Final Exam Cheat Sheet

December 11, 2008

1 L'Hôpital's Rule

L'Hôpital's rule states that when $f(x)$ and $g(x)$ are two functions both tending to zero at $x = a$, or both tending to $\pm\infty$ at $x = a$, and when $g'(x) \neq 0$ near (but not necessarily at) a, we have $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$. Here a can be finite or infinite.

When we encounter indeterminate forms other than $\frac{0}{0}$ and $\frac{\infty}{\infty}$, we convert them to one of those two forms. When we have an expression of the form $0 \cdot \infty$, we take the reciprocal of one of the two multiplicands; when we have an expression involving a power, such as 1^∞ and ∞^0, we write f^g as $e^{g \ln f}$, so that we can convert the power into a product.

2 Definite Integration

The definite integral $\int_a^b f(x)\,dx$ is the signed area below the curve $y = f(x)$, above the x-axis, left of the line $x = b$, and right of the line $x = a$. It is defined as either the lower sum or the upper sum of the function, assuming they exist and converge to the same number.

Here the lower sum with n ordinates corresponds to dividing the interval $[a, b]$ into n equal strips, taking the height of each strip to be equal to the minimum of f on the strip, and adding the areas of the strips; we then take the limit of the lower sum as $n \to \infty$. We may write the lower sum as $\lim_{n\to\infty}(\frac{b-a}{n} \sum_{i=1}^n \min\{f(x) : a + (i-1)\frac{b-a}{n} \leq x \leq a + i\frac{b-a}{n}\})$. The upper sum corresponds to taking the height of each strip to be the maximum; we may write it as $\lim_{n\to\infty}(\frac{b-a}{n} \sum_{i=1}^n \max\{f(x) : a+(i-1)\frac{b-a}{n} \leq x \leq a+i\frac{b-a}{n}\})$.

The left sum corresponds to taking the height of each strip to be the height of f at its left edge; this is equal to $\lim_{n\to\infty}(\frac{b-a}{n}\sum_{i=0}^{n-1} f(a + i\frac{b-a}{n}))$. The right sum is identical except that we use the right edge, which is equal to $\lim_{n\to\infty}(\frac{b-a}{n}\sum_{i=1}^{n} f(a+i\frac{b-a}{n}))$. When f is strictly increasing, the left sum is equivalent to the lower sum, and the right sum to the upper sum. When f is strictly decreasing, the left sum is equivalent to the upper sum, and the right sum to the lower sum.

A function is integrable if its lower and upper sum exist and are equal. Every continuous or piecewise continuous function is integrable; every strictly increasing or strictly decreasing function is integrable. Definite integrals satisfy the following rules:

- $\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
- $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$
- $\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$
- $\int_a^a f(x)\,dx = 0$
- $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

In the expression $\int_a^b f(x)\,dx$, we call $f(x)$ the integrand and a and b the limits of integration.

3 Indefinite Integration

An antiderivative (or indefinite integral) of $f(x)$, written $\int f(x)\,dx$, is a function $F(x)$ such that F is differentiable wherever f is continuous, and $F'(x) = f(x)$. The antiderivative is defined up to one constant for each interval that f is continuous on: for example, if f is continuous on $\mathbb{R}$, then any function of the form $F + c$ is an antiderivative, whereas if f is continuous on $\mathbb{R} \setminus \{0\}$, then the antiderivative is any function of the following form:

$$\int f(x)\,dx = \begin{cases} F(x) + c_1 & x < 0 \\ F(x) + c_2 & x > 0 \end{cases}$$

Most of the basic indefinite integral formulas are summarized below. We omit the constants of integration; they follow the above rule about intervals on which the integrand is continuous.

$f(x)$	$\int f(x)\,dx$
$x^n, n \neq -1$	$\frac{1}{n+1}x^{n+1}$
$\frac{1}{x}$	$\ln\lvert x\rvert$
e^x	e^x
$\ln x$	$x\ln x - x$
$\sin x$	$-\cos x$
$\cos x$	$\sin x$
$\tan x$	$-\ln\lvert\cos x\rvert$
$\cot x$	$\ln\lvert\sin x\rvert$
$\sec x$	$\ln\lvert\sec x - \tan x\rvert$
$\csc x$	$\ln\lvert\csc x - \cot x\rvert$
$\sec^2 x$	$\tan x$
$\csc^2 x$	$-\cot x$
$\tan x \sec x$	$\sec x$
$\cot x \csc x$	$-\csc x$
$\cosh x$	$\sinh x$
$\sinh x$	$\cosh x$
$\frac{1}{\sqrt{1-x^2}}$	$\arcsin x$
$\frac{1}{1+x^2}$	$\arctan x$
$\frac{1}{\sqrt{x^2-1}}$	$\operatorname{sign}(x)\operatorname{arcosh}\lvert x\rvert$
$\frac{1}{\sqrt{x^2+1}}$	$\operatorname{arsinh}\lvert x\rvert$

Definite and indefinite integration are related via the fundamental theorem of calculus. The theorem's first form states that if f is continuous on $[a,b]$, then $F(x) = \int_a^x f(t)\,dt$ is an antiderivative of $f(x)$ on $[a,b]$. Its second form states that if f is continuous on $[a,b]$ with antiderivative F, then $\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)$.

Using the fundamental theorem of calculus, we may differentiate functions defined by integrals. If g is continuous, then the formula for the derivative of $\int_a^{f(x)} g(t)\,dt$ is $f'(x)g(f(x))$. We may also differentiate functions of the form $\int_{f_a(x)}^{f_b(x)} g(t)\,dt$ as a difference of two functions, yielding the formula $f_b'(x)g(f_b(x)) - f_a'(x)g(f_a(x))$.

4 Integration and Geometry

The area below the curve $y = f(x)$ and above the curve $f = g(x)$, between $x = a$ and $x = b$, is $\int_a^b (f(x) - g(x))\,dx$. When we have to find the area between two curves without specified limits, we find the limits by solving the equation $f(x) = g(x)$; when we have to find the area between two curves without one clearly being above the other, we integrate $f(x) - g(x)$ when $f > g$ and $g(x) - f(x)$ when $g > f$.

The volume of the solid of revolution generated by rotating the area under the curve $y = f(x)$ around the x-axis is $\pi \int_a^b (f(x))^2\,dx$. Similarly, the volume of the solid generated by rotating the area between two curves $y = f(x)$ and $y = g(x)$, with $f(x) > g(x)$, is $\pi \int_a^b ((f(x))^2 - (g(x))^2)\,dx$. When we are not given explicit limits or told which curve is above which, we apply the same method as for finding the area between two curves.

The arc length of the differentiable curve $y = f(x)$ is given by $\int_a^b \sqrt{1 + (f'(x))^2}\,dx$. If we rotate it around the x-axis, we obtain a surface of revolution, whose surface area is $2\pi \int_a^b f(x)\sqrt{1 + (f'(x))^2}\,dx$.

5 Integration Rules

The substitution rule allows us to convert an integral of $f(x)$ to an integral of another function of u, a variable related to x. It is given by the formula $\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(x)\,dx$. To use the substitution rule, we write u as a function of x; then we write du as a function of x or u times du, and change the limits of integration accordingly. Alternatively, we write x as a function of u, and dx as du times a function of x or u.

Linear substitution involves changing variables from x to $u = cx + d$, with $c \neq 0$. In that case, we have $du = c\,dx$ and $x = \frac{u-d}{c}$ and we have $\int_a^b f(cx + d)\,dx = \frac{1}{c} \int_{\frac{a-d}{c}}^{\frac{b-d}{c}} f(u)\,du$. In the indefinite case, note that if $F(x)$ is an antiderivative of $f(x)$, then $\frac{1}{c}F(cx + d)$ is an antiderivative of $f(cx + d)$.

Other special cases of substitution include writing $\int \frac{f'(x)}{f(x)}\,dx$ as $\ln |f(x)|$. In addition, when faced with quadratic expressions under square root signs, we use trigonometric or hyperbolic substitution: with $\sqrt{a^2 - x^2}$ we try $x = a \cos u$ or $x = a \sin u$, with $\sqrt{x^2 - a^2}$ we try $x = a \sec u$ or $x = a \cosh u$, and with $\sqrt{x^2 + a^2}$ we try $x = a \tan u$ or $x = a \sinh u$.

Integration by parts allows us to convert integrals of products to integrals of related products: we have $\int uv' \, dx = uv - \int u'v \, dx$. When we are faced with $\int f(x)g(x) \, dx$, we have a choice as to which of f and g we choose to be u and differentiate, and which we choose to be v' and integrate. A helpful mnemonic in such cases is to differentiate the function that appears first on the following list and integrate the one that appears second:

- Logarithmic functions
- Inverse trigonometric/hyperbolic functions
- Algebraic functions
- Trigonometric/hyperbolic functions
- Exponential functions

This is referred to as LIATE.

Sometimes, it is useful to turn $\int f(x) \, dx$ into $\int 1 \cdot f(x) \, dx$ and integrate by parts with $u = f(x)$, $v' = 1$. This is most useful when $f(x)$ appears above 1 on the LIATE list, i.e. is logarithmic or inverse trigonometric.