FILTERED FLOER AND SYMPLECTIC HOMOLOGY VIA GROMOV–WITTEN THEORY

A DISSERTATION
SUBMITTED TO THE DEPARTMENT OF MATHEMATICS
AND THE COMMITTEE ON GRADUATE STUDIES
OF STANFORD UNIVERSITY
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

Luís Miguel Pereira de Matos Geraldes Diogo

August 2012

Abstract

We describe a procedure for computing Floer and symplectic homology groups, with action filtration and algebraic operations (coming from a version of Floer's equation on Riemann surfaces), in an important class of examples. Namely, we consider closed monotone symplectic manifolds X with smooth symplectic divisors Σ, Poincaré dual to a positive multiple of the symplectic form (satisfying a few more technical assumptions). We express the Floer homology of X and the symplectic homology of $X\backslash\Sigma$, for a special class of Hamiltonians, in terms of absolute and relative Gromov–Witten invariants of the pair (X, Σ), and some additional Morse-theoretic information. The key point of the argument is a relation between solutions of Floer's equation and pseudo-holomorphic curves, both defined on the symplectization of a pre-quantization bundle over Σ. As an application, we compute the symplectic homology rings of cotangent bundles of spheres, and compare our results with an earlier computation in string topology.

Acknowledgements

It is a great pleasure for me to thank the many people who have had a decisive impact in my life and work during these years as a graduate student.

I have to begin with my advisor, Yasha Eliashberg, who has taught me so much for so long. I look up to his generosity with his time and ideas, to his creativity, energy, humor, kindness, and to his example of persistence attacking problems until they are solved. Having the opportunity with learn so much from Yasha has been an enormous privilege, my appreciation of which has not ceased to increase with time.

A special mention is also due to Sam Lisi for his friendship, his patience in teaching me so many things, and for long hours of discussions about this project. The work presented in this thesis is at least as much his as mine (except for the mistakes, for which I claim sole authorship).

My work has benefited enormously from interactions with many mathematicians. In particular, Strom Borman, Frédéric Bourgeois, Kai Cieliebak, Tobias Ekholm, Oliver Fabert, Joel Fish, Janko Latschev, Mark McLean, Alex Oancea, Leonid Polterovich, Paolo Rossi, Nick Sheridan and Dimitri Zvonkine. I want to thank especially to Eleny Ionel, for her excellent classes and for always making time to answer my questions.

I had the privilege of starting my graduate studies at the University of Chicago, where I had great teachers and great friends. I want to thank them all, and in particular to Shmuel Weinberger. I also want to thank all my teachers and fellow students at Stanford University, for a fantastic learning environment. I was quite lucky to have had the opportunity to spend the year 2009/2010 at the Mathematical Sciences Research Institute. This was an extraordinary experience, for which I am very grateful. I also want to thank Leonid Polterovich and Strom Borman, at the University of Chicago, Frédéric Bourgeois and Samuel Lisi, at the Université Libre de Bruxelles, and Miguel Abreu, at Instituto Superior Técnico, for their wonderful hospitality.

I am extremely lucky for all the friends I have made, in Portugal, Chicago, Stanford, Berkeley and in many conferences in many places. I have to apologize for not naming them here, but I would certainly leave out many important people if I tried. I want to thank them in any case, and I am sure that they know that I am referring to them. I want to make one exception, though, and thank my dear friends Frank and Sylvia Soler, for letting me be part of their family.

I want to write a warm hug to my family, that I missed and miss so much, namely Zé, Alice, Augusto, Nuno, Carla, Bruna, Duarte, Maria, João, Rosa and Quim. Um grande abraço e obrigado por tudo!

The final words have to go to Ana, who makes me so happy, for her constant presence and for allowing things to make sense. This is for her.

This thesis was partly supported by the fellowship SFRH / BD / 28035 / 2006 of *Fundação para a Ciência e a Tecnologia*, Portugal.

FCT
Fundação para a Ciência e a Tecnologia
MINISTÉRIO DA EDUCAÇÃO E CIÊNCIA

POPH
QUALIFICAR É CRESCER.

QREN
QUADRO DE REFERÊNCIA ESTRATÉGICO NACIONAL
PORTUGAL 2007.2013

UNIÃO EUROPEIA
Fundo Social Europeu

Contents

List of Figures

Chapter 1

Introduction

Floer and symplectic homology groups are very important tools in the study of symplectic manifolds (M, ω), respectively closed and open. These groups are the Morse homologies of the symplectic action functional associated with a Hamiltonian function $H : S^1 \times M \to \mathbb{R}$, which is defined on (a cover of) the free loop space LM. These invariants have a very rich structure. On one hand, fixing H, there is a filtration of the Floer chain complex of H by the symplectic action. This can be used to define spectral invariants, which have many applications in symplectic topology, via, for instance, symplectic quasi-morphisms and quasi-states. On the other hand, these homology groups have a rich algebraic structure, defined in terms of spaces of solutions of elliptic equations over punctured Riemann surfaces. In particular, one can use the pair-of-pants to define a product on Floer and symplectic homology.

Despite the usefulness and richness of these invariants, they are frequently very hard to compute. One important reason is that, to define them rigorously, one often needs to study equations with perturbation terms that make them very hard to solve explicitly. The goal of this thesis is to prove a version of the following.

Theorem 1.1. *There is an explicit description of the Floer and symplectic homology groups, with their action filtration and algebraic structures, in a certain important class of examples. This description involves Gromov–Witten numbers, both absolute and relative, and some Morse-theoretic data.*

We will apply these techniques to compute symplectic homology rings of cotangent bundles of spheres, and recover a result of Cohen–Jones–Yan, in [CJY04]. One could object that the definition of Gromov–Witten invariants also often involves the study of solutions of perturbed equations. But there are many cases in which they can effectively be computed, using for example tools from algebraic geometry or complex analysis (see [Bea95] and [Zin11], for instance).

Before we give a more concrete description of our work, we should stress that all of it is joint with Samuel Lisi, and that a large portion of this material will appear in [DL12]. This is part of a larger project, joint with Strom Borman, Yakov Eliashberg, Samuel Lisi and Leonid Polterovich. Borman's upcoming thesis [Bor] contains a different approach to some of the topics discussed in this text.

We should also point out that this approach to Floer theory is very much inspired by the work of F. Bourgeois and A. Oancea, in [BO09b] and [BO09a], of F. Bourgeois, T. Ekholm and Y. Eliashberg, in [BEE09], and of Y. Eliashberg and L. Polterovich, in [EP10]. It is also related with what P. Seidel explains in Section 1 of [Sei02].

Let us now sketch the main steps in this work. We will consider the following setting: (X, ω) is a monotone closed symplectic manifold, with integral $[\omega] \in \operatorname{Image}\big(H^2(X;\mathbb{Z}) \to H^2(M;\mathbb{R})\big)$. $\Sigma \subset X$ is a monotone smooth symplectic submanifold of codimension 2, which is Poincaré-dual to $K[\omega]$, for some integer $K > 0$. We will also assume that Σ admits a perfect Morse function and that $H_1(\Sigma;\mathbb{R}) = 0$.

We will compute the Floer homology of X and the symplectic homology of W, the completion of $\widetilde{W} := X\backslash\Sigma$, for a certain class of Hamiltonians (which will be called S- and J-shaped). These Hamiltonians are degenerate, which is usually not the case in Floer theory. So, the first thing we need to do is describe what we mean by Floer theory for such Hamiltonians. This will involve a Morse–Bott version of the Floer chain complex, following work of [Bou02], [BO09b] and [BEE09].

The next step is to split X and W along certain contact-type hypersurfaces. Under our assumptions, we can identify a neighborhood of $\Sigma \subset X$ with a normal disk bundle of Σ. The boundary of this bundle is a contact-type hypersurface $Y \subset X$, and choices can be made so that Y is an S^1-bundle over Σ (Y is a *pre-quantization bundle*). In fact, there is an isosymplectic embedding of a piece of the symplectization of Y, $(a, b) \times Y \hookrightarrow X$, for some interval $(a, b) \subset \mathbb{R}$. We will split X along two parallel copies of Y, and W along one copy of Y, in a sense similar to that of symplectic field theory (see [BEH+03]). Therefore, if we start with X, we will have three pieces: W, the symplectization $\mathbb{R} \times Y$ and the normal bundle $N\Sigma$. If we start with W instead, we will have two pieces: W and $\mathbb{R} \times Y$. An argument similar to that in [BEH+03] allows us to describe what happens to the solutions of Floer's equation in X and W, as we split the manifolds. We get *split Floer trajectories*, with components in $\mathbb{R} \times Y$, and possibly also in W and $N\Sigma$. This gives an alternative description of the Floer and symplectic homology, which we refer to as *split Floer and symplectic homology* (see Figure 3.3 below).

A key point will be that, in both Floer and symplectic homologies, for the classes of Hamiltonians H under consideration, the supports of the Hamiltonian vector fields H will be contained in $\mathbb{R} \times Y$. Therefore, the components of split Floer trajectories that are contained in W and in $N\Sigma$ satisfy a (perturbed) pseudo-holomorphic curve equation, whereas the components in $\mathbb{R} \times Y$ satisfy Floer's equation.

Next, we show that, under a certain symmetry assumption on the almost complex structure J in $\mathbb{R} \times Y$, the components $\tilde{v}$ of split Floer trajectories contained in $\mathbb{R} \times Y$ are in bijective correspondence with (equivalence classes of) pairs $(\tilde{u}, f)$, where $\tilde{u}$ is a punctured pseudo-holomorphic curve in $\mathbb{R} \times Y$ and f is a function with values in $\mathbb{R} \times S^1$ that solves an auxiliary equation (we will say that f is a *cylinder solution*). Furthermore, when we restrict our attention to components of rigid Floer solutions, which are the ones used in the definition of the differential and operations in Floer theory, the corresponding cylinder solutions f can be understood rather explicitly. This reduces the problem of computing (rigid) Floer solutions to that of computing punctured pseudo-holomorphic curves in $\mathbb{R} \times Y$.

The final step in our description of Floer and symplectic homology is to relate pseudo-holomorphic curves in W, $\mathbb{R} \times Y$ and $N\Sigma$ with Gromov–Witten numbers. Pseudo-holomorphic curves in W, asymptotic at punctures to Reeb orbits of Y, can be equivalently described by maps from closed Riemann surfaces into X, intersecting Σ with certain tangency conditions (see Figure 5.4 below). These are precisely described by *relative Gromov–Witten numbers* of the pair (X, Σ).

As for pseudo-holomorphic curves $\tilde{u}$ in $\mathbb{R} \times Y$, they project to pseudo-holomorphic maps $w : \mathbb{C}P^1 \to \Sigma$. The reason why these are defined on all of $\mathbb{C}P^1$ is that punctures of $\tilde{u}$ asymptote to Reeb orbits of Y, which are fibers of the bundle $S^1 \to Y \to \Sigma$. Now, $\mathbb{R} \times Y$ can be thought of as the complement of the zero section on a complex line bundle $E \to \Sigma$. The pseudo-holomorphic curve $\tilde{u}$ then corresponds to a meromorphic section of the bundle $w^*E \to \mathbb{C}P^1$. Therefore, we reduce the problem of finding pseudo-holomorphic curves in $\mathbb{R} \times Y$ to that of finding maps $w : \mathbb{C}P^1 \to \Sigma$ and meromorphic sections of $w^*E \to \mathbb{C}P^1$. On one hand, the counts of maps w are precisely those that contribute to *Gromov–Witten numbers* of Σ. On the other hand, meromorphic sections of a holomorphic line bundle over $\mathbb{C}P^1$ are well understood: they form a $\mathbb{C}^*$-family, once we fix the positions and multiplicities of the zeros and poles. We can thus reduce the problem of finding pseudo-holomorphic maps $\tilde{u}$ to that of finding Gromov–Witten numbers of Σ.

Finally, we need to study pseudo-holomorphic maps into $N\Sigma$. In many cases, one can argue that the components in $N\Sigma$ of rigid configurations can only be simple covers of fibers of $N\Sigma \to \Sigma$. For more general configurations, an argument similar of that of the previous paragraph reduces the problem again to finding Gromov–Witten numbers of Σ. This completes the description of how to relate Floer trajectories in X and W with Gromov–Witten invariants of the pair (X, Σ).

We use the procedure outlined above to compute symplectic homology rings, in the case of $(X, \Sigma) = (Q_n, Q_{n-1})$, where Q_n is the n-dimensional complex projective quadric. We will review the topology of these manifolds, and collect the relevant Gromov–Witten numbers from [Bea95]. In this case, W is symplectomorphic to T^*S^n, and a theorem of A. Abbondandolo and M. Schwarz implies that the symplectic homology of W is isomorphic to the homology of the free loop space of S^n, as rings (see [AS10]). R. Cohen, J. Jones and J. Yan computed these rings (see [CJY04]), and our results match theirs. One interesting point is that our computation of the pair-of-pants product needs to include some *broken configurations* (as represented in Figure 5.8). One might at first hope that counts of pseudo-holomorphic pairs-of-pants in $\mathbb{R} \times Y$ might be enough to describe the Floer product, but that is not the case even in these simple examples. We will not compute Floer homology groups of closed manifolds in this thesis, but refer the interested reader to [EP10] for the case of $Q_2 = \mathbb{C}P^1 \times \mathbb{C}P^1$, which includes applications to quasi-states. We also refer to Borman's upcoming thesis [Bor] for computations in other examples.

Summing up our discussion, here is a schematic description of the argument:

$$\left\{\begin{array}{c}\text{non-degenerate}\\ \text{Floer}\\ \text{trajectories}\end{array}\right\} \overset{A}{\longleftrightarrow} \left\{\begin{array}{c}\text{degenerate}\\ \text{Floer}\\ \text{trajectories}\end{array}\right\} \overset{B}{\longleftrightarrow} \left\{\begin{array}{c}\text{split}\\ \text{Floer}\\ \text{trajectories}\end{array}\right\} \overset{C}{\longleftrightarrow}$$

$$\overset{C}{\longleftrightarrow} \left\{\begin{array}{c}\text{holomorphic curves}\\ \&\\ \text{cylinder solutions}\end{array}\right\} \overset{D}{\longleftrightarrow} \left\{\begin{array}{c}\text{GW numbers}\\ \&\\ \text{relative GW numbers}\end{array}\right\}$$

In Chapter 2, we will quickly review Floer and symplectic homologies, their algebraic structures and the action filtration. We will describe our setup in more detail and (briefly) explain correspondences A and B in Chapter 3. Correspondence C will be explained in Chapter 4. In Chapter 5, we will quickly review (absolute and relative) Gromov–Witten numbers and explain correspondence D. We will also sum up the argument with a description of the symplectic homology differential and pair-of-pants product in terms of Gromov–Witten theory. In Chapter 6, we illustrate our results with the computation of the (previously know) symplectic homology rings of cotangent bundles of spheres.

Chapter 2

Floer homology and symplectic homology

In this chapter, we will review the construction of Floer and symplectic homology, with their ring structures.

2.1 Filtered Floer homology groups

We begin with a review of Hamiltonian Floer homology. For details, we refer to [Sal99]. Let (M^{2n}, ω) be a closed symplectic manifold, so that $\omega \in \Omega^2(M)$ satisfies $d\omega = 0$ and ω^n is a volume form on M. We will assume our manifolds to be *monotone*, which means that there is a real constant $\lambda > 0$ such that

$$\langle \omega, A \rangle = \lambda \langle c_1(TX), A \rangle$$

for all $A \in H_2(X; \mathbb{Q})$.

Let J be an almost complex structure in M, compatible with ω. This means that $J \in End(TM)$, $J^2 = -Id$ and $\omega(., J.)$ is a Riemannian metric. Call a function $H : S^1 \times M \to \mathbb{R}$ a *Hamiltonian*. One can use H to define an S^1-dependent vector field X_{H_t} in M, by the relation $\omega(., X_{H_t}) = dH_t$. Abbreviate X_{H_t} to X_H. The goal of Floer theory is to study 1-periodic X_H-orbits in M.

The Floer complex is morally a Morse complex for the *action functional*:

$$\mathcal{A}_H : \widehat{LM} \to \mathbb{R}$$
$$(\gamma, u) \mapsto \int_{D^2} u^*\omega - \int_{S^1} H(t, \gamma(t))\mathrm{d}t.$$

where $\widehat{LM}$ is a cover of the space L_0M of contractible loops in M, given by pairs (γ, u), where $\gamma : S^1 \to M$ and $u : D^2 \to M$ is such that $u|_{\partial D^2} = \gamma \in L_0M$ (under a certain equivalence relation). The critical points of this functional are precisely the (capped) 1-periodic orbits of X_H. We fix, for each 1-periodic orbit γ, a capping plane u_γ. Denote by $\mathcal{P}_H$ the set of 1-periodic orbits orbits of X_H.

Remark 2.1. *There are also versions of Floer theory for non-contractible orbits. One could, for instance, consider pairs (γ, u) such that $u : S \to M$, where S is a compact surface with one boundary component and $u|_{\partial S} = \gamma$. One might also be interested in studying periodic orbits that define non-trivial elements of $\pi_1(M)$ or $H_1(M; \mathbb{Z})$. One could decompose the space of 1-periodic orbits into homotopy classes of free loops, or, put differently, conjugacy classes in $\pi_1(M)$, as is done in [BO09b]. These equivalence classes are preserved by the Floer homology differential. Another option would be to decompose the space of periodic orbits into homology classes, as done in [EGH00] in the context of symplectic field theory. If some orbits define torsion elements in $H_1(M; \mathbb{Z})$, then one could use a fractional grading for the elements of the Floer complex, as pointed out in Section 2.9.1 of [EGH00]. This is the approach that we will take in our setting.*

We need to specify the coefficient ring for the Floer chain complex. We will take the *Novikov ring*

$$\Lambda := \mathbb{Z}[t, t^{-1}]$$

of Laurent polynomials in t. We are now ready to define the *Floer chain complex* as

$$CF_*(H) = \Lambda\langle \mathcal{P}_H \rangle$$

by which we mean the free Λ-module generated by the 1-periodic orbits of X_H.

This complex has a *grading*, prescribed by

$$\deg(\gamma\, t^m) = \mu_{RS}(\gamma) + 2mN, \tag{2.1}$$

Here μ_{RS} is the *Conley–Zehnder index* (under the conventions specified by Robbin and Salamon in [RS93], hence our notation) of γ associated with the trivialization of $TM|_\gamma$ that is given by the capping u_γ (see [Sal99]). The number N is the *minimal Chern number* of M, given by the minimum of the set $\big\{\langle c_1(TM), A\rangle : A \in H_2(M;\mathbb{Z})\big\} \cap \mathbb{Z}_{>0}$. We should think of a monomial $\gamma\, t^m$ as the periodic orbit γ with the capping given by the connect sum of u_γ with a surface of Chern class mN. We also take

$$\mathcal{A}_H(\gamma\, t^m) = \mathcal{A}_H(\gamma, u_\gamma) + \lambda m N. \tag{2.2}$$

A justification for (2.1) and (2.2) will be given at the end of this section.

To define the differential in $CF_*(H)$, we count solutions of *Floer's equation*

$$\begin{gathered} V : \mathbb{R} \times S^1 \to M \\ \partial_s V + J(V)\big(\partial_t V - X_H\big) = 0 \end{gathered} \tag{2.3}$$

for the variables $(s,t) \in \mathbb{R} \times S^1$. This can be thought of as the *positive* gradient flow equation for the action $\mathcal{A}_H$. Given 1-periodic orbits γ_- and γ_+, let

$$\mathcal{M}(\gamma_+, \gamma_-) = \big\{V : \mathbb{R} \times S^1 \to M \,|\, V \text{ solves (2.3) and } \lim_{s\to\pm\infty} V(s,t) = \gamma_\pm(t)\big\}/\mathbb{R}$$

where we take a quotient by domain translations of the variable $s \in \mathbb{R}$. Since M is a monotone manifold, the $\mathcal{M}(\gamma_+, \gamma_-)$ are manifolds for generic choices of H and J. The space $\mathcal{M}(\gamma_+, \gamma_-)$ might have multiple components of different dimensions, which depend on the homology class in $H_2(M;\mathbb{Z})$ obtained by gluing the Floer cylinders to the capping disks u_{γ_-} and u_{γ_+} (with the opposite orientation on the latter). Denote by $\mathcal{M}\big((\gamma_+, u_+), (\gamma_-, u_-)\big)$ the space of $V \in \mathcal{M}(\gamma_+, \gamma_-)$ such that u_+ is homologous to $V \cup u_-$. Denote by $\mathcal{M}_0(\gamma_+, \gamma_-)$ the union of the components of $\mathcal{M}(\gamma_+, \gamma_-)$ of dimension zero. These zero-dimensional spaces turn out to be compact, and can be given an appropriate orientation, so that one can define their signed counts $\#\mathcal{M}_0(\gamma_+, \gamma_-)$.

These numbers are used to define the Floer differential:

$$\begin{aligned} \mathrm{d} : CF_k(H) &\to CF_{k-1}(H) \\ x \in \mathcal{P}_H &\mapsto \sum_{y \in \mathcal{P}_H} \#\mathcal{M}_0(x,y) \ . \ y\, t^{-j(x,y)} \end{aligned} \tag{2.4}$$

where $2j(x,y)N = -\big(\mu_{RS}(x) - \mu_{RS}(y) - 1\big)$. Given $V \in \mathcal{M}_0(x,y)$, we can also write $j(x,y)N = \langle c_1(TM), (-u_x) \cup V \cup u_y \rangle$.

Theorem. *(Floer) $d^2 = 0$. The* Floer homology *of M is $H_*(CF_*(H), d)$, and it does not depend on the generic choices of H, J. In fact, it is isomorphic to $H^{*-n}(M; \Lambda)$ (singular cohomology with Novikov coefficients).*

Denote the Floer homology of M with respect to the Hamiltonian H by $HF_*(H)$.

We have mentioned that (2.3) is the equation for the positive gradient flow of $\mathcal{A}_H$. In particular, $\mathcal{A}_H$ increases along solutions V, as s increases, and d decreases $\mathcal{A}_H$. This implies that one can filter the Floer complex $CF_*(H)$ by values of the action, and define, for $a \in \mathbb{R}$, subcomplexes

$$CF_k^a(H) = \left\{ \sum_{i,j} c_{i,j}\, x_i\, t^j \in CF_k(H) \,\middle|\, \forall_{i,j}\, \mathcal{A}_H(x_i\, t^j) \leqslant a \right\}$$

Denote the homology of this subcomplex by $HF_*^a(H)$. It is then the case that

$$HF_k(H) \cong \lim_{a \to \infty} HF_k^a(H).$$

Even though $HF_*(H)$ is isomorphic to $H^{*-n}(M; \Lambda)$, one can extract very useful symplectic (and not just topological) information from the Floer complex, when taking into account the action filtration. It is particularly useful to consider the *spectral invariants* of H, with which one can define, for instance, *symplectic quasi-morphisms* and *quasi-states* in M. For more details, see for example [Oh97] and [EP03].

We finish this section with a statement of the facts that motivate definitions (2.1) and (2.2), and the expressions for j in (2.4). For more details, see [MS04] and [Sal99].

Proposition. *Let $V : \mathbb{R} \times S^1 \to M$ be a solution of (2.3), connecting the 1-periodic* *orbits* γ_- *and* γ_+*, let* $u_\pm : D^2 \to M$ *be cappings for* $\gamma_\pm$ *and let* $V \cup u_-$ *be the capping for* γ_+ *that is induced by* V *and* u_-*. Then,*

- $\dim \mathcal{M}\big((\gamma_+, V \cup u_-), (\gamma_-, u_-)\big) = \mu_{RS}(\gamma_+, V \cup u_-) - \mu_{RS}(\gamma_-, u_-)$*;*
- $E(V) := \frac{1}{2}\int_{\mathbb{R}\times S^1} |\partial_s V|^2 + |\partial_t V - X_H|^2 ds\, dt = \mathcal{A}_H(\gamma_+, V \cup u_-) - \mathcal{A}_H(\gamma_-, u_-)$.

Now, let $A \in H_2(M;\mathbb{Z})$ *and denote by* $u_+\#A$ *the connect sum. Then,*

- $\mu_{RS}(\gamma_+, u_+\#A) = \mu_{RS}(\gamma_+, u_+) + 2\langle c_1(TM), A\rangle$*;*
- $\mathcal{A}_H(\gamma_+, u_+\#A) = \mathcal{A}_H(\gamma_+, u_+) + \langle \omega, A\rangle$.

2.2 Symplectic homology

Symplectic homology is a version of Hamiltonian Floer homology for (*completions* of) a certain class of symplectic manifolds with boundary, called *Liouville domains*. We will review the construction and some properties of this invariant, referring to [Oan04] [Sei08] for more details.

A Liouville domain is a symplectic manifold with boundary $(\widetilde{W}, \omega)$ with a vector field V pointing outward along $Y = \partial\widetilde{W}$, such that $\mathcal{L}_V\omega = \omega$. The 1-form $\alpha := (\iota_V\omega)|_Y$ is a *contact form* on Y. Write $\xi = \ker\alpha$ for the *contact structure* and R for the *Reeb vector field* of α, defined uniquely by the conditions $\iota_R d\alpha \equiv 0$ and $\alpha(R) \equiv 1$. One can use V to form the *completion* $W := \widetilde{W} \cup_Y [0,\infty) \times Y$. The *Liouville form* $\eta := \iota_V\omega$, a primitive for ω, can be extended to $(0,\infty)\times Y \subset W$ as $e^r\alpha$, where r is the coordinate on $(0,\infty)$. Denote this extension also by η and the symplectic form $d\eta \in \Omega^2(W)$ by ω. For technical reasons, we will further assume that $c_1(TW)$ is a (possibly vanishing) torsion element in $H^2(W;\mathbb{Z})$.

Let J be an almost complex structure in W, compatible with ω. We require that J preserve ξ and that $J\partial_r = R$, on $(0,\infty)\times Y$ (one can sometimes relax this condition and require J to be only asymptotically cylindrical). To define Floer homology, one needs a Hamiltonian in W. We consider $H : S^1 \times W \to \mathbb{R}$ such that:

- $H|_{\widetilde{W}}$ is an S^1-independent C^2-small Morse function;

- $H|_{(0,\infty)\times Y}$ is a small perturbation, near the 1-periodic orbits, of $h(e^r)$, for some function $h : \mathbb{R} \to \mathbb{R}$ such that $\lim_{\tau\to\infty} h'(\tau) = \infty$.

The symplectic homology of W is the Floer homology of such Hamiltonians. Since (W, ω) is exact and $c_1(TW)$ is torsion, we can use integer coefficients in the definition, instead of a Novikov ring. We can split the Floer complex into summands indexed by free homotopy classes of orbits, each of which admitting a $\mathbb{Z}$-grading, as in [BO09b]. In the case when all periodic orbits define torsion elements in $H_1(W;\mathbb{Z})$, one can define instead a $\mathbb{Q}$-grading, using the ideas in Section 2.9.1 of [EGH00]. The advantage of this is to reduce the number of choices necessary to grade the symplectic homology complexes, which is practical for computations. Symplectic homology can be shown to be independent of H and J. We denote it by $SH_*(W)$.

If (W_1, η_1) and (W_2, η_2) are two completions of Liouville domains (with Liouville forms η_i and symplectic forms $\mathrm{d}\eta_i$) for which there is a diffeomorphism $\phi : W_1 \to W_2$ such that $\phi^*\eta_2 = \eta_1$, then $SH_*(W_1) = SH_*(W_2)$ (see Section 7 in [Sei08]). Therefore, for completions W such that $H^1(W;\mathbb{R}) = 0$, as in the examples that we will consider in Chapter 6, symplectic homology is a symplectomorphism invariant.

2.3 Operations on Floer theory; relation with string topology

We recall now how to use spaces of solutions of elliptic equations defined over punctured Riemann surfaces to define operations on Floer and symplectic homology. A reference in the case of Floer homology is the thesis of Schwarz [Sch95]. For symplectic homology, a reference is Abbondandolo and Schwarz's [AS10]. We will use Seidel's approach to operations on Floer theory (see [Sei08] and [Rit11]).

Fix a Hamiltonian $H : S^1 \times M \to \mathbb{R}$ and an almost complex structure J in M. Let $\Gamma_F^+ = \{z_1^+, \ldots, z_{k_+}^+\}$ and $\Gamma_F^- = \{z_1^-, \ldots, z_{k_-}^-\}$ be two disjoint finite subsets of a fixed Riemann surface σ (which for our purposes will always be $\mathbb{C}P^1$). Write $\Gamma_F := \Gamma_F^+ \cup \Gamma_F^-$ and $S := \sigma\backslash\Gamma_F$. Fix conformal parametrizations $\varphi_i^\pm : \mathbb{R}_\pm \times S^1 \to S$ of neighborhoods of the $z_i^\pm \in \Gamma_F^\pm$. Choose a 1-form $\beta \in \Omega^1(S)$, such that $(\varphi_i^\pm)^*\beta = c_i^\pm\,\mathrm{d}t$, for some constants $c_i^{\pm} > 0$, and $\mathrm{d}\beta \leqslant 0$ (with respect to the conformal structure on S). Seidel's generalization of Floer's equation (2.3) is

$$\begin{gathered} v : S \to M \\ (\mathrm{d}v - X_H \otimes \beta)^{0,1} = 0 \end{gathered} \tag{2.5}$$

Note that, in the case when $S = \mathbb{R} \times S^1$ and $\beta = \mathrm{d}t$, this equation becomes (2.3). By counting rigid solutions of (2.5), one can define operations

$$HF_*(c_1^+ H) \otimes \ldots \otimes HF_*(c_{k_+}^+ H) \to HF_*(c_1^- H) \otimes \ldots \otimes HF_*(c_{k_-}^- H)$$

when M is closed. When M is a completion of a Lioville domain, the same procedure defines operations on symplectic homology. Composition of these operations corresponds to gluing of domains (see [Rit11]).

As a particular case, one can let $\sigma = \mathbb{C}P^1$, $\Gamma_F^+ = \{0, 1\}$, $\Gamma_F^- = \{\infty\}$, and $\beta = \psi^* \mathrm{d}t$, for a branched cover $\psi : \mathbb{C}P^1 \backslash \Gamma_F \to \mathbb{R} \times S^1$, and define an operation

$$HF_*(H) \otimes HF_*(H) \to HF_*(2H).$$

Using a continuation map (from an interpolation of Hamiltonians), we can construct an isomorphism $HF_*(H) \to HF_*(2H)$. Inverting this map, we get a product

$$HF_*(H) \otimes HF_*(H) \to HF_*(H).$$

A similar structure can be defined on symplectic homology. The continuation map argument is a bit more subtle in this context (see the Appendix 3 in [Rit11]).

These ring structures on Floer and symplectic homology are often related with other structures. Indeed, we have the following enhancement of Floer's theorem.

Theorem. *1. [PSS96] If (M, ω) is a closed semi-positive (a generalization of monotone) symplectic manifold, then, over $\mathbb{Q}$,*

$$HF_*(M) \cong QH^{*-n}(M)$$

as rings (we will recall later how to use Gromov–Witten invariants to define the quantum cohomology ring $QH^*(M)$, which is $H^*(M;\Lambda)$ as an abelian group).

2. *[AS10] If N is a closed spin manifold, then*

$$SH_*(T^*N) \cong H_*(LN)$$

as rings (where $H_*(LN)$ is the homology of the free loop space of N, with the Chas–Sullivan product; this is part of the string topology *of N).*

The string topology rings of some manifolds have been computed. As an example, the following was proven in [CJY04].

Theorem (Cohen-Jones-Yan)**.** *If $n > 1$, the ring $H_*(LS^n)$ is isomorphic to*

- $(\Lambda[b] \otimes \mathbb{Z}[a,v])\,/(a^2, ab, 2av)$, *for some $a \in H_0(LS^n)$, $b \in H_{n-1}(LS^n)$ and $v \in H_{3n-2}(LS^n)$, if n is even,*
- $\Lambda[a] \otimes \mathbb{Z}[u]$, *for $a \in H_0(LS^n)$ and $u \in H_{2n-1}(LS^n)$, if n is odd.*

If we shift the grading by $-n$, then the product preserves the grading. In Chapter 6, we will see how to use our techniques to give an alternative proof of this result.

Chapter 3

Split Floer homology

This chapter describes the assumptions that we make on our manifolds, and gives an indication of how to degenerate both the Hamiltonians and the manifolds. Our description of the degenerations will not contain most details, but is included to motivate what will be done in later chapters.

3.1 Symplectic divisors on monotone manifolds

We now describe the particular class of Liouville domains that will be of interest to us. Let (X, ω) be a closed connected symplectic manifold, with integral $[\omega] \in \mathrm{Image}\big(H^2(X;\mathbb{Z}) \to H^2(M;\mathbb{R})\big)$ and with a connected closed symplectic submanifold Σ of codimension 2. Assume that Σ is Poincaré-dual to $K[\omega]$, for some integer $K > 0$. Many interesting examples can be obtained by taking as X^{2n} a complete intersection in $\mathbb{C}P^{n+r}$, and as Σ the intersection of X with a projective hypersurface.

Note. *Donaldson showed that every symplectic manifold with an integral symplectic form admits a symplectic submanifold Poincaré-dual to* $K[\omega]$*, for* $K > 0$ *sufficiently large (see [Don96]). The examples we will consider in the last chapter are all* polarized Kähler manifolds*, in the sense of Biran (see [Bir01]). These consist of quadruples* (X, ω, J, Σ)*, such that* (X, ω, J) *is a Kähler manifold with integral symplectic form, and* Σ *is a smooth and reduced complex hypersurface Poincaré-dual to a positive integer multiple of* ω*. We should point out that we will work with almost complex* *structures J that are not necessarily integrable.*

Let us assume that (X, ω) is monotone, and let $\lambda_X > 0$ be such that $\langle \omega, A \rangle = \lambda_X \langle c_1(TX), A \rangle$ for all $A \in H_2(X; \mathbb{Q})$. We will denote the normal bundle to $\Sigma \subset X$ by $N\Sigma$ and the boundary of a disk tubular neighborhood of Σ by Y. This is an S^1-bundle over Σ.

Lemma 3.1. *For every $A \in H_2(\Sigma; \mathbb{Q})$,*

$$\langle \omega, A \rangle = \frac{\lambda_X}{1 - K\,\lambda_X} \langle c_1(TX), A \rangle.$$

Proof. Since the first Chern class is additive, $c_1(T\Sigma) = c_1(TX)|_\Sigma - c_1(N\Sigma)$. Given $A \in H_2(\Sigma; \mathbb{Q})$,

$$\begin{aligned} \langle c_1(T\Sigma), A \rangle = \langle c_1(TX), A \rangle - \langle c_1(N\Sigma), A \rangle = 1/\lambda_X \langle \omega, A \rangle - K \langle \omega, A \rangle = \\ = (1/\lambda_X - K) \langle \omega, A \rangle \end{aligned}$$

which is what we wanted to show. We have used the fact that $\langle c_1(N\Sigma), A \rangle = \#(\Sigma \cap A) = \langle K\omega, A \rangle$, based on the assumption that $\Sigma = \mathrm{PD}(K\omega)$. □

This result implies that (Σ, ω) is also monotone (with $\lambda_\Sigma = \frac{\lambda_X}{1-K\,\lambda_X}$), if $K\,\lambda_X < 1$.

Important assumptions. *Throughout the rest of this text, (X, ω) will be a closed connected monotone symplectic manifold, with integral $[\omega] \in H^2(X; \mathbb{Z})$, and with a monotone smooth connected symplectic divisor Σ, Poincaré-dual to $K[\omega]$ for some integer $K > 0$. We will further assume that Σ admits a perfect Morse function and that $H_1(\Sigma; \mathbb{R}) = 0$.*

The following result will also be useful.

Lemma 3.2. *Let $\widetilde{W} := X \backslash \Sigma$. Write $\lambda_X = p/q$ for some $p, q \in \mathbb{Z}_+$.*

- $[\omega]|_{\widetilde{W}} = 0 \in H^2(\widetilde{W}; \mathbb{R})$ *and $c_1(T\widetilde{W})$ is torsion in $H^2(\widetilde{W}; \mathbb{Z})$;*

- *if $H_1(X;\mathbb{Z})$ has no torsion, then $(Kp)\, c_1(T\widetilde{W}) = 0$;*
- *$\widetilde{W}$ is the interior of a Liouville domain.*

Proof. Let $A \in H_2(\widetilde{W};\mathbb{Q})$. Then,

$$\langle \omega, A\rangle = \frac{1}{K}\,\#(\Sigma \cap A) = 0$$

because A does not intersect Σ. Also, using the monotonicity of X,

$$\langle c_1(T\widetilde{W}), A\rangle = \langle c_1(TX), A\rangle = \frac{1}{\lambda_X}\,\langle \omega, A\rangle = 0.$$

This implies the first part of the Lemma.

For the second part, we write part of the long exact sequence for the pair (X, Σ):

$$H^2(X, W;\mathbb{Z}) \xrightarrow{\varphi} H^2(X;\mathbb{Z}) \xrightarrow{\psi} H^2(\widetilde{W};\mathbb{Z}).$$

Denoting by $\Phi \in H^2(N\Sigma, \Sigma;\mathbb{Z}) = H^2(X, W;\mathbb{Z})$ the Thom class of the bundle $N\Sigma$, we have $\varphi(\Phi) = \mathrm{PD}(\Sigma) = K[\omega] = \frac{Kp}{q} c_1(TX)$ (the first identity follows from Proposition 6.24 in [BT82]; the third identity is a consequence of the assumption that $H_1(X;\mathbb{Z})$ has no torsion, which implies that $H^2(X;\mathbb{Z})$ has no torsion). Thus, by exactness, $0 = (\psi \circ \varphi)(q\Phi) = (Kp)\, c_1(T\widetilde{W})$, as wanted.

We now show that $\widetilde{W}$ is the interior of a Liouville domain. One can think of $\widetilde{W}$ as the interior of a compact manifold $\overline{W}$, such that $\partial\overline{W}$ is diffeomorphic to Y. Since $\Sigma = \mathrm{PD}(K\omega)$, for some $K > 0$, $\partial\overline{W}$ is a *convex* boundary, which means that $\overline{W}$ has a Liouville vector field defined on a collar neighborhood $(0, 1] \times Y \to \overline{W}$ of $\partial\overline{W}$. This means that, on $(0, 1] \times Y$, we have a vector field V_1, such that $\mathcal{L}_{V_1}\omega = \omega$. Therefore, in that neighborhood of $\partial\overline{W}$, $\eta_1 := \iota_{V_1}\omega$ is a primitive for ω. On the other hand, since we just saw that $[\omega] = 0$, we know that $\omega = \mathrm{d}\eta_2$, for some global 1-form $\eta_2 \in \Omega^1(\overline{W})$. Now, choose a function $\beta : (0, 1] \to [0, 1]$ which is identically 0 near 0 and identically 1 near 1. Then,

$$\mathrm{d}\big(\beta\eta_1 + (1-\beta)\eta_2\big) = \omega + \mathrm{d}\beta \wedge (\eta_1 - \eta_2).$$

Observe that $\mathrm{d}(\eta_1 - \eta_2) = 0$ in $(0,1] \times Y$. Suppose that $\eta_1 - \eta_2 = \mathrm{d}f$, for some function $f : (0,1] \times Y \to \mathbb{R}$ (whose existence will be shown later). Take now $g : (0,1] \times Y \to \mathbb{R}$ such that $g \equiv f$ in $(\operatorname{supp} \mathrm{d}\beta) \times Y$, $g \equiv 0$ very near $\{1\} \times Y = \partial\overline{W}$ and $g \equiv 1$ very near $\{0\} \times Y$. Define $\eta := \beta\eta_1 + (1-\beta)\eta_2 - \beta \mathrm{d}g \in \Omega^1(\overline{W})$. Then,

$$\mathrm{d}\eta = \omega + \mathrm{d}\beta \wedge (\eta_1 - \eta_2) - \mathrm{d}\beta \wedge \mathrm{d}g = \omega.$$

Since $\eta \equiv \eta_1$ near $\partial\overline{W}$, η is a Liouville form on $\overline{W}$. To conclude the proof, we just need to show the existence of the function f above. This follows from the fact that $H^1(Y;\mathbb{R}) = 0$, which we now prove. The Gysin sequence for the fibration $S^1 \to Y \to \Sigma$ yields

$$0 \to H^1(\Sigma;\mathbb{R}) \to H^1(Y;\mathbb{R}) \to H^0(\Sigma;\mathbb{R}) \xrightarrow{\cup c_1(Y \to \Sigma)} H^2(\Sigma;\mathbb{R}).$$

By monotonicity of Σ, the map on the right is non-zero. Since Σ admits a perfect Morse function, $H^*(\Sigma;\mathbb{Z})$ has no torsion, so the map on the right is an injection. Therefore, $H^1(\Sigma;\mathbb{R}) \to H^1(Y;\mathbb{R})$ is an isomorphism. Since we assume in this text that $H_1(\Sigma;\mathbb{R}) = 0$, we conclude that $H^1(Y;\mathbb{R}) = 0$, which finishes the proof of the Lemma. □

This result implies that one can define symplectic homology of the completion W of $\widetilde{W}$, with integer coefficients and with a rational grading, without needing Novikov coefficients. If $H_1(X;\mathbb{Z})$ has no torsion and $K = p = 1$, as will be the case in Chapter 6, then $c_1(T\widetilde{W}) = 0$ and there is an integer grading on symplectic homology.

Remark 3.1. *It should be possible to extend the results that will be explained in this text to the general case of a complex projective manifold X with an* ample *(or* positive*) smooth divisor Σ (see [Huy05] for the definitions). In this case, $X \backslash \Sigma$ also has convex boundary (see Section 2.7 in [CE]).*

The assumptions that Σ has a perfect Morse function and that $H_1(\Sigma;\mathbb{R}) = 0$ are both used in the proof above, but in a rather weak way, and one should be able to do away with them, at least in some important cases.

Given X and Σ as above, we can choose a Hermitian metric on $N\Sigma$, for which a connection 1-form defines a contact form α in Y. We can further assume that the Reeb flow corresponds to flowing along the fibers of $S^1 \to Y \to \Sigma$ (say that Y is a *pre-quantization bundle*). By the symplectic tubular neighborhood theorem (see Section 9.3.2 in [EM02] and Section 2.1 in [Bir01]), there is an isosymplectic embedding $\big((a,b) \times Y, \mathrm{d}(e^r\alpha)\big) \hookrightarrow (X,\omega)$, for some interval $(a,b) \subset \mathbb{R}$. Biran has shown that, in the setting of a polarization, there is such an embedding with full volume in X (see [Bir01]). One should be able to extend this result to a symplectic (not necessarily Kähler) setting, using methods from [Gir02]. For any $x \in (a,b)$, ∂_r is a local Liouville vector field near $\{x\} \times Y$, so we say that this is a *contact-type hypersurface* in X. Therefore, $\{x\} \times Y$ separates X into two pieces: a convex filling of Y, corresponding to the side where $r < x$, and whose completion is symplectomorphic to W; and a concave filling, on the side where $r > x$, which is symplectomorphic to a disk normal bundle of Σ in X.

Remark 3.2. *Since we want the symplectization coordinate r in $\big(\mathbb{R} \times Y, d(e^r\alpha)\big)$ to grow as one approaches Σ, we will think of the bundle $S^1 \to Y \to \Sigma$ as having first Chern class equal to $-c_1(N\Sigma)$. Note that $\mathbb{R} \times Y$ can be thought of as the complement of the zero section on a complex line bundle $E \to \Sigma$ that is dual to $N\Sigma$.*

At this point, we would like to say some words about the monotonicity assumptions on X and Σ. On one hand, these are useful to ensure transversality for the spaces of pseudo-holomorphic curves and Floer trajectories that we will consider. On the other hand, monotonicity will also lie behind the fact that the Novikov parameter in the Floer homology differential counts intersections of Floer trajectories in X with the divisor Σ, as will be explained at the end of the next section.

3.2 Degenerating the Hamiltonian

Recall that we mentioned, when defining Floer homology and symplectic homology, that our Hamiltonians might need to be S^1-dependent, so that they can satisfy a certain non-degeneracy condition. Nonetheless, for our purposes, we will need to consider certain degenerate Hamiltonians, which will require an adjustment in our description of Floer theory. We now define the classes of Hamiltonians that we will use (see also Figure 3.1).

Definition 3.1. *A function $H : X \to \mathbb{R}$ is called* S-shaped*, if:*

1. *there are values $r_1, r_2 \in (a, b)$, where $(a, b) \times Y \hookrightarrow X \backslash \Sigma = \widetilde{W}$, such that the support of dH is contained in $(r_1, r_2) \times Y$;*
2. *on $(r_1, r_2) \times Y$, $H(r, Y) = h(e^r)$, for some monotone increasing function $h : \mathbb{R} \to \mathbb{R}$;*
3. *h' has one absolute maximum M, which is not an integer;*
4. *for all $0 < c < M$, there are exactly two values of $r \in (r_1, r_2)$ such that $h'(e^r) = c$.*

A function $H : W \to \mathbb{R}$ is called J-shaped *if*

1. *there is a value $r_1 \in (a, b)$, where $(a, b) \times Y \hookrightarrow X \backslash \Sigma = \widetilde{W}$, such that the support of dH is contained in $(r_1, \infty) \times Y \subset W$;*
2. *on $(r_1, \infty) \times Y$, $H(r, Y) = h(e^r)$, for some monotone increasing function $h : \mathbb{R} \to \mathbb{R}$;*
3. *h'' is positive on (r_1, ∞) and $\lim_{r \to \infty} h'(r) = \infty$;*
4. *for all $0 < c$, there is exactly one value of $r \in (r_1, \infty)$ such that $h'(e^r) = c$ (this actually follows from the previous conditions).*

We will want to define Floer and symplectic homology for S- and J-shaped Hamiltonians, respectively. Notice that, on the support of the derivative of such H, the Hamiltonian vector field is $X_H(r, y) = h'(e^r)R(y)$, where $R(y)$ is the Reeb vector field at $y \in Y$. Therefore, on the support of $\mathrm{d}H$, the 1-periodic orbits of X_H come in S^1-families (because H is time-independent) and correspond to Reeb orbits in Y. The S^1-families of X_H-orbits on the level $\{r\} \times Y$ correspond precisely to the Reeb orbits in Y of period $T = h'(e^r)$. Since the Reeb flow on the pre-quantization bundle

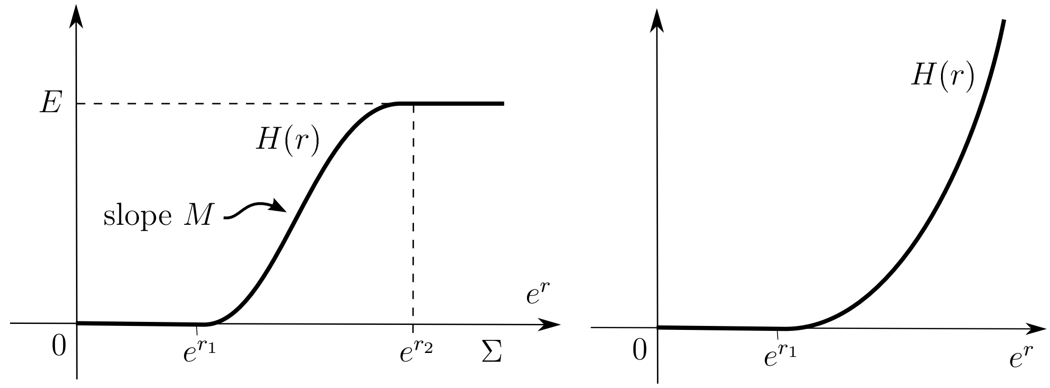

Figure 3.1: S- and J-shaped Hamiltonians

Y goes around the orbits of the fibration $S^1 \to Y \to \Sigma$, for each positive integer k there is a Y-family of (parametrized) Reeb orbits of period k.

To define the symplectic action and the Floer grading, it will be useful to specify cappings for our orbits. Recall that, given $A \in H_2(\Sigma;\mathbb{Z})$, we have $\#(A \cap \Sigma) = K\langle\omega, A\rangle$. Therefore, if we fix a point $p \in \Sigma$, and $A \in H_2(\Sigma;\mathbb{Z})$ such that $\langle\omega, A\rangle \neq 0$, then we can lift a representative of A that intersects Σ only at p (to order $K\langle\omega, A\rangle$) to a surface in Y whose only boundary component is a $(K\langle\omega, A\rangle)$-cover of the fiber over p. As a consequence, the $(K\langle\omega, A\rangle)$-multiple of every Reeb orbit in Y can be capped by a surface in $\widetilde{W}$, and it vanishes on $H_1(\widetilde{W};\mathbb{Z})$. For this reason, we will say that every Reeb orbit has a *fractional capping* inside $\widetilde{W}$. Notice also that every non-constant periodic orbit in X admits a capping via (a multiple of) a fiber of (a disk bundle of) $N\Sigma$. This capping disk is oriented in such a way that its intersection number with Σ is negative. It will be useful to think of (symplectic homology) orbits in W as having fractional cappings in W, and of non-constant (Floer homology) orbits in X as having both a fractional capping in $\widetilde{W}$ and a capping in X that intersects Σ negatively. Constant orbits have constant cappings.

Lemma 3.3. *The 1-periodic orbits of an S-shaped Hamiltonian H in X are of two types:*

- *constant: corresponding to the points $p \in X\backslash(\operatorname{supp} dH)$. When given a trivial capping, these orbits have action $\mathcal{A}_H = -H(p)$;*

- *non-constant: for each integer* $0 < k < M = \max h'$*, there are two values* $r^{\pm} \in (r_1, r_2)$ *such that* $k = h'(e^{r^{\pm}})$*. There is a* Y*-family of 1-periodic* X_H*-orbits contained in* $\{r^+\} \times Y$ *and another* Y*-family contained in* $\{r^-\} \times Y$*. When given a rational capping inside* $\widetilde{W}$*, these orbits have action* $\mathcal{A}_H = e^{r^{\pm}} h'(e^{r^{\pm}}) - h(e^{r^{\pm}})$*, respectively. When capped by a disk whose intersection number with the divisor* Σ *is* $-k$*, their action is* $\mathcal{A}_H = e^{r^{\pm}} h'(e^{r^{\pm}}) - h(e^{r^{\pm}}) - k/K$*.*

The 1-periodic orbits of a J-shaped Hamiltonian H *in* W *are of two types:*

- *constant: corresponding to the points* $p \in W \backslash (\operatorname{supp} dH)$*, with action* $\mathcal{A}_H = -H(p)$*;*

- *non-constant: for each integer* $k > 0$*, there is one value* $r \in (r_1, \infty)$ *such that* $k = h'(e^r)$*. There is a* Y*-family of 1-periodic* X_H*-orbits contained in* $\{r\} \times Y$*. These orbits have action* $\mathcal{A}_H = e^r h'(e^r) - h(e^r)$*, with respect to a fractional capping in* W*.*

Proof. We have already described the periodic X_H-orbits, and are only left with justifying the values of their actions. Start with the case of W, where ω is exact (recall Lemma 3.2), so we can write $\omega = \mathrm{d}\eta$ for some $\eta \in \Omega^1(W)$. Fix $A \in H_2(\Sigma; \mathbb{Z})$ such that $\langle \omega, A \rangle \neq 0$. Given a non-constant 1-periodic orbit γ of the Hamiltonian H, we saw above that the $(K\langle \omega, A \rangle)$-cover of γ, denoted by $\gamma_{K\langle \omega, A \rangle}$, is trivial on $H_1(W; \mathbb{Z})$. Let $u_{K\langle \omega, A \rangle} : S \to W$ be a capping for $\gamma_{K\langle \omega, A \rangle}$, where S is a surface with one boundary component and $u|_{\partial S} = \gamma_{K\langle \omega, A \rangle}$. Then

$$\int_S (u_{K\langle \omega, A \rangle})^* \omega - \int_{S^1} H \mathrm{d}t = \int_{S^1} (\gamma_{K\langle \omega, A \rangle})^* \eta - H(\gamma_{K\langle \omega, A \rangle}) \mathrm{d}t = $$
$$= K\langle \omega, A \rangle \Big(\int_{S^1} \gamma^* \eta - H(\gamma) \mathrm{d}t \Big)$$

and thus $\mathcal{A}_H(\gamma) = \int_{S^1} \gamma^* \eta - H \mathrm{d}t$, with respect to the fractional capping $\frac{1}{K\langle \omega, A \rangle} u_{K\langle \omega, A \rangle}$ (note that this is independent of capping). The non-constant orbits are contained in a half-infinite piece of $\mathbb{R} \times Y$, where $\eta = e^r \alpha$ (where α is the contact form on Y). For a 1-periodic X_H-orbit $\gamma(t) = \big(r, \tilde{\gamma}(Tt)\big) \subset \mathbb{R} \times Y$, where $\tilde{\gamma} : \mathbb{R}/T\mathbb{Z} \to Y$ is a closed

Reeb orbit of period $T = h'(e^r)$, we have

$$\mathcal{A}_H(\gamma) = \int_{S^1} e^r\alpha(\dot{\gamma}) - h(e^r)\mathrm{d}t = e^r h'(e^r) - h(e^r).$$

The fact that constant orbits with constant cappings have action given by the value of $-H$ is immediate from the definition of action.

Consider now the case of X. If we again choose fractional cappings contained inside $\widetilde{W}$ (where $\omega = \mathrm{d}\eta$) for the non-constant 1-periodic X_H-orbits, then the computation of the action of these orbits is the same as the one done above for symplectic homology. The exactness of $\omega|_{\widetilde{W}}$ again implies that the action is independent of the choice of fractional capping in $\widetilde{W}$. Now, let γ_k be an X_H-orbit corresponding to a Reeb orbit of multiplicity k and denote by u'_k the capping of γ by a plane that intersects $-k$ times Σ. There is a corresponding capping $u'_{kK\langle\omega,A\rangle}$ for a $(K\langle\omega, A\rangle)$-cover of γ_k, which also admits a capping $u_{kK\langle\omega,A\rangle} : S \to X$ inside $\widetilde{W}$. The difference in actions computed with respect to these cappings is given by the Proposition at the end of Section 2.1:

$$\begin{aligned}\frac{1}{K\langle\omega, A\rangle}\Big(\int_{D^2}(u'_{kK\langle\omega,A\rangle})^*\omega - \int_S (u_{kK\langle\omega,A\rangle})^*\omega\Big) &= \\ = \frac{1}{K\langle\omega, A\rangle}\Big\langle\omega, \big(u'_{kK\langle\omega,A\rangle} \cup (-u_{kK\langle\omega,A\rangle})\big)\Big\rangle &= \\ = \frac{1}{K\langle\omega, A\rangle}\frac{1}{K}\#\Big(\Sigma \cap \big(u'_{kK\langle\omega,A\rangle} \cup (-u_{kK\langle\omega,A\rangle})\big)\Big) &= \frac{-kK\langle\omega, A\rangle}{K^2\langle\omega, A\rangle} = -\frac{k}{K}.\end{aligned}$$

Therefore, $\mathcal{A}_H(\gamma, u') = \mathcal{A}_H(\gamma, u/l) - k/K$, as wanted. The computation of the actions of constant orbits (with constant cappings) is analogous to the one for W. □

Since the 1-periodic orbits of X_H come in manifold families, there are two approaches one can take to define Floer and symplectic homology chain complexes: either perturb H to a non-degenerate time-dependent Hamiltonian, or use a Morse–Bott version of Floer and symplectic homology. Both approaches should give isomorphic homology groups. We will use the latter. In [BO09b], Bourgeois and Oancea describe a Morse–Bott chain complex that computes symplectic homology for time-independent Hamiltonians, and show that it is isomorphic, as an abelian group, to symplectic homology, as defined for non-degenerate Hamiltonians. On one hand, we need a stronger form of their result, allowing for Hamiltonians that are non-degenerate not only because they are time-independent, but also because they are constant on large subsets of X and W, and because the Reeb flow itself is degenerate on the pre-quantization bundle Y. On the other hand, the manifolds that we consider are not as general as those considered in [BO09b], since they do not restrict their attention to Liouville domains whose boundaries are pre-quantization bundles. The fact that the Reeb flow is *Morse–Bott non-degenerate* (as in [Bou02]) is useful to achieve transversality for the relevant spaces of Floer trajectories and pseudo-holomorphic curves.

We will now describe the version of Floer theory that is appropriate for our setting, but without a justification of the construction or of the independence of the choices involved.

Recall that, by definition, if H is an S-shaped Hamiltonian, then there are two separating contact-type hypersurfaces $Y_1 = \{r_1\} \times Y$ and $Y_2 = \{r_2\} \times Y$, such that $\mathrm{d}H$ is supported between Y_1 and Y_2 (see Figure 3.1). Denote by A and B the two connected components of $X \backslash (Y_1 \cup Y_2)$ where H is constant, as in Figure 3.2. A is a tubular neighborhood of Σ and B is diffeomorphic to $\widetilde{W} = X \backslash \Sigma$. Similarly, a J-shaped Hamiltonian is constant on a subset of W that is diffeomorphic to $\widetilde{W}$, and that we also denote as B.

The closures $\overline{A}$ and $\overline{B}$ are manifolds with boundary, on which we choose auxiliary functions $f_A : \overline{A} \to \mathbb{R}$ and $f_B : \overline{B} \to \mathbb{R}$, that are constant on the boundaries. Suppose that these functions are Morse–Smale on the interiors A and B, respectively, and also that f_A attains its minimum on $\partial \overline{A} = Y_2$ and that f_B has its maximum on $\partial \overline{B} = Y_1$.[1] Lemma 3.3 and Morse–Bott homology (see [BH11] and [BO09b]) suggest that we define the chain complex $CF_*(H)$ for Floer homology of the S-shaped Hamiltonian

[1] Alternatively, we could choose a Morse–Smale function $f : X \to \mathbb{R}$ such that one of its level sets is a copy of Y, and whose maximum is attained at the connected component of $X \backslash Y$ that contains Σ.

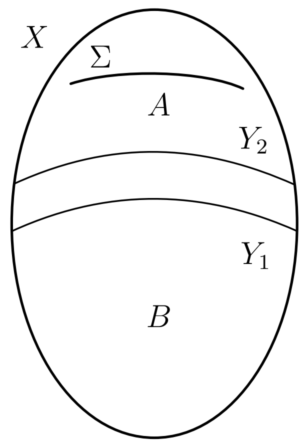

Figure 3.2: Various pieces in X

H (as an abelian group) as follows:

$$CF_*(H) = CM_*(-f_B)[-n] \oplus \Big(CC_*(Y)^{<M} \oplus CC_*(Y)^{<M}[1]\Big) \oplus \\ \oplus \Big(CC_*(Y)^{<M}[-1] \oplus CC_*(Y)^{<M}\Big) \oplus CM_*(-f_A)[-n] \quad (3.1)$$

where $CM_*(g)$ is the Morse complex of a function g (we say more about the gradings on the Morse complexes in (3.1) below, in Lemma 3.4). $CC_*(Y)^{<M}$ is the truncation of the chain complex for contact homology of Y, which is generated by Reeb orbits of period less than M.[2] Since $Y \to \Sigma$ is a pre-quantization bundle, and its (parametrized) periodic Reeb orbits come in Y-families, we take an auxiliary Morse-Smale function $f_\Sigma : \Sigma \to \mathbb{R}$ on Σ. Then $CC_*(Y)$ has a generator p_k for every critical point $p \in \Sigma$ and every multiplicity $k > 0$. For each multiplicity $0 < k < M$, there are two Y-families of 1-periodic X_H-orbits corresponding to Reeb orbits of period k, one on the concave part and one on the convex part of H. The function f_Σ lifts to a Morse–Bott function on Y, whose critical manifolds are circles. We take auxiliary Morse functions on these circles, with two critical points. This justifies our need for four copies of $CC_*(Y)^{<M}$. The degree shifts account for the degrees of the critical points of the auxiliary Morse functions on fibers of $S^1 \to Y \to \Sigma$, and for the fact that some orbits are located on

[2] We will see below that Y has no *bad orbits*, in the sense of symplectic field theory.

the convex part (those that generate $CC_*(Y)^{<M} \oplus CC_*(Y)^{<M}[1]$), and some on the concave part of H (those generating $CC_*(Y)^{<M}[-1] \oplus CC_*(Y)^{<M}$). Summing up, every critical point $p \in \Sigma$ and multiplicity $0 < k < M$ gives rise to four generators of $CF_*(H)$, denoted by $\check{p}_k^{cvx}$, $\hat{p}_k^{cvx}$, $\check{p}_k^{ccv}$ and $\hat{p}_k^{ccv}$. As in Lemma 3.3, we think of constant orbits as having constant cappings, and non-constant orbits as having fractional cappings inside $\widetilde{W}$, or disk cappings intersecting Σ negatively. We take all the pieces in the above direct sum to be generated over the Novikov ring $\Lambda = \mathbb{Z}[t, t^{-1}]$.

Similarly, we define the chain complex $CS_*(H)$ for Floer homology of a J-shaped Hamiltonian H (as an abelian group) as

$$CS_*(H) = CM_*(-f_B)[-n] \oplus CC_*(Y) \oplus CC_*(Y)[1]. \tag{3.2}$$

We now have two copies of $CC_*(Y)$, with unbounded periods, because for each $k > 0$, there is a Y-family of 1-periodic X_H-orbits corresponding to Reeb orbits of period k. This time, we can take coefficient over $\mathbb{Z}$, instead of Λ.

The following result tells us what the gradings are on these chain complexes.

Lemma 3.4. *The grading of the generators of $CF_*(H)$ is as follows:*

- *if $x \in \mathrm{Crit}(f_I)$, where $I = A$ or B, then $\deg(x) = \mathrm{ind}_{-f_I}(x) - n$;*
- *given $q_i \in \mathrm{Crit}(f_\Sigma)$ such that $\mathrm{ind}_{f_\Sigma}(q_i) = i$, let $q_{i,k}$ denote the corresponding Reeb orbits of multiplicity k. Then, for X_H-orbits on the convex part of H*

$$\deg(\check{q}_{i,k}^{cvx}) = 2\left(\frac{1}{K\lambda_X} - 1\right)k - n + 1 + i$$

and

$$\deg(\hat{q}_{i,k}^{cvx}) = 2\left(\frac{1}{K\lambda_X} - 1\right)k - n + 2 + i$$

with respect to a fractional capping inside $\widetilde{W}$. With respect to a capping by a disk intersecting Σ (negatively), we have instead

$$\deg(\check{q}_{i,k}^{cvx}) = -2k - n + 1 + i$$

and

$$\deg(\widehat{q}_{i,k}^{cvx}) = -2k - n + 2 + i$$

As for orbits on the concave part of H, we have $\deg(\check{q}_{i,k}^{ccv}) = \deg(\check{q}_{i,k}^{cvx}) - 1$ *and* $\deg(\widehat{q}_{i,k}^{ccv}) = \deg(\widehat{q}_{i,k}^{cvx}) - 1$.

The grading of the generators of $CS_*(H)$ is as follows:

- *if $x \in \mathrm{Crit}(f_B)$, then $\deg(x) = \mathrm{ind}_{-f_I}(x) - n$;*
- *given $q_i \in \mathrm{Crit}(f_\Sigma)$ such that $\mathrm{ind}_{f_\Sigma}(q_i) = i$, let $q_{i,k}$ denote the corresponding Reeb orbits of multiplicity k. Then, for X_H-orbits of H*

$$\deg(\check{q}_{i,k}) = 2\left(\frac{1}{K\lambda_X} - 1\right)k - n + 1 + i$$

and

$$\deg(\widehat{q}_{i,k}) = 2\left(\frac{1}{K\lambda_X} - 1\right)k - n + 2 + i$$

for a fractional capping inside W.

Proof. We will use a combination of results from other authors. As with the proof of Lemma 3.3, we begin with the case of symplectic homology of W. Recall that we assume constant orbits to have constant cappings and non-constant orbits to have fractional cappings inside W. The formula for constant orbits is given in Lemma 7.2 in [SZ92]. The formula for non-constant orbits can be explained as follows:

$$\deg(\check{q}_{i,k}) = \mu_{RS}(\check{q}_{i,k}) = \mu_{RS}(q_{i,k}).$$

The term on the right is the Robbin–Salamon index for a Reeb orbit (whereas the term in the middle is the Robbin–Salamon index for a Hamiltonian orbit). The formula is justified in Lemma 3.4 of [BO09b] (although some of our conventions are different). Similarly,

$$\deg(\widehat{q}_{i,k}) = \mu_{RS}(\widehat{q}_{i,k}) = \mu_{RS}(q_{i,k}) + 1.$$

Now, since $q_{i,k}$ is associated with a Morse–Bott family of (unparametrized) k-periodic Reeb orbits that is parametrized by Σ (which we denote as Σ_k), Lemma 2.4 in [Bou02] yields

$$\mu_{RS}(q_{i,k}) = \mu_{RS}(\Sigma_k) - (n-1) + i$$

where $\mu_{RS}(\Sigma_k)$ is the Robbin–Salamon index of a Reeb orbit in Σ_k. Therefore, the formulas for $\deg(\check{q}_{i,k})$ and $\deg(\hat{q}_{i,k})$ follow from the fact that $\mu_{RS}(\Sigma_k) = 2\left(\frac{1}{K\lambda_X} - 1\right)k$, under a fractional capping contained in Y (and thus in W), which we now justify.

Let γ_k be a Reeb orbit in Σ_k. To assign an index to γ_k, we will argue as in Section 2.9.1 of [EGH00] and Section 9.1 in [Bou02]. Take $A \in H_2(\Sigma;\mathbb{Z})$ such that $\langle\omega, A\rangle \neq 0$, and use it to construct a capping $u_{kK\langle\omega,A\rangle}$ for the multiple $\gamma_{kK\langle\omega,A\rangle}$, as explained before. Such capping lies inside Y, and we think of it as inside W. With respect to the induced trivialization of $TW|_{\gamma_{kK\langle\omega,A\rangle}}$, $\mu_{RS}(\Sigma_{kK\langle\omega,A\rangle}) = 2\langle c_1(T\Sigma), A\rangle k$ (the index vanishes with respect to the product framing, and it changes by c_1 under change of framing). We then take $\mu_{RS}(\Sigma_k) = 2\,\frac{\langle c_1(T\Sigma),A\rangle}{K\langle\omega,A\rangle}\,k$, which can sometimes be fractional. Now,

$$\frac{\langle c_1(T\Sigma), A\rangle}{K\langle\omega, A\rangle} = \frac{\langle\omega, A\rangle/\lambda_\Sigma}{K\langle\omega, A\rangle} = \frac{1}{K\lambda_\Sigma}$$

and the fact that $\lambda_\Sigma = \frac{\lambda_X}{1-K\lambda_X}$ (see Lemma 3.1) implies that $\frac{1}{K\lambda_\Sigma} = \frac{1}{K\lambda_X} - 1$. Therefore, $\mu_{RS}(\Sigma_k) = 2\left(\frac{1}{K\lambda_X} - 1\right)k$, as wanted.

The proof in the case of Floer homology of X is completely analogous. We can take a capping for the X_H-periodic orbit corresponding to $\check{q}_{i,k}$ by a disk u'_k that intersects $-k$ times the divisor Σ, and an analogous capping $u'_{kK\langle\omega,A\rangle}$ for a $(K\langle\omega, A\rangle)$-cover of this orbit. Using the Proposition at the end of Section 2.1, we see that the difference in indices given by the two choices of (fractional) cappings for $\check{q}_{i,k}$ is

$$\begin{aligned}\frac{1}{K\langle\omega,A\rangle}\,2\left\langle c_1(TX), u'_{kK\langle\omega,A\rangle}\cup(-u_{kK\langle\omega,A\rangle})\right\rangle = \\ &= \frac{1}{K\langle\omega,A\rangle}\,2\,\frac{1}{\lambda_X}\left\langle\omega, u'_{kK\langle\omega,A\rangle}\cup(-u_{kK\langle\omega,A\rangle})\right\rangle = \\ &= \frac{1}{K\langle\omega,A\rangle}\,2\,\frac{1}{K\lambda_X}\#\Big(\Sigma\cap\big(u'_{kK\langle\omega,A\rangle}\cup(-u_{kK\langle\omega,A\rangle})\big)\Big) = \\ &= \frac{1}{K\langle\omega,A\rangle}\,2\,\frac{1}{K\lambda_X}(-kK\langle\omega,A\rangle) = -\frac{2k}{K\lambda_X}.\end{aligned}$$

Adding this to the index formula with respect to the fractional capping inside $\widetilde{W}$, we get the (integer) index with respect to a capping by a disk intersecting $-k$ times Σ. The only point that still needs justification is the relation between the degrees of convex and concave generators. This is once again due to our Morse–Bott setting, and the fact that the second derivative of a concave function is negative. □

Remark 3.3. *Since the Floer differential connects elements with index difference one, the symplectic homology chain complex splits as a sum over the fractional parts of the indices of the generators. This is analogous to writing the symplectic homology complex as a sum over free homotopy classes of orbits (in which case each of the summands can be given an integer grading).*

Since the Hamiltonian H is Morse–Bott, the Floer differential should count *cascades*, with components solving Seidel's equation, connected at removable singularities to gradient flow lines of the auxiliary Morse functions that were chosen. One can also describe the pair-of-pants product for the Hamiltonian H in terms of cascades. We will not give a more explicit description of these configurations at this point, but will provide more details in Section 5.3.1 below.

Proposition 3.1. *The chain complex $CF_*(H)$ for an S-shaped Hamiltonian H computes the Floer homology $HF_*(X)$; the complex $CS_*(H)$ for a J-shaped Hamiltonian H computes the symplectic homology $SH_*(W)$.*

We will not present a proof of Proposition 3.1 here. A possible approach to proving this result would be via a continuation map constructed from the interpolation between a degenerate (S- or J-shaped) Hamiltonian and a non-degenerate small perturbation.

In the case of Floer homology of the closed manifold X, the choice of fractional cappings contained in $\widetilde{W}$, for the non-constant 1-periodic orbits in X, has an important consequence. It implies that the Novikov variable t (of degree $2N$) in the Floer homology differential keeps track of how many times a Floer cylinder intersects $\Sigma = \mathrm{PD}(K[\omega])$. More explicitly, one can decompose the Floer differential as

$$\mathrm{d} = \sum_{i \geqslant 0} \mathrm{d}_i \, t^{-\frac{i}{K \lambda_X N}} \tag{3.3}$$

where d_i counts Floer cylinders intersecting i times the divisor Σ (possibly after a small perturbation to ensure a transverse intersection). Recall from (2.4) that the exponent of t in the contribution of an orbit y to the Floer differential of an orbit x is $-\langle c_1(TX), (-u_x) \cup V \cup u_y \rangle / N$, where u_x and u_y are cappings, V is a Floer trajectory connecting y and x, and N is the minimal Chern number of X. Formula (3.3) now follows from monotonicity of X and the fact that $[\omega] = \mathrm{PD}(\Sigma/K)$. Compare this with Section 3 in [EP10]. If we chose instead cappings by disks intersecting Σ (negatively), then the exponents of t would also involve the multiplicities of the periodic orbits. If x corresponds to a Reeb orbit of period k and y corresponds to one of period l, then the contribution of y to $\mathrm{d}(x)$ can be written as

$$\sum_{i \geqslant 0} (\mathrm{d}_i(x), y) \, t^{-\frac{i+k-l}{K \lambda_X N}} \tag{3.4}$$

where $(\mathrm{d}_i(x), y)$ is a signed count of rigid Floer trajectories connecting y and x that intersect i times the divisor Σ.

3.3 Splitting the manifold

We have just described how to define Floer and symplectic homology for a class of degenerate Hamiltonians (omitting many details about the differential). Before we can relate these chain complexes with Gromov–Witten numbers, we need to also degenerate the manifold. This construction is inspired by the work of Bourgeois and Oancea (namely Section 5 of [BO09a]).

Let H be an S-shaped Hamiltonian on X. Recall that there are two values $r_1, r_2 \in (a, b)$ such that $\operatorname{supp} \mathrm{d}H \subset (r_1, r_2) \times Y$, and that we denote $Y_i = \{r_i\} \times Y$, for $i = 1, 2$ (see Figure 3.2). We wish to split X along the Y_i, in a way similar to symplectic field theory splitting, explained in Section 3.4 of [BEH$^+$03].

Recall that [BEH$^+$03] describes limits of pseudo-holomorphic curves on a symplectic manifold M (possibly with cylindrical ends), as one *stretches the neck* of M along a compact contact-type hypersurface V, or, more generally, a *stable Hamiltonian structure*. In the limit, one gets pseudo-holomorphic buildings, possibly including pieces in the symplectization $\mathbb{R} \times V$. In our case, we want to describe the limits of Floer trajectories as one splits X along Y_1 and Y_2. To this end, recall that Floer trajectories can themselves be thought of as pseudo-holomorphic curves (an idea inspired by Gromov's paper [Gro85]). For instance, a Floer cylinder $\mathbb{R} \times S^1 \to X$ is the same as a holomorphic section of $(\mathbb{R} \times S^1) \times X$, for a certain almost complex structure on the product (see [EKP06], Section 4.12.1). Then, one can try to apply the results of [BEH$^+$03] to the splitting of $(\mathbb{R} \times S^1) \times X$ along $(\mathbb{R} \times S^1) \times Y_1$ and $(\mathbb{R} \times S^1) \times Y_2$. The reason that this argument needs further justification, which we will not provide here, is that the $(\mathbb{R} \times S^1) \times Y_1$ are not compact, and so the compactness results in [BEH$^+$03] do not apply as stated. However, note that, since H is constant near the Y_i, the Floer equation actually coincides with the pseudo-holomorphic curve equation in those regions. This justifies that one should be able to argue as if one were splitting pseudo-holomorphic curves along compact contact-type hypersurfaces, as in [BEH$^+$03]. In his upcoming thesis [Bor], Borman describes a compactness result that applies to a setting similar to ours.

In order to identify spaces of Floer trajectories in X with spaces of split Floer trajectories, one needs, in addition to the compactness statement aluded to above, also a gluing argument. We will not provide the details of these arguments.

The process of splitting X with an S-shaped Hamiltonian H does not affect the non-constant 1-periodic X_H-orbits. We can thus say that a chain complex coinciding with (3.1) as an abelian group computes the Floer homology of X. The differential

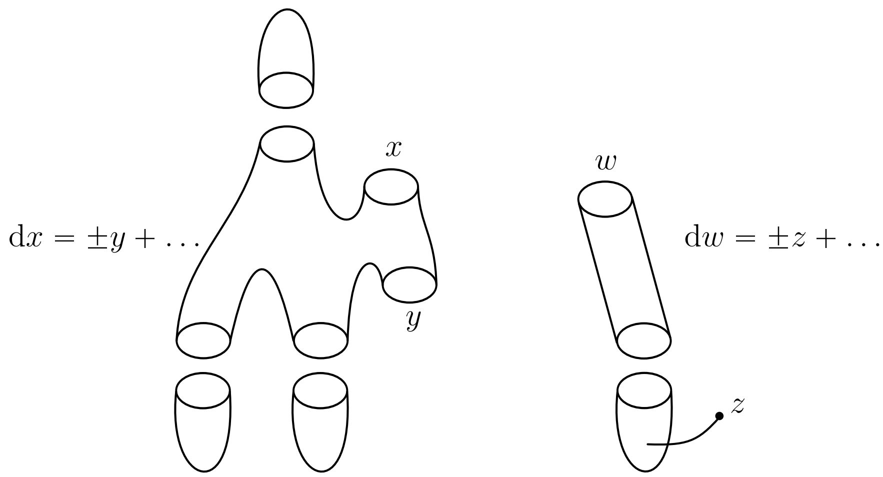

Figure 3.3: Two split Floer differentials

and the algebraic operations can now be defined in terms of symplectic-field-theory-type buildings, possibly with components in $\mathbb{R} \times Y$, in $N\Sigma$ and in W, connected with gradient flow lines of auxiliary Morse functions. Call these *split Floer trajectories* and refer to this description as *split Floer homology*. Figure 3.3 contains some split Floer trajectories contributing to the split Floer differential. In the picture on the left, the top disk represents a plane in $N\Sigma$, the intermediate piece is in $\mathbb{R} \times Y$ and the two bottom pieces are planes in W. On the right, the cylinder is in $\mathbb{R} \times Y$ and the rest is in W. The letters x, y and w represent non-constant periodic Floer orbits. The letter z represents a critical point of an auxiliary Morse function in the region where H is constant; the segment connecting z to a plane is a gradient flow line for the same function. The other periodic trajectories depicted are asymptotic Reeb orbits in Y. In Section 5.3.1, we will depict more split trajectories, but first we will need to relate them with pseudo-holomorphic curves.

We should point out that, when splitting, the function f_B is replaced by a function on $f_W : W \to \mathbb{R}$ that grows at infinity. Similarly, f_A is replaced by a function $f_{N\Sigma} : N\Sigma \to \mathbb{R}$, that decreases at infinity. In $\mathbb{R} \times Y$, gradient flow lines should be thought of as vertical lines.

There is an analogue of the above discussion in which one splits W along a single copy of Y. This leads to the *split symplectic homology* of W.

Chapter 4

An ansatz for split Floer trajectories

We will now relate split Floer trajectories for S- and J-shaped Hamiltonians (mentioned above) with pseudo-holomorphic curves. Recall that we split our manifolds in such a way that the only component with a non-constant Hamiltonian is the symplectization $\mathbb{R} \times Y$. In this chapter, we will show how Floer trajectories in $\mathbb{R} \times Y$ can be related with (perturbed) pseudo-holomorphic curves and solutions of an auxiliary equation. We will describe the relevant moduli spaces of pseudo-holomorphic curves and solutions of the auxiliary equation, and then relate them with Floer trajectories. We begin with a description of the various types of punctures and marked points on the domains of our maps to $\mathbb{R} \times Y$. These results will appear in [DL12], joint with Samuel Lisi.

4.1 Several types of punctures and marked points

Even though we will not be careful with these distinctions in subsequent sections, it is important to point out that the configurations we will consider have punctures of different types, which we now describe.

To define the differential and operations on Floer and symplectic homology, one considers solutions of Seidel's equation (2.5) on a Riemann surface $S = \mathbb{C}P^1 \backslash \Gamma_F$,

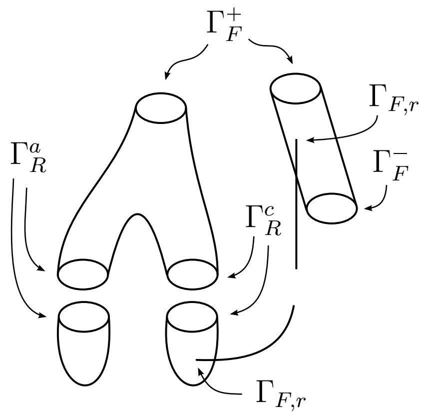

Figure 4.1: Types of punctures on a broken augmented pair-of-pants

where $\Gamma_F \subset \mathbb{C}P^1$ is a finite set. We decompose $\Gamma_F = \Gamma_F^+ \cup \Gamma_F^-$ into positive and negative punctures.

When we consider S- and J-shaped Hamiltonians, as in Section 3.2, the periodic X_H-orbits are not isolated, forming manifolds, possibly with boundary. For this reason, we need to replace counts of solutions of Seidel's equation with counts of *cascades*, in the sense of Morse–Bott homology (see [BO09a]). These have some components solving Seidel's equation, connected to gradient flow lines of auxiliary Morse functions on the manifolds of orbits. We will think of the points that connect to gradient flow lines as removable singularities, and denote the set of those by $\Gamma_{F,r}$.

When we split the manifold, some solutions of Seidel's equation split into different pieces, with new punctures asymptotic to Reeb orbits of Y. These new punctures are of two types. Some are capped by holomorphic planes in W or in $N\Sigma$, and we denote the set of such Reeb punctures by Γ_R^a (the superscript stands for 'augmentation', although one usually reserves this term for planes in the convex filling W, not on the concave filling $N\Sigma$). The other case is when the puncture affects the conformal structure of the domain. The set of those punctures is called Γ_R^c. Figure 4.1 sketches an example of a broken pair-of-pants where all types of punctures occur.

4.2 Pseudo-holomorphic curves in $\mathbb{R} \times Y$

To define (perturbed) pseudo-holomorphic curves in $\mathbb{R} \times Y$, we will choose some additional structure on the closed symplectic manifold (Σ, ω), and lift it to $\mathbb{R} \times Y$.

We will need a generic ω-compatible almost complex structure J in Σ. For $m \geqslant 3$, let $\mathcal{M}_m$ be the moduli space of stable Riemann surfaces S of genus 0 (nodal surfaces of genus 0, with at least three nodes and marked points on each irreducible component), and let $\mathcal{U}_m$ be the corresponding universal curve bundle. We will consider perturbation forms $\nu \in \Gamma\big(\mathcal{U}_m, \mathrm{Hom}^{0,1}(TS, T\Sigma)\big)$, or, put differently, $\mathcal{M}_m$-dependent forms $\nu \in \Omega^{0,1}(S, T\Sigma)$. We further assume that these perturbations are supported away from the nodes and marked points in S. We will see in Section 5.1.1 that the *Gromov–Witten numbers* of Σ can be defined using moduli spaces of maps $w : S \to \Sigma$ that solve the perturbed equation

$$\mathrm{d}w + J \circ \mathrm{d}w \circ j = \nu.$$

The form ν is chosen so that homologically trivial curves in Σ are not constant. This will allow us to choose only one Morse function f_Σ on Σ and, for generic J and ν, to have the transversality required for constructing fiber products of moduli spaces of holomorphic curves and stable/unstable manifolds of f_Σ, with respect to evaluation maps at various marked points. Since there are no constant holomorphic curves, there will be no trees of three or more gradient flow lines meeting at one point.

If $m < 3$, then we do not have a space of stable curves, so we cannot talk about the universal curve bundle. This is not a problem, because in this case it will be enough to consider solutions of $\mathrm{d}w + J \circ \mathrm{d}w \circ j = 0$, with no perturbation term ν.

The almost complex structure J in Σ can be lifted uniquely to a cylindrical almost complex structure $\tilde{J}$ on $\mathbb{R} \times Y$. Note that the projection $P : Y \to \Sigma$ is such that the pullback bundle $P^*T\Sigma$ coincides with the contact distribution ξ on Y. Therefore, $\tilde{J}$ is determined by the conditions $\tilde{J}|_\xi = P^*J$ and $\tilde{J}\partial_r = R$. We can also lift ν to $\tilde{\nu}$ in $\mathbb{R} \times Y$, by making $\tilde{\nu}$ trivial in the ∂_r and R directions. Let now $Z = \{z_1, \ldots, z_k\}$ be a finite set of points in $\mathbb{C}P^1$. We will be interested in studying maps

$$\begin{aligned} \tilde{u} = (a, u) : \mathbb{C}P^1 \backslash Z \to \mathbb{R} \times Y \\ \mathrm{d}\tilde{u} + \tilde{J} \circ \mathrm{d}\tilde{u} \circ i = \tilde{\nu} \end{aligned} \tag{4.1}$$

Transversality for ν-perturbed J-holomorphic curves in Σ will imply transversality for $\tilde{\nu}$-perturbed $\tilde{J}$-holomorphic curves in $\mathbb{R} \times Y$.

4.3 The cylinder equation

Let $\tilde{u} = (a, u) : S \backslash \{\text{other punctures}\} \to \mathbb{R} \times Y$ be a solution of (4.1), where $S = \mathbb{C}P^1 \backslash \Gamma_F$. Fix conformal parametrizations $\varphi_i : \mathbb{R}_\pm \times S^1 \to S$ of neighborhoods of the $z_i \in \Gamma_F$ and $\beta \in \Omega^1(S)$, as in the discussion of Seidel's equation (2.5).

We will be interested in solutions of the following auxiliary equation:

$$\begin{aligned} f = (f_1, f_2) : S \to \mathbb{R} \times S^1 \\ \mathrm{d}f_1 - \mathrm{d}f_2 \circ i + h'(e^{a+f_1})\beta \circ i = 0. \end{aligned} \tag{4.2}$$

Since f takes values in $\mathbb{R} \times S^1$, we call this the *cylinder equation*. Denote by $\mathcal{C}(\tilde{u})$ the space of cylinder solutions associated with $\tilde{u}$.

Proposition 4.1. *Let $\tilde{u}$ be a perturbed pseudo-holomorphic curve in $\mathbb{R} \times Y$. Then,*

1. *equation (4.2) is a Fredholm problem, of index*

$$\operatorname{ind}(\mathcal{C}(\tilde{u})) = 2 - k^-_{cvx} - k^+_{ccv},$$

where k^-_{cvx} is the number punctures in Γ^-_F converging to non-constant orbits in the region where H is convex, and k^+_{ccv} is the number punctures in Γ^+_F converging to non-constant orbits in the region where H is concave;

2. *if $\operatorname{ind}(\mathcal{C}(\tilde{u})) \leqslant 0$, then the kernel of the linearized operator is 1-dimensional, and is spanned by the generator of the S^1 action;*

3. *if* $\operatorname{ind}(\mathcal{C}(\tilde{u})) > 0$*, then the linearized operator is surjective, and its kernel includes the generator of the* S^1 *action.*

A proof of this statement will be given in [DL12]. The index formula follows from the study of the relevant asymptotic operators, obtained from the linearization of equation (4.2). The proof of second and third statements involves the study of the eigenvalues of the asymptotic operators and uses automatic transversality, namely Proposition 2.2 in [Wen10].

To study rigid Floer trajectories, it will also be very useful to understand solutions of equation (4.2) in families of low index.

Proposition 4.2. 1. *If* $\operatorname{ind}(\mathcal{C}(\tilde{u})) = 0$*, then* $\mathcal{C}(\tilde{u})$ *is cobordant to a single point.*

2. *If* $\operatorname{ind}(\mathcal{C}(\tilde{u})) = 1$*, then* $\mathcal{C}(\tilde{u})$ *is isomorphic to* $\mathbb{R}$*.*

Proof. We will leave the details of the proof of the first part to [DL12]. Let us just sketch the argument. We need to recall some information and introduce some notation. For each puncture $z \in \Gamma_F$, we have the data $(T_z, b_z) \in \mathbb{R}^2$ such that $h'(e^{b_z}) = T_z$. Then, we have in cylindrical coordinates near the puncture z,

$$a(s,t) \in T_z s + c_z + W^{1,p,\delta}(\mathbb{R}^{\pm} \times S^1),$$

for a constant c_z and for $\delta > 0$ small. Recall that in cylindrical coordinates, $\beta = \kappa_z \mathrm{d}t$, at least sufficiently close to the puncture. Cylinder solutions (f_1, f_2) are in the function space

$$f_1 \in -T_{\pm} s - c_{\pm} + r_{\pm} + W_c^{1,p,\delta}(S), \quad f_2 \in W_c^{1,p,\delta}(S).$$

Let $\mu : S \to \mathbb{R}$ be a smooth function supported near the punctures such that, in cylindrical coordinates, $\mu = -T_z s - c_z + b_z$ close enough to the puncture z. Let $g_1 = f_1 - \mu$ and $g_2 = f_2$. Then, our problem may be reformulated as:

$$\mathrm{d}g_1 - \mathrm{d}g_2 \circ j + \mathrm{d}\mu - h'(e^{(a+\mu)+g_1})\beta \circ j = 0.$$

The proof of the first part of this Proposition consists of studying the family of equations

$$\mathrm{d}g_1 - \mathrm{d}g_2 \circ j + \tau\big(\mathrm{d}\mu - h'(e^{(a+\mu)+g_1})\beta \circ j\big) = 0$$

for $\tau \in [0,1]$. One can show that the space of solutions is invariant under change in τ. When $\tau = 1$, we have the problem we are interested in. When $\tau = 0$, we get the standard Cauchy–Riemann equation. The asymptotic conditions imply that there is a unique solution when $\tau = 0$, and thus also when $\tau = 1$, as wanted.

Now we prove the second part of the Proposition. If $(a,u) : \mathbb{R} \times S^1 \to \mathbb{R} \times Y$ is a trivial cylinder, then it has the form

$$(a,u)(s,t) = \big(Ts + C, \gamma(Ts)\big)$$

for some Reeb orbit γ of period T and some constant $C \in \mathbb{R}$. Equation (4.2) can be written as

$$\begin{cases} \partial_s f_1 - \partial_t f_2 + h'(e^{a+f_1}) = 0 \\ \partial_t f_1 + \partial_s f_2 = 0 \end{cases}$$

for $(f_1, f_2) : \mathbb{R} \times S^1 \to \mathbb{R} \times S^1$ satisfying the appropriate asymptotic conditions. It will be more useful to consider instead the functions $(b,c) = (f_1 + a, f_2)$, which solve the system of equations

$$\begin{cases} \partial_s b - \partial_t c + h'(e^b) - T = 0 \\ \partial_t b + \partial_s c = 0 \end{cases} \tag{4.3}$$

with asymptotic conditions $\lim_{s\to\pm\infty}(b,c)(s,t) = (r_\pm, k_\pm)$, for some constants $r_\pm \in \mathbb{R}$ and $k_\pm \in S^1$.

We begin by showing that there can be no solution (b,c) such that $k_- \neq k_+$. Suppose that (b,c) solves (4.3), with the required asymptotics. Since $c : \mathbb{R} \times S^1 \to S^1$ is null-homotopic, we can find a lift $\tilde{c} : \mathbb{R} \times S^1 \to \mathbb{R}$. Define $I : \mathbb{R} \to \mathbb{R}$ such that $I(s) = \int_{S^1} \tilde{c}(s,t)\mathrm{d}t$. Then,

$$\frac{\mathrm{d}I}{\mathrm{d}s} = \int_{S^1} \partial_s \tilde{c}(s,t)\mathrm{d}t = \int_{S^1} -\partial_t b(s,t)\mathrm{d}t = 0.$$

and we conclude that I is a constant function. But

$$\lim_{s\to-\infty} I(s) \equiv k_-(\text{mod } 1) \text{ and } \lim_{s\to\infty} I(s) \equiv k_+(\text{mod } 1)$$

hence $k_- = k_+$, as wanted.

We continue with the study of (4.3). If we assumed $c \equiv k$ to be constant, and $b(s,t) = b(s)$ to be independent of t, then we would get the ordinary differential equation

$$\frac{\mathrm{d}b}{\mathrm{d}s} + h'(e^b) - T = 0. \tag{4.4}$$

The asymptotic conditions at $\pm\infty$ would imply $h'(e^{r_\pm}) = T$. Since H is S-shaped, the equation $h'(e^r) = T$ has only two solutions. Since Floer trajectories increase the symplectic action, it is necessarily the case that $r_- > r_+$. This implies that the Floer trajectories (b,v) given by these (a,u) and (f_1,f_2) will converge to an orbit on the concave part of H as $s \to -\infty$ and to an orbit on the convex part of H as $s \to +\infty$. Since (4.4) is autonomous, we get a family of solutions parametrized by $s \in \mathbb{R}$ (which is compatible with the fact that we want index 1-families of solutions (f_1,f_2)). We will show that these are indeed all the solutions to this problem.

Observe that the system of equations (4.3) is a Floer equation on $\mathbb{R} \times S^1$ for the Hamiltonian $H(x,y) = \int h'(e^x)\mathrm{d}x - Tx$ and the standard symplectic form and complex structure. Therefore, the solutions can be thought of as holomorphic curves $\mathbb{R} \times S^1 \to M := (\mathbb{R} \times S^1) \times (\mathbb{R} \times S^1)$, for some twisted almost complex structure on the target (by the trick of Gromov that was alluded to in Section 3.3). The projection of these curves to the first cylinder factor is the identity. Since the solutions of (4.4) come in $\mathbb{R}$-families and k is not restricted, we get a foliation of the open subset $(\mathbb{R} \times S^1) \times \big((r_+, r_-) \times S^1\big) \subset M$ by holomorphic curves. We want to show that these are all the solution of the system of differential equations. Call them *gradient solutions* (because they are solutions of an ordinary differential equation).

Now, suppose we fix a constant k and find a solution $G = (b_1, c_1)$ of the systems of equations that is not a gradient solution, with $\lim_{s\to\pm\infty} c_1(s,t) = k$. If $c_1 \equiv k$, then b would again solve (4.4) and G would be a gradient solution. Therefore, $c_1 \not\equiv k$, and the graph of G intersects the graph of a gradient solution $F = (b_2, c_2)$ with $c_2 \neq k$.

Take now another gradient solution $\tilde{F} = (b_3, c_3)$ such that $c_3 \equiv k$. We can homotope G to $\tilde{F}$, since they are both null-homotopic maps $\mathbb{R} \times S^1 \to \mathbb{R} \times S^1$. Since G and $\tilde{F}$ have the same asymptotics, we can make sure that the homotopy is C^0-small on neighborhoods of $\pm\infty \times S^1 \subset \mathbb{R} \times S^1$. This implies that the graph of G will be homotoped to the graph of $\tilde{F}$ in M, and that the intersections with the graph of F will remain in a compact region of M. So, we have an equality of signed counts

$$\#\big(Graph(G) \cap Graph(F)\big) = \#\big(Graph(\tilde{F}) \cap Graph(F)\big) = 0,$$

since F and $\tilde{F}$ have different asymptotics. But then positivity of intersection for holomorphic curves in 4-dimensions implies that the graphs of G and F do not intersect, which gives a contradition. Therefore, there is no such G, as we wanted to show. □

4.4 Floer trajectories in $\mathbb{R} \times Y$

We now study solutions of a perturbed version of Seidel's Floer equation, in $\mathbb{R} \times Y$:

$$\begin{gathered} \tilde{v} = (a, v) : S \backslash \{\text{punctures}\} \to \mathbb{R} \times Y \\ (\mathrm{d}\tilde{v} - X_H \circ \beta)^{0,1} = \tilde{\nu}. \end{gathered} \tag{4.5}$$

Consider the moduli space

$$\mathcal{H} = \Big\{\big((a,u),(f_1,f_2)\big) | (a,u) \text{ satisfies (4.1) and } (f_1,f_2) \text{ satisfies (4.2)}\Big\}/\mathbb{C}^*$$

where we take the quotient by the $\mathbb{C}^*$-action given by

$$e^{\rho+i\theta}.\big((a,u),(f_1,f_2)\big) = \big((a+\rho, \phi_\theta \circ u),(f_1-\rho, f_2-\theta)\big),$$

ϕ_θ being the Reeb flow on Y for time θ. Let also

$$\mathcal{F} = \big\{(b,v) \text{ satisfying (4.5)}\big\}.$$

Our goal is to prove the following result.

Theorem 4.1. *The map*

$$\begin{aligned} \Phi : \mathcal{H} &\to \mathcal{F} \\ (a,u),(f_1,f_2) &\mapsto (a+f_1, \phi_{f_2} \circ u) \end{aligned} \tag{4.6}$$

is well-defined and a diffeomorphism.

We will split the proof of this result into the following parts:

(i) Φ is well-defined;

(ii) Φ is a bijection;

(iii) Φ is differentiable and an immersion.

Proof of (i). It is clear that Φ is $\mathbb{C}^*$-equivariant. To see that Φ is well-defined, let $(\tilde{u}, f) = \big((a,u),(f_1,f_2)\big) \in \mathcal{H}$. Write $\tilde{v} = (b,v) := \Phi(\tilde{u}, f)$. We need to show that $\tilde{v} \in \mathcal{F}$. First, notice that, if we denote by π_1 and π_2 the projections associated with the splitting

$$T(\mathbb{R}\times Y) = \big(\mathbb{R}\langle\partial_r\rangle \oplus \mathbb{R}\langle R\rangle\big) \oplus \xi$$

then

$$\pi_1(\mathrm{d}\tilde{u} + \tilde{J}\circ \mathrm{d}\tilde{u}\circ i) = \partial_r \otimes (\mathrm{d}a - u^*\alpha\circ i) + R \otimes (\mathrm{d}a - u^*\alpha \circ i)\circ i = 0 \tag{4.7}$$

and

$$\pi_2(\mathrm{d}\tilde{u} + \tilde{J}\circ \mathrm{d}\tilde{u}\circ i) = \pi_2\mathrm{d}u + \tilde{J}\circ\pi_2\mathrm{d}u\circ i = \tilde{\nu} \tag{4.8}$$

Since H is S- or J-shaped, we have $X_H(r,y) = h'(e^r)R(y)$, so

$$(\mathrm{d}\tilde{v} - X_H\circ\beta)^{0,1} = \mathrm{d}\tilde{v} + \tilde{J}\circ \mathrm{d}\tilde{v}\circ i - h'(e^b)R\otimes\beta + h'(e^b)\partial_r\otimes\beta\circ i.$$

Therefore,

$$\begin{aligned} \pi_1\big((\mathrm{d}\tilde{v} - X_H\circ\beta)^{0,1}\big) = \partial_r &\otimes (\mathrm{d}b - v^*\alpha\circ i + h'(e^b)\beta\circ i) + \\ &+ R\otimes(\mathrm{d}b - v^*\alpha\circ i + h'(e^b)\beta\circ i)\circ i. \end{aligned}$$

Formula (4.6) implies that

$$v^*\alpha = \alpha \circ \mathrm{d}(\phi_{f_2} \circ u) = \mathrm{d}f_2 + u^*\alpha$$

so

$$\begin{aligned}\pi_1\big((\mathrm{d}\tilde{v} - X_H \circ \beta)^{0,1}\big) &= \partial_r \otimes (\mathrm{d}a + \mathrm{d}f_1 - \mathrm{d}f_2 \circ i - u^*\alpha \circ i + h'(e^{a+f_1})\beta \circ i) + \\ &\quad + R \otimes (\mathrm{d}a + \mathrm{d}f_1 - \mathrm{d}f_2 \circ i - u^*\alpha \circ i + h'(e^{a+f_1})\beta \circ i) \circ i = \\ &= \partial_r \otimes \Big((\mathrm{d}a - u^*\alpha \circ i) + (\mathrm{d}f_1 - \mathrm{d}f_2 \circ i + h'(e^{a+f_1})\beta \circ i)\Big) + \\ &\quad + R \otimes \Big((\mathrm{d}a - u^*\alpha \circ i) + (\mathrm{d}f_1 - \mathrm{d}f_2 \circ i + h'(e^{a+f_1})\beta \circ i)\Big) \circ i = 0\end{aligned}$$

because of (4.7) and (4.2). On the other hand,

$$\begin{aligned}\pi_2\big((\mathrm{d}\tilde{v} - X_H \circ \beta)^{0,1}\big) &= \pi_2 \mathrm{d}v + \tilde{J} \circ \pi_2 \mathrm{d}v \circ i = \pi_2 \mathrm{d}\phi_{f_2} \circ \mathrm{d}u + \tilde{J} \circ \pi_2 \mathrm{d}\phi_{f_2} \circ \mathrm{d}u \circ i = \\ &= \mathrm{d}\phi_{f_2} \circ (\pi_2 \mathrm{d}u + \tilde{J} \circ \pi_2 \mathrm{d}u \circ i) = \mathrm{d}\phi_{f_2} \circ \tilde{\nu} = \tilde{\nu}\end{aligned}$$

using the fact that $\mathrm{d}\phi_{f_2}$ commutes with π_2 and $\tilde{J}$, (4.8) and the fact that $\tilde{\nu}$ is R-invariant. This concludes the proof that $\tilde{v}$ solves (4.5). □

Now that we have seen that Φ is well-defined, we show that it is bijective.

Proof of (ii). Fix $\tilde{v} = (b, v) \in \mathcal{F}$.

Claim 1. *For every $f = (f_1, f_2)$ that solves*

$$df_1 - df_2 \circ i + h'(e^b)\beta \circ i = 0, \tag{4.9}$$

there is a unique solution $\tilde{u} = (a, u)$ of (4.1) such that $\Phi(\tilde{u}, f) = \tilde{v}$.

This implies that, to understand $\Phi^{-1}(\tilde{v})$, we should study the solutions of (4.9).

Proof of Claim 1. Such $\tilde{u}$ is uniquely determined by (4.6) to be

$$\tilde{u} = (a, u) = \big(b - f_1, \phi_{-f_2} \circ v\big).$$

The computations in the proof of (i) above also show that $\tilde{u}$ solves (4.1). □

The fact that Φ is a bijection will now follow from the fact that (4.9) always has a $\mathbb{C}^*$-family of solutions.

Claim 2. *Given $\tilde{v} \in \mathcal{F}$, there is a solution of (4.9), unique up to the $\mathbb{C}^*$-action*

$$e^{\rho+i\theta}.(f_1, f_2) = (f_1 - \rho, f_2 - \theta).$$

Proof of Claim 2. We want a solution (f_1, f_2) of (4.9) such that, near $z_i \in \Gamma$, with a parametrization $\varphi_i : \mathbb{R}_\pm \times S^1 \to \mathbb{C}P^1$,

$$(f_1, f_2)(s,t) \sim (\pm T_i s + c_i, d_i)$$

where T_i is the period of the Reeb orbit associated with the puncture z_i and the Seidel solution (b, v) and $c_i \in \mathbb{R}$, $d_i \in S^1$ are constants. To make sense of this, we fix functions $\psi_i \in C^\infty(S, \mathbb{R})$ such that $\psi_i(s,t) \equiv 1$ near z_i and $\psi_i \equiv 0$ away from a neighborhood of the z_i that is contained in $\varphi_i(\mathbb{R}_\pm \times S^1)$. Then, we say that

$$(g_1, g_2) := \Big(f_1(s,t) - \sum_i \psi_i(s,t)(\pm T_i s + c_{i,1}), f_2(s,t) - c_{i,2}\Big) \in W^{1,p,\delta}(S, \mathbb{R} \times S^1).$$

Denote $\eta(s,t) = \sum_i \psi_i(s,t)(\pm T_i s + c_{i,1})$. Finding (f_1, f_2) is equivalent to finding (g_1, g_2), for some constants $c_{i,1}, c_{i,2}$. Now, equation (4.9) is equivalent to

$$\mathrm{d}g_1 - \mathrm{d}g_2 \circ i = -h'(e^b)\beta \circ i + \mathrm{d}\eta.$$

The left side of this equation defines a Fredholm operator of index 2, whose kernel is precisely given by constants in $\mathbb{R}^2$. Therefore, the cokernel is trivial, and the operator is surjective. This implies that there is a required unique solution of this equation. □

The fact that Φ is a bijection follows immediately from the two claims, and from the definition of $\mathcal{H}$ as a quotient by $\mathbb{C}^*$. □

We already know that Φ is a bijection. We now prove that it is also a diffeomorphism.

Proof of (iii). We will show that Φ gives a bijection of tangent spaces. We first describe the tangent spaces to $\mathcal{F}$ and $\mathcal{H}$. These are both spaces of solutions of elliptic differential equations, so their tangent spaces are kernels of linearized operators.

J and ω define a Hermitian metric on $\mathbb{R}\times Y$. Denote the corresponding connection by ∇. With respect to this connection, the linearized operator associated with the pseudo-holomorphic curve equation (4.1), at a solution $\tilde{u}=(a,u)$, is (see [Dra04])

$$\begin{aligned} D^{holo}_{\tilde{u}} : W^{1,p,\delta}\big(S,\tilde{u}^*T(\mathbb{R}\times Y)\big) &\to L^{p,\delta}\big(S,\Lambda^{0,1}(T^*S)\otimes_J \tilde{u}^*T(\mathbb{R}\times Y)\big) \\ \zeta &\mapsto \nabla\zeta + J\nabla_{j.}\zeta + (\nabla_\zeta J)\,\mathrm{d}\tilde{u}\circ j - \nabla_\zeta\tilde{\nu} \end{aligned}$$

Writing $\zeta=(\lambda,\mu)$ in terms of the splitting $\tilde{u}^*\big(T(\mathbb{R}\times Y)\big)=\big(\mathbb{R}\langle\partial_r\rangle\oplus\mathbb{R}\langle R\rangle\big)\oplus u^*\xi$, we get

$$\zeta \mapsto (\nabla\lambda + i\nabla_{j.}\lambda) + \big(\nabla\mu + J\nabla_{j.}\mu - (\nabla_\mu J)\pi\circ\mathrm{d}u\circ j - \nabla_\mu\tilde{\nu}\big)$$

Denote $D^{holo}_{\tilde{u}} = D^{holo}_1 + D^{holo}_2$, under this splitting. Write $\lambda=\lambda_1+i\lambda_2$. Then

$$\begin{aligned} D^{holo}_1\lambda = \nabla\lambda + i\nabla_{j.}\lambda &= (\mathrm{d}\lambda_1 - \mathrm{d}\lambda_2\circ j) + i(\mathrm{d}\lambda_2 + \mathrm{d}\lambda_1\circ j) = \\ &= (\mathrm{d}\lambda_1 - \mathrm{d}\lambda_2\circ j) + i(\mathrm{d}\lambda_1 - \mathrm{d}\lambda_2\circ j)\circ j \end{aligned}$$

and

$$D^{holo}_2\mu = \nabla\mu + J\nabla_{j.}\mu + (\nabla_\mu J)\pi\circ\mathrm{d}u\circ j - \nabla_\mu\tilde{\nu}$$

The linearized operator associated with the cylinder equation (4.2) at a solution $f=(f_1,f_2)$ is

$$\begin{aligned} D^{cyl}_{(\tilde{u},f)} : W^{1,p,\delta}(S,\mathbb{C}) &\to L^{p,\delta}\big(S,\mathrm{Hom}(T^*S,\mathbb{R})\big) \\ (F_1,F_2) &\mapsto \mathrm{d}F_1 - \mathrm{d}F_2\circ j + h''(e^{a+f_1})e^{a+f_1}F_1\beta\circ j. \end{aligned}$$

Therefore, the tangent space to $\mathcal{H}$ at $[\tilde{u},f]$ is given by the quotient of $\ker D^{holo}_{\tilde{u}} \oplus \ker D^{cyl}_{(\tilde{u},f)}$ by the $\mathbb{R}^2$-action induced by the $\mathbb{C}^*$-action on pairs $(\tilde{u},f)$.

Finally, we consider the linearization of the operator corresponding to Seidel's equation (4.5), at a solution $\tilde{v} = (b, v)$:

$$\begin{aligned} D^{Floer}_{\tilde{v}} \colon W^{1,p,\delta}\big(S, \tilde{v}^*T(\mathbb{R}\times Y)\big) &\to L^{p,\delta}\big(S, \Lambda^{0,1}(T^*S)\otimes_J \tilde{v}^*T(\mathbb{R}\times Y)\big) \\ \zeta &\mapsto \nabla\zeta + J\nabla_{j\cdot}\zeta + (\nabla_\zeta J)\, \mathrm{d}\tilde{v}\circ j - \nabla_\zeta X_H \otimes \beta - \nabla_\zeta(JX_H)\otimes\beta\circ j - \nabla_\zeta \tilde{\nu} \end{aligned}$$

Writing $\zeta = (\rho, \sigma)$ in terms of the splitting $v^*\big(T(\mathbb{R}\times Y)\big) = \big(\mathbb{R}\langle\partial_r\rangle \oplus \mathbb{R}\langle R\rangle\big) \oplus v^*\xi$, we get

$$\begin{aligned} \zeta \mapsto \big(&\nabla\rho + i\nabla_{j\cdot}\rho - (\nabla_\rho h'(e^r)R)\otimes\beta + (\nabla_\rho h'(e^r)\partial_r)\otimes\beta\circ j\big) + \\ &+ \big(\nabla\sigma + J\nabla_{j\cdot}\sigma + (\nabla_\sigma J)\pi\circ \mathrm{d}v\circ j - \nabla_\sigma\tilde{\nu}\big) \end{aligned}$$

Denote $D^{Floer}_{\tilde{v}} = D^{Floer}_1 + D^{Floer}_2$, under this splitting. Write $\rho = \rho_1 + i\rho_2$. Then

$$\begin{aligned} D^{Floer}_1\rho &= \nabla\rho + i\nabla_{j\cdot}\rho - (\nabla_\rho h'(e^r)\partial_r)\otimes\beta - (\nabla_\rho h'(e^r)R)\otimes\beta\circ j = \\ &= (\mathrm{d}\rho_1 - \mathrm{d}\rho_2\circ j + \rho_1 h''(e^r)e^r\beta\circ j) + i(\mathrm{d}\rho_2 + \mathrm{d}\rho_1\circ j - \rho_1 h''(e^r)e^r\beta) = \\ &= (\mathrm{d}\rho_1 - \mathrm{d}\rho_2\circ j + \rho_1 h''(e^r)e^r\beta\circ j) + i(\mathrm{d}\rho_1 - \mathrm{d}\rho_2\circ j + \rho_1 h''(e^r)e^r\beta\circ j)\circ j \end{aligned}$$

and

$$D^{Floer}_2\sigma = \nabla\sigma + J\nabla_{j\cdot}\sigma + (\nabla_\sigma J)\pi\circ\mathrm{d}v\circ j - \nabla_\sigma\tilde{\nu}$$

We now write the differential of the map $\Phi : \mathcal{H} \to \mathcal{F}$ at a point $[\tilde{u}, f] = [(a,u),(f_1,f_2)]$, such that $\Phi[\tilde{u}, f] = \tilde{v} = (b, v)$. Under the splittings $\tilde{u}^*\big(T(\mathbb{R}\times Y)\big) = \big(\mathbb{R}\langle\partial_r\rangle \oplus \mathbb{R}\langle R\rangle\big) \oplus u^*\xi$ and $\tilde{v}^*\big(T(\mathbb{R}\times Y)\big) = \big(\mathbb{R}\langle\partial_r\rangle \oplus \mathbb{R}\langle R\rangle\big) \oplus v^*\xi$, as above, and writing $\lambda = \lambda_1 + i\lambda_2$, we have

$$D\Phi\big((\lambda,\mu),(F_1,F_2)\big) = \big((\lambda_1+F_1) + i(\lambda_2+F_2), D(\phi_{f_2})\mu\big).$$

Since $\mathcal{H}$ and $\mathcal{F}$ are manifolds of the same dimension, it is enough to show that $D\Phi$ is surjective. Consider then $(\rho,\sigma) \in T_{\tilde{v}}\mathcal{F} = \ker D^{Floer}_{\tilde{v}}$, where $\tilde{v} = \Phi[(a,u),(f_1,f_2)]$.

Take $(\lambda, \mu) = \big(0, D(\phi_{-f_2})\sigma\big)$ and $F_1 + iF_2 = \rho$. Note that

$$\begin{aligned} D^{holo}_{\tilde{u}}(\lambda, \mu) = D^{holo}_2 \mu &= \nabla \mu + J\nabla_{j.}\mu + (\nabla_\mu J)\pi \circ \mathrm{d}v \circ j - \nabla_\mu \tilde{\nu} = \\ &= \nabla D(\phi_{-f_2})\sigma + J\nabla_{j.} D(\phi_{-f_2})\sigma + \big(\nabla_{D(\phi_{-f_2})\sigma} J\big)\pi \circ \mathrm{d}v \circ j - \nabla_{D(\phi_{-f_2})\sigma}\tilde{\nu} = \\ &\overset{(\ddagger)}{=} D(\phi_{-f_2})\big(\nabla\sigma + J\nabla_{j.}\sigma + (\nabla_\sigma J)\pi \circ \mathrm{d}v \circ j - \nabla_\sigma \tilde{\nu}\big) = \\ &= D(\phi_{-f_2}) D^{Floer}_2 \sigma = 0 \end{aligned}$$

In (‡), we have used the fact that ∇ is the Levi-Civita connection for the metric $\omega(., J.)$ on $\mathbb{R} \times Y$, that the flow ϕ of the Reeb vector field on Y is by isometries of $\mathbb{R} \times Y$, and that J and ν are invariant under ϕ.

We also have

$$D^{cyl}_{(\tilde{u}, f)}(F_1, F_2) = \mathrm{d}F_1 - \mathrm{d}F_2 \circ j + h''(e^{a+f_1})e^{a+f_1}F_1\beta \circ j = \pi_1(D^{Floer}_1 \rho) = 0$$

so $(\lambda, \mu) \in \ker D^{holo}_{\tilde{u}} = T_{\tilde{u}}\mathcal{M}$ and $(F_1, F_2) \in \ker D^{cyl}_{(\tilde{u}, f)}$. Furthermore,

$$D\Phi\big((\lambda, \mu), (F_1, F_2)\big) = (\rho, \sigma)$$

which completes the proof that $D\Phi$ is surjective and that Φ is a diffeomorphism. □

4.5 Excluding unwanted solutions

According to Proposition 4.1 there are certain configurations in the split Floer homology differential of X whose associated cylinder equation is not transverse. Namely, if $k^-_{cvx} = 1 = k^+_{ccv}$, which implies that the index of the linearized cylinder equation is zero, and both kernel and cokernel are one-dimensional. We now explain how to exclude those configurations, with an argument involving the action functional.

The only configurations in the split Floer homology differential whose corresponding cylinder equation is not transverse are given by punctured cylinders

$$V = (b, v) : \mathbb{R} \times S^1 \backslash \{p\} \to \mathbb{R} \times Y$$

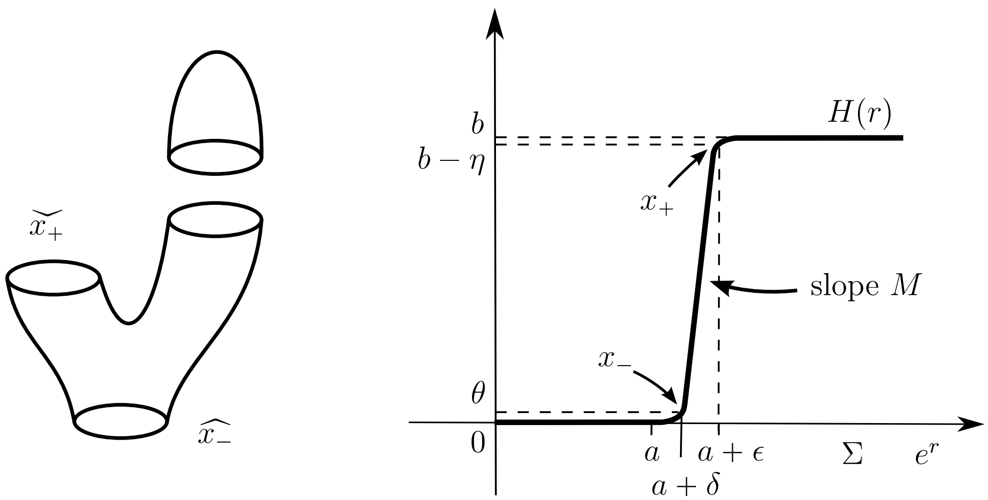

Figure 4.2: Unwanted configuration and steep Hamiltonian

where the puncture converges to a Reeb orbit at the $+\infty$ end of the symplectization, which is capped by a plane on a fiber of the normal bundle $N\Sigma$. In Figure 4.2, we sketch one such configuration, and an S-shaped Hamiltonian H with steep slope M, for which we will be able to show that such configurations cannot exist.

Suppose that $\lim_{s\to\pm\infty} v(s,t) = x_\pm(t)$, for some Reeb orbits $x_\pm$ in Y. One of the hypothetic configurations that we are trying to rule out would contribute with $\pm\, x_-\, t^{\frac{-1}{K\lambda_X N}}$ to the differential of x_+. The energy of such a configuration would be

$$E = \mathcal{A}_H(x_+) + 1/K - \mathcal{A}_H(x_-).$$

Suppose that x_+ is an l-cover of a fiber of $S^1 \to Y \to \Sigma$. Then, x_- is an $(l+1)$-cover of a fiber, since the component of the split Floer trajectory that is contained in $\mathbb{R}\times Y$ projects to a contractible map to Σ, by index reasons. Put differently, this component is a perturbation of a cover of a trivial cylinder.

Therefore, according to Lemma 3.3, and using the notation in Figure 4.2,

$$E = (a+\epsilon)l - b + \eta + 1/K - (a+\delta)(l+1) + \theta =$$
$$= -a + \epsilon l - b + \eta + 1/K - \delta(l+1) + \theta.$$

Since the energy of a non-constant Floer trajectory is positive, we get that

$$b < -a + 1/K + \eta + \theta - \delta + (\epsilon - \delta)l.$$

Therefore, we conclude that, if we fix small η, θ, δ and ϵ, and large L and b, we can conclude that such configurations cannot exist for $0 < l < L$. Heuristically, this means that if H is close to a step function and b is large, then we can exclude the existence of configurations as in Figure 4.2, as long as l is not too large.

Chapter 5

Relation with Gromov–Witten numbers

We saw above that Floer trajectories in $\mathbb{R}\times Y$ can be understood in terms of punctured pseudo-holomorphic curves in $\mathbb{R}\times Y$ and solutions of the auxiliary cylinder equation. In this chapter, we relate pseudo-holomorphic curves in $\mathbb{R}\times Y$, in $N\Sigma$ and in W with Gromov–Witten numbers of Σ and relative Gromov–Witten numbers of (X,Σ). Then, we explain how to use this information to express the differentials in split Floer and symplectic homology. We also give an indication of the analogous results for the pair-of-pants products.

5.1 Pseudo-holomorphic curves in $\mathbb{R}\times Y$ and $N\Sigma$, meromorphic sections of holomorphic line bundles and Gromov–Witten numbers of Σ

We will now see how to relate pseudo-holomorphic curves in $\mathbb{R}\times Y$ with meromorphic sections of line bundles over $\mathbb{C}P^1$ and with Gromov–Witten numbers of Σ.

5.1.1 Gromov–Witten numbers and quantum cohomology

Let us quickly review some basics about Gromov–Witten theory. Although we use a slightly different point of view (namely with respect to the perturbations that we take, and to the fact that we will use a Morse chain version), we refer the reader to [MS04] for more details.

Recall that (Σ^{2n-2}, ω) is a closed symplectic manifold on which we chose a generic ω-compatible almost complex structure J and a family ν of perturbation 1-forms, parametrized by the universal curve and supported away from nodes and marked points (see Section 4.2).

We are interested in maps $w : \mathbb{C}P^1 \to \Sigma$ that solve the perturbed equation

$$\mathrm{d}w + J \circ \mathrm{d}w \circ j = \nu.$$

The spaces of such maps realizing homology classes $A \in H_2(\Sigma)$, modulo domain automorphisms preserving m marked points, define moduli spaces $\mathcal{M}_{A,m}(\Sigma)$. These come equipped with m evaluation maps at the marked points, and (for generic J and ν) define pseudo-cycles in the product of m copies of Σ:

$$\mathrm{ev} : \mathcal{M}_{A,m}(\Sigma) \to \Sigma^m.$$

These pseudo-cycles have dimension $2n - 2 + 2\langle c_1(T\Sigma), A\rangle + 2m - 6$. To define Gromov–Witten invariants of Σ, one intersects such pseudo-cycles with homology classes in Σ. To that end, one can take pseudo-cycle representatives for generators of the homology of Σ (see for instance [MS04] and [Sch99]), and work with intersections of pseudo-cycles. We are interested in a slightly more general chain-level definition (which is why we use the term 'numbers' and not 'invariants'). Let $f_\Sigma : \Sigma \to \mathbb{R}$ be a Morse–Smale function with respect to a Riemannian metric g on Σ. The critical points of f_Σ generate a chain complex that computes the singular homology of Σ. Schwarz showed this fact in [Sch99], by proving that the unstable manifolds associated with a Morse cycle form a pseudo-cycle in Σ, which in turn defines a unique homology class. He also showed that, if we fix a pseudo-cycle P in Σ, then for generic pairs (f_Σ, g),

P intersects the Morse pseudo-cycles of (f_Σ, g) transversely, and in the interior of the top dimensional strata of the Morse pseudo-cyles (recall that a Morse pseudo-cycle is given by the union of stable or unstable manifolds of finitely many critical points, of possibly different indices). For details on this point, see Theorem 4.9 in [Sch99]. We can then argue that, if we fix a countable collection of pseudo-cycles P_i inside powers of Σ, then for a Baire set of pairs (f_Σ, g), products of stable and unstable manifolds of critical points of f intersect the P_i transversaly. To define Gromov–Witten numbers of Σ, we let the P_i be the pseudo-cycles associated with the relevant moduli spaces of perturbed pseudo-holomorphic curves, and intersect those with the stable and unstable manifolds of a generic pair (f_Σ, g). Given a homology class $A \in H_2(\Sigma)$ and Morse chains $C_1, \ldots, C_m$, we denote by $\mathrm{GW}^{\Sigma}_{A,m}(C_1, \ldots, C_m)$ the corresponding Gromov–Witten number, obtained by intersecting the pseudocycles given by $\mathcal{M}_{A,m}(\Sigma)$ and by $C_1 \times \ldots \times C_m$, in Σ^m.

Remark 5.1. *In this text, we made the simplifying assumption that Σ admits a perfect Morse function, which holds in the examples that we will consider. The main consequence is that there is no need to distinguish between Gromov–Witten numbers and invariants, which can be read from the quantum cohomology rings. See also Remarks 3.1 and 5.4 for other implications of this assumption.*

At this point, we should remark that, since we assume our symplectic manifolds to be monotone, in principle we would not need the perturbation term ν to achieve transversality for the spaces of perturbed pseudo-holomorphic curves, and to define Gromov–Witten numbers. A key feature of Gromov–Witten numbers is that, if we choose the C_i to be homology classes, instead of chains, then $\mathrm{GW}^{\Sigma}_{A,m}(C_1, \ldots, C_m)$ does not depend on generic J and ν (and is called an 'invariant'). In fact in Chapter 6, we will compute some Gromov–Witten invariants in a setting where $\nu = 0$. The reason why we introduce the perturbation terms ν in our definition is to have enough transversality of the evaluation maps at marked points to the stable and unstable manifolds of *a single* Morse function in Σ. For this to be the case, we need homologically trivial curves in Σ not to be constant, which can be achieved by the term ν in the equation.

If $m < 3$, then we do not have a space of stable curves, so we cannot talk about the universal curve bundle. But in this case, it turns out to be enough to consider *unperturbed* pseudo-holomorphic curves, which solve $\mathrm{d}w + J \circ \mathrm{d}w \circ j = 0$. Just as before, we can use moduli spaces of solutions of this equation to define Gromov–Witten numbers with one and two marked points.

One important property of Gromov–Witten numbers, which we will use, is the *divisor equation*. This states that, given $A \in H_2(\Sigma)$, homology classes $C_1, \ldots, C_m \in H_*(\Sigma)$ and $H \in H_{2n-2}(\Sigma)$,

$$\mathrm{GW}^{\Sigma}_{A,m}(C_1, \ldots, C_m) = \frac{1}{\#(A \cap H)}\, \mathrm{GW}^{\Sigma}_{A,m+1}(C_1, \ldots, C_m, H).$$

The meaning of this is that each curve in class A intersecting $C_1, \ldots, C_m$ will intersect $\#(A \cap H)$ times the class H, and will therefore contribute with the factor $\#(A \cap H)$ to the count of curves in homology class A intersecting $C_1, \ldots, C_m, H$.

Another important point about Gromov–Witten invariants (on homology, not on the chain level) is that they have also been defined using methods of algebraic geometry. The advantage of this approach is that it is often much more computable than its symplectic counterpart. It has been shown that in the case of complex projective Kähler varieties, where both the symplectic and the algebraic definitions make sense, the invariants are the same (see for instance [LT99]). The point of our work is precisely to describe, in certain examples, the Floer differential and pair-of-pants product in terms of Gromov–Witten numbers, that can sometimes be computed explicitly using tools from algebraic geometry (as in [Bea95] or [Zin11]).

Let us recall also how to use Gromov–Witten invariants to construct a deformation of the cup product on cohomology. Assume that the symplectic manifold (M, ω) is monotone, with minimal Chern number N. Consider a Novikov ring $\Lambda_{\mathbb{Q}} := \mathbb{Q}[t, t^{-1}]$, where t is a variable of degree $2N$, as in Section 2.1. Define a product on $QH^*(M) := H^*(M; \mathbb{Q}) \otimes \Lambda_{\mathbb{Q}}$ as follows: given $a \in H^k(M; \mathbb{Q})$ and $b \in H^l(M; \mathbb{Q})$,

$$a * b := \sum_{A \in H^2(M;\mathbb{Z})} (a * b)_A\, t^{\langle c_1(TM), A\rangle / N}$$

where $(a * b)_A \in H^*(M;\mathbb{Q})$ is specified by saying that, for any $c \in H^*(M;\mathbb{Q})$,

$$\int_M (a * b)_A \cup c = \mathrm{GW}^M_{A,3}(\mathrm{PD}(a), \mathrm{PD}(b), \mathrm{PD}(c)).$$

$QH^*(M;\mathbb{Q})$ with this ring structure is called the *quantum cohomology algebra*. As we have recalled in Section 2.3, quantum cohomology of monotone symplectic manifolds is isomorphic to Floer homology. In Chapter 6, we will use results about quantum cohomology algebras (from [Bea95]) to extract the Gromov–Witten invariants that will be necessary for our symplectic homology computations.

5.1.2 Meromorphic sections of holomorphic line bundles and pseudo-holomorphic curves in $\mathbb{R} \times Y$

As was pointed out in Remark 3.2, the symplectization $\mathbb{R} \times Y$ can be thought of as the complement of the zero section of a complex line bundle $E \to \Sigma$ (associated with the S^1-bundle $Y \to \Sigma$).

Let now $\tilde{u} = (a, u) : S \to \mathbb{R} \times Y$ be a pseudo-holomorphic curve, where S is the complement of a finite subset of $\mathbb{C}P^1$. One can project $\tilde{u}$ to the divisor Σ, and obtain a pseudo-holomorphic map $w : S \to \Sigma$. Since punctures of finite energy pseudo-holomorphic curves asymptote to Reeb orbits in Y, and these are multiple covers of the fibers of $Y \to \Sigma$, w extends to a map from $\mathbb{C}P^1$. Now, (a, u) can be identified with a section s of the bundle $w^*E \to \mathbb{C}P^1$, with prescribed zeros and poles at the points in $\mathbb{C}P^1 \backslash S$. This section is complex linear, by (4.7). Since every complex line bundle over $\mathbb{C}P^1$ admits a unique holomorphic strucutre (see Exercise 3.3.7 in [Huy05]), we conclude that s is actually a meromorphic section. This proves the following result.

Lemma 5.1. *Every pseudo-holomorphic curve $\tilde{u} = (a, u) : S \to \mathbb{R} \times Y$ defines a pseudo-holomorphic map $w : \mathbb{C}P^1 \to \Sigma$ and a meromorphic section of $w^*E \to \mathbb{C}P^1$ with zeros and poles on the finite set $\mathbb{C}P^1 \backslash S$.*

We can now reduce the question of counting punctured pseudo-holomorphic maps $\tilde{u}$ to that of counting pseudo-holomorphic curves $w : \mathbb{C}P^1 \to \Sigma$, together with meromorphic sections s of w^*E. The count of maps w is related with the computation of Gromov–Witten numbers of Σ. We can also give a complete description of the relevant meromorphic sections s. Recall that, given a divisor of points D and a holomorphic line bundle L over $\mathbb{C}P^1$, where both D and L have degree d, there is a $\mathbb{C}^*$-family of meromorphic sections of L such that the divisor associated with each section is D. One can justify this fact by reducing it to the simplest case of trivial L: use a trivialization of L over $\mathbb{C} \subset \mathbb{C}P^1$ to identify meromorphic sections of L with meromorphic functions on $\mathbb{C}P^1$ (see pages 342–345 of [Mir95]).

Remark 5.2. *The fact that meromorphic sections of line bundles over $\mathbb{C}P^1$ come in $\mathbb{C}^*$-families implies that the contact homology differential (without point constraints) should vanish for pre-quantization bundles (see [EGH00]), because the moduli spaces of non-constant holomorphic curves with cylindrical ends have an S^1-action without fixed points. This is not the case in our setting, though, because we impose marker conditions on our asymptotic limits (we should think of* non-equivariant *contact homology, as in Section 3.2 of [BO09a]).*

From this point on, let us restrict our attention to cylinders in $\mathbb{R} \times Y$ contributing to the split Floer or symplectic homology differential, possibly with punctures capped by planes in W or in $N\Sigma$. For an example, see Figure 5.1. In this picture and in all the ones that will follow, the periodic orbits represent asymptotic Reeb orbits. The configuration depicted here contains components in $N\Sigma$ (on top), in $\mathbb{R} \times Y$ (in the middle) and in W (on the bottom).

Given a non-vanishing Gromov–Witten number $\mathrm{GW}^{\Sigma}_{A,l}(C_1, \ldots, C_l)$, where the C_i are stable or unstable manifolds of critical points of a Morse function $f_\Sigma : \Sigma \to \mathbb{R}$, there are rigid pseudo-holomorphic maps $w : (\mathbb{C}P^1; z_1, \ldots, z_l) \to (\Sigma; C_1, \ldots, C_l)$, modulo automorphisms. Since we are describing the differential, we have $l \geqslant 2$. Let us try to describe which rigid maps

$$\tilde{u} : \mathbb{R} \times S^1 \backslash \{l - 2 \text{ punctures}\} \to \mathbb{R} \times Y,$$

with fixed markers at 0 and ∞, this gives rise to. Let l_+ be the number of punctures that are capped by a plane in $N\Sigma$ and let $l_- = l - 2 - l_+$ be the number of punctures capped in W. Fix a map w as above and assume without loss of generality that

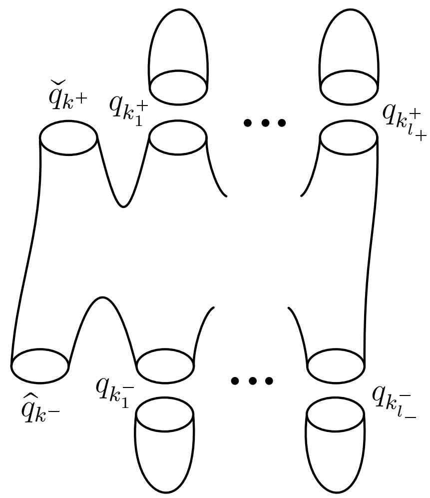

Figure 5.1: Configurations given by coefficients $c(k^+, k_1^+, \dots, k_{l_+}^+; k^-, k_1^-, \dots, k_{l_-}^-)$

$z_1 = 0$ and $z_2 = \infty$. Given $b \in S^1$, we can define another pseudo-holomorphic map $\tilde{w}$, such that $\tilde{w}(z) = w(bz)$. This gives us one degree of freedom to fix one marker, either at 0 or at ∞. Each of these maps $\tilde{w}$ defines a line bundle $\tilde{w}^*E \to \mathbb{C}P^1$, of degree $d(A) := \langle c_1(E \to \Sigma), A\rangle$. Since prescribing zeros and poles gives a $\mathbb{C}^*$-family of meromorphic sections, we have another S^1-parameter that can be used to fix another marker. This motivates the following definition (see Figure 5.1).

Definition 5.1. *Let $c(k^+, k_1^+, \dots, k_{l_+}^+; k^- k_1^-, \dots, k_{l_-}^-)$ be the number of pseudo-holomorphic curves $\tilde{u} : \mathbb{R} \times S^1 \backslash \{l-2 \text{ punctures}\} \to \mathbb{R} \times Y$, with one positive Floer puncture asymptotic to a Reeb orbit of multiplicity k^+, l_+ positive punctures (asymptotic to Reeb orbits of multiplicity $k_1^+, \dots, k_{l_+}^+$, respectively) to be capped by planes in $N\Sigma$, one negative Floer puncture asymptotic to a Reeb orbit of multiplicity k^-, and l_- negative punctures (asymptotic to Reeb orbits of multiplicity $k_1^-, \dots, k_{l_-}^-$, respectively) to be capped by planes in W, associated with a non-zero Gromov–Witten number* $\mathrm{GW}^{\Sigma}_{A,l}(C_1, \dots, C_l)$.

Note that we should have

$$d(A) = \langle c_1(E \to \Sigma), A\rangle = k^- + \sum_{i=1}^{l_-} k_i^- - k^+ - \sum_{j=1}^{l_+} k_j^+,$$

to obtain the correct difference of the number of zeros and poles. The following result will be useful in our computations of symplectic homology differentials.

Lemma 5.2. *$c(k+1;k,1) = 1$ and $c(k+|d|;k) = |d|$, where $d = d(A) < 0$ for some $A \in H_2(\Sigma;\mathbb{Z})$.*

Proof. We begin with the proof that $c(k+1;k,1) = 1$. This corresponds to saying that, given a rigid augmentation plane in W capping the simple Reeb orbit corresponding to the critical point $q \in \Sigma$, we have a contribution $\widehat{q}_k$ to the differential of $\check{q}_{k+1}$, for any multiplicity $k \geqslant 1$, coming from the configuration in Figure 5.2.

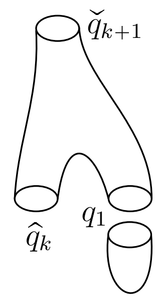

Figure 5.2: Configurations given by coefficients $c(k+1;k,1)$

The component $\tilde{u} : \mathbb{C}P^1 \backslash \{0, \infty, \lambda\} \to \mathbb{R} \times Y$ (where λ is the augmented puncture) projects to a null-homologous map $w : \mathbb{C}P^1 \to \Sigma$, so the line bundle $w^*E \to \mathbb{C}P^1$ is trivial. The relevant meromorphic sections are in this case meromorphic functions, which can be written explicitly as:

$$\begin{aligned} s : \mathbb{C}\backslash\{0, \lambda\} &\to \mathbb{C}^* \\ z &\mapsto az^{k-1}(z-\lambda) \end{aligned}$$

where $a = re^{i\theta} \in \mathbb{C}^*$. This extends to a map $s : \mathbb{C}P^1 \to \mathbb{C}P^1$. The zeros correspond to the points 0 and λ on the domain, and represent $\widehat{q}_k$ and q_1, respectively; the pole is attained at ∞ and corresponds to $\check{q}_{k+1}$. Over $\mathbb{C}P^1\backslash 0$, with holomorphic coordinate $y = 1/z$, the section s becomes

$$y \mapsto a\frac{1-\lambda y}{y^k}$$

Write $\lambda = \rho e^{i\varphi}$ and observe that we have four real degrees of freedom: r, θ, ρ and φ. We need to quotient out our space of configurations by $\mathbb{R}$-translations on the domain and on the target, so we can assume $r = \rho = 1$. Fixing two markers will rigidify these configurations. We will show that we indeed get a unique configuration (corresponding to unique values of θ and φ) when we fix markers.

To fix the markers at $\check{q}_{k+1}$ and $\hat{q}_k$, we can, for instance, force markers aligned with positive real directions on the domain to map to markers aligned with positive real directions on the target. For $\hat{q}_k$, let $z = t$, for $t > 0$ small. We get

$$s(z) = s(t) = e^{i\theta} t^{k-1}(t - e^{i\varphi}) \approx -t^{k-1} e^{i(\theta+\varphi)}.$$

For the argument of the image to coincide with the prescribed marker on the target, we get $\theta + \varphi = \pi$. For $\check{q}_{k+1}$, let $y = t$, for $t > 0$ small. We get

$$s(y) = s(t) = e^{i\theta} \frac{1 - e^{i\varphi} t}{t^k} \approx e^{i\theta}/t^{k-1}$$

and we can again force the argument to match with the corresponding prescribed marker on the target. This implies $\theta = 0$ and $\varphi = \pi$. Therefore, after prescribing the markers, we get unique values for θ and φ, which justifies $c(k+1; k, 1) = 1$.

We now wish to show that $c(k + |d|; k) = |d|$, where $d = \langle c_1(E \to \Sigma), A \rangle < 0$ for some $A \in H_2(\Sigma, \mathbb{Z})$. This means that, if $p, q \in \Sigma$ are critical points of f_Σ, then any rigid holomorphic sphere in Σ contributing to $\mathrm{GW}^{\Sigma}_{A,2}(W^u(p), W^s(q))$, gives a contribution of $(|d| \,.\, \hat{q}_k)$ to the differential of $\check{p}_{k+|d|}$. These terms correspond to pseudo-holomorphic cylinders $\tilde{u} : \mathbb{C}P^1 \backslash \{0, \infty\} \to \mathbb{R} \times Y$ projecting to maps $w : \mathbb{C}P^1 \to \Sigma$ such that $[w] = A$, and are represented in Figure 5.3. The line bundle w^*E has degree $d < 0$.

The maps $\tilde{u}$ are given by composing meromorphic sections of $w^*E \to \mathbb{C}P^1$ with reparametrizations

$$\mathbb{C}P^1 \to \mathbb{C}P^1$$
$$z \mapsto bz$$

for $|b| = 1$ (since in non-equivariant contact homology our cylinders have asymptotic

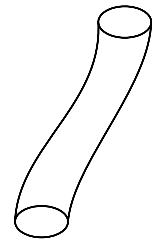

Figure 5.3: Configurations given by coefficients $c(k + |d|; k)$

markers, these domain rotations are not automorphisms we can mod out by).

Let us write down coordinates for the domain, $\mathbb{C}P^1 \backslash \{0, \infty\}$, and target, the degree $d < 0$ line bundle $w^*E \to \mathbb{C}P^1$. On the domain, take a coordinate z on $\mathbb{C} = \mathbb{C}P^1 \backslash 0$ and $y = 1/z$ on $\mathbb{C} = \mathbb{C}P^1 \backslash \infty$. On the target, take coordinates (ζ, u) on $(\mathbb{C}P^1 \backslash 0) \times (\mathbb{C}P^1 \backslash 0)$ and $(\xi, v) = (1/\zeta, u/\xi^{|d|})$ on $(\mathbb{C}P^1 \backslash \infty) \times (\mathbb{C}P^1 \backslash \infty)$. We will consider meromorphic sections of w^*E with a prescribed zero of order k (corresponding to a k-multiple of the fiber of $Y \to \Sigma$ over $q \in \Sigma$), say at 0, and a prescribed pole of order $k + |d|$ (corresponding to a $(k + |d|)$-multiple of the fiber of $Y \to \Sigma$ over $p \in \Sigma$), say at ∞. These sections can be written as

$$u(\zeta) = a\zeta^k \qquad \text{and} \qquad v(\xi) = u(\zeta)/\xi^{|d|} = a/\xi^{k+|d|}.$$

Therefore, we can write the map $\mathbb{C}P^1 \backslash \{0, \infty\} \to w^*E$ as

$$z \mapsto \big(bz, a(bz)^k\big) \qquad \text{and} \qquad y \mapsto \big(y/b, ab^{k+|d|}/y^{k+|d|}\big).$$

Since we quotient by $\mathbb{R}$-translation on the symplectization direction, which in our case corresponds to the radial direction on the fiber, we can assume $|a| = 1$. Fixing the two markers, say by requiring that the positive real lines map to the positive real lines, we get

$$ab^k = 1 \qquad \text{and} \qquad ab^{k+|d|} = 1$$

which implies that $b^{|d|} = 1$ and $a = b^{-k}$. Therefore, writing $\eta = e^{2\pi i/|d|}$,

$$(a, b) \in \{(1, 1), (\eta^{-k}, \eta), \ldots, (\eta^{-k(|d|-1)}, \eta^{|d|-1})\}.$$

These are the $|d|$ solutions we were after, and this is why $c(k + |d|; k) = |d|$. □

5.1.3 Pseudo-holomorphic curves in $N\Sigma$

Split Floer trajectories may also contain components in $N\Sigma$, where the Hamiltonian is constant. Therefore, components in $N\Sigma$ are pseudo-holomorphic curves, possibly connected with gradient flow lines of a Morse function $f_{N\Sigma}$ in $N\Sigma$. The simplest such components are cappings of punctures of pseudo-holomorphic curves in $\mathbb{R} \times Y$, as in Figure 5.1. These are pseudo-holomorphic curves in $N\Sigma$, converging to Reeb orbits in Y. For such cappings to be rigid, they should correspond to fibers of $N\Sigma \to \Sigma$. This implies that such planes should asymptote to simple Reeb orbits, since otherwise one would have non-rigid families of capping planes in $N\Sigma$, given by covers of fibers of this line bundle. As a consequence, we can conclude that 'positive augmentation' punctures of rigid holomorphic curves in $\mathbb{R} \times Y$ must converge to simple Reeb orbits.

There could also be configurations with more complicated components in $N\Sigma$. An argument similar to that of the previous section implies that the pseudo-holomorphic components of such configurations correspond to pseudo-holomorphic maps $w : \mathbb{C}P^1 \to \Sigma$, together with meromorphic sections of $w^*N\Sigma$.

5.2 Pseudo-holomorphic curves in W and relative Gromov–Witten numbers of (X, Σ)

When we split X and W, we get a piece that is symplectomorphic to W, with a constant Hamiltonian. So, as in $N\Sigma$, the components of split Floer trajectories contained in W are pseudo-holomorphic curves, possibly connected to gradient flow lines of a Morse function f_W in W. Pseudo-holomorphic curves in W for a J that is cylindrical at infinity can be identified with closed holomorphic curves in X intersecting Σ.

These are the curves described by relative Gromov-Witten numbers.

5.2.1 Relative Gromov–Witten numbers

Let us quickly review relative Gromov–Witten theory. For details, we refer the reader to [IP03]. Similarly to the absolute case, discussed above, we will use a Morse–Bott chain level version of relative Gromov–Witten numbers.

Let $f_\Sigma : \Sigma \to \mathbb{R}$ be a Morse function and $f_X : X \to \mathbb{R}$ be a Morse–Bott function. Suppose that Σ is a critical manifold of f_X, with no other critical manifolds of dimension greater than zero, and that the global maximum of f_Σ is attained at Σ. We will sometimes not be careful in distinguishing f_X from f_W. We also choose a Riemannian metric g in X, with respect to which we define the Morse flows of f_X and f_Σ. Denote these flows by $\phi^s_{f_X}$ and $\phi^s_{f_\Sigma}$, respectively, where time is measured by the variable $s \in \mathbb{R}$. The critical points of f_X and f_Σ should be thought of as the critical points of a Morse function on X, obtained by using f_Σ to perturb f_X near Σ (as in [BH11]). Given $x \in \mathrm{Crit}(f_\Sigma)$, let $W^s_{f_X}(x) := W^s_{f_\Sigma}(x)$ and

$$W^u_{f_X}(x) := W^u_{f_X}\left(W^u_{f_\Sigma}(x)\right) = \left\{a \in X \,|\, \lim_{s\to-\infty} \phi^s_{f_X}(a) \in W^u_{f_\Sigma}(x)\right\}.$$

As in Morse theory, one can use Morse–Bott chains formed by stable and unstable manifolds to define pseudo-cycles in X, which can be used to compute its singular homology (see [Fra04] and [BH11]).

We will also need an almost complex structure in X, such that Σ is an almost complex submanifold. Take one that extends the previously chosen J in Σ, and denote this extension also by J. Given $k \in \mathbb{Z}_{\geqslant 0}$, $\vec{s} = (s_1, \ldots, s_l) \in \mathbb{Z}^l_{>0}$ and $A \in H_2(X)$, we can define moduli spaces $\mathcal{M}_{A,k,\vec{s}}(X;\Sigma)$ of pseudo-holomorphic map $\mathbb{C}P^1 \to X$ (whose image is not entirely contained in Σ), with $k + l$ disjoint marked points, m of which mapping into Σ, with orders of tangency to Σ prescribed by the entries of the vector $\vec{s}$. These spaces have evaluation maps defining pseudo-cycles

$$\mathrm{ev} : \mathcal{M}_{A,k,\vec{s}}(X;\Sigma) \to X^k \times \Sigma^l$$

of dimension $2n + 2\langle c_1(TX), A\rangle + 2\big(k + l - \sum_{i=1}^{l} s_i\big) - 6$. Given Morse–Bott chains $C_1, \ldots, C_k$ in X and Morse chains $B_1, \ldots, B_l$ in Σ, we define the relative Gromov–Witten number $\mathrm{GW}^{X,\Sigma}_{A,k,\vec{s}}\big(C_1, \ldots, C_k; B_1, \ldots, B_l\big)$, to be the intersection number of the pseudo-cycles defined by $\mathcal{M}_{A,k,\vec{s}}(X;\Sigma)$ and by $C_1 \times \ldots \times C_k \times B_1 \times \ldots \times B_l$, in $X^k \times \Sigma^l$.

A useful property of relative Gromov–Witten numbers is that, when the intersections with Σ are all transverse, they can be expressed in terms of absolute Gromov–Witten numbers of X: if $\vec{s} = (1, \ldots, 1)$ and $\#(A \cap \Sigma) = |\vec{s}|$,[1] then

$$\mathrm{GW}^{X,\Sigma}_{A,k,\vec{s}}(C_1, \ldots, C_k; B_1, \ldots, B_{|\vec{s}|}) = \mathrm{GW}^{X}_{A,k+|\vec{s}|}(C_1, \ldots, C_k, B_1, \ldots, B_{|\vec{s}|}). \tag{5.1}$$

This requires a slight extension of the above description of Morse chain level absolute Gromov–Witten numbers, to the case when X has a Morse–Bott function and its critical manifolds have auxiliary Morse functions.

One important point in the proof of (5.1) is to justify that there are generically no relevant holomorphic curves contained inside Σ. These might contribute to the absolute Gromov–Witten number, but not to the relative number, by definition. The following result rules out this possibility.

Lemma 5.3. *Fix a vector* $\vec{s} = (1, \ldots, 1)$ *and* $A \in \mathrm{Image}(H_2(\Sigma;\mathbb{Z}) \to H_2(X;\mathbb{Z}))$ *such that*[2] $\#(A \cap \Sigma) \geqslant |\vec{s}|$. *Suppose that there are chains* C_i *in* X *and* B_j *in* Σ, *so that* $\mathrm{GW}^{X}_{A,k+|\vec{s}|}(C_1, \ldots, C_k, B_1, \ldots, B_{|\vec{s}|}) \neq 0$. *Generically, there are no maps* $u : \mathbb{C}P^1 \to X$ *contributing to this count, such that* $\mathrm{Image}(u) \subset \Sigma$.

Proof. If the absolute Gromov–Witten number in non-zero, then,

$$\dim \mathcal{M}_{A,k+|\vec{s}|}(X) = \sum_{i=1}^{k}(2n - \dim C_i) + \sum_{j=1}^{|\vec{s}|}(2n - \dim B_j)$$

[1] The assumption $\#(A \cap \Sigma) = |\vec{s}|$ cannot be removed: consider the example of $(X, \Sigma) = (\mathbb{C}P^3, \mathbb{C}P^2)$. Denoting the complex-oriented generator of $H_2(\mathbb{C}P^3;\mathbb{Z})$ by L, we have $\mathrm{GW}^{\mathbb{C}P^3}_{L,2}(pt, pt) = 1$, but $\mathrm{GW}^{\mathbb{C}P^3,\mathbb{C}P^2}_{L,0,(1,1)}(pt, pt) = 0$, because a holomorphic curve in homology class L going through two points in $\mathbb{C}P^2$ is contained in $\mathbb{C}P^2$ and therefore does not contribute to a relative Gromov–Witten number.

[2] The condition $\#(A \cap \Sigma) \geqslant |\vec{s}|$ does not hold in the example of the previous footnote.

and so

$$2n + 2\langle c_1(TX), A\rangle + 2(k + |\vec{s}|) - 6 = 2n(k + |\vec{s}|) - \sum_{i=1}^{k} \dim C_i - \sum_{j=1}^{|\vec{s}|} \dim B_j. \quad (5.2)$$

We wish to rule out the existence of holomorphic curves contained entirely in Σ that might contribute to this non-zero count. If such configurations existed, then we would have

$$\dim \mathcal{M}_{A,k+|\vec{s}|}(\Sigma) \geqslant \sum_{i=1}^{k} \big((2n-2) - (\dim C_i - 2)\big) + \sum_{j=1}^{|\vec{s}|} \big((2n-2) - \dim B_j\big).$$

Therefore,

$$2n - 2 + 2\langle c_1(T\Sigma), A\rangle + 2(k + |\vec{s}|) - 6 \geqslant 2n(k + |\vec{s}|) - \sum_{i=1}^{k} \dim C_i - \sum_{j=1}^{|\vec{s}|} \dim B_j - 2|\vec{s}|.$$

This fact, together with $\langle c_1(T\Sigma), A\rangle = \langle c_1(TX), A\rangle - \langle c_1(N\Sigma), A\rangle$, $\langle c_1(N\Sigma), A\rangle = \#(A \cap \Sigma) \geqslant |\vec{s}|$ and (5.2), implies a contradiction and proves the lemma. □

Remark 5.3. *As observed by Maulik and Pandharipande in [MP06], the relative Gromov–Witten numbers of a pair (X, Σ) can often be obtained from the (absolute) Gromov–Witten numbers of X and of Σ, and from the map $H^*(X) \to H^*(\Sigma)$.*

5.2.2 Pseudo-holomorphic curves in W

Let J be a generic almost complex structure in X, for which Σ is an almost complex submanifold and J is cylindrical near Σ. Such J also defines a cylindrical almost complex structure on W.

The following result tells us that pseudo-holomorphic curves in W can be described in terms of relative Gromov–Witten numbers of (X, Σ). Fix points $p_1, \ldots, p_m \in \Sigma$, corresponding to simple Reeb orbits $\gamma_1, \ldots, \gamma_m$ in Y.

Proposition 5.1. *Given positive integers $k_1, \ldots, k_m$, there is a bijective correspondence between J-holomorphic spheres in X that intersect Σ precisely at the points p_i*

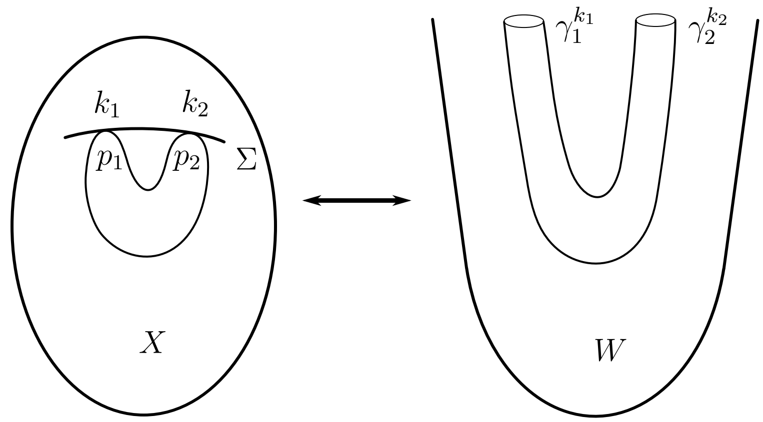

Figure 5.4: Pseudo-holomorphic curves in X and W

with order of tangency k_i, and J-holomorphic curves in W of genus 0 with punctures asymptoting to k_i-covers of the γ_i.

Relative Gromov–Witten numbers are important for our purposes for three reasons. The first is that, as a consequence of the previous Proposition, they count augmentation planes in W capping negative punctures of Floer and pseudo-holomorphic curves in $\mathbb{R} \times Y$. The second reason is that they contain information about the split Floer homology differential connecting non-constant orbits in $\mathbb{R} \times Y$ with constant orbits in W (see Figure 5.5). Finally, they also describe some broken configurations contributing to the pair-of-pants product, as we will see later. For the moment, let us focus on the second application of relative numbers, to the differential connecting non-constant and constant orbits.

Suppose that the relative number $\mathrm{GW}^{X,\Sigma}_{A,1,(k)}\left(W^u_{f_X}(x_j); W^u_{f_\Sigma}(q_i)\right)$ is non-vanishing, for certain critical points $q_i \in \Sigma$ and $x_j \in X \backslash \Sigma$. Let the pseudo-holomorphic map $w : \mathbb{C}P^1 \to \Sigma$, such that $w(0) \in W^u_{f_\Sigma}(q_i)$ and $w(\infty) \in W^u_{f_X}(x_j)$, contribute to this count. The next result is analogous to Lemma 5.2.

Lemma 5.4. *The map w gives a contribution $(k \ . \ x_j)$ to the split symplectic homology differential of $\check{q}_{i,k}$.*

Proof. For each $b \in S^1$, we have a new pseudo-holomorphic map $\tilde{w}$, such that $\tilde{w}(z) = w(bz)$. Since the symplectic homology differential fixes markers, as in non-equivariant

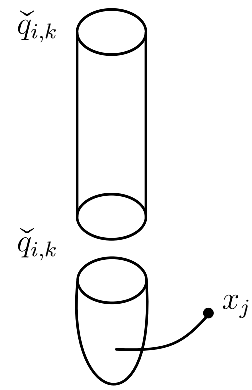

Figure 5.5: The differential $\mathrm{d}_{\vee M}$

contact homology, we need to determine for how many values of $b \in S^1$ we can fix the marker of $\breve{q}_{i,k}$. Since w intersects Σ with order of tangency k, it can be written, up to lower order terms, as

$$z \mapsto (z^k, 0 \ldots, 0).$$

near $z = 0 \in \mathbb{C}P^1$ (on the domain) and near the intersection point (on the target). We now see that we can impose a marker condition on $\breve{q}_{i,k}$ for k values of $b \in S^1$, which implies the result. □

5.3 Floer and symplectic homology via Gromov–Witten theory

We are now ready to write formulas for the differential and product in split symplectic and Floer homologies, in terms of holomorphic curves and gradient flow lines.

5.3.1 Symplectic homology

Assume $f_W : W \to \mathbb{R}$ is a Morse function and $f_\Sigma : \Sigma \to \mathbb{R}$ is a *perfect* Morse function on Σ with critical points

$$\mathrm{Crit}(f_\Sigma) = \{q_1, \ldots, q_m\}.$$

Recall that this means that the Morse differential vanishes. Since Y is a pre-quantization bundle over Σ, then (if M is not an integer),

$$CC_*(Y)^{<M} = \bigoplus_{i=1}^{m} \bigoplus_{k=1}^{\lfloor M \rfloor} \Lambda \langle q_{i,k} \rangle$$

is the truncation of the contact homology chain complex by orbits of period less than M. According to (3.2), the chain complex for split symplectic homology of W is

$$CS_*(W) = \widecheck{CH}_*(Y) \oplus \widehat{CH}_*(Y) \oplus CM_*(-f_W)[-n]$$

where $\widecheck{CH}_*(Y)$ is a copy of the chain complex $CH_*(Y)$ in which we denote the generators as $\check{q}_{i,k}$ and $\widehat{CH}_*(Y)$ is a copy of the chain complex $CH_*(Y)[1]$ (degree shift of $+1$) in which we denote the generators as $\hat{q}_{i,k}$. $CM_*(-f_W)[-n]$ is the Morse complex of $-f_W$ with a degree shift of $-n$.

The differential in $CS_*(W)$ is the 3×3 matrix

$$\mathrm{d} = \begin{pmatrix} 0 & 0 & 0 \\ \mathrm{d}_{\vee\wedge} & 0 & 0 \\ \mathrm{d}_{\vee M} & 0 & \mathrm{d}_M \end{pmatrix}$$

The vanishing of so many terms is justified by index considerations and by the fact that Y has no *bad orbits*, in the sense of [EGH00]. If such orbits existed, then there would be a non-zero term $\mathrm{d}_{\wedge\vee}$, on the first row and second column of the differential matrix (see Lemma 4.28 in [BO09b]). There are no bad orbits in our setting, as a consequence of the index computation in Lemma 3.4. Recall that, according to this result, if we choose trivializations along periodic Reeb orbits by using capping planes that intersect Σ, then the parity of the index of a Reeb orbit does not change if one changes the multiplicity of the orbit.

The term d_M is the Morse differential for the function $-f_W$ on W. The term $\mathrm{d}_{\vee\wedge} : \widecheck{CH}_*(Y) \to \widehat{CH}_*(Y)$ counts cascades of pseudo-holomorphic buildings. It has two types of contributions: from pseudo-holomorphic cylinders in $\mathbb{R} \times Y$ (possibly with punctures capped in W) and from Morse flow lines in the spaces of orbits (see Figure

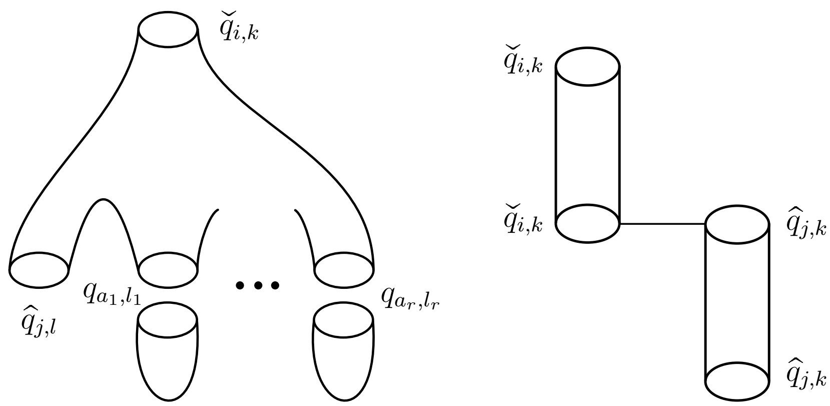

Figure 5.6: The differential $\mathrm{d}_{\vee\wedge}$

5.6). The term $\mathrm{d}_{\vee M} : \widetilde{CH}_*(Y) \to CM_*(-f_W)[-n]$ counts mixed curves connecting Reeb orbits to critical points of f_W (see Figure 5.5).

The term $\mathrm{d}_{\vee\wedge}$ can be written as follows:

$$\begin{aligned}
\mathrm{d}_{\vee\wedge}\, \check{q}_{i,k} = \sum_{q_j \in \mathrm{Crit}(f_\Sigma)} \Big(\sum_{l,r} \sum_{(q_{a_1},\dots,q_{a_r})} \sum_{A \in H_2(\Sigma)} \sum_{A_i \in H_2(X)} & \delta_{d(A),l+\sum l_\alpha - k} \,.\, c(k; l, l_1, \dots, l_r) \,. \\
&. \ \mathrm{GW}^{\Sigma}_{A,r+2}\left(W^u_{f_\Sigma}(q_i), W^s_{f_\Sigma}(q_j), W^s_{f_\Sigma}(q_{a_1}), \dots, W^s_{f_\Sigma}(q_{a_r})\right) \,. \\
&. \ \mathrm{GW}^{X,\Sigma}_{A_1,0,(l_1)}\left(\varnothing; W^u_{f_\Sigma}(q_{a_1})\right) \dots \mathrm{GW}^{X,\Sigma}_{A_r,0,(l_r)}\left(\varnothing; W^u_{f_\Sigma}(q_{a_r})\right) \widehat{q}_{j,l} - \\
&- \left\langle c_1(Y \to \Sigma), W^u_{f_\Sigma}(q_i) \cap W^s_{f_\Sigma}(q_j) \right\rangle \widehat{q}_{j,k} \Big)
\end{aligned}$$

where r indexes the number of augmentations and $d(A) = \langle c_1(Y \to \Sigma), A\rangle$. The Kronecker deltas ensure that we consider meromorphic sections of line bundles with the correct difference of number of zeros and poles. The coefficients $c(k; l, l_1, \dots, l_r)$ were described in Section 5.1.2. The last term in the formula accounts for (Morse–Bott) gradient flow lines in Y connecting $\check{q}_{i,k}$ with $\widehat{q}_{j,k}$.

Remark 5.4. *The assumption that f_Σ is a perfect Morse function simplifies our computations a bit. Otherwise we would need to include additional terms in the formula, namely terms $\mathrm{d}_{\vee\vee}$ and $\mathrm{d}_{\wedge\wedge}$ corresponding to rigid gradient flow lines in Σ (these terms would also appear in the contact homology differential of Y).*

Remark 5.5. *As will be seen in Chapter 6, some contributions to $\mathrm{d}_{\vee\wedge}$ consist of* *terms with more than one end corresponding to the same* $q_i \in \text{Crit}(f_\Sigma)$*. This is the case, for instance, when the relevant Gromov–Witten invariants count curves in class* $0 \in H_2(\Sigma, \mathbb{Z})$ *(we will see this, for example, in the terms* $d\check{m}_{k+1} = 2\widehat{m}_k + \ldots$ *in the differential of symplectic homology of* T^*S^2*). For these curves, the perturbation terms in the Cauchy–Riemann equation on* Σ *are crucial. They imply that one such configuration does not project to a single point in* Σ*, or, put differently, that it is not a cover of a trivial cylinder. This is what allows us to achieve transversality with a single Morse function* f_Σ*, instead of needing distinct functions for different punctures in the domains of our pseudo-holomorphic curves.*

To describe the terms $\mathrm{d}_{\vee M}$, denote first $\text{Crit}(f_W) = \{x_1, \ldots x_{m'}\}$. Then,

$$\mathrm{d}_{\vee M}\, \check{q}_{i,k} = k \;.\; \sum_{x_j \in \text{Crit}(f_W)} \sum_{A \in H_2(X)} \delta_{\#(\Sigma \cap A),k} \;.\; \text{GW}^{X,\Sigma}_{A,1,(k)} \left(W^u_{f_W}(x_j); W^u_{f_\Sigma}(q_i) \right) x_j$$

The presence of the coefficient k was justified in Lemma 5.4. The only relevant homology classes A are those in the image of the map $H_2(X \backslash \Sigma) \to H_2(X)$. The Kronecker delta forces the pseudo-holomorphic curves in X to intersect Σ at only one point, with order of tangency k. For a schematic representation, recall Figure 5.5.

There is a similar description of the pair-of-pants product. Products involving only non-constant orbits are given not just by rigid pseudo-holomorphic pairs-of-pants in $\mathbb{R} \times Y$, but also by certain *broken configurations*.

We begin with a description of (possibly augmented) pairs-of-pants in $\mathbb{R} \times Y$ (see Figure 5.7). Since $\mathbb{C}P^1 \backslash \{0, 1, \infty\}$ has no automorphisms, we can only fix one marker on a pseudo-holomorphic map $\mathbb{C}P^1 \backslash \{0, 1, \infty\} \to \mathbb{R} \times Y$, using the fact that J is S^1-invariant (contrary to the case of holomorphic cylinders, in which we could fix two markers). As a consequence, in the product we don't need to consider analogues of the coefficients $c(k; l, l_1, \ldots, l_r)$ that appeared in the differential (recall Definition 5.1). Suppose that we are interested in the product of $\check{q}_{i_1,k_1}$ by $\widehat{q}_{i_2,k_2}$, for example. Then, a rigid map $\left(\mathbb{C}P^1 \backslash \{0, 1, \infty\}\right) \backslash \{\text{augmentation punctures}\} \to \mathbb{R} \times Y$, lifting a pseudo-holomorphic sphere that contributes to a $(k+3)$-point Gromov–Witten number of Σ,

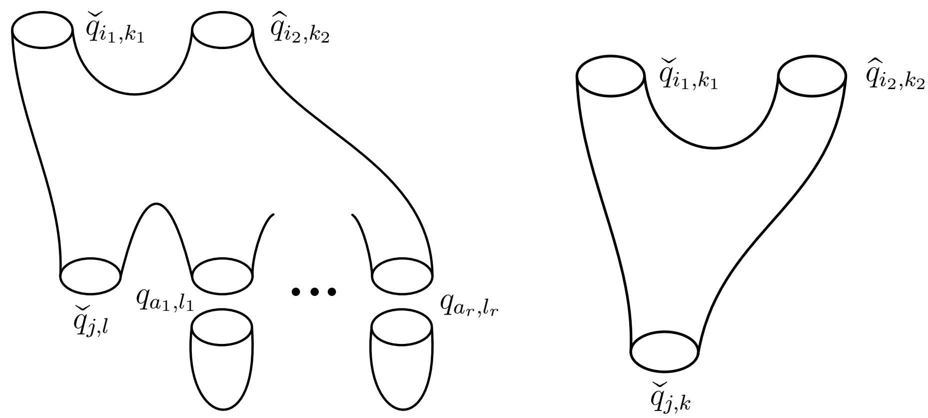

Figure 5.7: Augmented and non-augmented pairs-of-pants contributing to the product

would give

$$\begin{aligned}\check{q}_{i_1,k_1} \circ \widehat{q}_{i_2,k_2} = \sum_{q_j \in \mathrm{Crit}(f_\Sigma)} \sum_{l,r} \sum_{(q_{a_1},\ldots,q_{a_r})} \sum_{A \in H_2(\Sigma)} \sum_{A_i \in H_2(X)} \delta_{d(A),l+\sum l_\alpha - k_1 - k_2} \cdot \\ \cdot \ \mathrm{GW}^{\Sigma}_{A,r+3}\left(W^u_{f_\Sigma}(q_{i_1}), W^u_{f_\Sigma}(q_{i_2}), W^s_{f_\Sigma}(q_j), W^s_{f_\Sigma}(q_{a_1}), \ldots, W^s_{f_\Sigma}(q_{a_r})\right) \cdot \\ \cdot \ \mathrm{GW}^{X,\Sigma}_{A_1,0,(l_1)}\left(\varnothing; W^u_{f_\Sigma}(q_{a_1})\right) \ldots \mathrm{GW}^{X,\Sigma}_{A_r,0,(l_r)}\left(\varnothing; W^u_{f_\Sigma}(q_{a_r})\right) \check{q}_{j,l} + \ldots\end{aligned}$$

where $d(A) = \langle c_1(Y \to \Sigma), A \rangle$. As in the case of the differential, the Kronecker deltas keep track of zeros and poles of meromorphic sections of line bundles over $\mathbb{C}P^1$. There are similar contributions in the case when we multiply two generators in $\widehat{CH}$, and the output is in $\widehat{CH}$ (in which case the marker is fixed on the output orbit).

In the case when there are no augmentation planes, which often happens for degree reasons, we get

$$\begin{aligned}\check{q}_{i_1,k_1} \circ \widehat{q}_{i_2,k_2} = \sum_{q_j \in \mathrm{Crit}(f_\Sigma)} \sum_{l} \sum_{A \in H_2(\Sigma)} \delta_{d(A),l-k_1-k_2} \cdot \\ \cdot \ \mathrm{GW}^{\Sigma}_{A,3}\left(W^u_{f_\Sigma}(q_{i_1}), W^u_{f_\Sigma}(q_{i_2}), W^s_{f_\Sigma}(q_j)\right) \check{q}_{j,l} + \ldots\end{aligned}$$

There are also *broken configurations* contributing to the product. These are represented in Figure 5.8. They have several components: a cylinder in $\mathbb{R} \times Y$ that connects to another cylinder in W, with one removable singularity. At this point, it connects

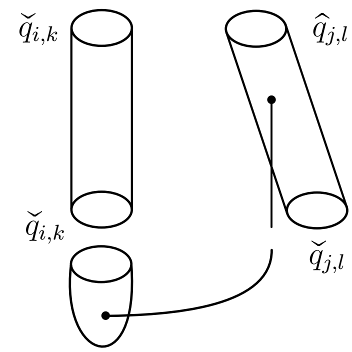

Figure 5.8: Broken pair-of-pants

to a gradient flow line of f_W that escapes to infinity and continues as a vertical line in $\mathbb{R}\times Y$. This vertical line intersects a pair-of-pants at a removable singularity. In principle, both holomorphic components in $\mathbb{R}\times Y$ could be something other than (covers of) non-trivial cylinders. But we will focus on the case when they are, which is the one relevant for our applications in Chapter 6. Note that in Figure 5.8 we drew the pair-of-pants with removable singularity in a slightly slanted manner. This is because the pair-of-pants has a stable domain, so our perturbation scheme from Section 4.2 implies that we get a lift of a perturbation of a constant map to Σ, in other words a perturbation of a cover of a trivial cylinder. These broken configurations can be counted as follows:

$$\check{q}_{i,k} \circ \check{q}_{j,l} = \sum_{q_j \in \mathrm{Crit}(f_\Sigma)} \sum_{l} \sum_{A \in H_2(\Sigma)} \delta_{\#(\Sigma\cap A),k} \cdot \mathrm{GW}^{X,\Sigma}_{A,1,(i)} \left(W^u_{f_W}(q_j); W^u_{f_\Sigma}(q_i)\right) \check{q}_{j,l} + \dots$$

The pseudo-holomorphic curves contributing to the relative Gromov–Witten number describe the pseudo-holomorphic planes in W, which are actually cylinders with a removable singularity, and the stable manifold for f_W contains the gradient flow lines going from the removable singularities to infinity in W. The Kronecker delta selects pseudo-holomorphic planes with only one puncture, asymptotic to a Reeb orbit of period k (or, equivalently, pseudo-holomorphic spheres in X with only one intersection with Σ, of order k).

Remark 5.6. *At this point, we should make a comment similar to that of Remark*

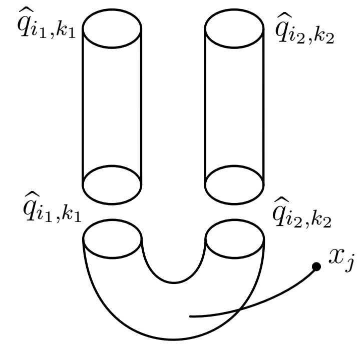

Figure 5.9: Constant orbit contributing to the product of two non-constant orbits

5.5. The fact that the pair-of-pants in $\mathbb{R} \times Y$ with removable singularity is only a perturbation of a cover of a trivial cylinder is crucial to achieve transversality of the evaluation maps at the punctures, for a single Morse function f_Σ.

The product of two non-constant orbits can also have a contribution from constant orbits (illustrated in Figure 5.9). The relevant part of these configurations is contained in W, where there is no J-preserving S^1-action. Therefore, no markers can be fixed. We get

$$\widehat{q}_{i_1,k_1} \circ \widehat{q}_{i_2,k_2} = \sum_{x_j \in \mathrm{Crit}(f_W)} \sum_{A \in H_2(X)} \delta_{\#(\Sigma \cap A), k_1+k_2} \cdot$$
$$\cdot \ \mathrm{GW}^{X,\Sigma}_{A,1,(k_1,k_2)} \left(W^u_{f_W}(x_j); W^u_{f_\Sigma}(q_{i_1}), W^u_{f_\Sigma}(q_{i_2}) \right) x_j + \dots$$

The product of critical points should coincide with the Morse chain level product. Usually, to define this operation, one needs more than one Morse function, for transversality reasons. This can be avoided at the expense of replacing gradient flow trees with gradient flow lines of one fixed Morse function, connected to a central (perturbed) pseudo-holomorphic curve.

These formulas will be used in Chapter 6, to compute the ring structure on symplectic homology groups of spheres.

5.3.2 Floer homology

The split Floer homology of an S-shaped Hamiltonian in X has a similar description. Let again $f_W : W \to \mathbb{R}$, $f_\Sigma : \Sigma \to \mathbb{R}$ and $f_{N\Sigma} : N\Sigma \to \mathbb{R}$ be Morse functions.

According to 3.1, the chain complex for split Floer homology of X is

$$\begin{aligned} HF_*(X) = \widetilde{CH}_*^{cvx}(Y) \oplus \widehat{CH}_*^{cvx}(Y) \oplus CM_*(-f_W)[-n] \oplus \\ \oplus \widetilde{CH}_*^{cve}(Y) \oplus \widehat{CH}_*^{cve}(Y) \oplus CM_*(-f_{N\Sigma})[-n] \end{aligned}$$

where the superscripts *cvx* and *cve* refer to the periodic orbits in the region where the Hamiltonian is convex and concave, respectively. The degree of a concave generator is 1 more than that of the corresponding convex generator. This complex is generated over the Novikov ring $\mathbb{Z}[t, t^{-1}]$. The Novikov variable t counts the number of positive punctures on split Floer differentials, capped by fibers of $N\Sigma \to \Sigma$.

One can now write a 6×6 matrix representing the split Floer differential, whose entries are explicitly described by absolute and relative Gromov–Witten numbers, as well as some Morse-theoretic information. There is a description of the product that is similar to the one given above for symplectic homology. An important difference when the manifold is closed (and thus not exact) is that there might be product terms that involve only critical points and that contain (broken) pseudo-holomorphic spheres whose energy is not small (which means that they intersect Σ). This is because the ring structure on Floer homology is isomorphic to quantum cohomology.

Chapter 6

The example of cotangent bundles of spheres

We will restrict our attention to pairs $(X, \Sigma) = (Q_n, Q_{n-1})$, where Q_n is the n-dimensional complex projective quadric. $Q_n \backslash Q_{n-1}$ is the n-dimensional affine complex quadric. When Q_n is equipped with the restriction of the Fubini-Study symplectic form that generates $H^2(\mathbb{C}P^{n+1}; \mathbb{Z})$, then $Q_n \backslash Q_{n-1}$ is symplectomorphic to the unit cotangent bundle of S^n. Therefore, W, the completion of $Q_n \backslash Q_{n-1}$, is symplectomorphic to T^*S^n (see Exercise 6.20 in [MS98]). We will use the results of the previous chapters to compute the symplectic homology rings $SH_*(T^*S^n)$, for $n > 1$.

As we will see in Proposition 6.1, Q_n is a monotone manifold, with $\lambda_{Q_n} = 1/n$, if $n > 1$, and $\lambda_{Q_1} = 1/2$. Also, $Q_{n-1} = \mathrm{PD}(\omega)$, if $n > 1$, so $K = 1$.

6.1 T^*S^2

The one and two dimensional quadrics Q_1, Q_2 are isomorphic to $\mathbb{C}P^1$ and $\mathbb{C}P^1 \times \mathbb{C}P^1$, respectively, as Kähler manifolds. Q_1 includes into Q_2 as the diagonal embedding $\Delta : \mathbb{C}P^1 \hookrightarrow \mathbb{C}P^1 \times \mathbb{C}P^1$. In this case, Y is diffeomorphic to $\mathbb{R}P^3$, and the Reeb flow corresponds to the flow along the fibers of the bundle $S^1 \to \mathbb{R}P^3 \to S^2$, of Chern class -2 (see Remark 3.2 above and Proposition 6.1, below).

6.1.1 Relevant Gromov–Witten numbers

Let us now compile some Gromov–Witten numbers that will be relevant for our computations. We will use the integrable complex structures in $\mathbb{C}P^1$ and $\mathbb{C}P^1 \times \mathbb{C}P^1$, with respect to which the moduli spaces of holomorphic spheres are transverse (see Lemma 3.3.1 and Example 3.3.6 in [MS04]). Call L the generator of $H_2(\mathbb{C}P^1;\mathbb{Z})$, corresponding to the complex orientation. Using the divisor equation, we have

$$\mathrm{GW}_{L,2}^{\mathbb{C}P^1}(pt, pt) = \frac{1}{\#(pt \cap L)} \mathrm{GW}_{L,3}^{\mathbb{C}P^1}(pt, pt, pt) = 1.$$

This is because there is a unique holomorphic curve in class L, mapping 0, 1 and ∞ to three generic points in $\mathbb{C}P^1$. Since (unperturbed) pseudo-holomorphic curves in homology class $0 \in H_2(\mathbb{C}P^1;\mathbb{Z})$ are constant, we have

$$\mathrm{GW}_{0L,3}^{\mathbb{C}P^1}\left(pt, \mathbb{C}P^1, \mathbb{C}P^1\right) = 1.$$

We also need some relative Gromov–Witten numbers of the pair $(\mathbb{C}P^1 \times \mathbb{C}P^1, \Delta)$. Denote by L_1 and L_2 the generators of $H_2(\mathbb{C}P^1 \times \mathbb{C}P^1;\mathbb{Z})$ corresponding to holomorphic spheres on each of the factors. Then

$$\mathrm{GW}_{L_i,0,(1)}^{\mathbb{C}P^1 \times \mathbb{C}P^1, \Delta}(\varnothing; pt) = \mathrm{GW}_{L_i,1}^{\mathbb{C}P^1 \times \mathbb{C}P^1}(pt) = 1$$

for $i = 1, 2$ (the point constraint is in Δ, not in $\mathbb{C}P^1 \times \mathbb{C}P^1$). This is because if one fixes a point $p \in \mathbb{C}P^1 \times \mathbb{C}P^1$, then there is exactly one holomorphic sphere in class L_i that goes through p.

We also have

$$\mathrm{GW}_{L_i,1,(1)}^{\mathbb{C}P^1 \times \mathbb{C}P^1, \Delta}(L_j; pt) = \mathrm{GW}_{L_i,2}^{\mathbb{C}P^1 \times \mathbb{C}P^1}(L_j, pt) = 1 - \delta_{i,j}.$$

for $i, j = 1, 2$. This means, for instance, that there is a unique vertical sphere in $\mathbb{C}P^1 \times \mathbb{C}P^1$ intersecting a horizontal sphere and a generic point in $\mathbb{C}P^1 \times \mathbb{C}P^1$, but that there is no horizontal sphere intersecting another horizontal sphere and a generic point in $\mathbb{C}P^1 \times \mathbb{C}P^1$. This is because two different horizontal spheres do not intersect.

The divisor equation also implies that

$$\mathrm{GW}^{\mathbb{C}P^1\times\mathbb{C}P^1}_{L_i,3}(pt, L_j, L_k) = \#(L_i \cap L_k)\,\mathrm{GW}^{\mathbb{C}P^1\times\mathbb{C}P^1}_{L_i,2}(pt, L_j) = (1-\delta_{i,k})(1-\delta_{i,j})$$

which will end up being useful in the study of the symplectic homology of T^*S^3.

Important are also the numbers

$$\begin{aligned}\mathrm{GW}^{\mathbb{C}P^1\times\mathbb{C}P^1,\Delta}_{L_i,1,(1)}(pt;\Delta) &= \mathrm{GW}^{\mathbb{C}P^1\times\mathbb{C}P^1}_{L_i,2}(pt,\Delta) = \mathrm{GW}^{\mathbb{C}P^1\times\mathbb{C}P^1}_{L_i,2}(pt, L_1+L_2) = \\ &= \mathrm{GW}^{\mathbb{C}P^1\times\mathbb{C}P^1}_{L_i,2}(pt, L_1) + \mathrm{GW}^{\mathbb{C}P^1\times\mathbb{C}P^1}_{L_i,2}(pt, L_2) = 1\end{aligned}$$

for $i = 1, 2$. This expresses the fact that, if we fix a generic point $p \in \mathbb{C}P^1 \times \mathbb{C}P^1$, then there is a unique holomorphic sphere in class L_i that goes through p and intersects Δ.

Finally, we have the following.

Lemma 6.1. *1.* $\mathrm{GW}^{\mathbb{C}P^1\times CP^1,\Delta}_{L_1+L_2,1,(2)}(pt;pt) = 1$*;*

2. $\mathrm{GW}^{\mathbb{C}P^1\times\mathbb{C}P^1,\Delta}_{L_1+L_2,1,(1,1)}(pt;pt,pt) = \mathrm{GW}^{\mathbb{C}P^1\times\mathbb{C}P^1}_{L_1+L_2,3}(pt,pt,pt) = 1$*;*

3. $\mathrm{GW}^{\mathbb{C}P^1\times\mathbb{C}P^1,\Delta}_{2L_i,1,(1,1)}(pt;pt,pt) = \mathrm{GW}^{\mathbb{C}P^1\times\mathbb{C}P^1}_{2L_i,3}(pt,pt,pt) = 0$*, for* $i = 1, 2$*.*

Proof. Even though there are effective ways of expressing relative Gromov–Witten numbers in terms of absolute numbers, we will compute these explicitly, using the fact that the integrable complex structure is generic enough.

For the first one, it is enough to show that, for instance, there is a unique holomorphic map $f : \mathbb{C}P^1 \to \mathbb{C}P^1 \times \mathbb{C}P^1$, such that $[f] = L_1 + L_2 \in H_2(\mathbb{C}P^1 \times \mathbb{C}P^1;\mathbb{Z})$, $f(\infty) = (\infty, 1)$, $f(0) = (0, 0)$, and such that f intersects the diagonal $\Delta \subset \mathbb{C}P^1 \times \mathbb{C}P^1$ in a non-transverse way. Since

$$f(\infty) = f\big([1;0]\big) = (\infty, 1) = \big([1;0],[1;1]\big),$$

we can write in homogeneous coordinates

$$f\big([z;1]\big) = \big([az+b;1],[z+c;z+d]\big)$$

which we will abbreviate as

$$f(z) = \left(az + b, \frac{z+c}{z+d}\right).$$

Since $f(0) = (0,0)$, we get $b = c = 0$, and so $f(z) = \big(az, z/(z+d)\big)$. Now, for the tangency condition, note that

$$f'(z) = \left(a, \frac{d}{(z+d)^2}\right)$$

and so $f'(0) = \big(a, 1/d\big)$. So f is tangent to the diagonal at $(0,0)$ precisely when $a = 1/d$. Therefore, our space of maps f can be identified with the space of $a \in \mathbb{C}^*$. Taking a quotient by the group of automorphisms of the domain $(\mathbb{C}P^1, \{0,\infty\})$, which is also $\mathbb{C}^*$, we get $\mathrm{GW}^{\mathbb{C}P^1 \times CP^1, \Delta}_{L_1+L_2,1,(2)}(pt; pt) = 1$, as wanted.

Now, to show that $\mathrm{GW}^{\mathbb{C}P^1 \times \mathbb{C}P^1}_{L_1+L_2,3}(pt, pt, pt) = 1$, it is enough to show that, for instance, there is a unique holomorphic map $f : \mathbb{C}P^1 \to \mathbb{C}P^1 \times \mathbb{C}P^1$ such that $[f] = L_1 + L_2$, $f(\infty) = (\infty, 1)$, $f(0) = (0,0)$ and $f(1) = (3,4)$. As we already saw, the first two point constraints imply that $f(z) = \big(az, z/(z+d)\big)$. Now,

$$f(1) = \left(a, \frac{1}{1+d}\right) = (3,4)$$

implies that $a = 3$ and $d = -3/4$. So, f is uniquely specified, which shows that the required Gromov–Witten number is 1.

Finally, the fact that $\mathrm{GW}^{\mathbb{C}P^1 \times \mathbb{C}P^1}_{2L_i,3}(pt, pt, pt) = 0$, for $i = 1, 2$, just encodes the fact holomorphic curves in classes $2L_i \in H_2(\mathbb{C}P^1 \times \mathbb{C}P^1; \mathbb{Z})$ are covers of either vertical or horizontal spheres, and therefore cannot go through three generic points in $\mathbb{C}P^1 \times \mathbb{C}P^1$. □

6.1.2 The group $SH_*(T^*S^2)$

$\Sigma = \mathbb{C}P^1$ admits a perfect Morse function with two critical points. Call the minimum m and the maximum M. $W = T^*S^2$ has a Morse function that grows at infinity and has two critical points, both located on the zero section. Call the minimum e and the saddle point c. Let this be the function f_W of Section 5.3.1. The split symplectic homology chain complex for a J-shaped Hamiltonian in $\mathbb{R} \times \mathbb{R}P^3$ is therefore

$$SC_*(T^*S^2) = \mathbb{Z}\left\langle e, c, \check{m}_k, \hat{m}_k, \widecheck{M}_k, \widehat{M}_k \right\rangle$$

where we take all integers $k > 0$.

We can apply the results in the previous chapter to compute the differential.

$$\begin{aligned}
\mathrm{d}\check{m}_{k+1} &= \mathrm{d}_{\vee\wedge}\check{m}_{k+1} + \mathrm{d}_{\vee M}\check{m}_{k+1} = \mathrm{d}_{\vee\wedge}\check{m}_{k+1} = \\
&= \Big(c(k+1;k,1) \;.\; \mathrm{GW}^{\mathbb{C}P^1}_{0L,3}(pt, \mathbb{C}P^1, \mathbb{C}P^1) \;.\; \mathrm{GW}^{\mathbb{C}P^1\times\mathbb{C}P^1,\Delta}_{L_1,0,(1)}(\varnothing; pt) + \\
&\qquad + c(k+1;k,1) \;.\; \mathrm{GW}^{\mathbb{C}P^1}_{0L,3}(pt, \mathbb{C}P^1, \mathbb{C}P^1) \;.\; \mathrm{GW}^{\mathbb{C}P^1\times\mathbb{C}P^1,\Delta}_{L_2,0,(1)}(\varnothing; pt) \Big) \,\hat{m}_k + \\
&\qquad + c(k+1;k-1) \;.\; \mathrm{GW}^{\mathbb{C}P^1}_{L,2}(pt, pt)\, \widehat{M}_{k-1} = \\
&= 2\,\hat{m}_k + 2\,\widehat{M}_{k-1}
\end{aligned}$$

$$\begin{aligned}
\mathrm{d}\widecheck{M}_k &= \mathrm{d}_{\vee\wedge}\widecheck{M}_k + \mathrm{d}_{\vee M}\widecheck{M}_k = \mathrm{d}_{\vee\wedge}\widecheck{M}_k = \\
&= \Big(c(k;k-1,1) \;.\; \mathrm{GW}^{\mathbb{C}P^1}_{0L,3}(pt, \mathbb{C}P^1, \mathbb{C}P^1) \;.\; \mathrm{GW}^{\mathbb{C}P^1\times\mathbb{C}P^1,\Delta}_{L_1,0,(1)}(\varnothing; pt) + \\
&\qquad + c(k;k-1,1) \;.\; \mathrm{GW}^{\mathbb{C}P^1}_{0L,3}(pt, \mathbb{C}P^1, \mathbb{C}P^1) \;.\; \mathrm{GW}^{\mathbb{C}P^1\times\mathbb{C}P^1,\Delta}_{L_2,0,(1)}(\varnothing; pt) \Big) \,\widehat{M}_{k-1} - \\
&\qquad - c_1(\mathbb{R}P^3 \to \mathbb{C}P^1)(\mathbb{C}P^1)\, \hat{m}_k = \\
&= 2\,\widehat{M}_{k-1} + 2\,\hat{m}_k
\end{aligned}$$

$$
\begin{aligned}
\mathrm{d}\check{m}_2 &= \mathrm{d}_{\vee\wedge}\check{m}_2 + \mathrm{d}_{\vee M}\check{m}_2 = \\
&= \Big(c(2;1,1)\ .\ \mathrm{GW}^{\mathbb{C}P^1}_{0L,3}(pt,\mathbb{C}P^1,\mathbb{C}P^1)\ .\ \mathrm{GW}^{\mathbb{C}P^1\times\mathbb{C}P^1,\Delta}_{L_1,0,(1)}(\varnothing;pt) + \\
&\quad + c(2;1,1)\ .\ \mathrm{GW}^{\mathbb{C}P^1}_{0L,3}(pt,\mathbb{C}P^1,\mathbb{C}P^1)\ .\ \mathrm{GW}^{\mathbb{C}P^1\times\mathbb{C}P^1,\Delta}_{L_2,0,(1)}(\varnothing;pt)\Big)\,\widehat{m}_1 + \\
&\quad + \mathrm{GW}^{\mathbb{C}P^1\times\mathbb{C}P^1,\Delta}_{L_1+L_2,1,(2)}(pt;pt)\,e = \\
&= 2\,\widehat{m}_1 + 2\,e
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{d}\breve{M}_1 &= \mathrm{d}_{\vee\wedge}\breve{M}_1 + \mathrm{d}_{\vee M}\breve{M}_1 = \\
&= c_1(\mathbb{R}P^3 \to \mathbb{C}P^1)(\mathbb{C}P^1)\,\widehat{m}_1 + \\
&\quad + \Big(\mathrm{GW}^{\mathbb{C}P^1\times\mathbb{C}P^1,\Delta}_{L_1,1,(1)}(pt;\Delta) + \mathrm{GW}^{\mathbb{C}P^1\times\mathbb{C}P^1,\Delta}_{L_2,1,(1)}(pt;\Delta)\Big)\,e = \\
&= 2\,\widehat{m}_1 + 2\,e
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{d}\check{m}_1 &= \mathrm{d}_{\vee\wedge}\check{m}_1 + \mathrm{d}_{\vee M}\check{m}_1 = \mathrm{d}_{\vee M}\check{m}_1 = \\
&= \Big(\mathrm{GW}^{\mathbb{C}P^1\times\mathbb{C}P^1,\Delta}_{L_1,1,(1)}\big(\overline{W^u_{f_W}(c)};pt\big) + \mathrm{GW}^{\mathbb{C}P^1\times\mathbb{C}P^1,\Delta}_{L_2,1,(1)}\big(\overline{W^u_{f_W}(c)};pt\big)\Big)\,c = \\
&= \Big(\mathrm{GW}^{\mathbb{C}P^1\times\mathbb{C}P^1,\Delta}_{L_1,1,(1)}(S^2;pt) + \mathrm{GW}^{\mathbb{C}P^1\times\mathbb{C}P^1,\Delta}_{L_2,1,(1)}(S^2;pt)\Big)\,c = \\
&= \Big(\mathrm{GW}^{\mathbb{C}P^1\times\mathbb{C}P^1,\Delta}_{L_1,1,(1)}(L_1-L_2;pt) + \mathrm{GW}^{\mathbb{C}P^1\times\mathbb{C}P^1,\Delta}_{L_2,1,(1)}(L_1-L_2;pt)\Big)\,c = \\
&= \Big(\mathrm{GW}^{\mathbb{C}P^1\times\mathbb{C}P^1,\Delta}_{L_1,1,(1)}(L_1;pt) - \mathrm{GW}^{\mathbb{C}P^1\times\mathbb{C}P^1,\Delta}_{L_1,1,(1)}(L_2;pt) + \\
&\quad + \mathrm{GW}^{\mathbb{C}P^1\times\mathbb{C}P^1,\Delta}_{L_2,1,(1)}(L_1;pt) - \mathrm{GW}^{\mathbb{C}P^1\times\mathbb{C}P^1,\Delta}_{L_2,1,(1)}(L_2;pt)\Big)\,c = \\
&= (0-1+1-0)\,c = 0
\end{aligned}
$$

where we assume that $S^2 \subset D^*S^2 \hookrightarrow \mathbb{C}P^1 \times \mathbb{C}P^1$ is oriented so as to define the homology class $(1,-1) \in H_2(\mathbb{C}P^1 \times \mathbb{C}P^1;\mathbb{Z})$ (the opposite choice of orientation would also give a vanishing result).

Summing up, we have

$$\begin{cases} \mathrm{d}\check{m}_{k+1} = 2\,\widehat{m}_k + 2\,\widehat{M}_{k-1} \\ \mathrm{d}\widecheck{M}_k = 2\widehat{m}_k + 2\,\widehat{M}_{k-1} \\ \mathrm{d}\check{m}_2 = 2\,\widehat{m}_1 + 2\,e \\ \mathrm{d}\widecheck{M}_1 = 2\,\widehat{m}_1 + 2\,e \end{cases}$$

for $k \geqslant 2$. Therefore,

$$SH_*(T^*S^2;\mathbb{Z}) = \mathbb{Z}\left\langle c, \check{m}_1, e, \widecheck{M}_k - \check{m}_{k+1}, \widehat{M}_k \right\rangle \oplus \mathbb{Z}/2\left\langle e + \widehat{m}_1, \widehat{M}_k + \widehat{m}_{k+1} \right\rangle$$

taking all $k \geqslant 1$.

We can compare these results with the computations of Cohen-Jones-Yan, which we recalled in Section 2.3. The indices on $SH_*(T^*S^2)$ can be computed using Lemma 3.4 and the fact that $K = 1$ and $\lambda_{Q_2} = 1/2$. The following tables show how our computations match the ones from algebraic topology. For the free part, we get

$SH_d(T^*S^2)$	c	$-\check{m}_1$	e	$\widecheck{M}_k - \check{m}_{k+1}$	$\widehat{M}_k$
$H_d(LS^2)$	a	b	1	bv^k	v^k
d	0	1	2	$2k+1$	$2k+2$

for $k \geqslant 1$. The choice of signs is motivated from the study of ring structure, which will be described below. For the $\mathbb{Z}/2$-torsion part:

$SH_d(T^*S^2)$	$e + \widehat{m}_1$	$\widehat{M}_k + \widehat{m}_{k+1}$
$H_d(LS^2)$	av	av^{k+1}
d	2	$2k+2$

for $k \geqslant 1$.

6.1.3 The ring $SH_*(T^*S^2)$

We now compute the pair-of-pants product on $SH_*(T^*S^2)$. To get the same result as in [CJY04], we need to show that

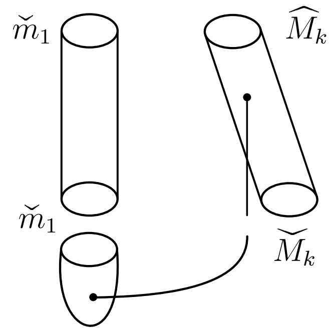

Figure 6.1: Broken pair-of-pants on T^*S^2

1. a annihilates everything except for 1 and for the fact that $a \circ v^k = av^k$;
2. $e = 1$ is the unit;
3. $b^2 = b \circ (bv^k) = (bv^k) \circ (bv^l) = 0$;
4. $b \circ v^k = bv^k$;
5. $v^k \circ v^l = v^{k+l}$;
6. $(bv^k) \circ v^l = bv^{k+l}$;
7. $(av^k) \circ (av^l) = 0$;
8. $(av^k) \circ v^l = av^{k+l}$.

The product of two orbits in $\widecheck{CH}$ is zero, since one cannot fix two markers on a pair-of-pants. This is the reason behind (3).

In the following, 3-point absolute Gromov–Witten numbers correspond to pseudo-holomorphic pairs-of-pants in $\mathbb{R} \times \mathbb{R}P^3$, whereas 2-point relative Gromov–Witten numbers correspond to either broken pairs-of-pants (as in the example of Figure 6.1) or to critical points contributing to the product of non-constant orbits, as explained in Section 5.3.1. The broken configurations are counted with a *negative sign*.

The product contains the pieces illustrated in Figure 6.1, and analogous ones with m_k in place of M_k. In order to apply the formulas from Section 5.3.1, which describe them in terms of Gromov–Witten numbers, we need to determine $W^u_{f_W}(p)$ for a generic point $p \in \Delta$. Such an unstable manifold defines a cycle $aL_1 + bL_2 \in H_2(\mathbb{C}P^1 \times \mathbb{C}P^1; \mathbb{Z})$ that intersects Σ precisely at p. This implies that

$$(aL_1 + bL_2) \ . \ (L_1 + L_2) = a + b = 1. \tag{6.1}$$

On the other hand, H is a deformation of a Morse–Bott function that grows radially on the fibers of T^*S^2. Since the point $x \in \Sigma$ corresponds to a simple orbit of the geodesic flow on S^2, the Morse–Bott manifold of p is given by one hemisphere of the zero section, glued to a copy of $S^1 \times [0, \infty)$ along the meridian in the zero section that corresponds to the orbit of the geodesic flow. The first factor on $S^1 \times [0, \infty)$ goes around that meridian and the second factor grows radially on the fibers. Such Morse–Bott manifold perturbs to a disk that intersects the zero section only once, say at the minimum e. Since the zero section defines $L_1 - L_2 \in H_2(\mathbb{C}P^1 \times \mathbb{C}P^1; \mathbb{Z})$ (for a certain choice of orientation), we have

$$(aL_1 + bL_2) \ . \ (L_1 - L_2) = -a + b = 1. \tag{6.2}$$

From (6.1) and (6.2), we conclude that $(a, b) = (0, 1)$, and that $aL_1 + bL_2 = L_2$.

We will not focus on the product formulas involving critical points, but check a direct consequence of (1), namely that $b \circ (av) = 0$:

$$\begin{aligned} b \circ (av) &= (-\check{m}_1) \circ (e + \hat{m}_1) = \\ &= -\check{m}_1 + \Big(\mathrm{GW}^{\mathbb{C}P^1 \times \mathbb{C}P^1, \Delta}_{L_1,1,(1)}(L_2; pt) + \mathrm{GW}^{\mathbb{C}P^1 \times \mathbb{C}P^1, \Delta}_{L_2,1,(1)}(L_2; pt) \Big) \check{m}_1 = \\ &= -\check{m}_1 + \check{m}_1 = 0 \end{aligned}$$

as wanted. For (4), observe that

$$\begin{aligned} b \circ v^k &= (-\check{m}_1) \circ \widehat{M}_k = \\ &= -\, \mathrm{GW}^{\mathbb{C}P^1}_{0L,3}(pt, \mathbb{C}P^1, \mathbb{C}P^1)\, \check{m}_{k+1} + \\ &\qquad + \Big(\mathrm{GW}^{\mathbb{C}P^1 \times \mathbb{C}P^1, \Delta}_{L_1,1,(1)}(L_2; pt) + \mathrm{GW}^{\mathbb{C}P^1 \times \mathbb{C}P^1, \Delta}_{L_2,1,(1)}(L_2; pt) \Big)\, \widecheck{M}_k = \\ &= -\check{m}_{k+1} + \widecheck{M}_k = bv^k. \end{aligned}$$

For (5), we have

$$\begin{aligned} v^k \circ v^l &= \widehat{M}_k \circ \widehat{M}_l = \mathrm{GW}_{0L,3}^{\mathbb{C}P^1}(\mathbb{C}P^1, \mathbb{C}P^1, pt)\, \widehat{M}_{k+l} = \\ &= \widehat{M}_{k+l} = v^{k+l}. \end{aligned}$$

For (6):

$$\begin{aligned} (bv^k) \circ v^l &= \left(\widecheck{M}_k - \check{m}_{k+1}\right) \circ \widehat{M}_l = \\ &= \mathrm{GW}_{0L,3}^{\mathbb{C}P^1}(\mathbb{C}P^1, \mathbb{C}P^1, pt)\, \widecheck{M}_{k+l} - \mathrm{GW}_{0L,3}^{\mathbb{C}P^1}(pt, \mathbb{C}P^1, \mathbb{C}P^1)\, \check{m}_{k+l+1} = \\ &= \widecheck{M}_{k+l} - \check{m}_{k+l+1} = bv^{k+l} \end{aligned}$$

For (7), there are three cases to consider: when $k = l = 1$

$$\begin{aligned} (av) \circ (av) &= (e + \hat{m}_1) \circ (e + \hat{m}_1) = \\ &= e \circ e + e \circ \hat{m}_1 + \hat{m}_1 \circ e + \hat{m}_1 \circ \hat{m}_1 = \\ &= e + \hat{m}_1 + \hat{m}_1 + \mathrm{GW}_{L_1+L_2,1,(1,1)}^{\mathbb{C}P^1 \times \mathbb{C}P^1, \Delta}(pt; pt, pt)\, e = \\ &= 2\, e + 2\, \hat{m}_1 = 0 \end{aligned}$$

When $k > 1$:

$$\begin{aligned} (av^k) \circ (av) &= (\widehat{M}_{k-1} + \hat{m}_k) \circ (e + \hat{m}_1) = \\ &= \widehat{M}_{k-1} \circ e + \widehat{M}_{k-1} \circ \hat{m}_1 + \hat{m}_k \circ e + \hat{m}_k \circ \hat{m}_l = \\ &= \widehat{M}_{k-1} + \mathrm{GW}_{0L,3}^{\mathbb{C}P^1}(\mathbb{C}P^1, pt, \mathbb{C}P^1)\, \hat{m}_k + \hat{m}_k + \mathrm{GW}_{L,3}^{\mathbb{C}P^1}(pt, pt, pt)\, \widehat{M}_{k-1} = \\ &= 2\, \widehat{M}_{k-1} + 2\, \hat{m}_k = 0 \end{aligned}$$

Finally, when $k, l > 1$:

$$\begin{aligned}(av^k) \circ (av^l) &= (\widehat{M}_{k-1} + \hat{m}_k) \circ (\widehat{M}_{l-1} + \hat{m}_l) = \\ &= \widehat{M}_{k-1} \circ \widehat{M}_{l-1} + \widehat{M}_{k-1} \circ \hat{m}_l + \hat{m}_k \circ \widehat{M}_{l-1} + \hat{m}_k \circ \hat{m}_1 = \\ &= \mathrm{GW}^{\mathbb{C}P^1}_{0L,3}(\mathbb{C}P^1, \mathbb{C}P^1, pt)\, \widehat{M}_{k+l-2} + \mathrm{GW}^{\mathbb{C}P^1}_{0L,3}(\mathbb{C}P^1, pt, \mathbb{C}P^1)\, \hat{m}_{k+l-1} + \\ &\qquad + \mathrm{GW}^{\mathbb{C}P^1}_{0L,3}(pt, \mathbb{C}P^1, \mathbb{C}P^1)\, \hat{m}_{k+l-1} + \mathrm{GW}^{\mathbb{C}P^1}_{L,3}(pt, pt, pt)\, \widehat{M}_{k+l-2} \\ &= 2\, \widehat{M}_{k+l-2} + 2\, \hat{m}_{k+l-1} = 0\end{aligned}$$

For (8), we need to check two cases: when $k = 1$

$$\begin{aligned}(av) \circ v^l &= (e + \hat{m}_1) \circ \widehat{M}_l \\ &= \widehat{M}_l + \mathrm{GW}^{\mathbb{C}P^1}_{0L,3}(pt, \mathbb{C}P^1, \mathbb{C}P^1)\, \hat{m}_{l+1} = \\ &= \widehat{M}_l + \hat{m}_{l+1} = av^{l+1}\end{aligned}$$

and, when $k > 1$,

$$\begin{aligned}(av^k) \circ v^l &= (\widehat{M}_{k-1} + \hat{m}_k) \circ \widehat{M}_l \\ &= \mathrm{GW}^{\mathbb{C}P^1}_{0L,3}(\mathbb{C}P^1, \mathbb{C}P^1, pt)\, \widehat{M}_{k+l-1} + \mathrm{GW}^{\mathbb{C}P^1}_{0L,3}(pt, \mathbb{C}P^1, \mathbb{C}P^1)\, \hat{m}_{k+l} \\ &= \widehat{M}_{k+l-1} + \hat{m}_{k+l} = av^{k+l}\end{aligned}$$

6.2 T^*S^n

For higher n, the topology of Q_n is more involved, so we need more preliminary work before we can compute $SH_*(T^*S^n)$.

6.2.1 The topology of Q_N

We will need perfect Morse functions on smooth complex projective quadrics $Q_N = \left\{\sum_{k=0}^N z_k^2 = 0\right\} \subset \mathbb{C}P^{N+1}$. We will use the following result (see Proposition 2.4.22 in [CG10]):

Proposition. *Let $X \hookrightarrow \mathbb{C}P^N$ be a smooth complex projective variety and assume* *that there is an algebraic action of* $\mathbb{C}^*$ *on* $\mathbb{C}P^N$ *which preserves* X *and has finitely many fixed points in* X. *Then* X *admits a perfect Morse function* F, *whose critical points are the fixed points of the action. The gradient flow lines of* F *with respect to the Kähler metric are the orbits of the* $\mathbb{R}_+$*-action on* X, *given by* $\mathbb{R}_+ < \mathbb{C}^*$.

The action of $S^1 \subset \mathbb{C}^*$ on X turns out to be Hamiltonian, for the induced Kähler form, and F is the corresponding moment map. The critical submanifolds are complex submanifods, so the indices of all critical points are even and F is perfect. The statement about Morse trajectories follows from the assumption that the action is holomorphic:

$$\begin{aligned} \langle \nabla F, . \rangle &= \mathrm{d}F = \omega\left(., X_F\right) = \omega\left(J., JX_F\right) = -\omega\left(JX_F, J.\right) = \\ &= -\left\langle JX_F, . \right\rangle = -\left\langle J\left(\frac{\partial}{\partial \theta}\right)_*, . \right\rangle = -\left\langle \left(\frac{\partial}{\partial r}\right)_*, . \right\rangle = \\ &\Longrightarrow -\nabla F = \left(\frac{\partial}{\partial r}\right)_* \end{aligned}$$

Now, to construct a perfect Morse function on Q_N, one just needs to show that it admits a $\mathbb{C}^*$-action with finitely many critical points, and which extends to $\mathbb{C}P^{N+1}$. We will consider separatly the cases of N even and N odd. Before we begin, note that, if we take homogeneous coordinates $z_0, \ldots, z_M$ on $\mathbb{C}P^M$, then under the change of variables $(z_k, z_l) \mapsto (u, v) = (z_k + iz_l, z_k - iz_l)$, the expression $z_k^2 + z_l^2$ becomes uv.

When $\underline{N = 2m}$ is even: Take coordinates $z_0, \ldots, z_{2m+1}$ on $\mathbb{C}P^{2m+1}$, and $Q_{2m} = \left\{ \sum_{k=0}^n z_k^2 = 0 \right\}$. Change variables $(z_{2k}, z_{2k+1}) \mapsto (u_k, v_k)$, $k \in \{0, \ldots, m\}$, as above, so that $Q_{2m} = \left\{ \sum_{k=0}^m u_k v_k = 0 \right\}$. Now, let $\mathbb{C}^*$ act on $\mathbb{C}P^{2m+1}$ by

$$\lambda \, . \, [u_0; v_0; u_1; v_1; \ldots; u_m; v_m] = [u_0; v_0; \lambda^{-1} u_1; \lambda v_1; \ldots; \lambda^{-m} u_m; \lambda^m v_m], \quad \lambda \in \mathbb{C}^*$$

This action clearly preserves Q_{2m}, and its $2m+2$ critical points are, all in Q_{2m}:

$$[1; 0; \ldots; 0], [0; 1; 0; \ldots; 0], \ldots, [0; \ldots; 0; 1]$$

We can apply that proposition and conclude that there is a perfect Morse function on Q_{2m}.

When $\underline{N = 2m-1}$ is odd: take coordinates $z_0, \ldots, z_{2m}$ on $\mathbb{C}P^{2m}$, and $Q_{2m-1} = \left\{ \sum_{k=0}^{n} z_k^2 = 0 \right\}$. Change variables $(z_{2k-1}, z_{2k}) \mapsto (u_k, v_k)$, $k \in \{1, \ldots, m\}$, as above, so that $Q_{2m-1} = \left\{ z_0^2 + \sum_{k=1}^{m} u_k v_k = 0 \right\}$. Now, let $\mathbb{C}^*$ act on $\mathbb{C}P^{2m}$ by

$$\lambda \,.\, [z_0; u_1; v_1; \ldots; u_m; v_m] = [z_0; \lambda^{-1} u_1; \lambda v_1; \ldots; \lambda^{-m} u_m; \lambda^m v_m], \quad \lambda \in \mathbb{C}^*$$

As above, this action clearly preserves Q_{2m-1}, and its $2m+1$ critical points are:

$$[1; 0; \ldots; 0], [0; 1; 0; \ldots; 0], \ldots, [0; \ldots; 0; 1]$$

The first one is not in Q_{2m-1}, so the quadric has only $2m$ critical points. The proposition implies again the existence of a perfect Morse function on Q_{2m}.

We will now write down explicitly the stable and unstable manifolds for these Morse functions, and compute the relevant intersections. In particular, we will describe the two dimensional spaces of gradient flow lines, and how the middle dimensional classes intersect, when N is even. Both pieces of information will be used in the symplectic homology computations.

When $\underline{N = 2m}$ is even: given $r \in \mathbb{R}_+$

$$r \,.\, [u_0; v_0; u_1; v_1; \ldots; u_m; v_m] = [u_0; v_0; r^{-1} u_1; r v_1; \ldots; r^{-m} u_m; r^m v_m]$$

so

$$\begin{aligned} A^1 := \overline{W^u([1; 0; \ldots; 0])} &= \overline{\left\{ x \in Q_n \,\middle|\, \lim_{t \to 0^+} r \,.\, x = [1; 0; \ldots; 0] \right\}} = \\ &= \left\{ [u_0; 0; 0; v_1; \ldots; 0; v_m] \right\} \end{aligned}$$

and

$$\overline{W^s([1;0;\ldots;0])} = \overline{\left\{x \in Q_n \middle| \lim_{r\to+\infty} r\ .\ x = [1;0;\ldots;0]\right\}} = \big\{[u_0;0;u_1;0;\ldots;u_m;0]\big\}.$$

This shows that $[1;0;\ldots;0]$ is a critical point of index N. Call it p_N^1. Similarly,

$$A^2 := \overline{W^u([0;1;\ldots;0])} = \big\{[0;v_0;0;v_1;\ldots;0;v_m]\big\}$$

and

$$\overline{W^s([0;1;\ldots;0])} = \big\{[0;v_0;u_1;0;\ldots;u_m;0]\big\}$$

so $[0;1;\ldots;0]$ is another critical point of index N. Call it p_N^2. For the remaining critical points, we have the following:

$$\overline{W^u([\underbrace{0;\ldots;0;1}_{2i+1};0;\ldots])} = \big\{[u_0;v_0;u_1;v_1;\ldots;u_i;v_i;0;v_{i+1};\ldots;0;v_m]\big\}$$

and

$$\overline{W^s([\underbrace{0;\ldots;0;1}_{2i+1};0;\ldots])} = \big\{[0;\ldots;0;u_i;0;u_{i+1};0;\ldots;u_m;0]\big\}$$

so $[\underbrace{0;\ldots;0;1}_{2i+1};0;\ldots]$ is a critical point of index $N+2i$. Call it p_{N+2i}. Also

$$\overline{W^u([\underbrace{0;\ldots;0;1}_{2i+2};0;\ldots])} = \big\{[0;\ldots;0;v_i;0;v_{i+1};\ldots;0;v_m]\big\}$$

and

$$\overline{W^s([\underbrace{0;\ldots;0;1}_{2i+2};0;\ldots])} = \big\{[u_0;\ldots;v_i;u_{i+1};0;\ldots;u_m;0]\big\}$$

therefore $[\underbrace{0;\ldots;0;1}_{2i+2};0;\ldots]$ is a critical point of index $N-2i$. Call it p_{N-2i}. Note that the closures of the unstable manifolds of these points consist of projective hyperplanes of complex dimension $m-i$, for $i>0$. On the other hand, the $\overline{W^u(p_{N+2i})}$ are hyperplane sections $Q_N \cap \mathbb{C}P^{m+i+1}$.

We thus have one critical point of every even index, except for index N, with two critical points. Let us now determine the trajectories that connect critical points of index difference 2:

$$\begin{aligned}\overline{W^u(p_{N-2i}) \cap W^s(p_{N-2i-2})} = \big\{[0;\ldots;0;v_i;0;v_{i+1};\ldots;0;v_m]\big\}\cap \\ \cap\big\{[u_0;\ldots;v_{i+1};u_{i+2};0;\ldots;u_m;0]\big\} = \\ = \big\{[0;\ldots;0;v_i;0;v_{i+1};0;\ldots;0]\big\} = \mathbb{C}P^1\end{aligned}$$

for $0 \leqslant i < m$, taking p_N^2 when $i = 0$. Also,

$$\begin{aligned}\overline{W^u(p_N^1) \cap W^s(p_{N-2})} = \big\{[u_0;0;0;v_1;\ldots;0;v_m]\big\}\cap \\ \cap\big\{[u_0;v_0;u_1;v_1;u_2;0;\ldots;u_m;0]\big\} = \\ = \big\{[u_0;0;0;v_1;0;\ldots;0]\big\} = \mathbb{C}P^1.\end{aligned}$$

Similarly,

$$\begin{aligned}\overline{W^u(p_{N+2i}) \cap W^s(p_{N+2i-2})} = \big\{[u_0;v_0;u_1;v_1;\ldots;u_i;v_i;0;v_{i+1};\ldots;0;v_m]\big\}\cap \\ \cap\big\{[0;\ldots;0;u_{i-1};0;u_i;0;\ldots;u_m;0]\big\} = \\ = \big\{[0;\ldots;0;u_{i-1};0;u_i;0;\ldots;0]\big\} = \mathbb{C}P^1\end{aligned}$$

for $0 < i \leqslant m$, taking p_N^1 when $i = 1$. Also,

$$\begin{aligned}\overline{W^u(p_{N+2}) \cap W^s(p_N^2)} = \big\{[u_0;v_0;u_1;v_1;0;v_2;\ldots;0;v_m]\big\}\cap \\ \cap\big\{[0;v_0;u_1;0;\ldots;u_m;0]\big\} = \\ = \big\{[0;v_0;u_1;0;\ldots;0]\big\} = \mathbb{C}P^1.\end{aligned}$$

We have concluded that if $\operatorname{ind} p - \operatorname{ind} q = 2$, then $\overline{W^u(p) \cap W^s(q)} = \mathbb{C}P^1$, the generator of $H_2(Q_N;\mathbb{Z})$ with complex orientation.

Now, we determine $A^i \,.\, A^j$, the (homology) intersection products. Recall that we are still assuming $N = 2m$. We will have to consider two cases separately: m odd and m even:

- m odd: It will be useful to introduce the following family of complex submanifolds of Q_N, for $s \in [0,1]$:

$$A^2_s := \{[sv_1; v_0; -sv_0; v_1; \dots; sv_m; v_{m-1}; -sv_{m-1}; v_m]\}$$

Note that $A^2_0 = A^2$ and $A^2_1 = \{[v_1; v_0; -v_0; v_1; \dots; v_m; v_{m-1}; -v_{m-1}; v_m]\}$. On homology,

$$\begin{aligned} A^1 \,.\, A^2 = A^1 \,.\, A^2_1 = \{[u_0; 0; 0; v_1; \dots; 0; v_m]\} \cap \\ \cap \{[v_1; v_0; -v_0; v_1; \dots; v_m; v_{m-1}; -v_{m-1}; v_m]\} = \\ = \{[1; 0; 0; 1; 0; \dots; 0]\} = pt \end{aligned}$$

Also

$$\begin{aligned} A^2 \,.\, A^2 = A^2 \,.\, A^2_1 = \{[0; v_0; \dots; 0; v_m]\} \cap \\ \cap \{[v_1; v_0; -v_0; v_1; \dots; v_m; v_{m-1}; -v_{m-1}; v_m]\} = \varnothing \end{aligned}$$

Similarly, one can show that $A^1 \,.\, A^1 = \varnothing$.

Remark 6.1. *We need these perturbations because, even though the Morse functions that we chose on the Q_N are perfect, they are not Morse–Smale. So, we need to make an additional slight perturbation to get the desired perfect Morse–Smale functions f_{Q_N}.*

- m even: Take now

$$A^2_s := \{[0; v_0; sv_2; v_1; -sv_1; v_2; \dots; sv_m; v_{m-1}; -sv_{m-1}; v_m]\}$$

Again, $A^2_0 = A^2$. Now, $A^2_1 = \{[0; v_0; v_2; v_1; -v_1; v_2; \dots; v_m; v_{m-1}; -v_{m-1}; v_m]\}$.

On homology,

$$\begin{aligned} A^1 \,.\, A^2 = A^1 \,.\, A^2_1 &= \\ &= \big\{[u_0; 0; 0; v_1; \ldots; 0; v_m]\big\} \cap \\ &\qquad \cap \big\{[0; v_0; v_2; v_1; -v_1; v_2; \ldots; v_m; v_{m-1}; -v_{m-1}; v_m]\big\} = \varnothing \end{aligned}$$

Also,

$$\begin{aligned} A^2 \,.\, A^2 = A^2 \,.\, A^2_1 &= \\ &= \big\{[0; v_0; \ldots; 0; v_m]\big\} \cap \\ &\qquad \cap \big\{[0; v_0; v_2; v_1; -v_1; v_2; \ldots; v_m; v_{m-1}; -v_{m-1}; v_m]\big\} = \\ &= \big\{[0; 1; 0; \ldots; 0]\big\} = pt \end{aligned}$$

Similarly, one can show that $A^1 \,.\, A^1 = pt$.

Another result we will need is that $B := A^1 + A^2$ is non-primitive (which means that it is Poincaré dual to a multiple of the Kähler class ω; see more about *primitive cohomology* at the beginning of Section 6.2.2), and that $C := A^1 - A^2$ is primitive. It is the case that $Q_k \subset Q_N$ is Poincaré dual to ω^{N-k}. The fact that C is primitive is equivalent to $C \,.\, Q_m = 0$. Also, proving that $PD(B) = \omega^m = PD(Q_m)$ amounts to showing that $B \,.\, A^i = Q_m \,.\, A^i$, for $i = 1, 2$.

$$\begin{aligned} Q_m &= Q_N \cap \mathbb{C}P^{m+1} \text{ inside } \mathbb{C}P^{N+1} = \\ &= Q_N \cap \big\{[u_0; v_0; u_1; 0; \ldots; u_m; 0]\big\} = \\ &= \big\{[u_0; 0; u_1; 0; \ldots; u_m; 0]\big\} \cup \big\{[0; v_0; u_1; 0; \ldots; u_m; 0]\big\} = \\ &= 2\,\mathbb{C}P^m \end{aligned}$$

so

$$Q_m \,.\, A^1 = \Big(\big\{[u_0; 0; u_1; 0; \ldots; u_m; 0]\big\} \cup \big\{[0; v_0; u_1; 0; \ldots; u_m; 0]\big\}\Big) \cap \\ \cap \big\{[u_0; 0; 0; v_1; \ldots; 0; v_m]\big\} = \big\{[1; 0; \ldots; 0]\big\} = pt$$

and

$$Q_m \,.\, A^2 = \Big(\big\{[u_0; 0; u_1; 0; \ldots; u_m; 0]\big\} \cup \big\{[0; v_0; u_1; 0; \ldots; u_m; 0]\big\}\Big) \cap \\ \cap \big\{[0; v_0; 0; v_1; \ldots; 0; v_m]\big\} = \big\{[0; 1; 0; \ldots; 0]\big\} = pt.$$

We can now see that

$$C \,.\, Q_m = (A^1 - A^2) \,.\, Q_m = 0$$

so C is primitive, and

$$B \,.\, A^1 = (A^1 + A^2) \,.\, A^1 = pt = Q_m \,.\, A^1$$

$$B \,.\, A^2 = (A^1 + A^2) \,.\, A^2 = pt = Q_m \,.\, A^2$$

so $B = PD(\omega^m)$ and $B = Q_m \in H_N(Q_N; \mathbb{Z})$.

Let us now consider the case when $\underline{N = 2m - 1}$ is odd: given $r \in \mathbb{R}_+$

$$r \,.\, [z_0; u_1; v_1; \ldots; u_m; v_m] = [z_0; r^{-1}u_1; rv_1; \ldots; r^{-m}u_m; r^m v_m]$$

As before

$$\overline{W^u([\underbrace{0; \ldots; 0; 1}_{2i}; 0; \ldots])} = \big\{[z_0; u_1; v_1; \ldots; u_i; v_i; 0; v_{i+1}; \ldots; 0; v_m]\big\}$$

and

$$\overline{W^s([\underbrace{0; \ldots; 0; 1}_{2i}; 0; \ldots])} = \big\{[0; \ldots; 0; u_i; 0; u_{i+1}; 0; \ldots; u_m; 0]\big\}$$

therefore $[\underbrace{0;\ldots;0;1}_{2i};0;\ldots]$ is a critical point of index $N+2i-1$. Call it p_{N+2i-1}. Also

$$\overline{W^u([\underbrace{0;\ldots;0;1}_{2i+1};0;\ldots])} = \{[0;\ldots;0;v_i;0;v_{i+1};\ldots;0;v_m]\}$$

and

$$\overline{W^s([\underbrace{0;\ldots;0;1}_{2i+1};0;\ldots])} = \{[z_0;\ldots;v_i;u_{i+1};0;\ldots;u_m;0]\}$$

so $[\underbrace{0;\ldots;0;1}_{2i+1};0;\ldots]$ is a critical point of index $N-2i+1$. Call it p_{N-2i+1}. Note that the unstable manifolds of these points consist of planes of complex dimension $m-i$, for $i>0$. On the other hand, the $W^u(p_{N+2i-1})$ are hyperplane sections.

As above, we can describe the 2-dimensional spaces of connecting trajectories:

$$\begin{aligned}\overline{W^u(p_{N-2i+1})\cap W^s(p_{N-2i-1})} &= \{[0;\ldots;0;v_i;0;v_{i+1};\ldots;0;v_m]\}\cap \\ &\cap\{[z_0;\ldots;v_{i+1};u_{i+2};0;\ldots;u_m;0]\} = \\ &= \{[0;\ldots;0;v_i;0;v_{i+1};0;\ldots;0]\} = \mathbb{C}P^1\end{aligned}$$

for $0<i<m$.

$$\begin{aligned}\overline{W^u(p_{N+1})\cap W^s(p_{N-1})} &= \{[z_0;u_1;v_1;0;v_2;\ldots;0;v_m]\}\cap \\ &\cap\{[z_0;u_1;v_1;u_2;0;\ldots;u_m;0]\} = \\ &= \{[z_0;u_1;v_1;0;\ldots;0]\} = Q_1 = 2\,\mathbb{C}P^1\end{aligned}$$

$$\begin{aligned}\overline{W^u(p_{N+2i-1})\cap W^s(p_{N+2i-3})} &= \{[z_0;u_1;v_1;\ldots;u_i;v_i;0;v_{i+1};\ldots;0;v_m]\}\cap \\ &\cap\{[0;\ldots;0;u_{i-1};0;u_i;0;\ldots;u_m;0]\} = \\ &= \{[0;\ldots;0;u_{i-1};0;u_i;0;\ldots;0]\} = \mathbb{C}P^1\end{aligned}$$

for $1<i\leqslant m$. We have concluded that if $\operatorname{ind} p - \operatorname{ind} q = 2$, then $\overline{W^u(p)\cap W^s(q)} = \mathbb{C}P^1$, the positive generator of $H_2(Q_N;\mathbb{Z})$, except if $\operatorname{ind} p = N+1$. In that case, $\overline{W^u(p)\cap W^s(q)} = 2\,\mathbb{C}P^1 \in H_2(Q_N;\mathbb{Z})$.

The last result we will need about the topology of quadrics is a description of relevant Chern classes. We need to fix some notation, first. Denote by NQ_N the normal bundle for the inclusion $Q_N \hookrightarrow Q_{N+1}$. Recall that Q_1 is biholomorphic to $\mathbb{C}P^1$ and denote by Q_1 the generator of $H_2(Q_1;\mathbb{Z})$ with the complex orientation. Also, Q_2 is biholomorphic to $\mathbb{C}P^1 \times \mathbb{C}P^1$; denote by L_i, $i \in \{1,2\}$, the generators of $H_2(Q_2;\mathbb{Z})$ given by the coordinate spheres, with complex orientation. These L_i are homologous under the inclusion $Q_2 \hookrightarrow Q_N$, for $N > 2$, and give a generator L of $H_2(Q_N;\mathbb{Z})$.

Proposition 6.1. *The following holds about the relevant first Chern classes:*

1. $\langle c_1(TQ_1), Q_1\rangle = 2$ *and* $\langle c_1(NQ_1), Q_1\rangle = 2$*;*
2. $\langle c_1(TQ_2), L_i\rangle = 2$ *and* $\langle c_1(NQ_2), L_i\rangle = 1$*, for* $i = 1, 2$*;*
3. $\langle c_1(TQ_N), L\rangle = N$ *and* $\langle c_1(NQ_N), L\rangle = 1$*, for* $N \geqslant 3$*.*

This is a consequence of the additivity of the first Chern class, applied to the inclusions $Q_N \hookrightarrow \mathbb{C}P^{N+1}$ and $Q_N \hookrightarrow Q_{N+1}$. The formulas for Chern classes of tangent bundles imply that the Q_N are monotone, with $\lambda_{Q_N} = 1/N$, if $N > 1$, and $\lambda_{Q_1} = 1/2$. The formulas for the Chern classes of normal bundles imply that $Q_{N-1} = \mathrm{PD}(\omega)$ in Q_N, if $N > 1$, so $K = 1$.

6.2.2 Gromov–Witten numbers of Q_N

We will extract the relevant Gromov–Witten numbers of complex projective quadrics from known computations of their quantum cohomology rings. Note that $Q_N \subset \mathbb{C}P^{N+1}$ is an example of a smooth *complete intersection*, which is a projective variety C of dimension N cut out by r polynomials (of degrees $d_1, \dots, d_r$) inside $\mathbb{C}P^{N+r}$. If $N \geqslant 3$, then the Lefschetz hyperplane theorem implies that $H^2(C;\mathbb{Z}) = \mathbb{Z}$. The additivity of the first Chern class implies that $c_1(C) = (N + r + 1 - \sum d_i)[\omega]$, where $[\omega] \in H^2(C;\mathbb{Z})$ is the hyperplane class (which we will, for ease of notation, not be careful to distinguish from the Kähler form ω). In particular, if $N + r + 1 - \sum d_i > 0$, then C is monotone. One can actually show that

$$H^k(C;\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } 0 \leqslant k \leqslant 2N \text{ even} \\ 0 & \text{for } 0 \leqslant k \leqslant 2N \text{ odd} \end{cases}$$

for $k \neq N$. Let

$$H^N(C;\mathbb{Q})_0 := \begin{cases} H^N(C;\mathbb{Q}) & \text{if } N \text{ odd} \\ \ker\left[(\omega^{N/2}|.) : H^N(C;\mathbb{Q}) \to \mathbb{Q}\right] & \text{if } N \text{ even} \end{cases}$$

where $(x|y) = \int_C x \cup y$ is the Poincaré pairing. Call $H^N(C;\mathbb{Q})_0$ the *primitive cohomology* of C. If N is even, then it turns out that $H^N(C;\mathbb{Q}) = \mathbb{Q}\langle\omega^{N/2}\rangle \oplus H^N(C;\mathbb{Q})_0$. For more on the topology of complete intersections, see Chapter 5 of [Dim92].

Denote the quantum product by $*$, quantum powers by x^{*k} and usual cup powers by x^k. We will use the following result (see [Bea95]).

Theorem (Beauville)**.** *Let $C \subset \mathbb{C}P^{N+r}$ be a smooth complete intersection of degree $(d_1, \ldots, d_r)$ and dimension $N \geqslant 3$, with $N \geqslant 2\sum(d_i - 1) - 1$. Let $d := d_1 \ldots d_r$ and $\delta := \sum(d_i - 1)$. The quantum cohomology algebra $QH^*(C;\mathbb{Q})$ is the algebra generated by the hyperplane class ω and the primitive cohomology $H^N(C;\mathbb{Q})_0$, with the relations:*

$$\omega^{*(N+1)} = d_1^{d_1} \ldots d_r^{d_r} \omega^{*\delta} t \qquad \omega * x = 0 \qquad x * y = (x|y)\frac{1}{d}\left(\omega^{*N} - d_1^{d_1} \ldots d_r^{d_r} \omega^{*(\delta-1)} t\right)$$

for $x, y \in H^N(C,\mathbb{Q})_0$.

We apply this result to the case of a quadric $C = Q_N$, which is a hypersurface of degree 2. We get the following.

Corollary. *The quantum cohomology algebra $QH^*(Q_N,\mathbb{Q})$ is the algebra generated by the hyperplane class ω and the primitive cohomology $H^N(Q_N,\mathbb{Q})_0$, with the relations:*

$$\omega^{*(N+1)} = 4\omega t \qquad \omega * x = 0 \qquad x * y = (x|y)\frac{1}{2}\left(\omega^{*N} - 4t\right)$$

for $x, y \in H^N(Q_N,\mathbb{Q})_0$.

Recall that

$$x * y = \sum_{A \in H_2(Q_N)} (x * y)_A \, t^{\langle c_1(TQ_N), A \rangle / N},$$

and that, according to Proposition 6.1, $\langle c_1(TQ_N), L \rangle = N$, for $N \geqslant 3$. Therefore, by degree reasons, $\omega * \omega^k = \omega^{k+1}$ for $0 \leqslant k \leqslant N-2$. Inductively, we get

$$\omega^{*k} = \omega^k$$

for $0 \leqslant k \leqslant N-1$. For similar degree reasons,

$$\omega^{*N} = \omega * \omega^{N-1} = \omega^N + l_0 t^L \qquad \text{and} \qquad \omega * \omega^N = l_1 \omega t^L$$

for some $l_0, l_1 \in \mathbb{Z}$. This implies that

$$\omega^{*(N+1)} = \omega * \omega^{*N} = \omega * (\omega^N + l_0 t^L) = (l_0 + l_1) \omega t^L.$$

By the Corollary above, we conclude that $l_0 + l_1 = 4$. Below, we will argue that $l_0 = l_1 = 2$.

The quantum product on cohomology contains the information about genus 0, 3-point Gromov–Witten numbers. The relation is given by the formula

$$\mathrm{GW}^C_{A,3}(x, y, z) = \big((x * y)_A | z\big)$$

for $x, y, z \in H^*(C)$. We will use Poincaré duality to write Gromov–Witten numbers with respect to homology: $\mathrm{GW}^C_{A,3}(\mathrm{PD}_C(x), \mathrm{PD}_C(y), \mathrm{PD}_C(z)) = \mathrm{GW}^C_{A,3}(x, y, z)$.

Let us now compute some Gromov–Witten numbers of Q_N. We will make extensive use of the associativity and (graded) commutativity of the quantum product. We begin with $\mathrm{GW}^{Q_N}_{L,3}(pt, L, H)$, where $pt \in Q_N$ is a point, $L \subset Q_N$ is a copy of $\mathbb{C}P^1$ and

$H = Q_N \cap \mathbb{C}P^N$ is a hyperplane section.

$$\begin{aligned}\mathrm{GW}^{Q_N}_{L,3}(pt, L, H) = \mathrm{GW}^{Q_N}_{L,3}\left(\frac{1}{2}\omega^N, \frac{1}{2}\omega^{N-1}, \omega\right) = \\ = \frac{1}{4}\left((\omega * \omega^N)_L | \omega^{N-1}\right) = \frac{1}{4}\int_{Q_N} l_1\omega^N = \frac{l_1}{2} = \\ = \frac{1}{4}\left((\omega * \omega^{N-1})_L | \omega^N\right) = \frac{1}{4}\int_{Q_N} l_0\omega^N = \frac{l_0}{2}\end{aligned}$$

Therefore, $l_0 = l_1 = 2$, since we have seen already that $l_0 + l_1 = 4$, and

$$\mathrm{GW}^{Q_N}_{L,3}(pt, L, H) = 1.$$

In Q_N, a holomorphic sphere u of class L intersects a generic hyperplane section H at a unique point. This Gromov–Witten invariant tells us that there is a unique such u intersecting a generic point and line. Using the divisor equation, we also get

$$\mathrm{GW}^{Q_N}_{L,2}(pt, L) = \frac{1}{\#(L \cap H)}\mathrm{GW}^{Q_N}_{L,3}(pt, L, H) = 1.$$

Let us compute now

$$\mathrm{GW}^{Q_N}_{L,3}(pt, pt, Q_N) = \mathrm{GW}^{Q_N}_{L,3}\left(\frac{1}{2}\omega^N, \frac{1}{2}\omega^N, 1\right) = \frac{1}{4}\left((\omega^N * 1)_L | \omega^N\right) = \frac{1}{4}\int_{Q_N} 0 = 0.$$

This is the expected answer: since generically there is a unique line through a point and a line in Q_N, there should be no line through two generic points in Q_N.

Now

$$\begin{aligned}\mathrm{GW}^{Q_N}_{2L,3}(pt, pt, pt) = \mathrm{GW}^{Q_N}_{2L,3}\left(\frac{1}{2}\omega^N, \frac{1}{2}\omega^N, \frac{1}{2}\omega^N\right) = \\ = \frac{1}{8}\left((\omega^N * \omega^N)_{2L} | \omega^N\right) = \frac{1}{8}\int_{Q_N} 4\omega^N = 1.\end{aligned}$$

We have used the fact that

$$\omega^N * \omega^N = (\omega^{*N} - 2q^L)^{*2} = \omega^{*2N} - 4\omega^{*N}q^L + 4q^{2L} =$$
$$= 4\omega q^L * \omega^{N-1} - 4\omega^{*N}q^L + 4q^{2L} = 4q^{2L}$$

Finally, if N is even and $C = \mathrm{PD}_{Q_N}(P)$, where $P \in H^N(Q_N, \mathbb{Q})_0$, then

$$\mathrm{GW}^{Q_N}_{L,3}(pt, C, C) = \mathrm{GW}^{Q_N}_{L,3}\left(\frac{1}{2}\omega^N, P, P\right) = \frac{1}{2}\left((P * P)_L|\omega^N\right)$$
$$= \frac{1}{2}\int_{Q_N}\left(\int_{Q_N} P \cup P \frac{1}{2}(l_0 - 4)\right)\omega^N = -\int_{Q_N} P \cup P.$$

This integral can be computed explicitly: if $P = PD(C) = PD(A^1 - A^2)$, as above, then

$$\int_{Q_N} P \cup P = C \ . \ C = (A^1 - A^2) \ . \ (A^1 - A^2) =$$
$$= A^1 \ . \ A^1 - 2A^1 \ . \ A^2 + A^2 = \begin{cases} -2 & \text{if } m \text{ odd} \\ 2 & \text{if } m \text{ even} \end{cases}$$

so

$$\mathrm{GW}^{Q_N}_{L,3}(pt, C, C) = \begin{cases} 2 & \text{if } m \text{ odd} \\ -2 & \text{if } m \text{ even} \end{cases}$$

On the other hand, for $B = A^1 + A^2 = PD(\omega^m)$,

$$\mathrm{GW}^{Q_N}_{L,3}(pt, B, B) = \mathrm{GW}^{Q_N}_{L,3}\left(\frac{1}{2}\omega^N, \omega^m, \omega^m\right) = \frac{1}{2}\left((\omega^m * \omega^m)_L|\omega^N\right) =$$
$$= \frac{1}{2}\left((\omega^{*N})_L|\omega^N\right) = \frac{1}{2}\left((\omega^N + l_0 q^L)_L|\omega^N\right) = \frac{l_0}{2}\int_{Q_N}\omega^N = 2$$

Also, since P is primitive,

$$\mathrm{GW}^{Q_N}_{L,3}(p, B, C) = \mathrm{GW}^{Q_N}_{L,3}\left(\frac{1}{2}\omega^N, \omega^m, P\right) = \frac{1}{2}\left((\omega^m * P)_L|\omega^N\right) = 0$$

One can now compute the remaining numbers:

$$\begin{aligned}
\mathrm{GW}^{Q_N}_{L,3}(pt,A^1,A^1) &= \mathrm{GW}^{Q_N}_{L,3}\left(pt,\frac{1}{2}(B+C),\frac{1}{2}(B+C)\right) = \\
&= \frac{1}{4}\Big(\mathrm{GW}^{Q_N}_{L,3}(pt,B,B) + 2\,\mathrm{GW}^{Q_N}_{L,3}(pt,B,C) + \mathrm{GW}^{Q_N}_{L,3}(pt,C,C)\Big) = \\
&= \begin{cases} 1 & \text{if } m \text{ odd} \\ 0 & \text{if } m \text{ even} \end{cases}
\end{aligned}$$

$$\begin{aligned}
\mathrm{GW}^{Q_N}_{L,3}(pt,A^2,A^2) &= \mathrm{GW}^{Q_N}_{L,3}\left(pt,\frac{1}{2}(B-C),\frac{1}{2}(B-C)\right) = \\
&= \frac{1}{4}\Big(\mathrm{GW}^{Q_N}_{L,3}(pt,B,B) - 2\,\mathrm{GW}^{Q_N}_{L,3}(pt,B,C) + \mathrm{GW}^{Q_N}_{L,3}(pt,C,C)\Big) = \\
&= \begin{cases} 1 & \text{if } m \text{ odd} \\ 0 & \text{if } m \text{ even} \end{cases}
\end{aligned}$$

$$\begin{aligned}
\mathrm{GW}^{Q_N}_{L,3}(pt,A^1,A^2) &= \mathrm{GW}^{Q_N}_{L,3}\left(pt,\frac{1}{2}(B+C),\frac{1}{2}(B-C)\right) = \\
&= \frac{1}{4}[\mathrm{GW}^{Q_N}_{L,3}(pt,B,B) + \mathrm{GW}^{Q_N}_{L,3}(pt,C,C)] = \\
&= \begin{cases} 0 & \text{if } m \text{ odd} \\ 1 & \text{if } m \text{ even} \end{cases}
\end{aligned}$$

so

$$\mathrm{GW}^{Q_N}_{L,3}(A^i,A^j,pt) = \begin{cases} \delta_{i,j} & \text{if } m \text{ odd} \\ 1-\delta_{i,j} & \text{if } m \text{ even} \end{cases} \qquad i,j \in \{1,2\}$$

To deal with the torsion terms of symplectic homology of T^*S^{N+1}, when $N =$

$2m-1$ is odd, we will also need

$$\begin{aligned}\mathrm{GW}^{Q_N}_{L,3}\big(W^u(p_{N-1}),pt,W^s(p_{N-1})\big) &= \mathrm{GW}^{Q_N}_{L,3}(\mathbb{C}P^{m-1},pt,Q_m) = \\ &= \mathrm{GW}^{Q_N}_{L,3}\left(\frac{1}{2}\omega^m,\frac{1}{2}\omega^N,\omega^{m-1}\right) = \frac{1}{4}\left((\omega^m * \omega^{m-1})_L|\omega^N\right) = \\ &= \frac{1}{4}\left((\omega^{*N})_L|\omega^N\right) = \frac{1}{4}\left((\omega^N + l_0 q^L)_L|\omega^N\right) = \frac{l_0}{4}\int_{Q_N}\omega^N = 1\end{aligned}$$

For completeness, let us also write down the Gromov–Witten numbers in homology class $0L$, which correspond to the intersection product that was computed above:

$$\mathrm{GW}^{Q_N}_{0L,3}(pt,Q_N,Q_N) = \#(pt\cap Q_N\cap Q_N) = 1$$

$$\mathrm{GW}^{Q_N}_{0L,3}(A^i,A^j,Q_N) = \#(A^i\cap A^j\cap Q_N) = \begin{cases} 1-\delta_{i,j} & \text{if } m \text{ odd} \\ \delta_{i,j} & \text{if } m \text{ even}\end{cases} \qquad i,j\in\{1,2\}$$

Finally, we will also need some relative Gromov–Witten numbers of (Q_N, Q_{N-1}), namely

$$\mathrm{GW}^{Q_N,Q_{N-1}}_{L,1,(1)}(pt;L) = \mathrm{GW}^{Q_N}_{L,2}(pt,L) = 1$$

and

$$\mathrm{GW}^{Q_N,Q_{N-1}}_{L,1,(1)}(L;pt) = \mathrm{GW}^{Q_N}_{L,2}(L,pt) = 1.$$

6.2.3 The group $SH_*(T^*S^n)$

The results of the previous sections can be used to compute $SH_*(T^*S^n)$, for $n>2$. As in the case of T^*S^2, we use a Morse function on T^*S^n that grows at infinity and has two critical points e (minimum) and c (saddle), to define the function f_B of Section 3.2. We begin with the case of even n. From the discussion above, we get the chain complex

$$SC_*(T^*S^n) = \mathbb{Z}\langle e,c,\check{q}_{i,k},\hat{q}_{i,k}\rangle$$

where we take all even integers $0 \leqslant i \leqslant 2n-2$ and all integers $k > 0$. For the differential, we have the following:

$$\begin{aligned}\mathrm{d}\check{q}_{0,l+1} &= \mathrm{d}_{\vee\wedge}\check{q}_{0,l+1} + \mathrm{d}_{\vee M}\check{q}_{0,l+1} = \mathrm{d}_{\vee\wedge}\check{q}_{0,l+1} = \\ &= c(l+1;l)\ .\ \mathrm{GW}^{Q_{n-1}}_{L,2}(L,pt)\,\widehat{q}_{2n-4,l} = \widehat{q}_{2n-4,l}\end{aligned}$$

for $l \geqslant 1$.

$$\begin{aligned}\mathrm{d}\check{q}_{2,l+1} &= \mathrm{d}_{\vee\wedge}\check{q}_{2,l+1} + \mathrm{d}_{\vee M}\check{q}_{2,l+1} = \mathrm{d}_{\vee\wedge}\check{q}_{2,l+1} = \\ &= -\langle c_1(Y \to Q_{n-1}), \mathbb{C}P^1\rangle\,\widehat{q}_{0,l+1} + \\ &\quad + c(l+1;l)\ .\ \mathrm{GW}^{Q_{n-1}}_{L,2}(L,pt)\,\widehat{q}_{2n-2,l} = \widehat{q}_{0,l+1} + \widehat{q}_{2n-2,l}\end{aligned}$$

for $l \geqslant 1$.

$$\begin{aligned}\mathrm{d}\check{q}_{2k,l} &= \mathrm{d}_{\vee\wedge}\check{q}_{2k,l} + \mathrm{d}_{\vee M}\check{q}_{2k,l} = \mathrm{d}_{\vee\wedge}\check{q}_{2k,l} = \\ &= -\langle c_1(Y \to Q_{n-1}), \mathbb{C}P^1\rangle\,\widehat{q}_{2k-2,l} = \widehat{q}_{2k-2,l}\end{aligned}$$

for $2 \leqslant k \leqslant n, k \neq \frac{n}{2}, l \geqslant 1$.

$$\begin{aligned}\mathrm{d}\check{q}_{n,l} &= \mathrm{d}_{\vee\wedge}\check{q}_{n,l} + \mathrm{d}_{\vee M}\check{q}_{n,l} = \mathrm{d}_{\vee\wedge}\check{q}_{n,l} = \\ &= -\langle c_1(Y \to Q_{n-1}), 2\,\mathbb{C}P^1\rangle\,\widehat{q}_{n-2,l} = 2\,\widehat{q}_{n-2,l}\end{aligned}$$

for $l \geqslant 1$.

$$\begin{aligned}\mathrm{d}\check{q}_{2,1} &= \mathrm{d}_{\vee\wedge}\check{q}_{2,1} + \mathrm{d}_{\vee M}\check{q}_{2,1} = \\ &= -\langle c_1(Y \to Q_{n-1}), \mathbb{C}P^1\rangle\,\widehat{q}_{0,1} + \mathrm{GW}^{Q_n,Q_{n-1}}_{L,1,(1)}(pt;L)\,e = \widehat{q}_{0,1} + e.\end{aligned}$$

Summing up, the differential in $SH_*(T^*S^n)$, for $n > 2$ even, is

$$\begin{cases} \mathrm{d}\check{q}_{0,l+1} = \widehat{q}_{2n-4,l} \\ \mathrm{d}\check{q}_{2,l+1} = \widehat{q}_{2n-2,l} + \widehat{q}_{0,l+1} \\ \mathrm{d}\check{q}_{4,l} = \widehat{q}_{2,l} \\ \vdots \\ \mathrm{d}\check{q}_{n-2,l} = \widehat{q}_{n-4,l} \\ \mathrm{d}\check{q}_{n,l} = 2\,\widehat{q}_{n-2,l} \\ \mathrm{d}\check{q}_{n+2,l} = \widehat{q}_{n,l} \\ \vdots \\ \mathrm{d}\check{q}_{2n-2,l} = \widehat{q}_{2n-4,l} \\ \mathrm{d}\check{q}_{2,1} = \widehat{q}_{0,1} + e \end{cases}$$

for $l \geqslant 1$. On the remaining generators, the differential vanishes.

The case of odd n is slightly different. Not surprisingly, the difference occurs near the middle dimensional homology classes of the divisor Q_{n-1}. We begin with the case $n > 3$, for ease of notation (the case $n = 3$ will be described below). The chain complex is

$$SC_*(T^*S^n) = \mathbb{Z}\left\langle e, c, \check{q}_{i,k}, \widehat{q}_{i,k}, \check{q}^1_{n-1,k}, \widehat{q}^1_{n-1,k}, \check{q}^2_{n-1,k}, \widehat{q}^2_{n-1,k} \right\rangle$$

where we take all even integers $0 \leqslant i \leqslant 2n-2$, $i \neq n-1$, and all integers $k > 0$. For the differential, we now have

$$\begin{aligned} \mathrm{d}\check{q}^i_{n-1,l} &= \mathrm{d}_{\vee\wedge}\check{q}^i_{n-1,l} + \mathrm{d}_{\vee M}\check{q}^i_{n-1,l} = \mathrm{d}_{\vee\wedge}\check{q}^i_{n-1,l} = \\ &= -\left\langle c_1(Y \to Q_{n-1}), \mathbb{C}P^1 \right\rangle \widehat{q}_{n-3,l} = \widehat{q}_{n-3,l} \end{aligned}$$

for $i = 0, 1$ and $l \geqslant 1$.

$$\begin{aligned}
\mathrm{d}\check{q}_{n+1,l} &= \mathrm{d}_{\vee\wedge}\check{q}_{n+1,l} + \mathrm{d}_{\vee M}\check{q}_{n+1,l} = \mathrm{d}_{\vee\wedge}\check{q}_{n+1,l} = \\
&= -\left\langle c_1(Y \to Q_{n-1}), \mathbb{C}P^1 \right\rangle \widehat{q}^1_{n-1,l} - \left\langle c_1(Y \to Q_{n-1}), \mathbb{C}P^1 \right\rangle \widehat{q}^2_{n-1,l} = \\
&= \widehat{q}^1_{n-1,l} + \widehat{q}^2_{n-1,l}
\end{aligned}$$

for $l \geqslant 1$.

Summing up, when $n > 3$ is odd, the differential is

$$\begin{cases}
\mathrm{d}\check{q}_{0,l+1} = \widehat{q}_{2n-4,l} \\
\mathrm{d}\check{q}_{2,l+1} = \widehat{q}_{2n-2,l} + \widehat{q}_{0,l+1} \\
\mathrm{d}\check{q}_{4,l} = \widehat{q}_{2,l} \\
\vdots \\
\mathrm{d}\check{q}^1_{n-1,l} = \widehat{q}_{n-3,l} \\
\mathrm{d}\check{q}^2_{n-1,l} = \widehat{q}_{n-3,l} \\
\mathrm{d}\check{q}_{n+1,l} = \widehat{q}^1_{n-1,l} + \widehat{q}^2_{n-1,l} \\
\mathrm{d}\check{q}_{n+3,l} = \widehat{q}_{n+1,l} \\
\vdots \\
\mathrm{d}\check{q}_{2n-2,l} = \widehat{q}_{2n-4,l} \\
\mathrm{d}\check{q}_{2,1} = \widehat{q}_{0,1} + e
\end{cases}$$

for $l \geqslant 1$. On the remaining generators, the differential vanishes.

We then get, for $n > 2$ even,

$$SH_*(T^*S^n) = \mathbb{Z}\left\langle c, \check{q}_{0,1}, e, \check{q}_{0,l+1} - \check{q}_{2n-2,l}, \widehat{q}_{0,l+1} \right\rangle \oplus \mathbb{Z}/2\mathbb{Z}\left\langle \widehat{q}_{n-2,l} \right\rangle$$

and, for $n > 3$ odd,

$$SH_*(T^*S^n) = \mathbb{Z}\left\langle c, \check{q}_{0,1}, e, \check{q}^1_{n-1,l} - \check{q}^2_{n-1,l}, \widehat{q}^1_{n-1,l}, \check{q}_{0,l+1} - \check{q}_{2n-2,l}, \widehat{q}_{0,l+1} \right\rangle$$

We can again compare this with the Cohen–Jones–Yan result, in Section 2.3. The indices are computed using Lemma 3.4. For $n > 2$ even, the free part is

$SH_d(T^*S^n)$	c	$-\check{q}_{0,1}$	e	$\check{q}_{2n-2,k} - \check{q}_{0,k+1}$	$-\widehat{q}_{0,k+1}$
$H_d(LS^n)$	a	b	1	bv^k	v^k
d	0	$n-1$	n	$(n-1)(2k+1)$	$(n-1)(2k+1)+1$

and for the $\mathbb{Z}/2\mathbb{Z}$ torsion we have

$SH_d(T^*S^n)$	$\widehat{q}_{n-2,k}$
$H_d(LS^n)$	av^k
d	$(n-1)2k$

If $n > 3$ is odd, then there is no torsion, and the free part is given by

$SH_d(T^*S^n)$	c	$-\check{q}_{0,1}$	e	$\pm(\check{q}^1_{n-1,k} - \check{q}^2_{n-1,k})$	$\pm\widehat{q}^1_{n-1,k}$	$\pm(\check{q}_{2n-2,k} - \check{q}_{0,k+1})$	$\pm\widehat{q}_{0,k+1}$
$H_d(LS^n)$	a	au	1	au^{2k}	u^{2k-1}	au^{2k+1}	u^{2k}
d	0	$n-1$	n	$(n-1)2k$	$(n-1)2k+1$	$(n-1)(2k+1)$	$(n-1)(2k+1)+1$

The signs in the table above are as follows:

$$au^{2k} = (-1)^{g(k,m)}(\check{q}^1_{n-1,k} - \check{q}^2_{n-1,k}) \quad \text{and} \quad u^{2k-1} = (-1)^{F(k,m)}\,\widehat{q}^1_{n-1,k},$$
$$au^{2k-1} = (-1)^{G(k,m)}(\check{q}_{2n-2,k-1} - \check{q}_{0,k}), \quad u^{2k} = (-1)^{f(k,m)}\,\widehat{q}_{0,k+1}$$

where $f, F, g, G : \mathbb{N}^2 \to \mathbb{Z}/2$ are such that

- if m is odd, then $f_m \equiv 1$, $G_m \equiv 0$ and $F_m = g_m \equiv 0$ (we could also choose $F_m = g_m \equiv 1$).
- if m is even, then $f_m(r) = G_m(r) = r + 1$ and $F_m(r) = g_m(r) = r$ (we could also choose $F_m(r) = g_m(r) = r + 1$);

Finally, the case $n = 3$ can be dealt with as above, but the differential has a slightly different lookx. The chain complex is

$$SC_*(T^*S^3) = \mathbb{Z}\left\langle e, c, \check{q}_{i,k}, \widehat{q}_{i,k}, \check{q}^1_{2,k}, \widehat{q}^1_{2,k}, \check{q}^2_{2,k}, \widehat{q}^2_{2,k} \right\rangle$$

where we take $i \in \{0, 4\}$, and all integers $k > 0$. The differential is

$$\begin{cases} \mathrm{d}\check{q}_{0,l+1} = \widehat{q}^1_{2,l} + \widehat{q}^2_{2,l} \\ \mathrm{d}\check{q}^1_{2,l+1} = \widehat{q}_{4,l} + \widehat{q}_{0,l+1} \\ \mathrm{d}\check{q}^2_{2,l+1} = \widehat{q}_{4,l} + \widehat{q}_{0,l+1} \\ \mathrm{d}\check{q}_{4,l} = \widehat{q}^1_{2,l} + \widehat{q}^2_{2,l} \\ \mathrm{d}\check{q}^1_{2,1} = \widehat{q}_{0,1} + e \\ \mathrm{d}\check{q}^2_{2,1} = \widehat{q}_{0,1} + e \end{cases}$$

for $l \geqslant 1$. On the remaining generators, the differential vanishes. Therefore,

$$SH_*(T^*S^3) = \mathbb{Z}\left\langle c, \check{q}_{0,1}, e, \check{q}^1_{2,l} - \check{q}^2_{2,l}, \widehat{q}^1_{2,l}, \check{q}_{0,l+1} - \check{q}_{4,l}, \widehat{q}_{0,l+1} \right\rangle$$

Comparing with the Cohen–Jones–Yan result, we get

$SH_d(T^*S^3)$	c	$-\check{q}_{0,1}$	e	$\check{q}^1_{2,k} - \check{q}^2_{2,k}$	$\widehat{q}^1_{2,k}$	$\check{q}_{4,k} - \check{q}_{0,k+1}$	$-\widehat{q}_{0,k+1}$
$H_d(LS^3)$	a	au	1	au^{2k}	u^{2k-1}	au^{2k+1}	u^{2k}
d	0	2	3	$4k$	$4k+1$	$4k+2$	$4k+3$

6.2.4 The ring $SH_*(T^*S^n)$

Let us now compute the pair-of-pants product. We start with the case of even n. To get a result matching [CJY04], we need to show the following:

1. a kills everything except for 1 and for the fact that $a \circ v^k = av^k$;
2. $e = 1$ is the unit;
3. $b^2 = b \circ (bv^k) = (bv^k) \circ (bv^l) = 0$;

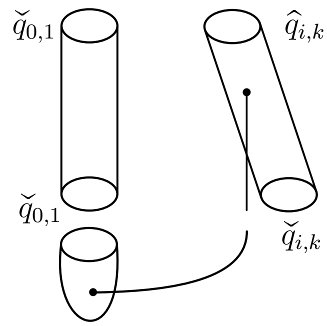

Figure 6.2: Broken pair-of-pants on T^*S^n

4. $b \circ v^k = bv^k$;

5. $v^k \circ v^l = v^{k+l}$;

6. $(bv^k) \circ v^l = bv^{k+l}$;

7. $av^k \circ v^l = av^{k+l}$.

We will not focus on the equations that involve constant orbits. As in the case of T^*S^2, the product of two orbits in $\widecheck{CH}$ is zero, since one cannot fix two markers on a pair-of-pants. This implies (3).

As in the case of T^*S^2, we use the description of the product given in Section 5.3.1. The 3-point absolute Gromov–Witten numbers correspond to holomorphic pairs-of-pants in $\mathbb{R} \times Y$, whereas 2-point relative Gromov–Witten numbers correspond to broken pairs-of-pants, which are counted with a *negative sign.* Figure 6.2 depicts the latter. In this case, given generic $p \in Q_{n-1}$, we have $[\overline{W^u_{f_W}(p)}] = L \in H_2(Q_n; \mathbb{Z})$.

For (4), observe that

$$\begin{aligned} b \circ v^k &= (-\check{q}_{0,1}) \circ (-\widehat{q}_{0,k+1}) = \\ &= \mathrm{GW}^{Q_{n-1}}_{L,3}(pt, pt, Q_{n-1})\, \check{q}_{0,k+1} + \mathrm{GW}^{Q_{n-1}}_{2L,3}(pt, pt, pt)\, \check{q}_{2n-2,k} - \\ &\qquad - \mathrm{GW}^{Q_N, Q_{N-1}}_{L,1,(1)}(L; pt)\, \check{q}_{0,k+1} = \\ &= 0\, \check{q}_{0,k+1} + \check{q}_{2n-2,k} - \check{q}_{0,k+1} = bv^k. \end{aligned}$$

For (5), we have

$$\begin{aligned} v^k \circ v^l &= (-\widehat{q}_{0,k+1}) \circ (-\widehat{q}_{0,l+1}) = \\ &= \mathrm{GW}^{Q_{n-1}}_{L,3}(pt, pt, Q_{n-1})\, \widehat{q}_{0,k+l+1} + \mathrm{GW}^{Q_{n-1}}_{2L,3}(pt, pt, pt)\, \widehat{q}_{2n-2,k+l} = \\ &= 0\, \widehat{q}_{0,k+l+1} - \widehat{q}_{0,k+l+1} = v^{k+l}. \end{aligned}$$

For (6):

$$\begin{aligned} (bv^k) \circ v^l &= (\check{q}_{2n-2,k} - \check{q}_{0,k+1}) \circ (-\widehat{q}_{0,l+1}) = \\ &= -\,\mathrm{GW}^{Q_{n-1}}_{0L,3}(Q_{n-1}, pt, Q_{n-1})\, \check{q}_{0,k+l+1} - \mathrm{GW}^{Q_{n-1}}_{L,3}(Q_{n-1}, pt, pt)\, \check{q}_{2n-2,k+l} + \\ &\quad + \mathrm{GW}^{Q_{n-1}}_{L,3}(pt, pt, Q_{n-1})\, \check{q}_{0,k+l+1} + \mathrm{GW}^{Q_{n-1}}_{2L,3}(pt, pt, pt)\, \check{q}_{2n-2,k+l} = \\ &= -\check{q}_{0,k+l+1} + 0\, \check{q}_{2n-2,k+l} + 0\, \check{q}_{0,k+l+1} + \check{q}_{2n-2,k+l} = bv^{k+l}. \end{aligned}$$

For (7):

$$\begin{aligned} (av^k) \circ v^l &= \widehat{q}_{n-2,k} \circ (-\widehat{q}_{0,l+1}) = -\,\mathrm{GW}^{Q_{n-1}}_{L,3}\left(W^u(p_{n-2}), pt, W^s(p_{n-2})\right) \widehat{q}_{n-2,k+l} + \ldots = \\ &= -\,\mathrm{GW}^{Q_{n-1}}_{L,3}(\mathbb{C}P^{n/2-1}, pt, Q_{n/2})\, \widehat{q}_{n-2,k+l} = -\widehat{q}_{n-2,k+l} = \widehat{q}_{n-2,k+l} = av^{k+l} \end{aligned}$$

(remember this is for the $\mathbb{Z}/2\mathbb{Z}$ torsion).

Now for the case of odd n. Write $m := \frac{n-1}{2}$. We need to show:

1. e is the unit;
2. $a^2 = a \circ (au^k) = (au^k) \circ (au^l) = 0$;
3. $a \circ u^k = au^k$ (special case: $k = 1$);
4. $u^k \circ u^l = u^{k+l}$;
5. $(au^k) \circ u^l = au^{k+l}$, $k > 1$;
6. $(au) \circ u^k = au^{k+1}$.

We will once again not focus on the formulas that involve critical points. As before, we expect that the product of two orbits in $\widetilde{CH}$ should vanish, and thus get the last identity in (2).

For (4), there are several cases to consider:

- $k = 2r, l = 2s$ even:

$$\begin{aligned} u^{2r} \circ u^{2s} &= \Big((-1)^{f(r,m)}\, \widehat{q}_{0,r+1}\Big) \circ \Big((-1)^{f(s,m)}\, \widehat{q}_{0,s+1}\Big) = \\ &= (-1)^{f(r,m)+f(s,m)} \Big(\mathrm{GW}_{L,3}^{Q_{n-1}}(pt, pt, Q_{n-1})\, \widehat{q}_{0,r+s+1} + \\ &\qquad + \mathrm{GW}_{2L,3}^{Q_{n-1}}(pt, pt, pt)\, \widehat{q}_{2n-2,r+s}\Big) = \\ &= 0\, \widehat{q}_{0,r+s+1} + (-1)^{f(r,m)+f(s,m)}\, \widehat{q}_{2n-2,r+s} = \\ &= (-1)^{f(r,m)+f(s,m)+1}\, \widehat{q}_{0,r+s+1} = (-1)^{f(r+s,m)}\, \widehat{q}_{0,r+s+1} = u^{2r+2s} \end{aligned}$$

  because $f(r,m) + f(s,m) + 1 = f(r+s,m)$.

- $k = 2r-1, l = 2s-1$ odd:

$$\begin{aligned} u^{2r-1} \circ u^{2s-1} &= \Big((-1)^{F(r.m)}\, \widehat{q}^{1}_{n-1,r}\Big) \circ \Big((-1)^{F(s,m)}\, \widehat{q}^{1}_{n-1,s}\Big) = \\ &= (-1)^{F(r,m)+F(s,m)} \Big(\mathrm{GW}_{0L,3}^{Q_{n-1}}(A^1, A^1, Q_{n-1})\, \widehat{q}_{0,r+s} + \\ &\qquad + \mathrm{GW}_{L,3}^{Q_{n-1}}(A^1, A^1, pt)\, \widehat{q}_{2n-2,r+s-1}\Big) = \\ &= (-1)^{F(r,m)+F(s,m)} \Big((A^1 \,.\, A^1)\, \widehat{q}_{0,r+s} + \\ &\qquad + \mathrm{GW}_{L,3}^{Q_{n-1}}(A^1, A^1, pt)\, \widehat{q}_{2n-2,r+s-1}\Big) = \\ &= \begin{cases} (-1)^{F(r,m)+F(s,m)} \big(0\, \widehat{q}_{0,r+s} + \widehat{q}_{2n-2,r+s-1}\big) & \text{if } m \text{ odd} \\ (-1)^{F(r,m)+F(s,m)} \big(\widehat{q}_{0,r+s} + 0\, \widehat{q}_{2n-2,r+s-1}\big) & \text{if } m \text{ even} \end{cases} = \\ &= (-1)^{F(r,m)+F(s,m)+m}\, \widehat{q}_{0,r+s} = (-1)^{f(r+s-1,m)}\, \widehat{q}_{0,r+s} = u^{2r+2s-2} \end{aligned}$$

  because $F(r,m) + F(s,m) + m = f(r+s-1,m)$.

- $k = 2r$ even, $l = 2s-1$ odd: it is useful to recall that in $H_m(Q_{n-1};\mathbb{Z})$

$$A^i \,.\, A^j = \begin{cases} 1-\delta_{i,j} & \text{if } m \text{ odd} \\ \delta_{i,j} & \text{if } m \text{ even} \end{cases}$$

$$\begin{aligned}
u^{2r} \circ u^{2s-1} &= \Big((-1)^{f(r,m)}\, \widehat{q}_{0,r+1}\Big) \circ \Big((-1)^{F(s,m)}\, \widehat{q}^{1}_{n-1,s}\Big) = \\
&= \begin{cases} (-1)^{f(r,m)+F(s,m)} \Big(\mathrm{GW}^{Q_{n-1}}_{L,3}(pt, A^1, A^1)\, \widehat{q}^{2}_{n-1,r+s} + \\ \qquad + \mathrm{GW}^{Q_{n-1}}_{L,3}(pt, A^1, A^2)\, \widehat{q}^{1}_{n-1,r+s}\Big) & \text{if } m \text{ odd} \\ (-1)^{f(r,m)+F(s,m)} \Big(\mathrm{GW}^{Q_{n-1}}_{L,3}(pt, A^1, A^1)\, \widehat{q}^{1}_{n-1,r+s} + \\ \qquad + \mathrm{GW}^{Q_{n-1}}_{L,3}(pt, A^1, A^2)\, \widehat{q}^{2}_{n-1,r+s}\Big) & \text{if } m \text{ even} \end{cases} = \\
&= \begin{cases} (-1)^{f(r,m)+F(s,m)} \big(\widehat{q}^{2}_{n-1,r+s} + 0\, \widehat{q}^{1}_{n-1,r+s}\big) & \text{if } m \text{ odd} \\ (-1)^{f(r,m)+F(s,m)} \big(0\, \widehat{q}^{1}_{n-1,r+s} + \widehat{q}^{2}_{n-1,r+s}\big) & \text{if } m \text{ even} \end{cases} = \\
&= (-1)^{f(r,m)+F(s,m)}\, \widehat{q}^{2}_{n-1,r+s} = (-1)^{f(r,m)+F(s,m)+1}\, \widehat{q}^{1}_{n-1,r+s} = \\
&= (-1)^{F(r+s,m)}\, \widehat{q}^{1}_{n-1,r+s} = u^{2r+2s-1}
\end{aligned}$$

  because $f(r,m) + F(s,m) + 1 = F(r+s,m)$.

We now consider equation (5) above. Again, there are several cases to consider:

- $k = 2r, l = 2s$ even:

$$(au^{2r}) \circ u^{2s} = (-1)^{g(r,m)}\left(\check{q}^1_{n-1,r} - \check{q}^2_{n-1,r}\right) \circ (-1)^{f(s,m)}\, \widehat{q}_{0,s+1} =$$

$$= \begin{cases} (-1)^{g(r,m)+f(s,m)}\Big(\mathrm{GW}^{Q_{n-1}}_{L,3}(A^1,pt,A^1)\,\check{q}^2_{n-1,r+s} + \\ \qquad + \mathrm{GW}^{Q_{n-1}}_{L,3}(A^1,pt,A^2)\,\check{q}^1_{n-1,r+s}\Big) + \\ \quad +(-1)^{g(r,m)+f(s,m)+1}\Big(\mathrm{GW}^{Q_{n-1}}_{L,3}(A^2,pt,A^1)\,\check{q}^2_{n-1,r+s} + & \text{if } m \text{ odd} \\ \qquad + \mathrm{GW}^{Q_{n-1}}_{L,3}(A^2,pt,A^2)\,\check{q}^1_{n-1,r+s}\Big) \\ \\ (-1)^{g(r,m)+f(s,m)}\Big(\mathrm{GW}^{Q_{n-1}}_{L,3}(A^1,pt,A^1)\,\check{q}^1_{n-1,r+s} + \\ \qquad + \mathrm{GW}^{Q_{n-1}}_{L,3}(A^1,pt,A^2)\,\check{q}^2_{n-1,r+s}\Big) + \\ \quad +(-1)^{g(r,m)+f(s,m)+1}\Big(\mathrm{GW}^{Q_{n-1}}_{L,3}(A^2,pt,A^1)\,\check{q}^1_{n-1,r+s} + & \text{if } m \text{ even} \\ \qquad + \mathrm{GW}^{Q_{n-1}}_{L,3}(A^2,pt,A^2)\,\check{q}^2_{n-1,r+s}\Big) \end{cases} =$$

$$= \begin{cases} (-1)^{g(r,m)+f(s,m)}\left(\check{q}^2_{n-1,r+s} + 0\,\check{q}^1_{n-1,r+s}\right) + \\ \quad +(-1)^{g(r,m)+f(s,m)+1}\left(0\,\check{q}^2_{n-1,r+s} + \check{q}^1_{n-1,r+s}\right) & \text{if } m \text{ odd} \\ \\ (-1)^{g(r,m)+f(s,m)}\left(0\,\check{q}^1_{n-1,r+s} + \check{q}^2_{n-1,r+s}\right) + \\ \quad +(-1)^{g(r,m)+f(s,m)+1}\left(\check{q}^1_{n-1,r+s} + 0\,\check{q}^2_{n-1,r+s}\right) & \text{if } m \text{ even} \end{cases} =$$

$$= (-1)^{g(r,m)+f(s,m)+1}(\check{q}^1_{n-1,r+s} - \check{q}^2_{n-1,r+s}) =$$

$$= (-1)^{g(r+s,m)}(\check{q}^1_{n-1,r+s} - \check{q}^2_{n-1,r+s}) = au^{2r+2s}$$

because $g(r,m) + f(s,m) + 1 = g(r+s,m)$.

- $k = 2r-1, l = 2s-1$ odd:

$$(au^{2r-1}) \circ u^{2s-1} = (-1)^{G(r,m)} \left(\check{q}_{2n-2,r-1} - \check{q}_{0,r}\right) \circ (-1)^{F(s,m)} \hat{q}^1_{n-1,s} =$$

$$= \begin{cases} (-1)^{G(r,m)+F(s,m)} \Big(\mathrm{GW}^{Q_{n-1}}_{0L,3}(Q_{n-1}, A^1, A^1)\, \check{q}^2_{n-1,r+s-1} + \\ \qquad + \mathrm{GW}^{Q_{n-1}}_{0L,3}(Q_{n-1}, A^1, A^2)\, \check{q}^1_{n-1,r+s-1} \Big) + \\ \quad + (-1)^{G(r,m)+F(s,m)+1} \Big(\mathrm{GW}^{Q_{n-1}}_{L,3}(pt, A^1, A^1)\, \check{q}^2_{n-1,r+s-1} + & \text{if } m \text{ odd} \\ \qquad + \mathrm{GW}^{Q_{n-1}}_{L,3}(pt, A^1, A^2)\, \check{q}^1_{n-1,r+s-1} \Big) \\ \\ (-1)^{G(r,m)+F(s,m)} \Big(\mathrm{GW}^{Q_{n-1}}_{0L,3}(Q_{n-1}, A^1, A^1)\, \check{q}^1_{n-1,r+s-1} + \\ \qquad + \mathrm{GW}^{Q_{n-1}}_{0L,3}(Q_{n-1}, A^1, A^2)\, \check{q}^2_{n-1,r+s-1} \Big) + \\ \quad (-1)^{G(r,m)+F(s,m)+1} \Big(\mathrm{GW}^{Q_{n-1}}_{L,3}(pt, A^1, A^1)\, \check{q}^1_{n-1,r+s-1} + & \text{if } m \text{ even} \\ \qquad + \mathrm{GW}^{Q_{n-1}}_{L,3}(pt, A^1, A^2)\, \check{q}^2_{n-1,r+s-1} \Big) \end{cases} =$$

$$= \begin{cases} (-1)^{G(r,m)+F(s,m)} \left(0\, \check{q}^2_{n-1,r+s-1} + \check{q}^1_{n-1,r+s-1}\right) + \\ \quad + (-1)^{G(r,m)+F(s,m)+1} \left(\check{q}^2_{n-1,r+s-1} + 0\, \check{q}^1_{n-1,r+s-1}\right) & \text{if } m \text{ odd} \\ \\ (-1)^{G(r,m)+F(s,m)} \left(\check{q}^1_{n-1,r+s-1} + 0\, \check{q}^2_{n-1,r+s-1}\right) + \\ \quad (-1)^{G(r,m)+F(s,m)+1} \left(0\, \check{q}^1_{n-1,r+s-1} + \check{q}^2_{n-1,r+s-1}\right) & \text{if } m \text{ even} \end{cases} =$$

$$= (-1)^{G(r,m)+F(s,m)} \left(\check{q}^1_{n-1,r+s-1} - \check{q}^2_{n-1,r+s-1}\right) =$$

$$= (-1)^{g(r+s-1,m)} \left(\check{q}^1_{n-1,r+s-1} - \check{q}^2_{n-1,r+s-1}\right) = au^{2r+2s-2}$$

because $G(r,m) + F(s,m) = g(r+s-1,m)$.

- $k = 2r$ even, $l = 2s-1$ odd:

$$
\begin{aligned}
(au^{2r}) \circ u^{2s-1} &= (-1)^{g(r,m)} \big(\breve{q}^{1}_{n-1,r} - \breve{q}^{2}_{n-1,r}\big) \circ (-1)^{F(s,m)}\, \widehat{q}^{1}_{n-1,s} = \\
&= (-1)^{g(r,m)+F(s,m)} \Big(\mathrm{GW}^{Q_{n-1}}_{0L,3}(A^1, A^1, Q_{n-1})\, \breve{q}_{0,r+s} + \\
&\qquad + \mathrm{GW}^{Q_{n-1}}_{L,3}(A^1, A^1, pt)\, \breve{q}_{2n-2,r+s-1} \Big) + \\
&\quad + (-1)^{g(r,m)+F(s,m)+1} \Big(\mathrm{GW}^{Q_{n-1}}_{0L,3}(A^2, A^1, Q_{n-1})\, \breve{q}_{0,r+s} + \\
&\qquad + \mathrm{GW}^{Q_{n-1}}_{L,3}(A^2, A^1, pt)\, \breve{q}_{2n-2,r+s-1} \Big) = \\
&= (-1)^{g(r,m)+F(s,m)} \begin{cases} 0\, \breve{q}_{0,r+s} + \breve{q}_{2n-2,r+s-1} - & \\ \qquad -\breve{q}_{0,r+s} + 0\, \breve{q}_{2n-2,r+s-1} & \text{if } m \text{ odd} \\ \breve{q}_{0,r+s} + 0\, \breve{q}_{2n-2,r+s-1} + & \\ \qquad +0\, \breve{q}_{0,r+s} - \breve{q}_{2n-2,r+s-1} & \text{if } m \text{ even} \end{cases} = \\
&= (-1)^{g(r,m)+F(s,m)+m+1} \big(\breve{q}_{2n-2,r+s-1} - \breve{q}_{0,r+s}\big) = \\
&= (-1)^{G(r+s,m)} \big(\breve{q}_{2n-2,r+s-1} - \breve{q}_{0,r+s}\big) = au^{2r+2s-1}
\end{aligned}
$$

because $g(r,m) + F(s,m) + m + 1 = G(r+s,m)$.

- $k = 2r-1$ odd, $l = 2s$ even:

$$
\begin{aligned}
(au^{2r-1}) \circ u^{2s} &= (-1)^{G(r,m)} \big(\breve{q}_{2n-2,r-1} - \breve{q}_{0,r}\big) \circ (-1)^{f(s,m)}\, \widehat{q}_{0,s+1} = \\
&= (-1)^{G(r,m)+f(s,m)} \Big(\mathrm{GW}^{Q_{n-1}}_{0L,3}(Q_{n-1}, pt, Q_{n-1})\, \breve{q}_{0,r+s} + \\
&\qquad + \mathrm{GW}^{Q_{n-1}}_{L,3}(Q_{n-1}, pt, pt)\, \breve{q}_{2n-2,r+s-1} \Big) + \\
&\quad + (-1)^{G(r,m)+f(s,m)+1} \Big(\mathrm{GW}^{Q_{n-1}}_{L,3}(pt, pt, Q_{n-1})\, \breve{q}_{0,r+s} + \\
&\qquad + \mathrm{GW}^{Q_{n-1}}_{2L,3}(pt, pt, pt)\, \breve{q}_{2n-2,r+s-1} \Big) = \\
&= (-1)^{G(r,m)+f(s,m)} \big(\breve{q}_{0,r+s} + 0\, \breve{q}_{2n-2,r+s-1}\big) + \\
&\qquad + (-1)^{G(r,m)+f(s,m)+1} \big(0\, \breve{q}_{0,r+s} + \breve{q}_{2n-2,r+s-1}\big) = \\
&= (-1)^{G(r,m)+f(s,m)+1} \big(\breve{q}_{2n-2,r+s-1} - \breve{q}_{0,r+s}\big) = \\
&= (-1)^{G(r+s,m)} \big(\breve{q}_{2n-2,r+s-1} - \breve{q}_{0,r+s}\big) = au^{2r+2s-1}
\end{aligned}
$$

because $G(r,m) + f(s,m) + 1 = G(r+s,m)$.

Finally, for (6) there are 2 cases:

- $k = 2r$ even:

$$\begin{aligned}(au) \circ u^{2r} &= (-\check{q}_{0,1}) \circ (-1)^{f(r,m)}\, \widehat{q}_{0,r+1} = \\ &= (-1)^{f(r,m)+1} \Big(\mathrm{GW}^{Q_{n-1}}_{L,3}(pt,pt,Q_{n-1})\, \check{q}_{0,r+1} + \mathrm{GW}^{Q_{n-1}}_{2L,3}(pt,pt,pt)\, \check{q}_{2n-2,r} - \\ &\qquad - \mathrm{GW}^{Q_N,Q_{N-1}}_{L,1,(1)}(L;pt)\, \check{q}_{0,r+1} \Big) = \\ &= (-1)^{f(r,m)+1} \big(0\, \check{q}_{0,r+1} + \check{q}_{2n-2,r} - \check{q}_{0,r+1} \big) = \\ &= (-1)^{G(r+1,m)} \big(\check{q}_{2n-2,r} - \check{q}_{0,r+1} \big) = au^{2r+1}\end{aligned}$$

because $f(r,m) + 1 = G(r+1,m)$.

- $k = 2r-1$ odd:

$$\begin{aligned}(au) \circ u^{2r-1} &= (-\check{q}_{0,1}) \circ (-1)^{F(r,m)}\, \widehat{q}^{1}_{n-1,r} = \\ &= \begin{cases} (-1)^{F(r,m)+1} \Big(\mathrm{GW}^{Q_{n-1}}_{L,3}(pt,A^1,A^1)\, \check{q}^{2}_{n-1,r} + \\ \quad + \mathrm{GW}^{Q_{n-1}}_{L,3}(pt,A^1,A^2)\, \check{q}^{1}_{n-1,r} - & \text{if } m \text{ odd} \\ \quad - \mathrm{GW}^{Q_N,Q_{N-1}}_{L,1,(1)}(L;pt)\, \check{q}^{1}_{n-1,r} \Big) \\ \\ (-1)^{F(r,m)+1} \Big(\mathrm{GW}^{Q_{n-1}}_{L,3}(pt,A^1,A^1)\, \check{q}^{1}_{n-1,r} + \\ \quad + \mathrm{GW}^{Q_{n-1}}_{L,3}(pt,A^1,A^2)\, \check{q}^{2}_{n-1,r} - & \text{if } m \text{ even} \\ \quad - \mathrm{GW}^{Q_N,Q_{N-1}}_{L,1,(1)}(L;pt)\, \check{q}^{1}_{n-1,r} \Big) \end{cases} = \\ &= \begin{cases} (-1)^{F(r,m)+1} \big(\check{q}^{2}_{n-1,r} + 0\, \check{q}^{1}_{n-1,r} - \check{q}^{1}_{n-1,r} \big) & \text{if } m \text{ odd} \\ (-1)^{F(r,m)+1} \big(0\, \check{q}^{1}_{n-1,r} + \check{q}^{2}_{n-1,r} - \check{q}^{1}_{n-1,r} \big) & \text{if } m \text{ even} \end{cases} = \\ &= (-1)^{F(r,m)} \big(\check{q}^{1}_{n-1,r} - \check{q}^{2}_{n-1,r} \big) = (-1)^{g(r,m)} \big(\check{q}^{1}_{n-1,r} - \check{q}^{2}_{n-1,r} \big) = au^{2r}\end{aligned}$$

because $F(r,m) = g(r,m)$. We conclude that our description of the ring structure on $SH_*(T^*S^n)$ matches that of $H_*(LS^n)$, as computed in [CJY04].

Bibliography

[AS10] A. Abbondandolo and M. Schwarz. Floer homology of cotangent bundles and the loop product. *Geom. Topol.*, 14(3):1569–1722, 2010.

[Bea95] A. Beauville. Quantum cohomology of complete intersections. *Mat. Fiz. Anal. Geom.*, 2(3-4):384–398, 1995.

[BEE09] F. Bourgeois, T. Ekholm, and Y. Eliashberg. Effect of Legendrian Surgery, 2009. Available at http://arxiv.org/abs/0911.0026.

[BEH+03] F. Bourgeois, Y. Eliashberg, H. Hofer, K. Wysocki, and E. Zehnder. Compactness results in Symplectic Field Theory. *Geom. Topol.*, 7(3):799–888, 2003.

[BH11] A. Banyaga and D. Hurtubise. Cascades and perturbed Morse–Bott functions, 2011. Available at http://arxiv.org/abs/1110.4609.

[Bir01] P. Biran. Lagrangian barriers and symplectic embeddings. *Geom. Funct. Anal.*, 11(3):407–464, 2001.

[BO09a] F. Bourgeois and A. Oancea. An exact sequence for contact- and symplectic homology. *Invent. Math.*, 175(3):611–680, 2009.

[BO09b] F. Bourgeois and A. Oancea. Symplectic homology, autonomous Hamiltonians, and Morse-Bott moduli spaces. *Duke Math. J.*, 146(1):71–174, 2009.

[Bor] M. S. Borman. *Computations in filtered Hamiltonian Floer theory using symplectic field theory.* PhD thesis, The University of Chicago. (in preparation).

[Bou02] F. Bourgeois. *A Morse–Bott approach to contact homology.* PhD thesis, Stanford University, 2002. Available at http://homepages.ulb.ac.be/~fbourgeo/pspdf/thesis.pdf.

[BT82] R. Bott and L. Tu. *Differential forms in algebraic topology*, volume 82 of *Graduate Texts in Mathematics.* Springer-Verlag, New York–Berlin, 1982.

[CE] K. Cieliebak and Y. Eliashberg. *From Stein to Weinstein and back: the symplectic geometry of Stein manifolds.* (in preparation).

[CG10] N. Chris and V. Ginzburg. *Representation theory and complex geometry.* Modern Birkhäuser Classics. Birkhäuser Boston, Boston, MA, 2010.

[CJY04] R. Cohen, D. Jones, and J. Yan. The loop homology algebra of spheres and projective spaces. In *Categorical decomposition techniques in algebraic topology (Isle of Skye, 2001)*, volume 215 of *Progr. Math.*, pages 77–92. Birkhäuser, Basel, 2004.

[Dim92] A. Dimca. *Singularities and topology of hypersurfaces.* Universitext. Springer–Verlag, New York, 1992.

[DL12] L. Diogo and S. Lisi. Floer solutions in the symplectization of a prequantization space, 2012. (in preparation).

[Don96] S. Donaldson. Symplectic submanifolds and almost-complex geometry. *J. Differential Geom.*, 44(4):666–705, 1996.

[Dra04] D. Dragnev. Fredholm theory and transversality for noncompact pseudo-holomorphic maps in symplectizations. *Comm. Pure Appl. Math.*, 57(6):726–763, 2004.

[EGH00] Y. Eliashberg, A. Givental, and H. Hofer. Introduction to symplectic field theory. *Geom. Funct. Anal. Special Volume, Part II*, pages 560–673, 2000.

[EKP06] Y. Eliashberg, S. S. Kim, and L. Polterovich. Geometry of contact transformations and domains: orderability versus squeezing. *Geom. Topol.*, 10:1635–1747, 2006.

[EM02] Y. Eliashberg and N. Mishachev. *Introduction to the h-principle*, volume 48 of *Graduate Studies in Mathematics*. Amer. Math. Soc., Providence, RI, 2002.

[EP03] M. Entov and L. Polterovich. Calabi quasimorphism and quantum homology. *Int. Math. Res. Not.*, 30:1635–1676, 2003.

[EP10] Y. Eliashberg and L. Polterovich. Symplectic quasi-states on the quadric surface and Lagrangian submanifolds, 2010. Available at http://arxiv.org/abs/1006.2501.

[Fra04] U. Frauenfelder. The Arnold–Givental conjecture and moment Floer homology. *Int. Math. Res. Not.*, 42:2179–2269, 2004.

[Gir02] E. Giroux. Géométrie de contact: de la dimension trois vers les dimensions supérieures. In *Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002)*, pages 405–414. Higher Ed. Press, Beijing, 2002.

[Gro85] M. Gromov. Pseudo holomorphic curves in symplectic manifolds. *Invent. Math.*, 82(2):307–347, 1985.

[Huy05] D. Huybrechts. *Complex geometry. An introduction.* Universitext. Springer, Berlin, 2005.

[IP03] E. Ionel and T. Parker. Relative Gromov–Witten invariants. *Ann. of Math.*, 157(1):45–96, 2003.

[LT99] J. Li and G. Tian. Comparison of algebraic and symplectic Gromov–Witten invariants. *Asian J. Math.*, 3(3):689–728, 1999.

[Mir95] R. Miranda. *Algebraic curves and Riemann surfaces*, volume 5 of *Graduate Studies in Mathematics*. Amer. Math. Soc., Providence, RI, 1995.

[MP06] D. Maulik and R. Pandharipande. A topological view of Gromov–Witten theory. *Topology*, 45(5):887–918, 2006.

[MS98] D. McDuff and D. Salamon. *Introduction to Symplectic Topology*. Oxford Mathematical Monographs. Oxford University Press, second edition, 1998.

[MS04] D. McDuff and D. Salamon. *J-holomorphic curves and symplectic topology*, volume 52 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2004.

[Oan04] A. Oancea. A survey of Floer homology for manifolds with contact type boundary or symplectic homology. *Ensaios Mat.*, 7:51–91, 2004.

[Oh97] Y.-G. Oh. Symplectic topology as the geometry of action functional. I. Relative Floer theory on the cotangent bundle. *J. Differential Geom.*, 46(3):499–577, 1997.

[PSS96] S. Piunikhin, D. Salamon, and M. Schwarz. Symplectic Floer–Donaldson theory and quantum cohomology. In *Contact and symplectic geometry (Cambridge, 1994)*, volume 8 of *Publ. Newton Inst.*, pages 171–200. Cambridge Univ. Press, Cambridge, 1996.

[Rit11] A. Ritter. Topological quantum field theory structure on symplectic homology, 2011. Available at http://arxiv.org/abs/1003.1781.

[RS93] J. Robbin and D. Salamon. The Maslov index for paths. *Topology*, 32(4):827–844, 1993.

[Sal99] D. Salamon. Lectures on Floer homology. In *Symplectic geometry and topology (Park City, UT, 1997)*, volume 7 of *IAS/Park City Math. Ser.*, pages 143–229. Amer. Math. Soc., Providence, RI, 1999.

[Sch95] M. Schwarz. *Cohomology Operations from S^1-Cobordisms in Floer Homology*. PhD thesis, ETH Zürich, 1995. Available at http://www.math.uni-leipzig.de/~schwarz/diss.pdf.

[Sch99] M. Schwarz. Equivalences for Morse homology. In *Geometry and topology in dynamics (Winston-Salem, NC, 1998/San Antonio, TX, 1999)*, volume 246 of *Contemp. Math.*, pages 197–216. Amer. Math. Soc., Providence, RI, 1999.

[Sei02] P. Seidel. Fukaya categories and deformations, 2002. Available at http://arxiv.org/abs/math/0206155.

[Sei08] P. Seidel. A biased view of symplectic cohomology. In *Current developments in mathematics, 2006*, pages 211–253. Int. Press, Sommerville, MA, 2008.

[SZ92] D. Salamon and E. Zehnder. Morse theory for periodic solutions of Hamiltonian systems and the Maslov index. *Comm. Pure Appl. Math.*, 45(10):1303–1360, 1992.

[Wen10] C. Wendl. Automatic transversality and orbifolds of punctured holomorphic curves in dimension four. *Comment. Math. Helv.*, 85(2):347–407, 2010.

[Zin11] A. Zinger. The genus 0 Gromov–Witten invariants of projective complete intersections, 2011. Available at http://arxiv.org/abs/1106.1633.

Luís Miguel Pereira de Matos Geraldes Diogo

I certify that I have read this dissertation and that, in my opinion, it is fully adequate in scope and quality as a dissertation for the degree of Doctor of Philosophy.

(Yakov Eliashberg) Principal Adviser

I certify that I have read this dissertation and that, in my opinion, it is fully adequate in scope and quality as a dissertation for the degree of Doctor of Philosophy.

(Eleny Ionel)

I certify that I have read this dissertation and that, in my opinion, it is fully adequate in scope and quality as a dissertation for the degree of Doctor of Philosophy.

(Søren Galatius)

Approved for the University Committee on Graduate Studies
