears in  $\partial x_1$ , so

$$h = \text{gr}(\rho_{12})^{-1} \lambda^{-1}(1; 0, 2; 0) = (-\tfrac{3}{2}; -1, 2; 0).$$

As in the case of  $\widehat{CFD}(T)$ , the gradings on  $CFA^-(C)$  can be calculated from the gradings on the algebra and the algebraic form of this module displayed earlier, combined with the normalization that  $\text{gr}(a) = (0; 0, 0; 0)$ . We find the (enhanced) grading on the type  $A$  module is given by:

$$\begin{aligned} \text{gr}(a) &= \langle g \rangle \setminus (0; 0, 0; 0) \\ \text{gr}(b_1) &= \langle g \rangle \setminus (\tfrac{1}{2}; \tfrac{1}{2}, -\tfrac{1}{2}; -1) \\ \text{gr}(b_2) &= \langle g \rangle \setminus (-\tfrac{1}{2}; \tfrac{1}{2}, -\tfrac{1}{2}; 0), \end{aligned}$$

where

$$g = (-\tfrac{1}{2}; 0, 1; 2).$$

The gradings of elements in the tensor product lie in the double-coset space

$$(-\tfrac{1}{2}; 0, 1; 2) \setminus G / (-\tfrac{3}{2}; -1, 2; 0).$$

Indeed, choosing appropriate coset representatives, we find that

$$\begin{aligned} \text{gr}(ax_1) &= (2; 0, 0; -4) \\ \text{gr}(ax_2) &= (1; 0, 0; -2) \\ \text{gr}(ax_3) &= (0; 0, 0; 0) \\ \text{gr}(b_1y_1) &= (6; 0, 0; -7) \\ \text{gr}(b_2y_1) &= (5; 0, 0; -6) \\ \text{gr}(b_1y_2) &= (0; 0, 0; -1) \\ \text{gr}(b_2y_2) &= (-1; 0, 0; 0) \end{aligned}$$

This allows us to compute Maslov and Alexander gradings of elements in the product complex. Specifically, taking homologies, we find that the Poincaré polynomial for the tensor product, after setting  $U = 0$ , and rescaling by  $t^3$ , is given by

$$\sum_{m,s} q^m \cdot t^s \cdot \text{rank } H_*(CFA^-(C) \boxtimes \widehat{CFD}(T)) = t^{-3}q^6 + t^{-2}q^5 + q^2 + t^2q + t^3.$$

Bearing in mind that the  $Q$ -grading here is the difference  $M - 2A$ , it is easy to see that this is in agreement with the calculations of Hedden [11, 12] for knot Floer homology groups of cables of the trefoil.

## 11. APPENDIX: BIMODULES AND CHANGE OF FRAMING

The modules  $\widehat{CFA}(Y)$  and  $\widehat{CFD}(Y)$  depend (up to quasi-isomorphism) not only on the 3-manifold  $Y$  but also on the parametrization of  $\partial Y$  by a reference surface  $F$ ; this dependence can already be seen for the modules associated to solid tori computed in Section 10.2. The result of reparametrization (e.g., change of framing) is captured by certain bimodules. Furthermore, the two modules  $\widehat{CFA}(Y)$  and  $\widehat{CFD}(Y)$  are related to each other by certain dualizing bimodules. These topics are explored in detail in [23]. In the present section we present the main results from that paper, and then exhibit the bimodules relevant to the case of torus boundary. When combined with Theorem 10.17, this gives a complete computation of the bordered Floer invariants of knot complements in  $S^3$ , with any framing.

**11.1. Statement of results.** Let  $F$  denote an oriented surface,  $q$  a point in  $F$ ,  $f_\partial: F \rightarrow \mathbb{R}$  a minimal self-indexing Morse functions with index 2 critical point  $q$ , and  $v \in T_q F$  a vector not tangent to any ascending flow from an index 1 critical point. (The basepoint  $z$  corresponds to  $v$ .) The data  $(F, f, v)$  up to homotopy is equivalent to a pointed matched circle, and so we have associated to  $(F, f, v)$  a DGA  $\mathcal{A}(F, f, v)$ .

Recall from Section 4.1 that a *bordered 3-manifold with boundary  $F$*  is a pair  $(Y, \phi)$  where  $Y$  is a compact, oriented 3-manifold with boundary and  $\phi: F \rightarrow \partial Y$  is an orientation-preserving diffeomorphism. (Heretofore, we have suppressed  $\phi$  from the notation, denoting the bordered 3-manifold simply by  $Y$ .) We have associated to  $(Y, \phi)$  quasi-isomorphism classes of modules  $\widehat{CFA}(Y, \phi)$  and  $\widehat{CFD}(Y, \phi)$  over  $\mathcal{A}(F, f, v)$  and  $\mathcal{A}(-F, f, v)$  respectively.

Given a bordered 3-manifold  $(Y, \phi)$  and a diffeomorphism  $\psi: (F, v) \rightarrow (F, v)$ , we have a new bordered 3-manifold  $(Y, \phi)\psi = (Y, \phi \circ \psi)$ .

**Theorem 11.1.** *Given a diffeomorphism  $\psi: (F, v) \rightarrow (F, v)$  and Morse functions  $f_\partial$  and  $g_\partial$  on  $F$  there is a differential bimodule  ${}_{\mathcal{A}(F, f_\partial, v)} \widehat{CFDA}(\psi)_{\mathcal{A}(F, g_\partial, v)}$ , well-defined up to quasi-isomorphism, such that for any  $(Y, \phi)$  as above, we have*

$$\begin{aligned} \widehat{CFD}((Y, \phi)\psi) &\simeq \widehat{CFDA}(\psi^{-1}) \widetilde{\otimes}_{\mathcal{A}(-F, f_\partial, v)} \widehat{CFD}(Y, \phi) \\ \widehat{CFA}((Y, \phi)\psi) &\simeq \widehat{CFA}(Y, \phi) \widetilde{\otimes}_{\mathcal{A}(F, f_\partial, v)} \widehat{CFDA}(\psi). \end{aligned}$$

(As we will see presently,  $\widehat{CFDA}(\psi)$  is defined in terms of a Heegaard diagram with two boundary components, one of which is treated in type  $D$  fashion and the other of which is treated in type  $A$  fashion.)

Next we turn to the duality between  $\widehat{CFD}$  and  $\widehat{CFA}$ . By  $M_{R,S}$  we mean a module  $M$  with commuting right actions by rings  $R$  and  $S$ ; similarly, by  ${}_{R,S}M$  we mean a module  $M$  with commuting left actions by  $R$  and  $S$ .

**Theorem 11.2.** *For any pointed matched circle  $\mathcal{Z} = (F, f_\partial, v)$  there are differential bimodules  $\widehat{CFAA}(\mathbb{I})_{\mathcal{A}(\mathcal{Z}), \mathcal{A}(-\mathcal{Z})}$  and  ${}_{\mathcal{A}(\mathcal{Z}), \mathcal{A}(-\mathcal{Z})} \widehat{CFDD}(\mathbb{I})$  such that for any bordered  $(Y, \phi)$ ,*

$$\begin{aligned} \widehat{CFD}(Y, \phi) &\simeq \widehat{CFA}(Y, \phi) \widetilde{\otimes}_{\mathcal{A}(\mathcal{Z})} \widehat{CFDD}(\mathbb{I}) \\ \widehat{CFA}(Y, \phi) &\simeq \widehat{CFAA}(\mathbb{I}) \widetilde{\otimes}_{\mathcal{A}(\mathcal{Z})} \widehat{CFD}(Y, \phi). \end{aligned}$$

As the notation suggests, these dualizing modules correspond to the identity map  $\mathbb{I}: F \rightarrow F$ . In fact, for any mapping class  $\phi: F \rightarrow F$  there are associated modules  $\widehat{CFDD}(\phi)$  and  $\widehat{CFAA}(\phi)$ . These are calculated in [24].

**11.2. Sketch of the construction.** Next, we outline the construction of the bimodules discussed above. Details can be found in [23], although the constructions are close enough to the construction of  $\widehat{CFD}$  and  $\widehat{CFA}$  that the reader might find it amusing to fill them in on his or her own.

**11.2.1. Bordered Heegaard diagrams for diffeomorphisms.** The module  $\widehat{CFDA}(\phi)$  is associated to a bordered Heegaard diagram for  $\phi$ . Such a diagram is a slight generalization of the notion from Section 4 to allow more than one boundary component. (These diagrams already made a brief appearance in Section 10.4.)

Specifically, an *arced bordered Heegaard diagram with two boundary components* is a quadruple  $\mathcal{H} = (\overline{\Sigma}, \overline{\alpha}, \beta, z)$  where

- $\overline{\Sigma}$  is an oriented surface of genus  $g$ , with two boundary components  $\partial_L \overline{\Sigma}$  and  $\partial_R \overline{\Sigma}$ ;
- $\overline{\alpha} = \{\overline{\alpha}_1^{a,L}, \dots, \overline{\alpha}_{2k}^{a,L}, \overline{\alpha}_1^{a,R}, \dots, \overline{\alpha}_{2k}^{a,R}, \alpha_1^c, \dots, \alpha_{g-2k}^c\}$  where the  $\overline{\alpha}_i^{a,L}$  are embedded arcs with endpoints on  $\partial_L \overline{\Sigma}$ , the  $\overline{\alpha}_i^{a,R}$  are embedded arcs with endpoints on  $\partial_R \overline{\Sigma}$ , and the  $\alpha_i^c$  are embedded circles in  $\Sigma$ ; and all of the  $\overline{\alpha}_i^{a,L}$ ,  $\overline{\alpha}_i^{a,R}$  and  $\alpha_i^c$  are pairwise disjoint, and  $\Sigma \setminus \overline{\alpha}$  is connected;
- $\beta = \{\beta_1, \dots, \beta_g\}$  are circles in  $\Sigma$  such that  $\Sigma \setminus \beta$  is connected; and
- $z$  is an arc in  $\overline{\Sigma} \setminus (\overline{\alpha} \cup \beta)$  connecting  $\partial_L \overline{\Sigma}$  and  $\partial_R \overline{\Sigma}$ .

As for ordinary bordered Heegaard diagrams, we let  $\Sigma = \overline{\Sigma} \setminus \partial \overline{\Sigma}$ ; we will be sloppy about the distinction between  $\Sigma$  and  $\overline{\Sigma}$ .

The data  $\mathcal{H}$  specifies a 3-manifold  $Y$  with two boundary components in a manner exactly analogous to Section 4.1. Moreover,  $\mathcal{H}$  specifies a homeomorphism  $\phi_i$  from a fixed reference surface  $F_i$  to each component  $\partial_i Y$  ( $i \in \{L, R\}$ ) of  $Y$ , as well as a framed arc connecting the two boundary components. We call  $\mathcal{H}$  a bordered Heegaard diagram for  $(Y, F_L, F_R)$ .

Assume that  $Y$  is homeomorphic to  $[0, 1] \times F$  and that  $F_L = -F_R$  (i.e., that the matchings on the two components of  $\partial \bar{\Sigma}$  agree, modulo reversal of orientation). There is a homeomorphism  $\Phi: [0, 1] \times F_R \xrightarrow{\cong} Y$  extending  $\phi_L$ . Moreover, this homeomorphism is unique up to homeomorphisms  $\Xi: [0, 1] \times F \rightarrow [0, 1] \times F$  such that  $\Xi|_{\{0\} \times F}$  is the identity. It follows that  $\Xi|_{\{1\} \times F}$  is well-defined up to homotopy, and hence (because  $F$  is a surface) up to isotopy. Consequently,  $\phi_R^{-1} \circ (\Phi|_{\{1\} \times F})$  is a homeomorphism  $F_R \rightarrow F_R$ , well-defined up to isotopy, i.e., an element  $\phi(\mathcal{H})$  of the mapping class group  $MCG(F)$ . Paying attention to the basepoint  $z$ , we obtain a map

$$\phi: \{\text{bordered Heegaard diagrams for } ([0, 1] \times F, F_R, F_R)\} \rightarrow MCG_0(F_R),$$

where  $MCG_0$  denotes the strongly based mapping class group, i.e., the mapping class group of diffeomorphisms fixing a small disk in  $F_R$ .

It is not hard to see that every element of  $MCG_0(F)$  arises this way. Indeed, given a diffeomorphism  $\psi: F_R \rightarrow F_R$ , we can explicitly construct a bordered Heegaard diagram realizing it as follows. Given a pointed matched circle  $\mathcal{Z}$ , let  $\mathbb{I}_{\mathcal{Z}} = (\bar{\Sigma}_{\mathcal{Z}}, \bar{\alpha}, \beta, z)$  be the bordered Heegaard diagram (with no  $\alpha$ -circles) where  $\bar{\alpha}_i^{a,L}$  and  $\bar{\alpha}_i^{a,R}$  each intersect  $\beta_i$  in a single point and are disjoint from  $\beta_j$  for  $i \neq j$ . (See Figure 53.) Then  $\mathbb{I}_{\mathcal{Z}}$  is a bordered Heegaard diagram for the identity map of  $F(\mathcal{Z})$ . Further, a regular neighborhood of

$$\bar{\alpha}_1^{a,R} \cup \dots \cup \bar{\alpha}_{2k}^{a,R} \cup \partial_R \bar{\Sigma}$$

in  $\bar{\Sigma}$  is canonically homeomorphic to  $F_R$  minus two disks. By removing the arc  $z$  from this neighborhood, we obtain  $F_R$  minus one disc. Hence  $\psi \in MCG_0(F_R)$  can be extended by the identity to a homeomorphism  $\bar{\psi}: \Sigma \rightarrow \Sigma$ . Then,  $(\bar{\Sigma}, \bar{\alpha}, \bar{\psi}^{-1}(\beta), z)$  is a bordered Heegaard diagram for  $\psi$ .

For the constructions that follow, it will not be important that  $F_L = -F_R$ . Then, instead of the mapping class group, one works with a *mapping class groupoid*. See [23] for further details.

**11.2.2. Holomorphic curves defining  $\widehat{CFDA}$ .** Let  $\mathcal{H} = (\bar{\Sigma}_g, \bar{\alpha}, \beta, z)$  be a bordered Heegaard diagram for  $\psi \in MCG_0(F_R)$ . The module  $\widehat{CFDA}(\psi) = \widehat{CFDA}(\mathcal{H})$  is defined by counting holomorphic curves in  $\Sigma \times [0, 1] \times \mathbb{R}$ . As usual, the two boundary components of  $\bar{\Sigma}$  become cylindrical ends of  $\Sigma$ , and hence of  $\Sigma \times [0, 1] \times \mathbb{R}$ . We call the end corresponding to  $\partial_L \bar{\Sigma}$  *west infinity* and the end corresponding to  $\partial_R \bar{\Sigma}$  *east infinity*.<sup>4</sup>

The module  $\widehat{CFDA}(\mathcal{H})$  is defined by treating west infinity of  $\Sigma \times [0, 1] \times \mathbb{R}$  as a Type *D* end (i.e., analogously to Section 6) and east infinity as a Type *A* end (i.e., analogously to Section 7). More precisely, let  $\mathfrak{S}(\mathcal{H})$  denote the set of  $g$ -tuples of points  $\mathbf{x}$  in  $\Sigma$  so that exactly one  $x_i$  lies on each  $\alpha$ - and each  $\beta$ -circle, and no two  $x_i$  lie on the same  $\alpha$ -arc.

<sup>4</sup>In particular, east infinity is the end whose induced orientation agrees with the orientation of  $F_R$ .

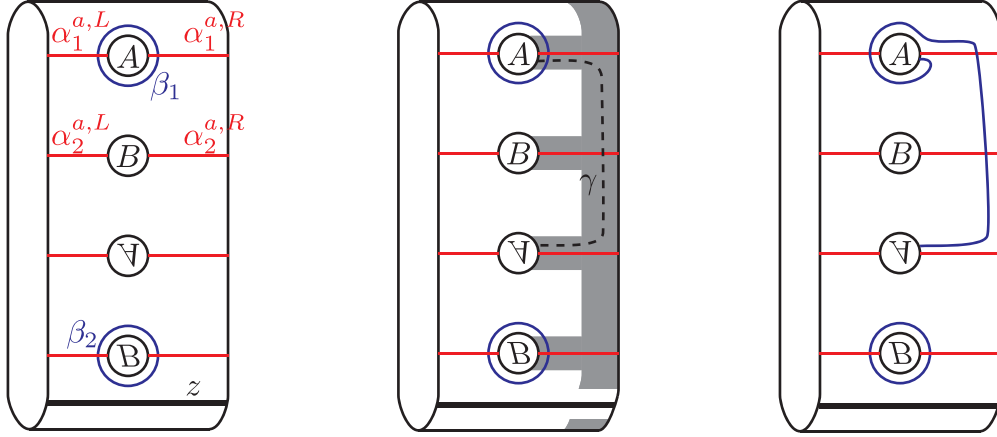


FIGURE 53. **Bordered Heegaard diagrams for diffeomorphisms.**

Left: the identity bordered Heegaard diagram for the genus 1 surface.

Center: the same diagram, with a regular neighborhood of  $\alpha_1^{a,R} \cup \dots \cup \alpha_{2k}^{a,R} \cup \partial_R \bar{\Sigma} \setminus \{z\}$  in dark gray and a closed curve  $\gamma$  as a dashed line.

Right: a bordered Heegaard diagram for the Dehn twist around the curve  $\gamma$ .

Given  $\mathbf{x} \in \mathfrak{S}(\mathcal{H})$ , let  $o_L(\mathbf{x})$  denote the set of  $\alpha^L$ -arcs occupied by  $\mathbf{x}$  and  $o_R(\mathbf{x})$  the set of  $\alpha^R$ -arcs occupied by  $\mathbf{x}$ . Let  $X(\mathcal{H})$  be the  $\mathbb{F}_2$ -vector space spanned by  $\mathfrak{S}(\mathcal{H})$ . Let  $I_{L,D}(\mathbf{x}) = I([2k] \setminus o^L(\mathbf{x}))$  and  $I_{R,A} = o^R(\mathbf{x})$ . We define both left and right actions of  $\mathcal{I}(F_R)$  on  $\mathfrak{S}(\mathcal{H})$  by

$$I(s) \cdot \mathbf{x} \cdot I(t) := \begin{cases} \mathbf{x} & I(s) = I_{L,D}(\mathbf{x}) \text{ and } I(t) = I_{R,A}(\mathbf{x}) \\ 0 & \text{otherwise,} \end{cases}$$

where  $s$  and  $t$  are  $k$ -element subsets of  $[2k]$ , as in Formula (3.10).

As a left module,  $\widehat{CFDA}(\mathcal{H})$  is just  $\mathcal{A}(F_L) \otimes_{\mathcal{I}} X(\mathcal{H})$ . Note that, for the first time in this paper, we have a module on which  $\mathcal{A}(\mathcal{Z}, i)$  may act non-trivially for  $i \neq 0$ .

Our next task is to define the  $\mathcal{A}_\infty$ -bimodule structure on  $\widehat{CFDA}(\mathcal{H})$ . (See Keller [16, Section 3] for basic notions relating to  $\mathcal{A}_\infty$ -bimodules.) The left and right actions strongly commute, i.e.,

$$m_{i+1+j}(a_1, \dots, a_i, \mathbf{x}, b_1, \dots, b_j) = 0$$

if  $i, j > 0$ , so we only need to describe the right action. The module  $\widehat{CFDA}(\mathcal{H})$  is also strictly unital, i.e.,

$$\begin{aligned} m_2(\mathbf{x}, 1) &= \mathbf{x} \\ m_{n+1}(\mathbf{x}, \dots, 1, \dots) &= 0, \quad n > 1. \end{aligned}$$

The definition of  $m_{n+1}(\mathbf{x}, a(\boldsymbol{\rho}_1), \dots, a(\boldsymbol{\rho}_n))$  uses holomorphic curves. Let

$$\mathcal{M}^B(\mathbf{x}, \mathbf{y}; \vec{\rho}^D; \vec{\rho}^A)$$

denote the moduli space of embedded holomorphic curves in the homology class  $B$ , asymptotic to the sequence of Reeb chords in  $\vec{\rho}^D$  at west  $\infty$  (with no height constraints), and to the ordered partition  $\vec{\rho}^A = (\rho_1^A, \dots, \rho_n^A)$  of Reeb chords at east  $\infty$ . The interesting multiplications are defined by

$$m_{n+1}(\mathbf{x}, a(\rho_1), \dots, a(\rho_n)) := \sum_{\mathbf{y} \in \mathfrak{S}(\mathcal{H})} \sum_{\substack{B \in \pi_2(\mathbf{x}, \mathbf{y}) \\ \{\vec{\rho}^D | \text{ind}(B; \vec{\rho}^D; \vec{\rho}^A) = 1\}}} \# \left( \mathcal{M}^B(\mathbf{x}, \mathbf{y}; \vec{\rho}^D; \vec{\rho}^A) \right) a(-\vec{\rho}^D) \mathbf{y}.$$

(Compare Definitions 6.3 and 7.2. The case  $n = 0$  is essentially the differential on  $\widehat{CFD}$ .)

We prove in [23] that this does, indeed, define an  $\mathcal{A}_\infty$ -bimodule. Both of the proofs of the Pairing Theorem (Sections 8 and 9) extend easily to prove the first part of Theorem 11.1.

*Remark 11.3.* Just as our differential modules have been graded by left or right  $G$ -sets, bimodules are graded by sets with left and right actions by  $G$ . The module  $\widehat{CFDA}(\psi)$  is graded by  $G(F_R)$ , thought of as a set with commuting left and right actions by  $G(F_R) \cong G(F_L)$ . The construction of the grading is a simple modification of the discussions in Sections 6.3 and 7.3.

11.2.3. *Holomorphic curves defining  $\widehat{CFDD}(\mathbb{I})$  and  $\widehat{CFAA}(\mathbb{I})$ .* The modules  $\widehat{CFDD}(\mathbb{I})$  and  $\widehat{CFAA}(\mathbb{I})$  are both defined in terms of holomorphic curves with respect to the identity Heegaard diagram  $\mathbb{I}_Z = (\Sigma, \alpha, \beta, z)$ , i.e., holomorphic curves in

$$(\Sigma \times [0, 1] \times \mathbb{R}, \alpha \times \{1\} \times \mathbb{R} \cup \beta \times \{0\} \times \mathbb{R})$$

where  $(\Sigma, \bar{\alpha}, \beta, z) = \mathbb{I}_Z$ . The module  $\widehat{CFDD}(\mathbb{I})$  is defined by treating both east infinities as Type  $D$  boundary components, and the module  $\widehat{CFAA}(\mathbb{I})$  is defined by treating both west infinities as Type  $A$  boundary components.

More precisely,  $\widehat{CFDD}(\mathbb{I})$  is defined as follows. Recall that  $-\mathcal{Z}$  is the pointed matched circle obtained by reversing the orientation on  $\mathcal{Z}$ . The module  $\widehat{CFDD}(\mathbb{I})$  is generated over  $\mathcal{A}(\mathcal{Z}) \times \mathcal{A}(-\mathcal{Z})$  by the  $2^{2k} = \sum_i \binom{2k}{i}$  ways of choosing  $i$  points from  $\{\alpha_1^{L,a} \cap \beta_1, \alpha_2^{L,a} \cap \beta_2, \dots, \alpha_k^{L,a} \cap \beta_k\}$  (and the  $2k - i$  points on the complementary  $\alpha_i^{R,a}$ ); we can identify this with the set  $\mathfrak{S}(\mathbb{I}_Z)$  from Section 11.2.2 by looking at  $o^L$ , the set of strands occupied on the left. There are left actions of  $\mathcal{I}(-\mathcal{Z})$  and  $\mathcal{I}(\mathcal{Z})$  on  $X(\mathbb{I}_Z)$  by, respectively,

$$I(s) \bullet \mathbf{x} := \begin{cases} \mathbf{x} & s = [2k] \setminus o^L(\mathbf{x}) \\ 0 & \text{otherwise,} \end{cases}$$

$$I(s) \circ \mathbf{x} := \begin{cases} \mathbf{x} & s = [2k] \setminus o^R(\mathbf{x}) \\ 0 & \text{otherwise.} \end{cases}$$

(Note that  $o^R(\mathbf{x}) = [2k] \setminus o^L(\mathbf{x})$ .) These two actions commute. Consequently, extending scalars from the idempotents to the whole algebra gives two commuting left actions of  $\mathcal{A}(\mathcal{Z})$ ; this is the module structure on  $\widehat{CFDD}(\mathbb{I})$ .

The differential on  $\widehat{CFDD}(\mathbb{I})$  is given as follows. Let

$$\mathcal{M}^B(\mathbf{x}, \mathbf{y}; \vec{\rho}^L, \vec{\rho}^R)$$

denote the moduli space of embedded holomorphic curves in the homology class  $B$ , asymptotic to the sequence of Reeb chords  $\vec{\rho}^L$  on the left and  $\vec{\rho}^R$  on the right. (We do not constrain the relative ordering of Reeb chords on the left and right.) Then

$$\partial \mathbf{x} := \sum_{\mathbf{y} \in \mathfrak{S}(\mathcal{H})} \sum_{\substack{B \in \pi_2(\mathbf{x}, \mathbf{y}) \\ \{\vec{\rho}^L, \vec{\rho}^R\} |_{\text{ind}(B; \vec{\rho}^L; \vec{\rho}^R)} = 1}} \#(\mathcal{M}^B(\mathbf{x}, \mathbf{y}; \vec{\rho}^L; \vec{\rho}^R)) a(-\vec{\rho}^L) \bullet a(-\vec{\rho}^R) \circ \mathbf{y}.$$

Next we turn to  $\widehat{CFAA}(\mathbb{I})$ . As a  $\mathbb{F}_2$ -vector space, the module  $\widehat{CFAA}(\mathbb{I})$  is just  $X(\mathbb{I}_{\mathcal{Z}})$ . There are commuting right actions of  $\mathcal{I}(-\mathcal{Z})$  and  $\mathcal{I}(\mathcal{Z})$  on  $X(\mathbb{I}_{\mathcal{Z}})$  by

$$\begin{aligned} \mathbf{x} \bullet I(s) &:= \begin{cases} \mathbf{x} & s = o^L(\mathbf{x}) \\ 0 & \text{otherwise,} \end{cases} \\ \mathbf{x} \circ I(s) &:= \begin{cases} \mathbf{x} & s = o^R(\mathbf{x}) \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

The module  $\widehat{CFAA}(\mathbb{I})$  is strictly unital, i.e.,

$$\begin{aligned} m_2(\mathbf{x}; 1; ) &:= m_2(\mathbf{x}; ; 1) := \mathbf{x} \\ m_{m+n+1}(\mathbf{x}; \dots, 1, \dots) &:= 0, \quad m+n > 1. \end{aligned}$$

The other higher multiplications are defined using holomorphic curves. Let

$$\mathcal{M}^B(\mathbf{x}, \mathbf{y}; \vec{\rho}^L; \vec{\eta}^R)$$

denote the moduli space of embedded holomorphic curves in the homology class  $B$ , asymptotic to the ordered partition  $\vec{\rho} = (\rho_1, \dots, \rho_m)$  at  $\partial_L \overline{\Sigma} \times [0, 1] \times \mathbb{R}$  and asymptotic to the ordered partition  $\vec{\eta} = (\eta_1, \dots, \eta_n)$  at  $\partial_R \overline{\Sigma} \times [0, 1] \times \mathbb{R}$ . Then define

$$\begin{aligned} m_{m+n+1}(\mathbf{x}; a(\rho_1), \dots, a(\rho_m); a(\eta_1), \dots, a(\eta_n)) &:= \\ \sum_{\mathbf{y} \in \mathfrak{S}(\mathcal{H})} \sum_{\substack{B \in \pi_2(\mathbf{x}, \mathbf{y}) \\ \text{ind}(B; \vec{\rho}; \vec{\eta}) = 1}} \#(\mathcal{M}^B(\mathbf{x}, \mathbf{y}; \vec{\rho}; \vec{\eta})) \mathbf{y}. \end{aligned}$$

*Remark 11.4.* The gradings on  $\widehat{CFDD}(\mathbb{I})$  and  $\widehat{CFAA}(\mathbb{I})$  are defined analogously to Sections 6.3 and 7.3 respectively.

*Remark 11.5.* For each of  $\widehat{CFDA}(\mathcal{H})$ ,  $\widehat{CFDD}(\mathbb{I})$  and  $\widehat{CFAA}(\mathbb{I})$ , the summands  $\mathcal{A}(\mathcal{Z}, i)$  inside  $\mathcal{A}(\mathcal{Z})$  act non-trivially even for  $i \neq 0$ , and the modules decompose as direct sums

$$\begin{aligned} \widehat{CFDA}(\mathcal{H}) &= \bigoplus_{i=-k}^k \widehat{CFDA}(\mathcal{H}, i) \\ \widehat{CFDD}(\mathbb{I}) &= \bigoplus_{i=-k}^k \widehat{CFDD}(\mathbb{I}, i) \\ \widehat{CFAA}(\mathbb{I}) &= \bigoplus_{i=-k}^k \widehat{CFAA}(\mathbb{I}, i). \end{aligned}$$

Of course, only the summands corresponding to  $i = 0$  contribute to tensor products with  $\widehat{CFD}$  or  $\widehat{CFA}$  as in Theorem 11.1. The other summands do arise naturally if one studies self-gluing of Heegaard diagrams, which in turn relates to the Floer homology of open books.

**11.3. Computations for 3-manifolds with torus boundary.** Here we will state the relevant bimodules for three-manifolds with torus boundary. We will then deduce a version of Theorem 10.17 for arbitrary integral framings of knots, Theorem 11.7. (In fact, the data from this section can be used to calculate the result for arbitrary framings; but that will be left to the interested reader.) The computations of the bimodules are given in [23].

There is a unique pointed matched circle representing the surface of genus one, so much of the notation for the torus algebra can be simplified. Moreover, it is only the summand in  $i = 0$  which is relevant to the case of bordered manifolds with connected boundary. We refer to this summand of the algebra simply as  $\mathcal{A}$ , and label its generators as in Section 10.

The mapping class group is generated by Dehn twists  $\tau_\mu$  and  $\tau_\lambda$  along meridian and longitude respectively (i.e.,  $\tau_\mu$  takes an  $n$ -framed knot complement to an  $n + 1$ -framed knot complement, or  $\mu$  is the twist on a curve dual to  $\alpha_2$  from Figure 40). We describe the  $i = 0$  part  $\widehat{CFDA}(\cdot, 0)$  of the type  $DA$  modules for Dehn twists about these two curves and their inverses. (Again, the  $i = 0$  part is the only part relevant to 3-manifolds with torus boundary.)

Heegaard diagrams for  $\tau_\mu$ ,  $\tau_\mu^{-1}$ ,  $\tau_\lambda$  and  $\tau_\lambda^{-1}$  are illustrated in Figure 54. Each of the four bimodules  $\widehat{CFDA}(\tau_\mu, 0)$ ,  $\widehat{CFDA}(\tau_\mu^{-1}, 0)$ ,  $\widehat{CFDA}(\tau_\lambda, 0)$ , and  $\widehat{CFDA}(\tau_\lambda^{-1}, 0)$  is projectively generated by three elements, denoted  $\mathbf{p}$ ,  $\mathbf{q}$ , and either  $\mathbf{r}$  or  $\mathbf{s}$ . (In fact, the proof of Theorem 11.7 only uses  $\widehat{CFDA}(\tau_\mu, 0)$ .) We give the algebraic structure on each of these bimodules in turn.

The generators are compatible with the idempotents as follows:

$$\iota_0 \cdot \mathbf{p} \cdot \iota_0 = \mathbf{p} \quad \iota_1 \cdot \mathbf{q} \cdot \iota_1 = \mathbf{q} \quad \iota_1 \cdot \mathbf{r} \cdot \iota_0 = \mathbf{r} \quad \iota_0 \cdot \mathbf{s} \cdot \iota_1 = \mathbf{s}.$$

The module  $\widehat{CFDA}(\tau_\mu, 0)$  is generated by  $\mathbf{p}$ ,  $\mathbf{q}$  and  $\mathbf{r}$ . The idempotents act as above; the other non-trivial algebra actions are given as follows:

$$\begin{aligned} m_2(\mathbf{p}, \rho_1) &= \rho_1 \otimes \mathbf{q} \\ m_2(\mathbf{p}, \rho_{12}) &= \rho_{123} \otimes \mathbf{r} \\ m_2(\mathbf{p}, \rho_{123}) &= \rho_{123} \otimes \mathbf{q} \\ m_3(\mathbf{p}, \rho_3, \rho_2) &= \rho_3 \otimes \mathbf{r} \\ m_3(\mathbf{p}, \rho_3, \rho_{23}) &= \rho_3 \otimes \mathbf{q} \\ m_2(\mathbf{q}, \rho_2) &= \rho_{23} \otimes \mathbf{r} \\ m_2(\mathbf{q}, \rho_{23}) &= \rho_{23} \otimes \mathbf{q} \\ m_1(\mathbf{r}) &= \rho_2 \otimes \mathbf{p} \\ m_2(\mathbf{r}, \rho_3) &= \mathbf{q}. \end{aligned}$$

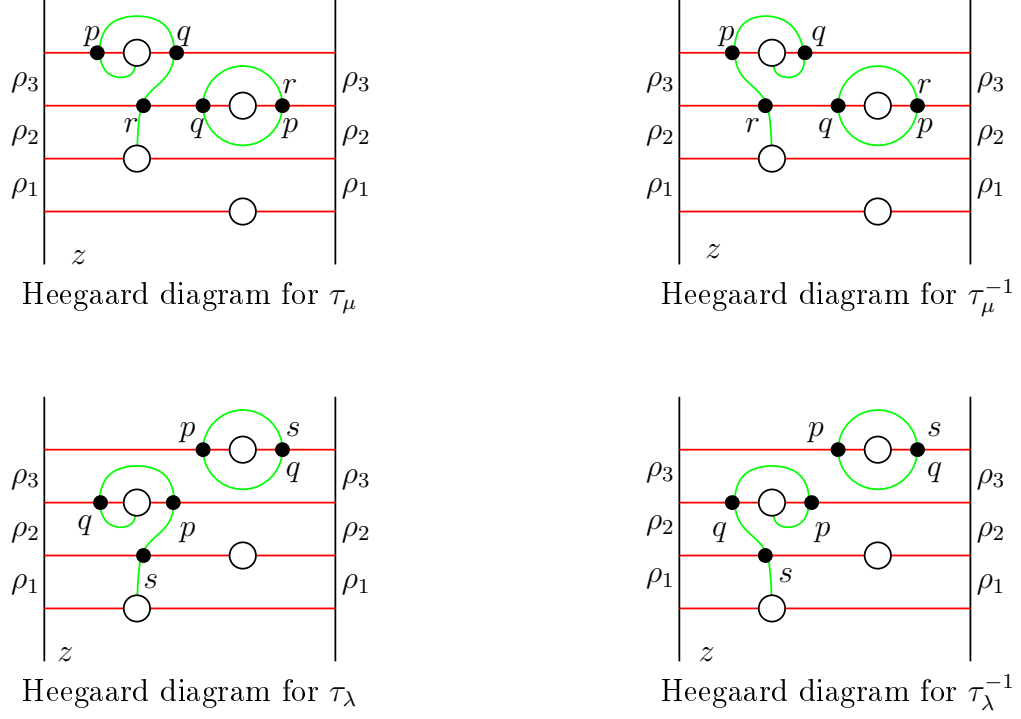


FIGURE 54. **Heegaard diagrams for mapping class group elements.** We have illustrated here genus two bordered diagrams for the generators of the genus one mapping class group and their inverses, as indicated. In each of the four diagrams, there are three generators in the  $i = 0$  summand.

The module  $\widehat{CFDA}(\tau_\mu^{-1}, 0)$  is generated by  $\mathbf{p}$ ,  $\mathbf{q}$  and  $\mathbf{r}$ . The idempotents act as above; the other non-trivial algebra actions are given as follows:

$$\begin{aligned}
 m_1(\mathbf{p}) &= \rho_3 \otimes \mathbf{r} \\
 m_2(\mathbf{p}, \rho_1) &= \rho_1 \otimes \mathbf{q} \\
 m_2(\mathbf{p}, \rho_{12}) &= \rho_1 \otimes \mathbf{r} \\
 m_2(\mathbf{p}, \rho_{123}) &= \rho_{123} \otimes \mathbf{q} \\
 m_3(\mathbf{p}, \rho_{123}, \rho_2) &= \rho_{12} \otimes \mathbf{p} \\
 m_2(\mathbf{q}, \rho_2) &= \mathbf{r} \\
 m_2(\mathbf{q}, \rho_{23}) &= \rho_{23} \otimes \mathbf{q} \\
 m_3(\mathbf{q}, \rho_{23}, \rho_2) &= \rho_2 \otimes \mathbf{p} \\
 m_2(\mathbf{r}, \rho_3) &= \rho_{23} \otimes \mathbf{q} \\
 m_3(\mathbf{r}, \rho_3, \rho_2) &= \rho_2 \otimes \mathbf{p}.
 \end{aligned}$$

The module  $\widehat{CFDA}(\tau_\lambda, 0)$  is generated by  $\mathbf{p}$ ,  $\mathbf{q}$  and  $\mathbf{s}$ . The idempotents act as above; the other non-trivial algebra actions are given as follows:

$$\begin{aligned}
m_3(\mathbf{q}, \rho_2, \rho_1) &= \rho_2 \otimes \mathbf{s} \\
m_3(\mathbf{q}, \rho_2, \rho_{12}) &= \rho_2 \otimes \mathbf{p} \\
m_3(\mathbf{q}, \rho_2, \rho_{123}) &= \rho_{23} \otimes \mathbf{q} \\
m_2(\mathbf{p}, \rho_1) &= \rho_{12} \otimes \mathbf{s} \\
m_2(\mathbf{p}, \rho_{12}) &= \rho_{12} \otimes \mathbf{p} \\
m_2(\mathbf{p}, \rho_{123}) &= \rho_{123} \otimes \mathbf{q} \\
m_2(\mathbf{p}, \rho_3) &= \rho_3 \otimes \mathbf{q} \\
m_1(\mathbf{s}) &= \rho_1 \otimes \mathbf{q} \\
m_2(\mathbf{s}, \rho_2) &= \mathbf{p} \\
m_2(\mathbf{s}, \rho_{23}) &= \rho_3 \otimes \mathbf{q}.
\end{aligned}$$

The module  $\widehat{CFDA}(\tau_\lambda^{-1}, 0)$  is generated by  $\mathbf{p}$ ,  $\mathbf{q}$  and  $\mathbf{s}$ . The idempotents act as above; the other non-trivial algebra actions are given as follows:

$$\begin{aligned}
m_1(\mathbf{q}) &= \rho_2 \otimes \mathbf{s} \\
m_2(\mathbf{p}, \rho_1) &= \mathbf{s} \\
m_2(\mathbf{p}, \rho_{12}) &= \rho_{12} \otimes \mathbf{p} \\
m_2(\mathbf{p}, \rho_{123}) &= \rho_{123} \otimes \mathbf{q} \\
m_2(\mathbf{p}, \rho_3) &= \rho_3 \otimes \mathbf{q} \\
m_3(\mathbf{p}, \rho_{12}, \rho_1) &= \rho_1 \otimes \mathbf{q} \\
m_2(\mathbf{s}, \rho_2) &= \rho_{12} \otimes \mathbf{p} \\
m_2(\mathbf{s}, \rho_{23}) &= \rho_{123} \otimes \mathbf{q} \\
m_3(\mathbf{s}, \rho_2, \rho_1) &= \rho_1 \otimes \mathbf{q}.
\end{aligned}$$

*Remark 11.6.* The computations leading to the results above are eased by noting that

- $CFDA(\tau_\mu^{-1}, 0)$  is inverse to  $CFDA(\tau_\mu, 0)$  (i.e.,  $CFDA(\tau_\mu^{-1}, 0) \boxtimes CFDA(\tau_\mu, 0) \simeq CFDA(\mathbb{I}, 0)$ );
- the Heegaard diagram for  $\tau_\lambda$  is obtained from that for  $\tau_\mu^{-1}$  by a horizontal reflection (across the  $x$ -axis); and
- the Heegaard diagram for  $\tau_\lambda^{-1}$  is obtained from that for  $\tau_\mu$  by a horizontal reflection.

Thus there is essentially only one computation to do.

To compute the type  $AA$  and  $DD$  identity bimodules for the torus, it is convenient to use the Heegaard diagram pictured in Figure 55. Note that the boundary segments on both sides have been labeled to be consistent with the type  $A$  conventions.

The resulting type  $AA$  bimodule is pictured in Figure 56. Note that algebra elements acting on one side are denoted  $\sigma_i$  while on the other side, they are denoted  $\rho_i$  (as indicated in Figure 55).

The type  $DD$  bimodule can also be calculated using the same Heegaard diagram, to give the module indicated in Figure 57. (We have relabeled the  $\rho_i$  and  $\sigma_i$  to

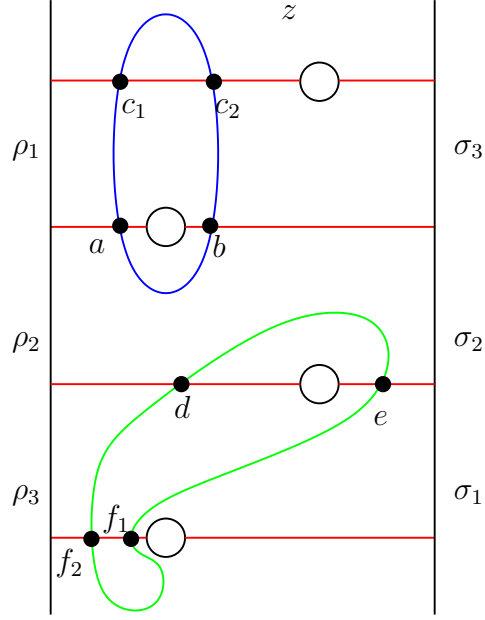
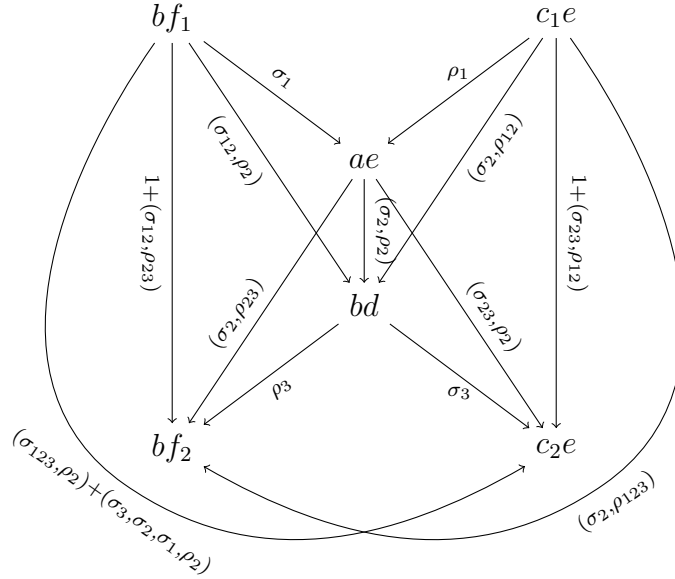


FIGURE 55. Heegaard diagram for the identity bimodule.

FIGURE 56. The type AA bimodule  $\widehat{CFAA}(\mathbb{Z}, 0)$ . The labels on the arrows indicate the  $\mathcal{A}_\infty$  operations; for instance, the label  $(\sigma_{23}, \rho_2)$  on the arrow from  $ae$  to  $c_2e$  means that  $m_2(ae, \sigma_{23}, \rho_2)$  contains a term  $c_2e$ .

conform to our conventions for type  $D$  boundary.) This simplifies to a module with two generators  $x$  and  $y$  with

$$\begin{aligned}\partial x &= (\rho_1\sigma_3 + \rho_3\sigma_1 + \rho_{123}\sigma_{123}) \otimes y \\ \partial y &= (\rho_2\sigma_2) \otimes x.\end{aligned}$$

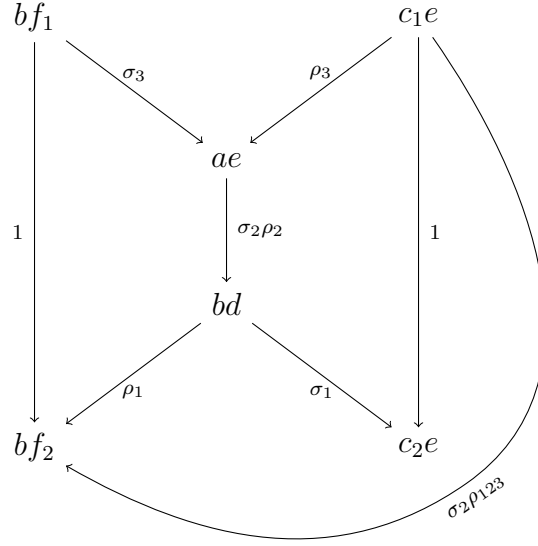


FIGURE 57. The type  $DD$  bimodule  $\widehat{CFDD}(\mathbb{I}, 0)$ . The labels on the arrows indicate differentials; for instance, the arrow from  $ae$  to  $bd$  signifies a term of  $\rho_2 \cdot \sigma_2 \otimes bd$  in  $\partial ae$ .

Finally, as the reader may well have expected, up to homotopy equivalence the type  $DA$  bimodule  $\widehat{CFDA}(\mathbb{I}, 0)$  associated to the identity map is just  $\mathcal{A}$  as an  $\mathcal{A}$ - $\mathcal{A}$  bimodule.

**11.4. From  $HF$  to  $\widehat{CFD}$  for arbitrary integral framings.** With the above bimodules in place, we turn now to the generalization of Theorem 10.17 for arbitrary (integral) framings.

**Theorem 11.7.** *Let  $CFK^-(K)$  be a model for a chain complex for a knot in  $S^3$  which is reduced. Let  $Y$  be the bordered three-manifold  $S^3 - \text{nb}d K$ , given any integral framing  $n$ . The associated type  $D$  module  $\widehat{CFD}(Y)$  can be extracted from  $CFK^-(K)$  using the procedure described in Theorem 10.17, except for the unstable chain, whose precise form depends on the framing parameter  $n$ . There are three different cases. When  $n < 2\tau(K)$ , it has the form*

$$\xi_0 \xrightarrow{D_1} \gamma_1 \xleftarrow{D_{23}} \gamma_2 \xleftarrow{D_{23}} \cdots \xleftarrow{D_{23}} \gamma_m \xleftarrow{D_3} \eta_0,$$

where  $m = 2\tau(K) - n$ . When  $n = 2\tau(K)$ , the unstable chain has the form

$$\xi_0 \xrightarrow{D_{12}} \eta_0.$$

Finally, when  $n > 2\tau(K)$  it has the form

$$\xi_0 \xrightarrow{D_{123}} \gamma_1 \xrightarrow{D_{23}} \gamma_2 \cdots \xrightarrow{D_{23}} \gamma_m \xrightarrow{D_2} \eta_0,$$

where  $m = n - 2\tau(K)$ .

*Proof.* In view of Theorem 11.1, we need only see what happens as we tensor the  $D$  module for a knot complement with sufficiently small framing parameter, as calculated in Theorem 10.17, with iterated copies of the bimodule  $\widehat{CFDA}(\tau_\mu, 0)$ .

We denote the generators in the idempotent  $\iota_0$  for  $\widehat{CFD}(Y)$  by  $x_i$ , and the generators in idempotent  $\iota_1$  by  $y_j$ . In particular, we write  $x_0$  and  $x_1$  for the elements  $\xi_0$  and  $\eta_0$  in the statement of the above theorem.

In the tensor product, each generator  $x_i$  gives rise to a pair of generators  $\mathbf{p} \boxtimes x_i$  and  $\mathbf{r} \boxtimes x_i$ , while each generator  $y_j$  gives rise to a generator  $\mathbf{q} \boxtimes y_j$ . Note that the generators  $\mathbf{p} \boxtimes x_i$  are the generators in the  $\iota_0$ -idempotent for the tensor product.

If we think of the  $x_i$  as the generators in a horizontally-simplified basis in the sense of Definition 10.15, then if there are  $2k+1$  basis elements,  $k$  of them are initial points of horizontal arrows. For each initial point of a horizontal arrow  $x_i$ , we can cancel  $\mathbf{r} \boxtimes x_i$  with some corresponding element of the form  $\mathbf{q} \boxtimes y_1$ ; i.e., if we have a chain

$$x_i \xrightarrow{D_3} y_1 \xrightarrow{D_{23}} \cdots \xrightarrow{D_{23}} y_\ell \xrightarrow{D_2} x_j;$$

this is transformed to

$$\begin{array}{ccccccc} \mathbf{p} \boxtimes x_i & \xrightarrow{D_3} & \mathbf{q} \boxtimes y_2 & \xrightarrow{D_{23}} & \mathbf{q} \boxtimes y_3 & \xrightarrow{D_{23}} & \cdots \xrightarrow{D_{23}} \mathbf{q} \boxtimes y_\ell \xrightarrow{D_{23}} \mathbf{r} \boxtimes x_j \\ D_2 \uparrow & & D_{23} \uparrow & & & & \downarrow D_2 \\ \mathbf{r} \boxtimes x_i & \xrightarrow{D_0} & \mathbf{q} \boxtimes y_1 & & & & \mathbf{p} \boxtimes x_j \end{array}$$

Setting  $y'_i = \mathbf{q} \boxtimes y_{i+1}$  for  $i = 1 \dots \ell - 1$  and  $y'_\ell = \mathbf{r} \boxtimes x_j$  and dropping the canceling pair  $\mathbf{r} \boxtimes x_i$  and  $\mathbf{q} \boxtimes y_1$ , our new chain has the same form as the original chain.

Vertical chains are taken to chains of the same form (without any need for cancellation).

Finally, we must observe the behavior of the “unstable chain” under tensor product. To this end, recall that the generator  $x_0$  which is neither the initial nor the final intersection point of a horizontal arrow. There is then a chain of the form

$$x_0 \xrightarrow{D_3} y_1 \xrightarrow{D_{23}} y_2 \xrightarrow{D_{23}} y_3 \xrightarrow{D_{23}} \cdots \xrightarrow{D_{23}} y_m \xleftarrow{D_1} x_1$$

with  $m > 1$ . Under the tensor product, canceling a differential from  $\mathbf{r} \boxtimes x_0$  to  $\mathbf{q} \boxtimes y_1$  as before, we see that this chain is carried to a similar chain from with length one less, starting at  $\mathbf{p} \boxtimes x_0$  and going on to  $\mathbf{p} \boxtimes x_1$ .

In the case where  $m = 1$ , we have the chain

$$x_0 \xrightarrow{D_3} y_1 \xleftarrow{D_1} x_1.$$

This chain is transformed to another configuration where we can once again cancel the pair  $\mathbf{r} \boxtimes x_0$  and  $\mathbf{q} \boxtimes y_1$ , to get the chain

$$\mathbf{p} \boxtimes x_0 \xleftarrow{D_{12}} \mathbf{p} \boxtimes x_1.$$

Now if we start with a chain of the form

$$x_0 \xleftarrow{D_{12}} x_1,$$

that in turn is transformed into

$$\mathbf{p} \boxtimes x_0 \xleftarrow{D_2} \mathbf{r} \boxtimes x_0 \xleftarrow{D_{123}} \mathbf{p} \boxtimes x_1 \xleftarrow{D_2} \mathbf{r} \boxtimes x_1.$$

(The arrow from  $\mathbf{r} \boxtimes x_1$  should be viewed as part of another chain.)

We declare the remaining elements to lie in an unstable chain of the following form

$$x_0 \xleftarrow{D_2} y_1 \xleftarrow{D_{23}} y_2 \cdots \xleftarrow{D_{23}} y_m \xleftarrow{D_{123}} x_1$$

with  $m = 1$  (after renaming variables). Now it is easy to see inductively that tensoring with our bimodule takes such an unstable chain with parameter  $m$  to one with parameter  $m + 1$ .

The precise relationship between the length (and type) of the unstable chain, the framing parameter, and  $\tau$  follows as in the proof of Theorem 10.17. (Indeed, it is a formal consequence of the above proof, together with the relationship between these three quantities for sufficiently large framing parameter as established in Theorem 10.17).  $\square$

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