SYMPLECTIC HOMOLOGY OF COMPLEMENTS OF SMOOTH DIVISORS

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1. Introduction

Symplectic homology is a version of Hamiltonian Floer homology for a class of open symplectic manifolds with contact boundary, including Liouville manifolds [FH94, Vit99]. Some of the applications of symplectic homology include special cases of the Weinstein conjecture [Vit99], and a proof that there are infinitely many distinct symplectic structures in $\mathbb{R}^{2n}$, for $n \geqslant 4$ [McL09].

One reason why symplectic homology is so relevant is the fact that it relates to other important objects of symplectic topology, including contact homology [BO09a], Rabinowitz Floer homology [CFO10] and the Fukaya category [Abo10, Gan12], as well as to string topology [AS12]. In this paper, we explore a relation between symplectic homology and quantum cohomology. For other relations between these two invariants, see Seidel [Sei16] as well as Borman and Sheridan [BS].

Symplectic homology has a wealth of algebraic structures, including a product and a Batallin–Vilkovisky operator, as observed by Seidel [Sei08]. This paper will not address the computation of these algebraic structures, but we intend to explore that direction in future work using the methods developed here.

Despite its great relevance, symplectic homology is often very hard to compute. Among the few known results is the fact that it vanishes for subcritical Weinstein manifolds (including $(\mathbb{R}^{2n}, \omega_{std})$) [Cie02] and for flexible Weinstein domains [?HERE]. Symplectic homology of the cotangent bundle of a closed, orientable spin manifold M is isomorphic as a ring to the Chas–Sullivan string topology ring of M [AS12, Abo15]. There is also a surgery formula for the symplectic homology of a Weinstein manifold [BEE12, BEE11], which has been used to compute new examples [EN15, EL15].

In this paper, we explore the problem of computing the symplectic homology of a special class of symplectic manifolds, namely complements of certain symplectic hypersurfaces Σ in closed symplectic manifolds X. An important source of examples comes from smooth ample divisors in projective varieties. Allowing for Σ to be a normal crossings divisor would be a significant improvement in generality, but is currently beyond our reach. Important progress has been made in that direction, including [Pas13, Ngu14, GP16, McL].

In this paper, we work under the following assumptions:

- (X^{2n}, ω) is a closed symplectic manifold;
- X has a symplectic submanifold Σ^{2n-2} such that $[\Sigma] \in H_{2n-2}(X; \mathbb{R})$ is Poincaré-dual to $[K\omega]$ for some $K > 0$ (in particular, $[K\omega]$ admits a lift to $H^2(X; \mathbb{Z})$);
- X is *spherically monotone*, so there is a constant $\tau_X > 0$ such that

$$\langle c_1(TX), A \rangle = \tau_X \langle [\omega], A \rangle$$

for any spherical homology class A (in the image of the Hurewicz map $\pi_2(X) \to H_2(X;\mathbb{Z})$);
- $\tau_X - K > 0$.

Given a spherical homology class A in Σ, a simple use of the direct sum property for Chern classes implies that $\langle c_1(T\Sigma), A\rangle = (\tau_X - K)\,\omega(A)$, hence (Σ, ω_Σ) is also spherically monotone (where ω_Σ is the restriction of ω to Σ), with

$$\tau_\Sigma = \tau_X - K. \tag{1.1}$$

In particular, one can make sense of the (absolute) Gromov–Witten invariants of (Σ, ω_Σ), as in [MS04].

Our assumptions also imply that the tuple (X, Σ, ω) is *strongly semi-positive*, in the sense of [TZ14, Definition 4.7(2)], hence its genus 0 relative Gromov–Witten invariants can be defined without virtual techniques, by taking an almost complex structure generic in a suitable sense [IP03, LR01, TZ14].

Under these assumptions on X and Σ, the complement $X\backslash\Sigma$ is the interior of a Liouville domain, hence its symplectic homology can be defined. The goal of this paper is to compute the symplectic homology groups of such complements, in terms of counts of rigid pseudoholomorphic spheres in Σ and X. These counts can sometimes be expressed as absolute Gromov–Witten invariants of (Σ, ω_Σ) and relative Gromov–Witten invariants of (X, Σ, ω).

The strategy of the proof is inspired by [BO09a], which relates the symplectic homology of a symplectic manifold with contact boundary with the contact homology of the boundary. The idea consists of stretching the neck along the boundary Y of a tubular neighborhood of Σ in X, and keeping track of the degenerations of Floer cascades that contribute to the differential. The limit configurations will include pieces in the symplectization of the contact manifold Y, and pieces in (the completion of) $X\backslash\Sigma$. The former project to pseudohomolorphic spheres in Σ. These are related to Gromov–Witten invariants of (Σ, ω_Σ). The curves in $X\backslash\Sigma$ are related to pseudoholomorphic spheres in X with tangency constraints in Σ. Under suitable hypotheses, they are counted by relative Gromov–Witten invariants of (X, Σ, ω). The following is the simplest formulation of our main result. We call a Morse function *perfect* if the corresponding Morse differential vanishes.

Theorem 1.1. *If Σ and $X\backslash\Sigma$ admit perfect Morse functions, then there is a chain complex computing the symplectic homology of $X\backslash\Sigma$, whose differential is expressed in terms of Gromov–Witten invariants of (Σ, ω_Σ), relative Gromov–Witten invariants of the (X, Σ, ω) and the Morse differentials in Y and in $X\backslash\Sigma$.*

For more general and precise statements, see Theorem 12.1 and Lemmas 12.4 and 12.5.

An important step in the argument is the proof of a correspondence between moduli spaces of solutions to Floer's equation in $\mathbb{R} \times Y$ and moduli spaces of punctured pseudoholomorphic spheres in $\mathbb{R} \times Y$. This is done by showing that the *difference* $\mathbf{e}$ (in a sense we make precise) between a Floer solution and a corresponding pseudoholomorphic curve solves a certain PDE, and then proving that solutions to this PDE exist and are unique (see Sections 9 and 10).

Remark 1.2. As mentioned earlier, this paper drew much inspiration from [BO09a] and its use of neck-stretching. We should point out some of the main differences between the two papers. In [BO09a], the authors consider a more general situation, in which the symplectic manifold is not necessarily obtained from the complement of a smooth symplectic divisor. For this reason, the contact boundaries Y they consider are not necessarily prequantization bundles over Σ, as in our case. The fact that our Y are so special has a number of advantages which we explore. Specifically, this enables us to obtain a bijection between moduli spaces of Floer trajectories and of pseudoholomorphic curves (rather than just a continuation map relating symplectic homology and a non-equivariant version of contact homology), which is then used to compute the symplectic homology differential explicitly. Also, the special nature of Y is useful to achieve transversality of moduli spaces, adapting standard geometric arguments (particularly using the monotonicity condition and automatic transversality) [MS04, Bou06, Dra04, Wen10]. For more on the relation between this work and [BO09a], see Remark 12.10.

Remark 1.3. The contact boundaries Y of the Liouville domains $X\backslash\Sigma$ are such that the Reeb flow is a free S^1-action. As explained in [Sei08, (3.2)], there is a spectral sequence converging to symplectic homology, whose first page consists of homology groups of Y and $X\backslash\Sigma$. One can think of this paper as computing the differential on that page of the spectral sequence (and there happen to be no higher order differentials). This is also related to Remark 12.10.

This paper is organized as follows. Part 1 provides more details about our geometric setup and the almost complex structures that we use. Part 2 describes the chain complexes that compute symplectic homology, before and after stretching the neck. We use a Morse–Bott approach, so the differentials count *cascades*. This part includes a discussion of transversality of moduli spaces and of evaluation maps, and the relevant compactness results. In Part 3, we compute Conley–Zehnder indices and use them to determine Fredholm indices of Floer cascades. We use monotonicity to show that only a limited collection of simple cascades can contribute to the differential. In Part 4, we explain why counting Floer cylinders with cascades in the differential is equivalent to counting punctured pseudohomolorphic curves with cascades, and how the latter can be expressed in terms of counts of pseudoholomorphic spheres in Σ and in X. We also relate these with Gromov–Witten invariants. Finally, in Part 5, we illustrate this computation scheme by calculating the symplectic homology of T^*S^2 (which, as we mentioned earlier, is isomorphic to the well-known homology of the free loop space of S^2, see for instance [CJY04]).

Part 1. **Setup, almost complex structures and neck-stretching**

2. Almost complex structures and neck-stretching

2.1. **Symplectic hyperplane sections.** We adopt some of our terminology from [BK13], with minor modifications.

Let (X,ω) be a closed symplectic manifold such that $[K\omega]$ admits a lift to $H^2(X;\mathbb{Z})$ for some $K > 0$. We refer to a closed codimension 2 symplectic submanifold $\Sigma^{2n-2} \overset{\iota}{\subset} X$ as a *symplectic divisor*. Write $\omega_\Sigma = \iota^*\omega$.

Let $\pi\colon E \to \Sigma$ be a complex line bundle over Σ such that $c_1(E) = [K\omega_\Sigma]$ in $H^2(\Sigma;\mathbb{R})$. Choose a Hermitian metric on E. Then, there exists a connection 1-form $\Theta \in \Omega^1(E\backslash\Sigma)$ such that $d\Theta = -K\pi^*\omega_\Sigma$. The restriction of $\alpha := -\Theta$ to the unit sphere bundle Y in E is a contact form.[1] (Y, α) is a *prequantization bundle* associated to $(\Sigma, K\omega_\Sigma)$. If the coordinate ρ stands for the norm in E measured with respect to the Hermitian metric, and $f\colon [0, +\infty) \to (0, 1]$ is a smooth function with $f(0) = 1$, $f' < 0$, then

$$\omega_E := d\left(\frac{f(\rho^2)}{K}\alpha\right) = \frac{2\rho f'(\rho^2)}{K} d\rho \wedge \alpha + f(\rho^2)\pi^*\omega_\Sigma \tag{2.1}$$

is a symplectic form on $E \backslash 0$ that extends smoothly to all of E. Its restriction to the 0-section is ω_Σ. Following [BK13], we will explicitly use $f(\rho^2) = \mathrm{e}^{-\rho^2}$. Observe that if μ is a closed 1-form on Σ, we may take a different connection form by $\Theta - \pi^*\mu$ and change the contact form correspondingly by taking $\alpha + \pi^*\mu|_Y$. This does not change the Reeb vector field, nor does it change the symplectic structure of the normal bundle.

Say that a *symplectic tubular neighborhood* of the symplectic divisor $\Sigma \overset{\iota}{\subset} X$ is an extension of ι to a symplectomorphism $\varphi\colon U \to X$, where U is a disk neighborhood of the zero section on a complex line bundle E over Σ, equipped with a Hermitian metric and a symplectic form ω_E as above. It is well known that every symplectic divisor has a symplectic tubular neighborhood (see [MS98]), where E is the symplectic normal bundle to Σ in X.

Definition 2.1. The symplectic divisor Σ is a *weak symplectic hyperplane section* of X if $[\Sigma]$ is Poincaré dual (over $\mathbb{Q}$) to $[K\omega]$ for some $K > 0$.

The symplectic divisor Σ is a *symplectic hyperplane section* if furthermore, there exists a a symplectic tubular neighborhood $\varphi\colon U \to X$ of Σ such that $(X\backslash\varphi(U), \omega)$ is a Weinstein domain.

In the following, we will focus primarily on the case where the divisor Σ is a *weak* symplectic hyperplane section Σ. The stronger assumption is used only in Proposition 7.6. Sources of examples of (weak) symplectic hyperplane sections are for instance smooth ample divisors in smooth complex projective varietes [Bir01] and Donaldson divisors in symplectic manifolds with rational $[\omega]$ [Gir02].

We now observe that the complement of the neighbourhood of the weak hyperplane section is a Liouville domain.

Lemma 2.2. *The restriction of ω to $X\backslash\varphi(U)$ is exact. There exists a 1-form λ with $d\lambda = K\omega$, and there exists a contact form α on Y induced by a connection 1-form, so that so that $\lambda|_Y = \mathrm{e}^{-1}\alpha$.*

Proof. First, we apply Poincaré Duality in $H^2(X;\mathbb{R})$, $H^2(W;\mathbb{R})$ to conclude that $[\omega|_W] = 0 \in H^2(W;\mathbb{R})$. Thus, $K\omega|_W = d\lambda_0$ for some 1-form λ_0 on W.

Fix a contact form α_0 on Y, coming from the above construction. Then, as coming from Equation (2.1) and using the explicit function $\mathrm{e}^{-\rho^2}$, we obtain

$$\omega|_Y = \frac{\mathrm{e}^{-1}}{K} d\alpha_0.$$

Thus, $d\lambda_0|_Y = \mathrm{e}^{-1} d\alpha_0$, and thus $\mathrm{e}\,\lambda_0|_Y = \alpha_0 + \nu + df$, where ν is a closed 1-form on Y and $f\colon Y \to \mathbb{R}$. From the Gysin sequence, however, we have

$$0 \to H^1(\Sigma;\mathbb{R}) \xrightarrow{\pi^*} H^1(Y;\mathbb{R}) \xrightarrow{\pi_*} H^0(\Sigma;\mathbb{R}) \xrightarrow{\wedge K\omega} H^2(\Sigma;\mathbb{R}).$$

[1]In [BK13], Θ is referred to as α^∇.

The last map is an injection from $\mathbb{R} \cong H^0(\Sigma;\mathbb{R})$ to $H^2(\Sigma;\mathbb{R})$ and so the closed 1-form $\nu = \pi^*\mu + dg$ for some closed 1-form μ on Σ and some function $g\colon Y \to \mathbb{R}$.

Extend now f, g to give smooth functions $f, g\colon W \to \mathbb{R}$. It follows then that $\lambda = \lambda_0 - \mathrm{e}^{-1}\, d(f+g)$ is a primitive of $K\omega$ whose restriction to Y is $\mathrm{e}^{-1}\,\alpha$, where $\alpha \coloneqq \alpha_0 + \pi^*\mu$, which is indeed a contact form induced by a connection 1-form on Y. □

Remark 2.3. Note that this implies that the vector field dual to λ points outwards at the boundary, and is thus a Liouville vector field and λ is a Liouville form for $K\omega$. This fact was also pointed out to us by McLean, using a different argument.

Notice also that the lemma forces the choice of a connection 1-form.

We fix a complex line bundle E with Hermitian metric and connection form Θ as above, such that, for the corresponding symplectic form ω_E, there is a symplectic tubular neighborhood $\varphi\colon U' \to X$ for some neighborhood U' of the zero-section in E. Let U be an open neighbourhood of the zero section so that $\overline{U} \subset U'$. The complement of $\varphi(U)$ therefore contains an open set, and we will allow our almost complex structures to be perturbed there.

2.2. **Almost complex structures.** We say that an almost complex structure J_Σ on Σ that is compatible with ω_Σ is *regular* if somewhere injective J_Σ-holomorphic spheres have surjective linearized Cauchy–Riemann operators. The set of regular almost complex structures is co-meagre in the space of almost complex structures compatible with ω_Σ (see [MS04]).

Since the Hermitian connection on E defines a horizontal distribution, any almost complex structure J_Σ on (Σ, ω_Σ) may be lifted to an almost complex structure J_E by requiring that J_E preserve the horizontal and vertical subspaces, is i on the vertical subspaces and is the lift of J_Σ to the horizontal subspaces. In particular, if $\pi\colon E \to \Sigma$ is the bundle projection map, we have $d\pi \cdot J_E = J_\Sigma \cdot d\pi$.

Definition 2.4. Fix a symplectic tubular neighbourhood of Σ, $\varphi\colon U \to X$. An almost complex structure J_X on (X, ω), compatible with ω, is called *admissible* if, in a neighbourhood of Σ, $J_X = \varphi_* J_E$, where J_E is the lift of an almost complex structure J_Σ on Σ. In particular, Σ has J_X invariant tangent spaces (i.e. is J_X-holomorphic).

In the following, we fix this tubular neighbourhood.

In order to compute the symplectic homology of $X \backslash \Sigma$, we will need to study Floer trajectories in the completion W of $X \backslash \Sigma$ (which we construct below). This will involve SFT-style splitting (neck-stretching) along a contact-type hypersurface in W. We will construct a suitable family of almost complex structures on W to implement this process.

Remark 2.5. The idea of this construction is very simply illustrated by an example where this becomes trivial: let $X = S^2$ with area 1, and let Σ be a point. It follows that $X \backslash \Sigma$ is symplectomorphic to the open unit disk. Its completion involves attaching $[0, +\infty) \times S^1$ to this open disk to give $\mathbb{C}$ with its standard symplectic form.

However, X can be identified with $\mathbb{CP}^1$ and thus $X \backslash \Sigma$ is, holomorphically, the complex plane, which, as we argued, is the completion of the disk.

More generally, this same phenomenon occurs. First, $X \backslash \Sigma$ must be completed by attaching a cylindrical end. This completion is then biholomorphic (but not symplectomorphic) to $X \backslash \Sigma$. Our construction is to start with an admissible almost complex structure J_X in X, complete $X\backslash\Sigma$ and equip it with a family of almost complex structures that allow us to simultaneously define symplectic homology and stretch the neck. In the trivial example of $X = \mathbb{CP}^1$, we stretch along a copy of S^1. The resulting split almost complex manifold (as in [BEH+03]) will consist of two levels: a lower level that is $\mathbb{C}$ and an upper level that is $\mathbb{R}\times S^1$. The lower level will then be biholomorphic to $X\backslash\Sigma$ with J_X, and the upper level will be biholomorphic to the normal bundle of Σ with the 0-section removed.

Let $Y \subset E$ be the unit circle bundle (with respect to the Hermitian metric). This is then a contact-type hypersurface in E, with contact form $\alpha = -\Theta$. Recall that $K\omega_E = d(\mathrm{e}^{-\rho^2}\alpha)$ on $E\backslash\Sigma$. Denote by R the Reeb vector field of α. Recall also that, given a choice of almost complex structure J_Σ on (Σ, ω_Σ), there is an induced almost complex structure J_E that is compatible with ω_E.

We now observe that there is a *symplectomorphism*

$$\begin{aligned} \psi_1\colon (E\backslash\Sigma, K\omega_E) &\to ((-\infty, 0)\times Y, d(\mathrm{e}^r\alpha)) \\ v &\mapsto (-|v|^2, \overline{v}/|v|) \end{aligned}$$

Since Σ has a neighborhood in X modelled after a neighborhood of the zero section in E, we can use ψ_1 to glue a copy of $\big([0,\infty)\times Y, d(\mathrm{e}^r\alpha)\big)$ to $(X\backslash\Sigma, K\omega)$. The resulting symplectic manifold is denoted by W and called the *completion* of $X\backslash\Sigma$. W is a Liouville manifold. Denote by λ a Liouville form on W that agrees with $\mathrm{e}^r\alpha$ on $[0,\infty)\times Y$. Notice that the restriction of this λ to $X\backslash\Sigma$ may be taken to be the 1-form λ given by Lemma 2.2.

Lemma 2.6. *Let Σ be a weak symplectic hyperplane section of (X,ω) and let J_X be an admissible almost complex structure on X. Fix a choice of Hermitian structure on $N\Sigma$ and symplectic neighborhood of Σ.*

Then, there exists a parametric family of almost complex structures J_R on the completion W of $X\backslash\Sigma$, which are cylindrical outside a compact set, so that (W, J_R) converge in the SFT sense to a split almost complex manifold whose bottom level is biholomorphic to $(X\backslash\Sigma, J_X)$ and whose upper level is $\mathbb{R}\times Y$ with an $\mathbb{R}$ translation and Reeb invariant J_Y.

Remark 2.7. Observe that $J_1 \coloneqq \psi_{1*}J_E$ becomes singular as $r\to 0$. Indeed, if R is the Reeb vector field on Y for the contact form α, we obtain:

$$J_1\partial_r = -\frac{1}{2r}R.$$

Consider now the diffeomorphism (that is not symplectic!)

$$\begin{aligned} \psi_2\colon E\backslash\Sigma &\to \mathbb{R}\times Y \\ v &\mapsto (-\ln|v|, \overline{v}/|v|) \end{aligned}$$

$J_Y \coloneqq (\psi_2)_*J_E$ is a cylindrical almost complex structure on $\mathbb{R}\times Y$, adapted to the contact form α (where we re-interpret the $U(1)$-connection 1-form as a contact form). J_Y is both $\mathbb{R}$–translation invariant and Reeb-flow invariant since these two actions correspond to the real scaling and $U(1)$ components of the $\mathbb{C}^*$ action on the hermitian line bundle E. Even though ψ_2 is not a symplectomorphism, J_Y is compatible with the symplectic form $d(\mathrm{e}^r\alpha)$.

Having in mind to stretch the neck of W along $\{0\}\times Y$ (in the sense of SFT compactness [BEH+03]), we will define a family J_R, for $R >> 0$, of almost complex

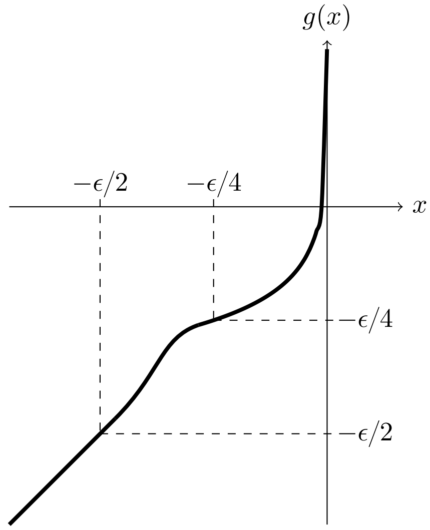

FIGURE 2.1. The function g

structures on W. Fix a small number $\epsilon > 0$ such that $(-\epsilon, \epsilon) \times Y \subset W$. We construct an almost complex structure on W which, on the symplectization piece $(-\epsilon, \infty) \times Y \subset W$, interpolates between J_1 and J_Y. Let $g\colon (-\infty, 0) \to \mathbb{R}$ be a diffeomorphism graphed Figure 2.1, such that

$$\begin{cases} g(x) = x \text{ in } (-\infty, -\epsilon/2) \\ g(-\epsilon/4) = -\epsilon/4 \\ g' > 0 \\ g'(x) = -\frac{1}{2x} \text{ for } r \geqslant -\epsilon/4 \end{cases}$$

Let $G : (-\infty, -\epsilon/4) \times Y \to (-\infty, -\epsilon/4) \times Y$ be the diffeomorphism $G(r, x) = (g(r), x)$. Define an almost complex structure J_W on W as follows:

$$J_W := \begin{cases} J_X & \text{on } W \backslash \big([-\epsilon/2, \infty) \times Y\big) \\ G_* J_1 & \text{on } [-\epsilon/2, -\epsilon/4) \times Y \\ J_Y & \text{on } [-\epsilon/4, \infty) \times Y \end{cases}$$

Since $g'(x) = 1$ if $x \leqslant -\epsilon/2$, we have

$$J_W \partial_r = -\frac{1}{2rg'(r)} R \tag{2.2}$$

on $[-\epsilon, 0) \times Y$,

We can now construct a family of almost complex structures J_R on W (the positive real parameter R is not to be confused with the Reeb vector field). For each $R \geqslant \epsilon/4$ take a diffeomorphism

$$f_R : [-\epsilon/4, \epsilon/4] \to [-R, R]$$

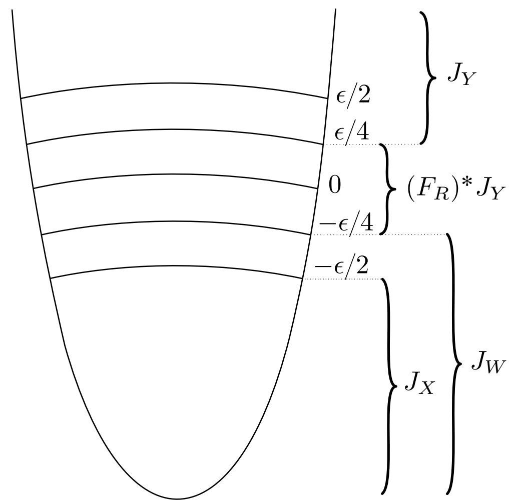

FIGURE 2.2. The almost complex structure J_R in (2.3)

such that $f_R' = 1$ near $\pm\epsilon/4$ and $f_R' > 0$. Define the diffeomorphisms

$$\begin{aligned} F_R : [-\epsilon/4, \epsilon/4] \times Y &\to [-R, R] \times Y \\ (r, x) &\mapsto (f_R(r), x) \end{aligned}$$

Let

$$J_R := \begin{cases} J_W & \text{on } W \backslash \big([-\epsilon/4, \infty) \times Y\big) \\ (F_R)^* J_Y & \text{on } [-\epsilon/4, \epsilon/4] \times Y \\ J_Y & \text{on } (\epsilon/4, \infty) \times Y \end{cases} \tag{2.3}$$

See Figure 2.2. Note that if $R = \epsilon/4$ and $f_R(r) = r$, then $J_{\epsilon/4} = J_W$.

By SFT compactness [BEH$^+$03], a sequence of finite energy J_R-holomorphic curves u_R in W has a subsequence converging to a pseudoholomorphic building with levels mapping to either $(\mathbb{R} \times Y, J_Y)$ or (W, J_W).

To define symplectic homology of W, we will use an almost complex structure J_R, for large enough R.

The following result compares J_W-holomorphic planes with J_X-holomorphic spheres, and will be useful to relate the symplectic homology differential with relative Gromov–Witten invariants.

Lemma 2.8. *There is a diffeomorphism $\psi \colon W \to X \backslash \Sigma$ such that $\psi_* J_W = J_X$. This map defines a bijection between J_W-holomorphic planes in W asymptotic to Reeb orbits of multiplicity k in Y and J_X-holomorphic spheres in X intersecting Σ at precisely one point, with order of tangency k.*

Proof. Define

$$\psi = \begin{cases} \mathrm{id} & \text{on } W \backslash \big([-\epsilon, \infty) \times Y\big) \\ \tilde{G} & \text{on } [-\epsilon, \infty) \times Y \end{cases}$$

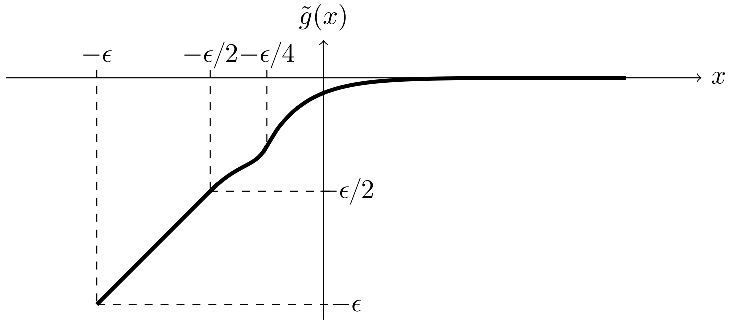

FIGURE 2.3. The function $\tilde{g}$

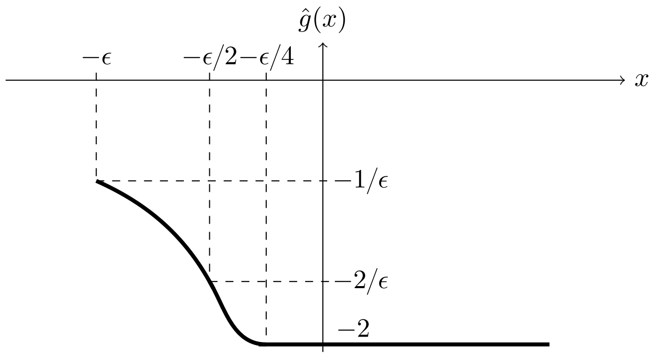

FIGURE 2.4. The function $\hat{g}$

where $\tilde{G}(r, x) = (\varphi \circ \psi_1^{-1})(\tilde{g}(r), x)$ for the diffeomorphism $\tilde{g} : [-\epsilon, \infty) \to [-\epsilon, 0)$ graphed in Figure 2.3, such that

$$\begin{cases} \tilde{g}(x) = x \text{ near } -\epsilon \\ \tilde{g}'(x) = \frac{1}{xg'(x)}\tilde{g}(x) \end{cases} \tag{2.4}$$

Recall that $\hat{g}(x) := \frac{1}{xg'(x)}$ (graphed in Figure 2.4) satisfies

$$\begin{cases} \hat{g}(x) = \frac{1}{x} \text{ near } [-\epsilon, -\epsilon/2) \\ \hat{g}(x) < 0 \text{ on } [-\epsilon, -\epsilon/4) \\ \hat{g}(x) = -2 \text{ near } -\epsilon/4 \end{cases}$$

hence we can extend $\hat{g}$ to $[-\epsilon, \infty)$ by setting it equal to -2 on $[-\epsilon/4, \infty)$. The ordinary differential equation (2.4) has a solution $\tilde{g}$. Note that, since $\hat{g}(x) = -2$ near $-\epsilon/4$, 2.4 becomes $\tilde{g}'(x) = -2\tilde{g}(x)$, hence $\tilde{g}(x) = C\,\mathrm{e}^{-2x}$ for some constant $C < 0$ and for all $x \geqslant -\epsilon/4$. This explains why the right endpoint of the target of $\tilde{g}$ is 0.

We are left with showing that $\psi_* J_W = J_X$. This is clear on $W \backslash \big([-\epsilon, \infty) \times Y\big)$. To try to minimize confusion, call the radial variable on $[-\epsilon, \infty) \times Y \subset W$ by r_1 and call the radial variable on $(\varphi \circ \psi_1^{-1})\big([-\epsilon, 0) \times Y\big) \subset X$ by r_2. Note that under

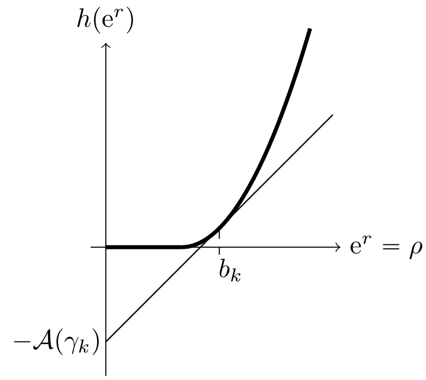

FIGURE 3.1. An admissible Hamiltonian and the graphical procedure for computing the action of a periodic orbit

ψ we have $r_2 = \tilde{g}(r_1)$. On $[-\epsilon, \infty) \times Y$, we need to check that

$$(\psi_* J_W)\left((\varphi \circ \psi_1^{-1})_* \frac{\partial}{\partial r_2}\right) = J_X\left((\varphi \circ \psi_1^{-1})_* \frac{\partial}{\partial r_2}\right) = (\varphi_* J_E)\left((\varphi \circ \psi_1^{-1})_* \frac{\partial}{\partial r_2}\right) =$$
$$= ((\psi_1)_* J_E)\frac{\partial}{\partial r_2} = J_1 \frac{\partial}{\partial r_2} = -\frac{1}{2r_2} R$$

This follows from the computation, in which we use 2.4 and 2.2

$$(\psi_* J_W)\left((\varphi \circ \psi_1^{-1})_* \frac{\partial}{\partial r_2}\right) = J_W\left((\psi^{-1} \circ \varphi \circ \psi_1^{-1})_* \frac{\partial}{\partial r_2}\right) = J_W\left((\tilde{g}^{-1})_* \frac{\partial}{\partial r_2}\right) =$$
$$= J_W\left(\frac{1}{\tilde{g}'(r_1)} \frac{\partial}{\partial r_1}\right) = \frac{r_1 g'(r_1)}{\tilde{g}(r_1)} J_W \frac{\partial}{\partial r_1} =$$
$$= \frac{r_1 g'(r_1)}{\tilde{g}(r_1)} \frac{-1}{2r_1 g'(r_1)} R = -\frac{1}{2\tilde{g}(r_1)} R = -\frac{1}{2r_2} R$$

□

Part 2. Moduli spaces and symplectic chain complex

3. The chain complex

We will describe two chain complexes associated to W whose generators are (essentially) the same, but for which the differentials are *a priori* different. Our main theorem is then a comparison of a these differentials. We will now define the group underlying these chain complexes.

Definition 3.1. Consider a function $h\colon (0, +\infty) \to \mathbb{R}$ with the following properties:

(i) $h(\rho) = 0$ for $\rho \leqslant 2$;
(ii) $h'(\rho) > 0$ for $\rho > 2$;
(iii) $h'(\rho) \to +\infty$ as $\rho \to \infty$;
(iv) $h''(\rho) > 0$ for $\rho > 2$;
(v) $h''(\rho) > C$ for some constant $C > 0$ and all $\rho \geqslant \rho_0$, where $h'(\rho_0) = 1$, and $h'''(\rho) \geqslant -\frac{h''(\rho)}{\rho}$.

From this, we obtain a Hamiltonian function H on either $\mathbb{R} \times Y$ or on W by setting $H(r, y) = h(\mathrm{e}^r)$ on $\mathbb{R}^+ \times Y$ and extending it by 0 elsewhere. We will refer to such Hamiltonians as *admissible*. See Figure 3.1.

Condition (v) will give us control over asymptotic operators associated to the Floer equation.

Since $\omega = d(\mathrm{e}^r \alpha)$ on $\mathbb{R}_+ \times Y$, the Hamiltonian vector field associated to H is $h'(\mathrm{e}^r)R$, where R is the Reeb vector field associated to α. Recall that (Y, α) is a prequantization bundle over (Σ, ω_Σ), and that the corresponding periodic Reeb orbits correspond to covers of the S^1-fibres of $Y \to \Sigma$. The periods of these orbits are positive integers, giving the multiplicities of the covers. The 1-periodic orbits of H are thus of two types:

(1) constant orbits: one for each point in
$$\overline{W} := \{w \in W \mid (dH)_w = 0\} = W \setminus \operatorname{supp}(dH);$$
(2) non-constant orbits: for each $k \in \mathbb{Z}_+$, there is a Y-family of 1-periodic X_H-orbits, contained in the level set $Y_k := \{b_k\} \times Y$, for the unique $b_k > \log 2$ such that $h'(\mathrm{e}^{b_k}) = k$. Each point in Y_k is the starting point of one such orbit.

Remark 3.2. It is important to observe that these Hamiltonians are Morse–Bott non-degenerate *except* at the boundary $\partial \operatorname{supp}(dH)$. In Section 5, we prove that the moduli spaces of curves we consider will not interact with these degenerate, constant orbits.

Recall that a family of periodic Hamiltonian orbits for a time-dependent Hamiltonian vector field is said to be Morse–Bott non-degenerate if the parametrized 1-periodic orbits form a manifold, and the tangent space of the family of orbits at a point is given by the eigenspace of 1 for the corresponding Poincaré return map. (Morse non-degeneracy requires these periodic orbits to be isolated, and the return map not to have 1 as an eigenvalue.)

By an abuse of notation, we will write $\pi_\Sigma \colon E \to \Sigma$ to denote the bundle projection, but will also denote its restrictions as $\pi_\Sigma \colon \mathbb{R} \times Y \to \Sigma$, $\pi_\Sigma \colon Y \to \Sigma$.

We also fix some auxiliary data, consisting of Morse functions and vector fields. Fix throughout a Morse function $f_\Sigma \colon \Sigma \to \mathbb{R}$ and a gradient-like vector field $Z_\Sigma \in \mathfrak{X}(\Sigma)$, which means that $\frac{1}{c}|df_\Sigma|^2 \leqslant df_\Sigma(Z_\Sigma) \leqslant c|df_\Sigma|^2$ for some constant $c > 0$. Denote the time-t flow of Z_Σ by $\varphi^t_{Z_\Sigma}$. Given $p \in \operatorname{Crit}(f_\Sigma)$, its stable and unstable manifolds (or ascending and descending manifolds, respectively) are

$$W^s_\Sigma(p) := \left\{ q \in \Sigma | \lim_{t \to \infty} \varphi^{-t}_{Z_\Sigma}(q) = p \right\}, W^u_\Sigma(p) := \left\{ q \in \Sigma | \lim_{t \to -\infty} \varphi^{-t}_{Z_\Sigma}(q) = p \right\}. \tag{3.1}$$

Notice the sign of time in the flow. We further require that (f_Σ, Z_Σ) be a Morse–Smale pair, i.e. that all stable and unstable manifolds of Z_Σ intersect transversally.

The contact distribution ξ defines an Ehresmann connection on the circle bundle $S^1 \hookrightarrow Y \to \Sigma$. Denote the horizontal lift of Z_Σ by $\pi^*_\Sigma Z_\Sigma \in \mathfrak{X}(Y)$. We fix a Morse function $f_Y \colon Y \to \mathbb{R}$ and a gradient-like vector field $Z_Y \in \mathfrak{X}(Y)$ such that (f_Y, Z_Y) is a Morse–Smale pair and the vector field $Z_Y - \pi^*_\Sigma Z_\Sigma$ is vertical (tangent to the S^1-fibers). Under these assumptions, flow lines of Z_Y project under π_Σ to flow lines of Z_Σ.

Observe that critical points of f_Y must lie in the fibres above the critical points of f_Σ (and these are zeros of Z_Y and Z_Σ respectively). For notational simplicity, we suppose that f_Y has two critical points in each fibre. In the following, given a critical point for f_Σ, $p \in \Sigma$, we denote the two critical points in the fibre above p by $\widehat{p}$ and $\check{p}$, the fibrewise maximum and fibrewise minimum of f_Y, respectively.

We will denote by $M(p)$ the Morse index of a critical point $p \in \Sigma$ of f_Σ, and by $\tilde{M}(\tilde{p}) = M(p) + i(\tilde{p})$ the Morse index of the critical point $\tilde{p} = \widehat{p}$ or $\tilde{p} = \check{p}$ of f_Y. The fibrewise index has $i(\widehat{p}) = 1$ and $i(\check{p}) = 0$.

Since H is admissible, we can identify $W \backslash \overline{W}$ with $(\log 2, \infty) \times Y$. Fix also a Morse function f_W and a gradient-like vector field Z_W on W, such that (f_W, Z_W) is a Morse–Smale pair and Z_W restricted to $[-\epsilon/4, \infty) \times Y$ is the constant vector field ∂_r, where r is the coordinate function on the first factor and $\epsilon > 0$ is as in Section 2.2. Use the diffeomorphism in Lemma 2.8 to define a pair (f_X, Z_X) on $X \backslash \Sigma$ from the pair (f_W, Z_W).

We now define the Morse–Bott symplectic chain complex of W and H. Recall that for every $k > 0$, each point in $Y_k := \{b_k\} \times Y \subset \mathbb{R}^+ \times Y$ is the starting point of a 1-periodic orbit of X_H, which cover k times their underlying Reeb orbits. For each critical point $\tilde{p} = \widehat{p}$ or $\tilde{p} = \check{p}$ of f_Y, there is a generator corresponding to the pair $(k, \tilde{p})$. We will denote this generator by $\tilde{p}_k$. The complex is then given by:

$$SC_*(W, H) = \left(\bigoplus_{k>0} \bigoplus_{p_k \in \mathrm{Crit}(f_\Sigma)} \mathbb{Z}\langle \check{p}_k, \widehat{p}_k \rangle \right) \oplus \left(\bigoplus_{x \in \mathrm{Crit}(f_W)} \mathbb{Z}\langle x \rangle \right) \tag{3.2}$$

Recall that λ is a Liouville form on W.

Definition 3.3. The *Hamiltonian action* of a loop $\gamma \colon S^1 \to W$ is

$$\mathcal{A}(\gamma) = \int \gamma^*(\lambda - H dt).$$

In particular, for any constant orbit $\gamma \in \overline{W}$, $\mathcal{A}(\gamma) = 0$ and for any orbit $\gamma_k \in Y_k$, we have

$$\mathcal{A}(\gamma_k) = \mathrm{e}^{b_k}\, h'(\mathrm{e}^{b_k}) - h(\mathrm{e}^{b_k}) > 0, \tag{3.3}$$

where b_k is as above. The action of γ_k is the negative of the y-intersept of the tangent line to the graph of h at e^{b_k}. See Figure 3.1. Note that the convexity of h implies that $\mathcal{A}(\gamma_k)$ is monotone increasing in k.

We will now define the gradings of the generators. These definitions will be justified in Section 7.1.3. For a critical point $\widetilde{p}$ of f_Y, and a multiplicity k, we define

$$|\widetilde{p}_k| = \tilde{M}(\widetilde{p}) + 1 - n + 2\frac{\tau_X - K}{K} k \in \mathbb{R}, \tag{3.4}$$

where we recall that τ_X is the monotonicity constant of X and $c_1(N\Sigma) = [K\omega_\Sigma]$.

Remark 3.4. If there is a spherical homology class $A \in H_2(X; \mathbb{Z})$ such that $\langle [\omega], A \rangle \neq 0$, then τ_X and K are rationally dependent and we can say that $|\tilde{p}_k| \in \mathbb{Q}$. If X is symplectically aspherical, the monotonicity condition is vacuously satisfied for any $\tau_X > K$.

For a critical point x of f_W, we define

$$|x| = n - M(x). \tag{3.5}$$

4. Floer moduli spaces before and after splitting

In this section, we describe the moduli spaces of cascades that contribute to the differential in the Morse–Bott symplectic homology of W, before and after stretching the neck.

We also define auxiliary moduli spaces of spherical "chains of pearls" in Σ and in X. (These are familiar objects, reminiscent of ones considered in the literature for Floer homology of compact symplectic manifolds [BC09, Oh96, PSS96].)

4.1. **Floer cascades before splitting.** Consider $H\colon W \to \mathbb{R}$ as defined in Section 3, together with the auxiliary data of (f_Σ, Z_Σ), (f_Y, Z_Y) and (f_W, Z_W). Fix an almost complex structure J_R as in (2.2) for a large $R > 0$.

Definition 4.1. A map $\tilde{v}\colon \mathbb{R} \times S^1 \to W$ is a *Floer cylinder* if

$$\partial_s \tilde{v} + J_R(\partial_t \tilde{v} - X_H(\tilde{v})) = 0. \tag{4.1}$$

Let $\mathcal{M}(x_-, x_+)$ denote the space of all such Floer cylinders such that, for fixed 1-periodic orbits $x_\pm$ of X_H, $\lim_{s\to\pm\infty} \tilde{v}(s, .) = x_\pm$.

Let S_- and S_+ denote connected spaces of 1-periodic X_H-orbits (either $\overline{W}$ or Y_k). Then we define

$$\mathcal{M}(S_-, S_+) = \bigcup_{x_-\in S_-, x_+\in S_+} \mathcal{M}(x_-, x_+).$$

Given a Floer cylinder $\tilde{v}$, we denote its asymptotic limits by $\tilde{v}(+\infty) = x_+$ and $\tilde{v}(-\infty) = x_-$.

Note that $\mathcal{A}(\tilde{v}(s_0, .)) \leqslant \mathcal{A}(\tilde{v}(s_1, .))$ if $s_0 \leqslant s_1$.

Definition 4.2. The *energy* of a Floer cylinder $\tilde{v}\colon \mathbb{R} \times S^1 \to W$ is given by:

$$E(\tilde{v}) = \int_{\mathbb{R}\times S^1} \tilde{v}^* d\lambda - \tilde{v}^* dH \wedge dt = \int_{\mathbb{R}\times S^1} ||\tilde{v}_s||^2_{J_R}\, ds \wedge dt,$$

where $||\tilde{v}_s||^2_{J_R} = d\lambda(\tilde{v}_s, J_R \tilde{v}_s)$.

Since the symplectic form is exact, a Floer cylinder $\tilde{v}$ with $\tilde{v}(\pm\infty) = x_\pm$ has

$$E(\tilde{v}) = \mathcal{A}(x_+) - \mathcal{A}(x_-) = \int x_+^*(\lambda - h(\mathrm{e}^r)dt) - \int x_-^*(\lambda - h(\mathrm{e}^r)dt).$$

A Floer cylinder is non-trivial if $E(\tilde{v}) > 0$, or equivalently, if $\tilde{v}$ is not of the form $(s, t) \mapsto \gamma(t)$ for some 1-periodic X_H-orbit γ.

Let us recall some convergence properties of finite energy Floer trajectories. This is a combination of statements of several theorems from the literature, with slightly different hypotheses. In the Morse–Bott case, we refer to [BO09b, Bou02, HWZ96a]. In the non-degenerate case, the relevant ideas have appeared in [RS01, HWZ96b], and of course the original work was done by Floer [Flo88].

Lemma 4.3. *Suppose that $\tilde{v}\colon \mathbb{R} \times S^1 \to W$ is a finite energy Floer cylinder contained in a compact subset of W. Then, for any sequence $s_k \to +\infty$ [resp., $s_k \to -\infty$] there is a subsequence we also denote $(s_k)_{k=1}^\infty$ and a 1-periodic orbit $\gamma(t)$ of the Hamiltonian vector field so that $\tilde{v}(s_k, t) \to \gamma(t)$ in C^∞.*

(1) *If γ is a Morse–Bott non-degenerate orbit, then the limit γ does not depend on the initial choice of sequence, and furthermore, we have the much stronger result that $\tilde{v}(s, t)$ converges to γ exponentially fast in s;*

(2) If γ is a non-isolated degenerate orbit, the limit may depend on the initial sequence $(s_k)_{k=1}^{\infty}$. Any two limits are connected by a family of periodic orbits of the same action.

Furthermore, if the curve converges exponentially fast, at a negative [resp., positive] puncture the rate of convergence is governed by the smallest positive [largest negative] eigenvalue of the corresponding asymptotic operator [Sie08].

Remark 4.4. There are several constructions of examples where the limit is not unique in the degenerate case. In particular, Siefring has carefully constructed such examples in the case of pseudoholomorphic curves in symplectizations [Sie16]. Similarly, a gradient trajectory of a smooth function that is not Morse–Bott can fail to converge to a single critical point, and may contain sequences converging to different critical points.

Remark 4.5. In the case of a Morse–Bott limit, a cylinder contained in a compact subset of W has finite energy if and only if it converges to 1-periodic Hamiltonian orbits at its punctures, see [Sal99, Proposition 1.21; BEH$^+$03, Proposition 5.13].

Yoel Groman pointed out to us that the assumption that $\tilde{v}$ has finite energy already implies that its image is contained in a compact subset of W, so the assumptions of Lemmas 4.3 and 4.6 can be simplified. This is because the action functional for our Hamiltonians satisfies the Palais–Smale condition [Gro15].

The next result shows that there are no non-trivial Floer cylinders whose positive asymptote x_+ is in $\overline{W}$. Note that if such a cylinder has asymptotic limits at $+\infty$ and $-\infty$, then it is contained in a compact subset of W, by the maximum principle.

Lemma 4.6. *Let $\tilde{v}\colon \mathbb{R} \times S^1 \to W$ be a solution of* (4.1) *that has finite energy and is contained in a compact subset of W. If there is a sequence $s_k^+ \to \infty$ such that the C^0-limit $\lim_{k\to\infty} \tilde{v}(s_k^+, .) = x_+ \in \overline{W}$, then $\tilde{v}$ is constant.*

Proof. Take any sequence $s_k^- \to -\infty$. Lemma 4.3 implies that there are subsequences (denoted also by $s_k^\pm$) such that $\tilde{v}(s_k^\pm, .)$ converge in C^1 (actually C^∞) to 1-periodic X_H-orbits $x_\pm$, with $x_+ \in \overline{W}$. Then,

$$\begin{aligned} 0 \leqslant E(\tilde{v}) &= \lim_{k\to\infty} E(\tilde{v}|_{[s_k^-, s_k^+]\times S^1}) = \lim_{k\to\infty} \big(\mathcal{A}(\tilde{v}(s_k^+, .)) - \mathcal{A}(\tilde{v}(s_k^-, .))\big) = \\ &= \mathcal{A}(x_+) - \mathcal{A}(x_-). \end{aligned}$$

Since $\mathcal{A}(x_+) = 0$ and $\mathcal{A}(\gamma) \geqslant 0$ for every 1-periodic X_H-orbit, we conclude that $E(\tilde{v}) = 0$. This implies the result. □

Definition 4.7. Fix $N \geqslant 0$. Let $S_0, S_1, \ldots, S_N$ be a collection of connected spaces of orbits, which can either be one of the Y_k or $\overline{W}$. Let (f_i, Z_i), $i = 0, \ldots, N+1$ be the pair of Morse function and gradient-like vector field of $f_i = f_Y$, $Z_i = Z_Y$ if $S_i = Y_k$ for some k, and $f_i = -f_W$, $Z_i = -Z_W$ if $S_i = \overline{W}$.

Let p be a critical point of f_N and q a critical point of f_0 (so p and q are generators of the chain complex (3.2)).

A *Floer cylinder with* 0 *cascades* ($N = 0$), with positive end at p and negative end at q, consists of a positive gradient trajectory $\nu\colon \mathbb{R} \to S_0$, such that $\nu(-\infty) = q$, $\nu(+\infty) = p$ and $\dot{\nu} = Z_0(\nu)$.

A *Floer cylinder with N cascades*, $N \geqslant 1$, with positive end at p and negative end at q, consists of the following data:

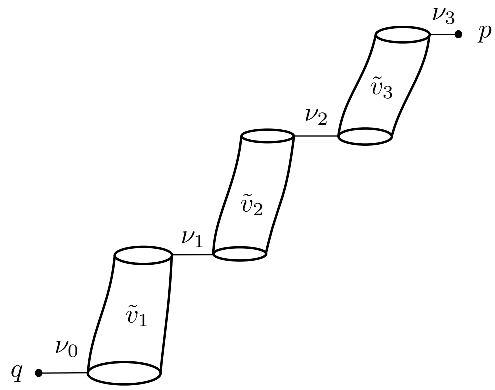

FIGURE 4.1. A Floer cylinder with 3 cascades, as in Definition 4.7, with positive end at p and negative end at q.

(i) $N-1$ length parameters $l_i > 0, i = 1, \ldots, N-1$;
(ii) Two half-infinite gradient trajectories, $\nu_0 \colon (-\infty, 0] \to S_0$ and $\nu_N \colon [0, +\infty) \to S_N$ with $\nu_0(-\infty) = q$, $\nu_N(+\infty) = p$ and $\dot{\nu}_i = Z_i(\nu_i)$ for $i = 0$ or N;
(iii) $N-1$ gradient trajectories ν_i defined on intervals of length l_i, $\nu_i \colon [0, l_i] \to S_i$ for $i = 1, \ldots, N-1$ such that $\dot{\nu}_i = Z_i(\nu_i)$;
(iv) N non-trivial Floer cylinders $\tilde{v}_i \colon \mathbb{R} \times S^1 \to W$, $i = 1, \ldots, N$, satisfying equation (4.1) where (defining $l_0 = 0$)

$$\tilde{v}_i(+\infty, \cdot) = \nu_i(0) \in S_i, \quad \tilde{v}_i(-\infty, \cdot) = \nu_{i-1}(l_{i-1}) \in S_{i-1}.$$

In the case of a Floer cylinder with $N \geqslant 1$ cascades, we refer to the non-trivial Floer cylinders $\tilde{v}_i$ as *sublevels*. See Figure 4.1 for a schematic illustration.

The energy of a Floer cylinder with N cascades is the sum of the energies of each of its N sublevels.

Lemma 4.8. *Floer cylinders with cascades do not contain sublevels $\tilde{v}_i$ such that $\lim_{s \to +\infty} \tilde{v}_i(s, .) \in \overline{W}$. If $\lim_{s \to -\infty} \tilde{v}_i(s, .) = x_- \in \overline{W}$, then $i = 1$ and $x_- \notin \partial \overline{W}$.*

Proof. The case $s \to +\infty$ follows from Lemma 4.6. For the case $s \to -\infty$, observe that ν_{i-1} is a negative flow line of Z_W, which agrees with ∂_r on $[-\epsilon/4, \infty) \times Y$ (in particular on $\partial \overline{W}$). If $i > 0$, then the sublevel $\tilde{v}_{i-1}$ must be such that $\lim_{s \to +\infty} \tilde{v}_{i-1}(s, .) \in \overline{W}$, which as we saw is impossible. If $i = 1$, then $\nu_0(-\infty) \in \mathrm{Crit}(f_W)$, and all critical points of f_W are "below $\partial \overline{W}$". Therefore, $x_- \notin \partial \overline{W}$. □

Remark 4.9. The points in $\partial \overline{W}$, which were ruled out as asymptotic orbits of sublevels by the previous Lemma, correspond to constant orbits of X_H that are not of Morse–Bott type. If finite energy solutions of Floer's equation were allowed to converge to orbits that were not of Morse–Bott type, there would be no guarantee of uniqueness of their asymptotic limits. In particular, one would not be able to claim exponential convergence to orbits, which is crucial to setup a Fredholm problem.

Lemma 4.10. *Let $S_0, S_1, \ldots, S_N$ be the spaces of orbits of a Floer cylinder with N cascades. Let $k(S_i) = k$ if $S_i = Y_k$ and let $k(S_i) = 0$ if $S_i = \overline{W}$. Then, $k(S_i)$ is monotone increasing in i.*

Proof. This follows immediately from the fact that $0 < E(\tilde{v}_i) = \mathcal{A}(\tilde{v}_i(+\infty)) - \mathcal{A}(\tilde{v}_i(-\infty))$ and from the monotonicity of $\mathcal{A}$ in the multiplicity k. □

Remark 4.11. These lemmas have a number of important consequences. In particular, a Floer cylinder with cascades between two critical points of f_W must have 0 cascades, which is to say that it consists of a flow line of $-Z_W$.

Furthermore, a Floer cylinder with cascades whose positive end is on a Y_k, $k \geqslant 1$, and whose negative end is on $\overline{W}$ must have the first sublevel connecting an orbit in a Y_l to a (constant) orbit in the interior of $\overline{W}$. This is the unique sub-level in the cascade that converges to a point in $\overline{W}$.

We are now able to define a differential on the chain complex (3.2). Given generators p, q, denote by

$$\mathcal{M}_{H,N}(q, p; J_R)$$

the space of Floer cylinders with N cascades from q to p (i.e. with negative end at q and positive end at p).

We then define

$$\partial_{\text{pre}}\, p = \sum_{|q|=|p|-1} \#\left(\mathcal{M}_{H,0}(q,p;J_R)/\mathbb{R}\right) q + \sum_{|q|=|p|-1} \sum_{N=1}^{\infty} \#\left(\mathcal{M}_{H,N}(q,p;J_R)/\mathbb{R}^N\right) q. \tag{4.2}$$

Observe that if $N = 0$, the Floer cylinder with 0 cascades is a flow line, and the $\mathbb{R}$ action is given by its standard reparametrization. If, instead, $N \geqslant 1$, the $\mathbb{R}^N$ action is given by domain translation on the N sublevels.

We call ∂_{pre} the *presplit* Floer differential, to distinguish it from the *split* differential we will define next.

Remark 4.12. To give a complete definition of ∂_{pre}, we should discuss the orientations of the moduli spaces involved and the associated signs. Instead, we will define another differential in the next section, and discuss in details the associated orientation issues in Section 8

4.2. **Floer cascades after splitting.** Recall the definition of the almost complex structures J_R, J_Y and J_W in Section 2.3. Given a sequence $\tilde{v}_{R_n}$ of finite energy J_{R_n}-Floer cylinders in W, with $R_n \to \infty$, SFT compactness [BEH+03] implies that there is a subsequence converging to an SFT building. Observe that the Hamiltonian H is supported in $(\log 2, \infty) \times Y$, which is above the hypersurface $\{0\} \times Y$ along which we stretch the neck. This makes it possible to apply the usual SFT compactness argument. A gluing argument gives that a transverse SFT-type building can be glued to obtain a "presplit" Floer cylinder, as in the previous section.

This Hamiltonian induces a function in $\mathbb{R} \times Y$, vanishing in $\mathbb{R}_- \times Y$, which we also denote by H. The non-constant orbits again form manifolds Y_k, where $k \geqslant 1$. The critical points of the Morse function $f_W : W \to \mathbb{R}$ are below the neck-stretching region. We get a different description of the differential on the chain complex (3.2) (defined above in terms of Floer cylinders with cascades) by counting *split* Floer cylinders with cascades. We begin by defining split Floer cylinders whose limits are non-constant orbits.

Definition 4.13. Let $x_\pm \in Y_{k_\pm}$ be 1-periodic orbits of X_H. A *split Floer cylinder* from x_- to x_+ consists of a map $\tilde{v} = (b, v)\colon \mathbb{R} \times S^1 \backslash \Gamma \to \mathbb{R} \times Y$, where $\Gamma =$

$\{z_1, \ldots, z_k\} \subset \mathbb{R} \times S^1$ is a (possibly empty) finite subset, such that

$$\partial_s \tilde{v} + J_Y(\partial_t \tilde{v} - X_H(\tilde{v})) = 0 \tag{4.3}$$

and in addition

- $\lim_{s \to \pm\infty} \tilde{v}(s, .) = x_\pm$;
- if $\Gamma \neq \emptyset$, then, for conformal parametrizations $\varphi_i : (-\infty, 0] \times S^1 \to \mathbb{R} \times S^1 \backslash \{z_1, \ldots, z_k\}$ of neighborhoods of the z_i, $\lim_{s \to -\infty} \tilde{v}(\varphi_i(s, .)) = (-\infty, \gamma_i)$, where the γ_i are periodic Reeb orbits in Y;
- for each Reeb orbit γ_i above, there is a J_W-holomorphic plane $U_i : \mathbb{C} \to W$ that is asymptotic to $(+\infty, \gamma_i)$.

Define a *split Floer cylinder with cascades* between two non-constant generators of the chain complex (3.2) by repeating Definition 4.7, replacing all Floer cylinders with split Floer cylinders.

See Figure 4.2 for a representation of a cascade built out of split Floer cylinders. We consider now split Floer cylinders between non-constant and constant orbits.

Definition 4.14. Let $x_+ \in Y_k$ and $x_- \in W$. A *split Floer cylinder* from x_- to x_+ is a pair $(\tilde{v}_1, \tilde{v}_2)$, where $\tilde{v}_1 = (b, v)\colon \mathbb{R} \times S^1 \backslash \Gamma \to \mathbb{R} \times Y$, $\Gamma = \{z_1, \ldots, z_k\} \subset \mathbb{R} \times S^1$ is a (possibly empty) finite subset, solving equation (4.3). In addition

- $\lim_{s \to +\infty} \tilde{v}_1(s, .) = x_+$;
- $\lim_{s \to -\infty} \tilde{v}_1(s, .) = (-\infty, \gamma)$, for some Reeb orbit γ in Y;
- if $\Gamma \neq \emptyset$, then, for conformal parametrizations $\varphi_i : (-\infty, 0] \times S^1 \to \mathbb{R} \times S^1 \backslash \{z_1, \ldots, z_k\}$ of neighborhoods of the z_i, $\lim_{s \to -\infty} \tilde{v}(\varphi_i(s, .)) = (-\infty, \gamma_i)$, where the γ_i are periodic Reeb orbits in Y;
- for each Reeb orbit γ_i above, there is a J_W-holomorphic plane $U_i : \mathbb{C} \to W$ that is asymptotic to $(+\infty, \gamma_i)$.

$\tilde{v}_2 : \mathbb{R} \times S^1 \to W$ is J_W-holomorphic and

- $\lim_{s \to +\infty} \tilde{v}_2(s, .) = (+\infty, \gamma)$;
- $\lim_{s \to -\infty} \tilde{v}_2(s, .) = x_-$.

Define a *split Floer cylinder with cascades* between a non-constant generator and a constant generator of the chain complex (3.2) by repeating Definition 4.7, replacing all Floer cylinders with split Floer cylinders.

Definition 4.15. We refer to the punctures Γ appearing in Definitions 4.13 and 4.14 as *augmentation punctures.* The corresponding J_W holomorphic planes, $U_i\colon \mathbb{C} \to W$ are referred to as *augmentation planes.*

This terminology is by analogy to linearized contact homology, where rigid such planes give an (algebraic) augmentation of the full contact homology differential. We make no such claim here.

For split Floer cylinders in $\mathbb{R} \times Y$, we introduce a new form of energy, a hybrid between the standard Floer energy and the Hofer energy used in symplectizations. Recall that the Hofer energy of a punctured pseudoholomorphic curve $\tilde{u}$ in the symplectization of Y with contact form α is given by:

$$\sup\{\int \tilde{u}^* d(\psi\alpha) \,|\, \psi\colon \mathbb{R} \to [0, 1] \text{ smooth and nondecreasing}\}.$$

In a symplectic manifold either compact or convex at infinity, the standard Floer energy of a cylinder $\tilde{v}\colon \mathbb{R} \times S^1 \to W$ is given by

$$\int \tilde{v}^*\omega - \tilde{v}^* dH \wedge dt.$$

In our situation, however, the target manifold is $\mathbb{R} \times Y$, which has a concave end. We therefore need to combine these two types of energy.

Definition 4.16. Consider a Hamiltonian $H\colon \mathbb{R} \times Y \to \mathbb{R}$ so that dH has support in $[R, \infty) \times Y$, for some $R \in \mathbb{R}$.

Let ϑ_R be the set of all non-decreasing smooth functions $\psi\colon \mathbb{R} \to [0, \infty)$ such that $\psi(r) = \mathrm{e}^r$ for $r \geqslant R$.

The *hybrid energy* E_R of $\tilde{v}\colon \mathbb{R} \times S^1 \backslash \Gamma \to \mathbb{R} \times Y$ solving Floer's equation 4.3 is then given by:

$$E_R(\tilde{v}) = \sup_{\psi \in \vartheta} \int_{\mathbb{R} \times S^1} \tilde{v}^* \left(d(\psi\alpha) - dH \wedge dt \right). \tag{4.4}$$

Notice that this is equivalent to partitioning $\mathbb{R} \times S^1 \backslash \Gamma = S_0 \cup S_1$, so that $S_0 = \tilde{v}^{-1}([R, +\infty) \times Y)$ and $S_1 = S \backslash S_0$. Then, $\tilde{v}$ has finite hybrid energy if and only if $\tilde{v}|_{S_0}$ has finite Floer energy and $\tilde{v}|_{S_1}$ has finite Hofer energy.

Equivalently, $\tilde{v}$ has finite hybrid energy if and only if the punctures $\{\pm\infty\} \cup \Gamma$ can be partitioned into Γ_F and Γ_C (with $+\infty \in \Gamma_F$, $\Gamma \subset \Gamma_C$ and $-\infty$ either in Γ_F or in Γ_C), such that in a neighbourhood of each puncture in Γ_F, the map $\tilde{v}$ is asymptotic to a Hamiltonian trajectory and in a neighbourhood of each puncture in Γ_C, the map is proper and negatively asymptotic to an orbit cylinder in $\mathbb{R} \times Y$. This follows from a variation on the arguments in [BEH+03, Proposition 5.13, Lemma 5.15]. By a slight abuse of notation, we will use the following notation to denote the Hamiltonian orbits to which such a punctured cylinder $\tilde{v}$ is asymptotic:

$$\begin{aligned} \tilde{v}(+\infty, t) &= \lim_{s \to \infty} \tilde{v}(s, t) \\ \tilde{v}(-\infty, t) &= \lim_{s \to -\infty} \tilde{v}(s, t) \quad \text{if} \quad -\infty \in \Gamma_F. \end{aligned}$$

Instead, if $-\infty \in \Gamma_C$, we will write $v(-\infty, t) = \lim_{s \to -\infty} v(s, t)$ for the Reeb orbit in Y that the curve converges to. Notice that since the cylinder is parametrized, the asymptotic limit is parametrized as well. Since there is an ambiguity of the S^1 parametrization of the Reeb orbits to which v is asymptotic at punctures $z \in \Gamma$, we will avoid using the analogous notation at punctures in Γ.

Remark 4.17. Lemmas 4.8 and 4.10 and Remark 4.11 have natural analogues for split Floer cascades.

We can now define a new differential ∂ on the chain complex (3.2). Given generators p, q, denote by

$$\mathcal{M}_{H,N}(q, p; J_Y, J_W)$$

the space of *split* Floer cylinders with N cascades from q to p (with negative end at q and positive end at p). We use again formula (4.2) to define the differential ∂, replacing the spaces $\mathcal{M}_{H,N}(q, p; J_R)$ of presplit Floer cylinders with cascades with the spaces $\mathcal{M}_{H,N}(q, p; J_Y, J_W)$ of split Floer cylinders with cascades.

We call ∂ the *split* Floer differential on (3.2). We will obtain later that this differential is of degree -1 (corresponding to our homology conventions). See Remark 6.10.

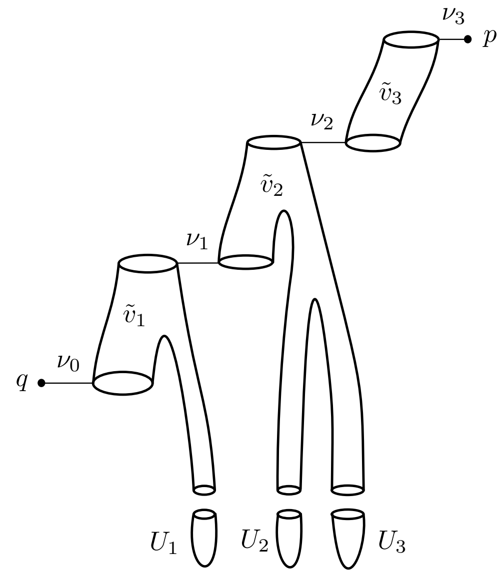

FIGURE 4.2. A split Floer cylinder with 3 cascades.

5. SKETCH OF ISOMORPHISM OF THE TWO FLOER COMPLEXES

Proposition 5.1. *For a generic choice of almost complex structures of the cylindrical type J_R, J_W and J_Y described in Section 2.2, and for $R > 0$ large enough, the presplit and split chain complexes are well-defined and are chain isomorphic. Furthermore, these complexes compute the symplectic homology of W.*

We will provide a sketch of the proof of this proposition. The main steps of such a proof are as follows:

- Show that these complexes are well-defined. For this, we need to obtain transversality for generic almost complex structures in the (very restrictive) class we consider, which are both $\mathbb{R}$ and S^1 (Reeb) invariant in the cylindrical end. This problem is similar in both the presplit and split complexes, and we address it in detail in Section 6 in the more difficult setting of the split complex. (The additional difficulty in the split complex comes from Floer cylinders with cascades that have punctures capped with planes in W. The planes can be made transverse by a perturbation of J supported in $\overline{W}$, where the symmetry condition on J is vacuous. Such a perturbation does not make the upper level(s) transverse. In the presplit case, such a building corresponds to a single cylinder with "toes" that dip in to $\overline{W}$, and thus a perturbation in $\overline{W}$ suffices to obtain transversality. See Figure 5.1.)
- For these complexes to be well-defined, we also need to establish that there are no curves counted either in ∂ or in the proof of $\partial^2 = 0$ that are asymptotic to the degenerate constant orbits at $\partial\overline{W}$. As we pointed out in Remark 4.9, these orbits fail to even be Morse–Bott, and thus represent a break-down of the standard analytical theory. We saw in Lemma 4.8 why these orbits don't appear in ∂. A similar argument (applying Lemma 4.6) implies that they also don't need to be considered when proving $\partial^2 = 0$.

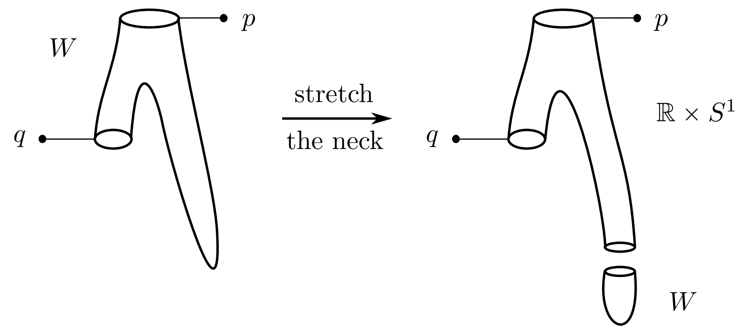

FIGURE 5.1. By perturbing J in $\overline{W}$ (where the symmetry condition is vacuous), the curve on the left can be made transverse. Such a perturbation only makes the plane in W transverse for the curve on the right.

- Compactness and gluing arguments allow us to identify the presplit and split moduli spaces for sufficiently large stretching parameter. Notice that by formula (2.3), each of the almost complex structures J_R used in stretching is biholomorphic to J_W, though the support of H is moved towards $+\infty$ by the biholomorphism. Thus, a slight modification of the standard SFT compactness theorem works in our setting. It is important to point out that after stretchig the neck, the component of a Floer cylinder that is contained in $\mathbb{R} \times Y$ is connected, which is why the split Floer cascades we described above contain information about all the presplit Floer cascades. This follows from an argument in [BO09a, Step 1 in proof of Proposition 5]. By a gluing argument, for each subcomplex of the split complex given by a bound in action, there exists a sufficiently large stretching parameter for which we get an identification with the corresponding subcomplex of the presplit complex.
- Show that the presplit complex is quasi-isomorphic to a symplectic homology complex obtained from a non-degenerate Hamiltonian. This involves constructing a continuation map Φ connecting two chain complexes. The delicate part of the proof that this is a chain map stems again from the failure of H to be Morse–Bott non-degenerate along the boundary $\partial\overline{W}$. We address this difficulty by means of the Abouzaid-Seidel Lemma 4.8 below. A usual action filtration argument gives that this is a chain isomorphism.

We now address the difficulty in the proof of the fact that a continuation map between presplit symplectic homology and a Morse perturbed version is a chain map. Recall that we defined a differential ∂_{pre} on $SC_*(W, H)$, as in (3.2), which we called presplit differential. We could construct a non-degenerate Hamiltonian $\tilde{H}\colon S^1 \times W \to \mathbb{R}$ such that $\tilde{H}(t, w) = H(w)$ outside a small neighbourhood of the periodic orbits of X_H. We require that $\tilde{H}$ be C^2-small and Morse on $W \setminus \big((-\infty, \varepsilon) \times Y\big)$ for some $\varepsilon > 0$. In addition, $\tilde{H}$ is a small time-dependent perturbation of H near the non-constant periodic orbits of X_H, using auxiliary Morse functions on the manifolds of orbits in a manner similar to [BO09b, page 73]. Picking a generic almost complex structure on W, we get a chain complex $SC_*(W, \tilde{H})$ that computes the symplectic homology of W. Denote its differential by $\tilde{\partial}$.

We could construct a chain map

$$\Phi\colon SC_*(W, H) \to SC_*(W, \tilde{H})$$

as follows. Let $\overline{H}\colon \mathbb{R} \times S^1 \times W \to \mathbb{R}$ be such that

- $\overline{H}(s, t, w) = H(w)$ if $s > 1$;
- $\overline{H}(s, t, w) = \tilde{H}(t, w)$ if $s < -1$;
- $\partial_s \overline{H} \leqslant 0$.

We will impose two more technical conditions on $\overline{H}$, on Lemma 5.4. Picking a domain-dependent almost complex structure interpolating between the ones used in the two chain complexes, we could define Φ via counts of index 0 Floer cylinders with cascades for $\tilde{H}$ (satisfying (5.1) below with $\beta = dt$).

Showing that Φ is a chain map depends on being able to glue cascades contributing to Φ with cylinders contributing to $\tilde{\partial}$, obtain 1-parameter families of cylinders connecting generators of the two complexes, and then show that the other boundary component of this family contributes to either $\Phi \circ \partial_{pre}$ or to $\tilde{\partial} \circ \Phi$ (and doing the same when gluing a contribution to ∂_{pre} with a contribution to Φ).

The presence of the degenerate constant orbits of X_H along $\partial\overline{W}$ could, in principle, be problematic for both gluing and compactness. These problems are overcome using a combination of the convergence of sequences in Lemma 4.3 and of the following result originally due to Abouzaid and Seidel (see [AS10, Lemma 7.2] and also [Rit13, Lemma 19.3]), and useful to prove a "no escape" lemma. We will state the result in more generality than necessary. In the following lemma, for instance, the annulus $S = (-\epsilon, 0] \times S^1$ should be thought of as a subset of the punctured cylinder, and β will then be the restriction of the form dt to this annulus.

Lemma 5.2. *Let Y be a co-oriented contact manifold with contact form α and denote the corresponding Reeb vector field by R.*

Let S be the annulus $S = (-\epsilon, 0] \times S^1$ with complex structure i.

Let J be an S-dependent family of almost complex structures on $[r_0, r_0 + \delta] \times Y$ and let $H\colon S \times [r_0, r_0 + \delta] \times Y \to \mathbb{R}$ be an S-dependent family of Hamiltonians with the following locally radial behaviour:

(1) $J(z, r, y)\partial_r = R$, $J(z, r, y) \ker \alpha = \ker \alpha$,

(2) $H(z, r, y) = h(z, \mathrm{e}^r)$, for some function $h(z, \rho)$.

Let β be a closed 1-form on S.

Suppose that $\tilde{v}\colon S \to [r_0, r_0 + \delta) \times Y$ satisfies the following equation:

$$0 = 2(d\tilde{v} - X_H \otimes \beta)^{0,1} = d\tilde{v} + J(z, \tilde{v})d\tilde{v} \circ i - X_H \otimes \beta - J(z, \tilde{v})X_H \otimes \beta \circ i, \tag{5.1}$$

and $\tilde{v}(\{0\} \times S^1) \subset \{r_0\} \times Y$.

Then, taking $\lambda = \mathrm{e}^r \alpha$,

$$\int_{\{0\}\times S^1} \tilde{v}^*\lambda - H(z, \tilde{v}(z))\beta \leqslant \int_{\{0\}\times S^1} (\lambda(X_H) - H)\beta = \\ = \int_{\{0\}\times S^1} \left(\mathrm{e}^{r_0} \frac{\partial h}{\partial \rho}(z, \mathrm{e}^{r_0}) - h(z, \mathrm{e}^{r_0})\right) \beta$$

Proof. Applying λ to (5.1) and using $\lambda \circ J = \mathrm{e}^r\, dr$ and $\lambda(JX_H) = 0$, we obtain

$$\tilde{v}^*\lambda + \mathrm{e}^r\, dr(d\tilde{v} \circ i) = \lambda(X_H)\beta.$$

By hypothesis, we have that $r(\tilde{v}(0,t)) \equiv r_0$. Furthermore, since the image of the annulus S is not below $r = r_0$, we have $dr(d\tilde{v} \circ i)(\frac{\partial}{\partial t}) \geqslant 0$ along $\{0\} \times S^1$, hence

$$\tilde{v}^*\lambda(\frac{\partial}{\partial t}) \leqslant \lambda(X_H)\beta(\frac{\partial}{\partial t}).$$

The result now follows. □

Following [AS10, Rit13], we introduce the following terminology:

Definition 5.3. Let (S, j) be a Riemann surface and let β be a 1-form on S so $d\beta \leqslant 0$. Let (W, ω) be a symplectic manifold, and let J be an S-dependent family of almost complex structures on W, compatible with ω. Let $H\colon S \times W \to \mathbb{R}$ be as S-dependent Hamiltonian. We write dH to indicate the (total) differential of H and $d_W H$ to denote the S-dependent family of 1-forms on W given by $d_W H|_{z \in S} = d(H(z, \cdot))$.

For a map $\tilde{v}\colon S \to W$, we define its

- *Topological energy*: $E_{\text{topo}}(\tilde{v}) = \int_S \tilde{v}^*(\omega - d(H\beta))$;
- *Geometric energy*: $E_{\text{geom}}(\tilde{v}) = \int_S \tilde{v}^*\omega - (\tilde{v}^* d_W H) \wedge \beta$.

Note that for any solution to Floer's equation (5.1), $E_{\text{geom}}(\tilde{v}) = \int_S \|\partial_s \tilde{v}\|^2 \geqslant 0$ is (pointwise) non-negative. We say that the triple (β, H, J) is *monotone* if the topological energy dominates the geometric energy pointwise (in the sense that the integrand in the geometric energy is equal to the integrand in the topological energy times a function $f \leqslant 1$). Observe that they are equal if H is z-independent and $\beta = dt$.

Lemma 5.4. *Let $(W, d\lambda)$ be an exact symplectic manifold with cylindrical end $((r_0, +\infty) \times Y, d(\mathrm{e}^r\,\alpha))$.*

Let (β, H, J) be a monotone triple on the cylinder $(\mathbb{R} \times S^1, i)$ with 1-form $\beta = dt$ and with $H_{(+\infty, t)}$ an admissible Hamiltonian as in Definition 3.1.

Let γ_+ be a 1-periodic orbit of $H_{(+\infty,t)}$ contained in the level $\{r_1\} \times Y$ for some $r_0 < r_1 \leqslant \log 2$ and let γ_- be a 1-periodic orbit of $H_{(-\infty,t)}$, contained entirely below the the level $\{r_1\} \times Y$.

Suppose furthermore that $H\colon \mathbb{R} \times S^1 \times W \to \mathbb{R}$ and J are locally radial (in the sense of Lemma 5.2) on $(r_1 - \delta, r_1 + \delta) \times Y$, with the additional condition that $H(s, t, r, y) = h(s, \mathrm{e}^r)$ for some function $h(s, \rho)$, with

$$\mathrm{e}^r \frac{\partial h}{\partial \rho}(s, \mathrm{e}^r) - h(s, \mathrm{e}^r) \geqslant 0$$

$$\partial_s\Big(\mathrm{e}^r \frac{\partial h}{\partial \rho}(s, \mathrm{e}^r) - h(s, \mathrm{e}^r)\Big) \leqslant 0.$$

Then, there does not exist any $\tilde{v}\colon \mathbb{R} \times S^1 \to W$ solving Floer's equation (5.1) for which there exists a sequence $s_k \to +\infty$ along which $\tilde{v}(s_k, t) \to \gamma_+(t) \subset \{r_1\} \times Y$ and for which $\tilde{v}(s, t) \to \gamma_-(t), s \to -\infty$.

Proof. Suppose that there was a $\tilde{v}$ as in the statement. Consider a smooth function $\tilde{r}\colon W \to \mathbb{R}$ that is equal to the first coordinate r on $(r_1 - \delta/2, r_1 + \delta/2) \times Y$ and constant outside of $(r_1 - \delta, r_1 + \delta) \times Y$. We can assume that $\delta > 0$ is small enough that $\tilde{r} \circ \gamma_-(t) < r_1 - \delta/2$. Take a regular value $r_1 - c$ of $\tilde{r} \circ \tilde{v}\colon \mathbb{R} \times S^1 \to \mathbb{R}$ in the interval $(r_1 - \delta/2, r_1)$.

By hypothesis, there exists a k_0 sufficiently large so that for each $k \geqslant k_0$, $\min_{t \in S^1} \tilde{r}(\tilde{v}(s_k, t)) > r_1 - c/2$. Fix such a k.

Define the subdomain S_k of $\mathbb{R} \times S^1$ by

$$S_k = \{(s,t) \in (-\infty, s_k) \times S^1 \mid \tilde{r}(\tilde{v}(s,t)) > r_1 - c\}.$$

Note that while this subdomain may be disconnected, its closure in $\mathbb{R} \times S^1$ is compact, due the fact that $\tilde{v}(s,t) \to \gamma_-(t)$ as $s \to -\infty$ and that $\tilde{r} \circ \gamma_-(t) < r_1 - \delta/2$. Since $k \geqslant k_0$, this set is non-empty.

By the fact that $r_1 - c$ is a regular value of $\tilde{r} \circ \tilde{v}$, we have that ∂S_k is a collection of smooth, embedded closed curves. Let Γ be the collection of embedded closed curves in $\mathbb{R} \times S^1$ so that the boundary of S_k is $\partial S_k = \Gamma \cup \{s_k\} \times S^1$. Let Γ be oriented with the orientation induced as the boundary of S_k. As an aside, note that the projection of $\Gamma \subset \mathbb{R} \times S^1$ to S^1 has total degree -1, since it separates the cylinder.

By construction, $\tilde{v}(\Gamma) \subset \{r_1 - c\} \times Y$, and by definition $\tilde{r}(\tilde{v}(S_k)) \subset (r_1 - c, r_1 + \delta]$. Combining Stokes's Theorem with Lemma 5.2 (and implicitly using biholomorphisms between neighborhoods of the connected components of Γ in S_k and annuli of the form $(-\epsilon, 0] \times S^1$), we obtain

$$\begin{aligned}
E_{\text{topo}}(\tilde{v}|_{S^k}) &= \int_{\{s_k\} \times S^1} \tilde{v}^*\lambda - H(s,t,\tilde{v}(s,t))dt + \int_{\Gamma} \tilde{v}^*\lambda - H(s,t,\tilde{v}(s,t))dt \\
&\leqslant \int_{s=s_k} \tilde{v}^*\lambda - H(s_k,t,\tilde{v}(s_k,t))dt + \int_{\Gamma} (\mathrm{e}^{r_1-c}\frac{\partial h}{\partial \rho}(s,\mathrm{e}^{r_1-c}) - h(\mathrm{e}^{r_1-c}))dt \\
&= \int_{s=s_k} \tilde{v}^*\lambda - H(s_k,t,\tilde{v}(s_k,t))dt - \\
&\qquad\qquad - \int_{s=s_k} (\mathrm{e}^{r_1-c}\frac{\partial h}{\partial \rho}(s,\mathrm{e}^{r_1-c}) - h(\mathrm{e}^{r_1-c}))dt + \\
&\qquad\qquad + \int_{S_k} \partial_s\big(\mathrm{e}^{r_1-c}\frac{\partial h}{\partial \rho}(s,\mathrm{e}^{r_1-c}) - h(\mathrm{e}^{r_1-c})\big)\, ds \wedge dt \\
&\leqslant \int_{s=s_k} \tilde{v}^*\lambda - H(s_k,t,\tilde{v}(s_k,t))dt.
\end{aligned}$$

As the parameter $k \to +\infty$, the right hand side converges to the action of γ_+, which is 0. Furthermore, since the Floer data is assumed to be monotone, $E_{\text{geom}}(\tilde{v}|_{S_k}) \leqslant E_{\text{topo}}(u|_{S_k})$ and since the S_k are nested, $E_{\text{geom}}(\tilde{v}|_{S_k})$ is monotone non-decreasing in k. It follows that $E_{\text{geom}}(\tilde{v}|_{S_k}) = 0$ for all k. As S_k has non-empty interior, we conclude that $\partial_s \tilde{v} = 0$, which contradicts the fact that $\tilde{v}(s,t) \to \gamma_-$ as $s \to -\infty$. □

We now explain how the previous result is useful in showing that Φ is a chain map. We may arrange for $\overline{H}$ to satisfy the radial conditions of Lemma 5.4. The only way that Φ could fail to be a chain map is if there were a family of finite energy Floer cylinders for $\overline{H}$ with a subsequence converging to a Floer cylinder $\tilde{v}\colon \mathbb{R} \times S^1 \to W$ with the following property: there is a sequence $s_k \to \infty$ such that the sequence of loops $\tilde{v}(s_k, .)$ converges to $\partial \overline{W}$ as in Lemma 4.3. But this situation is ruled out by Lemma 5.4.

Combining these compactness results with the standard constructions in symplectic homology, we obtain that our chain complex $SC_*(W, H)$ is chain homotopic to $SC_*(W, \tilde{H})$, which we know computes symplectic homology.

6. Transversality for the Floer and holomorphic moduli spaces

In this section, we will build the transversality theory needed for the Floer cascades in Section 4.2. In the process, we will also discuss transversality for pseudoholomorphic curves in X and in Σ, which will be necessary for the proof of our main result.

6.1. Statements of transversality results.

Before stating the main result of this section, we will introduce some definitions.

Recall from Section 4.2 that the differential ∂ counts elements in moduli spaces $\mathcal{M}_{H,N}(y, x; J_Y, J_W)$ of split Floer cylinders with cascades (modulo automorphisms), where J_Y is a lift to $\mathbb{R} \times Y$ of an almost complex structure J_Σ on Σ and J_W is an almost complex structure in W that agrees with J_Y outside of a compact subset, as in Section 2.2, and y, x are critical points of f_Y together with a multiplicity, or are critical points of f_W.

Definition 6.1. Denote the space of almost complex structures in Σ that are compatible with ω_Σ by $\mathcal{J}_\Sigma$.

Let $\mathcal{J}_W$ denote the the space almost complex structures J_W as described in Section 2.2, interpreted either as an almost complex structure on W or as an almost complex structure on X.

There is a surjective map $P\colon \mathcal{J}_W \to \mathcal{J}_\Sigma$, obtained by restricting J_W to J_Y, and then projecting to J_Σ.

Let $\varphi(U) \subset X$ be the fixed open neighbourhood of Σ on which any $J_W \in \mathcal{J}_W$ is standard (where φ is the tubular neighbourhood of Σ of Definition 2.1).

(In particular, none of the spheres in X is permitted to be constant.)

Notice that P is an open map since the almost complex structures in $\mathcal{J}_W$ have a standard form in the fixed open neighborhood of Σ in X, with respect to the diffeomorphism in Lemma 2.8. Also notice that Lemma 2.8 allows us to simultaneously view $J_W \in \mathcal{J}_W$ as an almost complex structure on W and as an almost complex structure on X.

Lemma 6.2. *Let $\tilde{v}\colon \mathbb{R} \times S^1 \backslash \Gamma \to \mathbb{R} \times Y$ be a finite hybrid energy Floer cylinder in $\mathbb{R} \times Y$ (as in Definition 4.16), converging to Hamiltonian orbits in the manifolds Y_+ and Y_- at $\pm\infty$, and with finitely many punctures at $\Gamma \subset \mathbb{R} \times S^1$ converging to Reeb orbits in $\{-\infty\} \times Y$. Then, the projection $\pi_\Sigma \circ \tilde{v}$ extends to a smooth J_Σ-holomorphic sphere $\pi_\Sigma \circ \tilde{v}\colon \mathbb{C}P^1 \to \Sigma$.*

Proof. The projection $\pi_\Sigma \circ \tilde{v}$ is J_Σ-holomorphic on $\mathbb{R} \times S^1$, since H is admissible (as in Definition 3.1) The result now follows from Gromov's removal of singularities theorem together with the finiteness of the energy of $\pi_\Sigma \circ \tilde{v}$. □

Similarly, by using the biholomorphism constructed in Lemma 2.8, we obtain the following.

Lemma 6.3. *Let $u\colon \mathbb{C} \to W$ be a J_W-holomorphic plane with finite Hofer energy. Then, under the identification of W with $X \backslash \Sigma$ from Lemma 2.8, u admits an extension to a J_X-holomorphic sphere $u\colon \mathbb{C}P^1 \to X$.*

In order to describe the projection to Σ of the levels of a split Floer cylinder with N cascades that map to $\mathbb{R} \times Y$, we introduce the following:

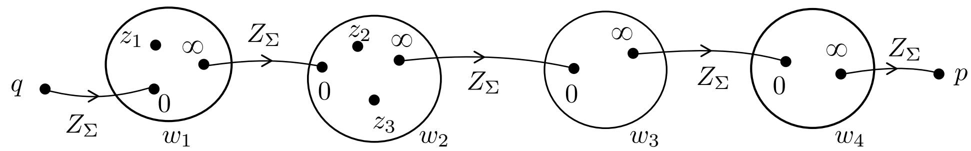

FIGURE 6.1. A chain of 4 pearls from q to p with 3 marked points.

Definition 6.4. A *chain of pearls* from q to p, where p and q are critical points of f_Σ, consists of the following:

- $N \geqslant 0$ parametrized J_Σ-holomorphic spheres w_i in Σ with two distinguished marked points at 0 and ∞ and a possibly empty collection of additional marked points $z_1, \ldots, z_k$ on the union of the N domains (distinct from 0 or ∞ in each of the N spherical domains); the spheres are either non-constant or contain at least one additional marked point;
- if $N = 0$, an infinite positive flow trajectory of Z_Σ from q to p; if $N \geqslant 1$, a half-infinite trajectory of Z_Σ from $w_N(\infty)$ to p, a half-infinite trajectory of Z_Σ from q to $w_1(0)$ and positive finite length trajectories of Z_Σ connecting $w_i(\infty)$ to $w_{i+1}(0)$, $i = 1, \ldots, N-1$.

See Figure 6.1. If such a chain of pearls is the projection to Σ of the components in $\mathbb{R} \times Y$ is a split Floer cylinder, then the additional marked points in the pseudoholomorphic spheres correspond to punctures in the Floer cylinders, where they converge to cylinders over Reeb orbits that are capped by planes in W.

Notice that the geometric configuration of two spheres touching at a critical point of f_Σ admits an interpretation as a chain of pearls in Σ, since the critical point is the image of any positive length flow line with that initial condition.

A *chain of pearls with a sphere in* X from q to p, where p is a critical point of f_Σ and q is a critical point of f_X, consists of the following:

- $N \geqslant 1$ parametrized J_Σ-holomorphic spheres w_i in Σ with two distinguished marked points at 0 and ∞ and a possibly empty collection of additional marked points $z_1, \ldots, z_k$ on the union of the N domains (distinct from 0 or ∞);
- a parametrized non-constant J_X-holomorphic sphere v in X;
- a half-infinite trajectory of Z_Σ from $w_N(\infty)$ to p, a half-infinite trajectory of $-Z_X$ from q to $v(0)$ (recall that Z_X is the push-forward of Z_W by the inverse of the map from Lemma 2.8);
- positive length trajectories of Z_Σ from $w_i(\infty)$ to $w_{i+1}(0)$ for $i = 1, \ldots, N-1$;
- the sphere in X touches the first sphere in Σ: $w_1(0) = v(+\infty)$;
- the spheres $w_2, \ldots, w_N$ satisfy the stability condition that they are either non-constant or contain at least one of the additional marked points (v is automatically non-constant and w_1 is allowed to be unstable).

An *augmented chain of pearls* [or an *augmented chain of pearls with a sphere in* X] from q to p is a chain of pearls [or chain of pearls with a sphere in X] together with k J_X-holomorphic spheres $v_i \colon \mathbb{CP}^1 \to X$, $i = 1, \ldots, k$, with the following additional properties:

- for each $z \in \mathbb{CP}^1$, $v_i(z) \in \Sigma$ if and only if $z = \infty$;

- if the puncture z_i is in the domain of the holomorphic sphere $w_{j_i} \colon \mathbb{CP}^1 \to \Sigma$, then $w_{j_i}(z_i) = v_i(\infty)$.

From the Lemmas above and the fact that the trajectories of Z_Y cover trajectories of Z_Σ, it follows that a Floer cylinder with N cascades projects to a chain of pearls or a chain of pearls with a sphere in X. Additionally, if any of the sublevels have augmentation planes, then those correspond to spheres in X passing through Σ at the images of the corresponding marked points in the chain of pearls.

Observe that we allow the sphere w_1 to be unstable in the definition of a chain of pearls in Σ with a sphere in X. The case in which w_1 is a constant curve without marked points corresponds to the situation in which the corresponding Floer cascade contains a non-trivial Floer cylinder $\tilde{v}_1$ contained in a single fibre of $\mathbb{R} \times Y \to \Sigma$, and has the asymptotic limits $\tilde{v}_1(+\infty, t)$ on a Hamiltonian orbit and $\tilde{v}_1(-\infty, t)$ on a closed Reeb orbit in $\{-\infty\} \times Y$ (see Figure 7.3 below, where this case is considered in detail). The Floer cylinder $\tilde{v}_1$ in $\mathbb{R} \times Y$ is non-trivial and hence stable, whereas the corresponding sphere w_1 in Σ is unstable. Since we do not quotient by automorphisms (yet), this does not pose a problem.

Definition 6.5. A chain of pearls in Σ is *simple* if each sphere is either simple (i.e. not multiply covered, [MS04, Section 2.5]) or is constant, and if the image of no sphere is contained in the image of another. If the chain of pearls has a sphere v in X, we require v to be somewhere injective.

An augmented chain of pearls is simple if the chain of pearls is simple and the augmentation spheres in X are somewhere injective, none has image contained in the open neighbourhood $\varphi(U)$ (as in Definition 6.1), and no sphere in X has image contained in the image of another sphere in X.

Remark 6.6. Recall that a chain of pearls with a sphere in X has a distinguished sphere v in X for which $v(0)$ is on the descending manifold of a critical point q of f_W. By the construction of f_W, this forces the image of v to intersect the complement of the tubular neighbourhood of Σ. As we will see in Section 7.2, Fredholm index considerations related to monotonicity will force the augmentation planes/spheres to leave the tubular neighbourhood.

Remark 6.7. Notice that our condition on a simple chain of pearls is slightly different than the condition imposed in [MS04, Section 6.1], with regard to constant spheres. For a chain of pearls to be simple by our definition, constant spheres may not be contained in another sphere, constant or not. In [MS04], there is no such condition on constant spheres.

Definition 6.8. Given a finite hybrid energy Floer cylinder with N cascades, we obtain an augmented chain of pearls (possibly with a sphere in X) by the following construction:

(1) cylinders in $\mathbb{R} \times Y$ are projected to Σ: by Lemma 6.2 these form holomorphic spheres in Σ;
(2) planes in W are interpreted as spheres in X by Lemma 6.3;
(3) flow lines of the gradient-like vector field Z_Y are projected to flow lines of Z_Σ.

We refer to this augmented chain of pearls in Σ (possibly with a sphere in X) as the projection of the Floer cylinder with N cascades.

A finite hybrid energy Floer cylinder with N cascades is *simple* if the resulting chain of pearls is simple.

Given generators x, y of the chain complex (3.2), denote by

$$\mathcal{M}^*_{H,N}(y, x; J_Y, J_W)$$

the space of *simple* split Floer cylinders with N cascades from y to x. Recall that if x or y is in in $\mathbb{R} \times Y$, the corresponding generator is described by a critical point $\widetilde{p}$ of f_Y (which can be either $\check{p}$ or $\hat{p}$), together with a multiplicity k. If instead, x or y is in W, it corresponds to a critical point of f_W.

Proposition 6.9. *There exists a residual set $\mathcal{J}_W^{reg} \subset \mathcal{J}_W$ of almost complex structures such that for each $J_W \in \mathcal{J}_W^{reg}$, $\mathcal{M}^*_{H,N}(y, x; J_Y, J_W)$ is a manifold.*

If $N = 0$ and thus x, y are generators in $\mathbb{R} \times Y$, then $x = \widetilde{p}_k$, $y = \widetilde{q}_k$ for the same multiplicity k, and

$$\dim_{\mathbb{R}} \mathcal{M}^*_{H,0}(y, x; J_Y, J_W) = |\widetilde{p}_k| - |\widetilde{q}_k|.$$

If $N \geqslant 1$, and q, p are generators in $\mathbb{R} \times Y$, then $x = \widetilde{p}_{k_+}$, $y = \widetilde{q}_{k_-}$ and

$$\dim_{\mathbb{R}} \mathcal{M}^*_{H,N}(y, x; J_Y, J_W) = |x| - |y| + N - 1$$

Finally, if $y \in W$ and $x \in \mathbb{R} \times Y$, then $x = \widetilde{p}_k$ and $y = q \in \mathrm{Crit}(f_W)$ and

$$\dim_{\mathbb{R}} \mathcal{M}^*_{H,N}(y, x; J_Y, J_W) = |x| - |y| + N.$$

Furthermore, the image $P(\mathcal{J}_W^{reg}) \subset \mathcal{J}_\Sigma$ (recall Definition 6.1) is residual and consists of almost complex structures that are regular for simple pseudoholomorphic spheres in Σ.

The two different formulas involving N reflect the fact that N counts the number of cylinders in $\mathbb{R} \times Y$. In the case of a Floer cascade that descends to W, there are therefore $N + 1$ cylinders in the cascade.

Remark 6.10. These index formulas imply that the split symplectic homology differential is of degree -1. Indeed, observe that the case $N = 0$ corresponds to a pure Morse configuration and doesn't depend on any almost complex structure. We count rigid flow lines modulo the $\mathbb{R}$-action, and thus require $|x| - |y| = 1$. For generators $y, x \in \mathbb{R} \times Y$, we consider these N cylinders modulo the $\mathbb{R}$-action on each one, giving an $\mathbb{R}^N$-action. From this, a rigid configuration has $|x| - |y| + N - 1 = N$. For the case with $y \in W$, we have $N + 1$ cylinders in the Floer cascade, so we have a rigid configuration modulo the $\mathbb{R}^{N+1}$-action when $N + 1 = |x| - |y| + N$.

Note that when both x, y are generators in $\mathbb{R} \times Y$, the moduli space $\mathcal{M}^*_{H,N}(y, x; J_Y, J_W)$ will depend on J_W only insofar as augmentation planes appear, otherwise it depends only on J_Y.

The split Floer differential ∂, introduced at the end of Section 4.2, was defined by counting elements in $\mathcal{M}_{H,N}(y, x; J_Y, J_W)$. We will see in Propositions 7.4 and 7.5 that our standing assumptions imply that this is equivalent to counting simple configurations in $\mathcal{M}^*_{H,N}(y, x; J_Y, J_W)$.

The rest of this section will be devoted to the proof of Proposition 6.9. It will proceed in the following steps:

- Section 6.2 describes the Fredholm set-up for Floer cascades. In Section 6.2.1 and in Appendix A, we discuss the necessary function spaces and linear theory for the Morse–Bott problems. Then, Section 6.2.2 splits the

linearization of the Floer operator in such a way as to split the transversality problem into two problems. The first is a Cauchy–Riemann-type operator acting on sections of a complex line bundle, and it is transverse for topological reasons (automatic transversality). The second is a transversality problem for a Cauchy–Riemann operator in Σ.
- Section 6.3 adapts the transversality arguments from [MS04] in order to obtain transversality for chains of pearls in Σ.
- Section 6.4 shows transversality for the components of the cascades contained in W. This problem is translated into the equivalent problem of obtaining transversality for spheres in X with order of contact conditions at Σ, together with evaluation maps. The main technical point is an extension of the transversality results from [CM07].
- Finally, Section 6.5 uses the splitting from Section 6.2.2 to lift the transversality results in Σ to obtain transversality for Floer cylinders *with cascades*, and to finish the proof of Proposition 6.9.

6.2. **A Fredholm theory for Floer cascades.**

6.2.1. *Sobolev spaces for the Morse–Bott problem.* The first step in the proof of Proposition 6.9 is to set-up the appropriate Fredholm theory. Recall that one of the important technical difficulties is that the periodic Hamiltonian orbits and the periodic Reeb orbits that we take as asymptotic boundary conditions for our punctured cylinders come in Morse–Bott families. Given a Floer solution $\tilde{v}\colon \mathbb{R} \times S^1\backslash\Gamma\colon \mathbb{R}\times Y$, we need to consider exponentially weighted Sobolev spaces of sections of the pull-back bundle $\tilde{v}^*T(\mathbb{R}\times Y)$, so that the relevant linearized operators are Fredholm. For $\delta > 0$, we denote by $W^{1,p,\delta}(\mathbb{R}\times S^1\backslash\Gamma, v^*T(\mathbb{R}\times Y))$ the space of sections that decay exponentially like $\mathrm{e}^{-\delta|s|}$ near the punctures (also in cylindrical coordinates near the punctures Γ). (In certain circumstances, we will find it useful to consider sections with exponential growth, and denote these by taking $\delta < 0$.)

In order to consider a parametric family of punctured cylinders in which the asymptotic limits move in their Morse–Bott families, we let $\mathbf{V}$ be a collection of vector spaces, associating to each puncture $z \in \Gamma \cup \{\pm\infty\}$ a vector subspace V_z of the tangent space to the corresponding Morse–Bott family of orbits. For $\delta > 0$, we then consider the space of sections $W^{1,p,\delta}_{\mathbf{V}}(\mathbb{R}\times S^1\backslash\Gamma, v^*T(\mathbb{R}\times Y))$ that converge exponentially at each puncture z to a vector in the corresponding vector space V_z.

These vector spaces also admit a description as the kernels of asymptotic operators associated to the linearized Floer operator at $\tilde{v}$, and can also be described by the kernel of a differential operator associated to the linearized Hamiltonian/Reeb flow. (See, for instance, [Sie08, Section 2.1] and the discussion in the Appendix A.)

These function spaces and background on Cauchy–Riemann operators on Hermitian vector bundles over punctured surfaces are discussed in greater detail in Appendix A.

Using this point of view, and the geometry of our problem, we will now proceed to split the linearized problem into two sub-problems, each of which can be addressed separately.

Remark 6.11. In this paper, we will not always be careful to specify how small δ has to be. It is worth pointing out that there is no value of δ that works for all moduli spaces of (possibly perturbed) holomorphic curves. The reason is that we need $|\delta|$ to be smaller than the absolute value of all eigenvalues in the spectra of the relevant linearized operators. As we can see in Table 2 in Appendix A, some of these asymptotic operators have smallest positive eigenvalue $\frac{1}{2}\left(-C+\sqrt{C^2+16\pi^2}\right)$, which becomes arbitrarily small as $C \to \infty$. The relevant value for C here is $h''(e^{b_k})e^{b_k}$, which can become arbitrarily large as the multiplicity $k \to \infty$. Since we are only ever interested in curves connecting orbits of bounded multiplicities, we can always choose δ sufficiently small.

6.2.2. *The linearization at a Floer solution.* We now adapt an observation first used in [Dra04, Bou06], to show that the linearization of the Floer operator is upper triangular with respect to the splitting of $T(\mathbb{R}\times Y)$ as $(\mathbb{R}\oplus\mathbb{R}R)\oplus\xi$. We then describe the non-zero blocks in this upper triangular presentation of the operator. The two diagonal terms are of special importance: one will be a Cauchy–Riemann-type operator acting on sections of a complex line bundle, and the other can be identified with the linearization of the Cauchy–Riemann operator for spheres in Σ.

We now explain this construction in more detail, starting with some notation. Let $\tilde v\colon \mathbb{R}\times S^1\backslash\Gamma \to \mathbb{R}\times Y$ be a Floer solution with punctures Γ, of finite hybrid energy, asymptotic to a closed Hamiltonian orbit at $\tilde v(+\infty,t)$, either asymptotic to a closed Hamiltonian orbit or with a negative end converging to a Reeb orbit at $\tilde v(-\infty,t)$, and with negative ends converging to Reeb orbits at the punctures in Γ. Let $w=\pi_\Sigma\circ\tilde v\colon \mathbb{C}P^1\to\Sigma$ be the smooth extension of the projection of $\tilde v$ to the divisor (as given by Lemma 6.2). The linearized projection $d\pi_\Sigma$ induces an isomorphism of complex vector bundles

$$\tilde v^*\big(T(\mathbb{R}\times Y)\big)\cong(\mathbb{R}\oplus\mathbb{R}R)\oplus w^*T\Sigma.$$

To see this, note that for each point $p\in Y$, $d\pi_\Sigma$ induces a symplectic isomorphism $(\xi_p,d\alpha)\cong(T_{\pi_\Sigma(p)}\Sigma,K\omega_\Sigma)$. By the Reeb invariance of the almost complex structure (and thus S^1-invariance under rotation in the fibre), this then gives a complex vector bundle isomorphism.

Let $\mathbf{V}$ associate to each puncture $z\in\Gamma\cup\{\pm\infty\}$ the tangent space to Y if the corresponding limit of $\tilde v$ is a closed Hamiltonian orbit and the tangent space to $\mathbb{R}\times Y$ if the corresponding limit of $\tilde v$ is a closed Reeb orbit at $-\infty$. As will be clearer shortly, this is associating to each puncture the entirety of the kernel of the corresponding asymptotic operator. Let

$$D_{\tilde v}\colon W^{1,p,\delta}_{\mathbf{V}}(\tilde v^*T(\mathbb{R}\times Y))\to L^{p,\delta}(\mathrm{Hom}^{0,1}(T(\mathbb{R}\times S^1),\tilde v^*T(\mathbb{R}\times Y))) \tag{6.1}$$

be the linearization of the nonlinear Floer operator at the solution $\tilde v$, for $\delta>0$ sufficiently small. The vector spaces $\mathbf{V}$ correspond to allowing the asymptotic limits to move in their Morse–Bott families. We have then a linearized evaluation map at the punctures with values in $\oplus_{z\in\{\pm\infty\}\cup\Gamma}V_z$.

Let

$$D^\Sigma_w\colon W^{1,p}(w^*T\Sigma)\to L^p(\mathrm{Hom}^{0,1}(T\mathbb{C}P^1,w^*T\Sigma))$$

be the linearized Cauchy–Riemann operator in Σ at the holomorphic sphere w. We also have the linearized Cauchy–Riemann operator $\dot D^\Sigma_w$ at the holomorphic cylinder $s+it\mapsto w(\mathrm{e}^{2\pi(s+it)})=\pi_\Sigma(\tilde v(s,t))$. Then, $(\pi_\Sigma\circ\tilde v)^*T\Sigma=w^*T\Sigma|_{\mathbb{R}\times S^1}$ is a Hermitian vector bundle over $\mathbb{R}\times S^1\backslash\Gamma$. Let $\mathbf{V_\Sigma}$ be the kernels of the asymptotic operators of $\dot D^\Sigma_w$ at each of the punctures, $\pm\infty$ and Γ. (These are explicitly given by $V_\Sigma(-\infty)=T_{w(0)}\Sigma$, $V_\Sigma(+\infty)=T_{w(\infty)}\Sigma$, $V_\Sigma(z)=T_{w(z)}\Sigma$ for each marked point $z\in\Gamma$.) We consider this operator acting on the space of sections

$$\dot D^\Sigma_w\colon W^{1,p,\delta}_{\mathbf{V_\Sigma}}(w^*T\Sigma|_{\mathbb{R}\times S^1})\to L^{p,\delta}(\mathrm{Hom}^{0,1}(\mathbb{R}\times S^1,w^*T\Sigma|_{\mathbb{R}\times S^1})).$$

The operator D^{Σ}_w is Fredholm independently of the weight, but $\dot{D}^{\Sigma}_w$ is only Fredholm when the weight δ is not an integer multiple of 2π. Furthermore, by combining [Wen16b, Proposition 3.15] with Lemma A.10, for $0 < \delta < 2\pi$, these operators have the same Fredholm index and their kernels and cokernels are isomorphic by the map induced by restricting a section of $w^*T\Sigma$ to the punctured cylinder.

Finally, define $D^{\mathbb{C}}_{\tilde{v}}$ by

$$\begin{aligned} D^{\mathbb{C}}_{\tilde{v}} \colon W^{1,p,\delta}_{\mathbf{V_0}}(\mathbb{R}\times S^1\backslash\Gamma,\mathbb{C}) &\to L^{p,\delta}(\mathrm{Hom}^{0,1}(T(\mathbb{R}\times S^1),\mathbb{C})) \\ (D^{\mathbb{C}}_{\tilde{v}}F)(\partial_s) &= F_s + iF_t + \begin{pmatrix} h''(\mathrm{e}^b)\,\mathrm{e}^b & 0 \\ 0 & 0\end{pmatrix} F \end{aligned} \tag{6.2}$$

where $\mathbf{V_0}$ associates the vector space $i\mathbb{R}$ to the punctures at which $\tilde{v}$ converges to a closed Hamiltonian orbit (i.e. at $+\infty$ and possibly at $-\infty$), and associates the vector space $\mathbb{C}$ at punctures at which $\tilde{v}$ converges to a closed Reeb orbit, i.e. at Γ and possibly at $-\infty$. Notice that again these are chosen so that they precisely give the kernels of the corresponding asymptotic operators of $D^{\mathbb{C}}_{\tilde{v}}$.

Lemma 6.12. *The isomorphism $\tilde{v}^*T(\mathbb{R}\times Y) \cong (\mathbb{R}\oplus\mathbb{R}R)\oplus w^*T\Sigma$ induces a decomposition:*

$$D_{\tilde{v}} = \begin{pmatrix} D^{\mathbb{C}}_{\tilde{v}} & M \\ 0 & \dot{D}^{\Sigma}_w \end{pmatrix}$$

where M is a multiplication operator that evaluates on ∂_s to a fibrewise linear map $M\colon w^*T\Sigma \to \mathbb{R}\oplus\mathbb{R}R$. (In particular, M is compact.) Furthermore, if $w = \pi_\Sigma\circ\tilde{v}$ is non-constant, then M is pointwise surjective except at finitely many points.

Proof. In our setting, the nonlinear Floer operator takes the form:

$$d\tilde{v} + J_Y(\tilde{v})d\tilde{v}\circ i - h'(\mathrm{e}^r)R\otimes dt + h'(\mathrm{e}^r)\partial_r\otimes ds = 0.$$

Write $\tilde{v} = (b,v)\colon \mathbb{R}\times S^1\backslash\Gamma \to \mathbb{R}\times Y$. If we apply dr to the previous equation, and use the fact that $dr\circ J_Y = -\alpha$, we get:

$$db - v^*\alpha\circ i + h'(\mathrm{e}^b)ds = 0$$

Denoting by $\pi_\xi\colon TY\to\xi$ the projection along the Reeb vector field, we get

$$\pi_\xi d\tilde{v} + J_Y(\tilde{v})\pi_\xi d\tilde{v}\circ i = 0. \tag{6.3}$$

Let g be the metric on $\mathbb{R}\times Y$ given by $g = dr^2 + \alpha^2 + d\alpha(\cdot, J_Y\cdot)$. This metric is J_Y-invariant. Let $\tilde{\nabla}$ be the Levi-Civita connection for g. Let ∇ be the Levi-Civita connection on $T\Sigma$ for the metric $\omega_\Sigma(\cdot, J_\Sigma\cdot)$.

Then, it follows that the linearization $D_{\tilde{v}}$ applied to a section ζ of $\tilde{v}^*T(\mathbb{R}\times Y)$ satisfies

$$\begin{aligned} D_{\tilde{v}}\zeta\,(\partial_s) &= \tilde{\nabla}_s\zeta + J_Y(\tilde{v})\tilde{\nabla}_t\zeta + \big(\tilde{\nabla}_\zeta J_Y(\tilde{v})\big)\partial_t\tilde{v} - \tilde{\nabla}_\zeta(J_YX_H)(\tilde{v}) \\ &= \tilde{\nabla}_s\zeta + J_Y(\tilde{v})\tilde{\nabla}_t\zeta + \big(\tilde{\nabla}_\zeta J_Y(\tilde{v})\big)\partial_t\tilde{v} + \tilde{\nabla}_\zeta(h'(\mathrm{e}^r)\partial_r)_{r=b}. \end{aligned} \tag{6.4}$$

Notice that $\tilde{\nabla}\partial_r = 0$ since g is a product metric. We have then

$$\tilde{\nabla}_\zeta(h'(\mathrm{e}^r)\partial_r)|_{r=b} = h''(\mathrm{e}^b)\,\mathrm{e}^b\,dr(\zeta)\partial_r.$$

Observe also that for any vector field V in $T\Sigma$, there is a unique horizontal lift $\tilde{V}$ to Y with the property $\alpha(\tilde{V}) = 0$. For any two vector fields V and W in $T\Sigma$, since $d\alpha(\tilde{V},\tilde{W}) = K\omega_\Sigma(V,W)$, we have the following

$$[\tilde{V},\tilde{W}] = \widetilde{[V,W]} - K\omega_\Sigma(V,W)R.$$

From this, it follows that the Levi-Civita connection $\tilde{\nabla}$ satisfies the following identities:

$$\begin{aligned}
\tilde{\nabla}_{\tilde{V}}\tilde{W} &= \widetilde{\nabla_V W} - \frac{K}{2}\omega_\Sigma(V,W)R \\
\tilde{\nabla}_R R &= 0 \\
\tilde{\nabla}_R \tilde{V} &= -\frac{1}{2}J_Y\tilde{V}.
\end{aligned}$$

A simple computation using the Reeb-flow invariance of J_Y and the torsion-free property of the connection gives

$$\tilde{\nabla}_{\partial_r}J_Y = 0 = \tilde{\nabla}_R J_Y.$$

We will now compute $D_{\tilde{v}}\zeta\,(\partial s)$, first when $\zeta = \zeta_1\partial_r + \zeta_2 R = (\zeta_1 + i\zeta_2)\partial_r$, and then when ζ is a section of $\tilde{v}^*\xi$.

For the first computation, it suffices to notice the following two identities

$$\begin{aligned}
D_{\tilde{v}}\partial_r\,(\partial_s) &= h''(\mathrm{e}^b)\,\mathrm{e}^b\,\partial_r \\
D_{\tilde{v}}R\,(\partial_s) &= 0.
\end{aligned}$$

It follows then from the Leibniz rule that we have

$$D_{\tilde{v}}(\zeta_1 + i\zeta_2)\,\partial_r\,(\partial_s) = (\zeta_s + i\zeta_t) + \big(h''(e^b)e^b\zeta_1\big)\,\partial_r = D^{\mathbb{C}}_{\tilde{v}}(\zeta_1 + i\zeta_2)\,\partial_r(\partial_s).$$

Now consider the case when ζ is a section of $\tilde{v}^*\xi$, and is thus the lift $\zeta = \tilde{\eta}$ of a section η of $w^*T\Sigma$. We compute

$$\begin{aligned}
\tilde{\nabla}_s R &= \tilde{\nabla}_{\pi_\xi v_s}R = -\frac{1}{2}J_Y\pi_\xi v_s \\
\tilde{\nabla}_s\zeta &= \widetilde{\nabla_{w_s}\eta} - \frac{K}{2}\omega_\Sigma(w_s,\eta)R - \frac{1}{2}\alpha(v_s)J_Y\zeta,
\end{aligned}$$

and similarly for $\tilde{\nabla}_t$. We then obtain the following covariant derivatives of J_Y, where $\tilde{W}$ is a section of $\tilde{v}^*\xi$:

$$\begin{aligned}
(\tilde{\nabla}_\zeta J_Y)\partial_r &= \tilde{\nabla}_\zeta R - J_Y\tilde{\nabla}_\zeta\partial_r = -\frac{1}{2}J_Y\zeta \\
(\tilde{\nabla}_\zeta J_Y)R &= -\tilde{\nabla}_\zeta\partial_r - J_Y\tilde{\nabla}_\zeta R = -\frac{1}{2}\zeta \\
(\tilde{\nabla}_\zeta J_Y)\tilde{W} &= \tilde{\nabla}_\zeta(J_Y\tilde{W}) - J_Y\tilde{\nabla}_\zeta\tilde{W} \\
&= \widetilde{\nabla_\eta J_\Sigma W} - \frac{K}{2}\omega_\Sigma(\eta, J_\Sigma W)R - J_Y\left(\widetilde{\nabla_\eta W} - \frac{K}{2}\omega_\Sigma(\eta,W)R\right) \\
&= (\widetilde{\nabla_\eta J_\Sigma})W - \frac{K}{2}\omega_\Sigma(\eta, J_\Sigma W)R - \frac{K}{2}\omega_\Sigma(\eta,W)\partial_r.
\end{aligned}$$

It follows then

$$\begin{aligned}
D_{\tilde{v}}\zeta\,(\partial_s) &= \tilde{\nabla}_s\zeta + J_Y\tilde{\nabla}_t\zeta + (\tilde{\nabla}_\zeta J_Y)\tilde{v}_t \\
&= \widetilde{\nabla_s}\eta - \frac{1}{2}\alpha(v_s)J_Y\zeta - \frac{K}{2}\omega_\Sigma(w_s,\eta)R + J_Y\widetilde{\nabla_t}\eta + \frac{1}{2}\alpha(v_t)\zeta + \frac{K}{2}\omega_\Sigma(w_t,\eta)\partial_r \\
&\quad - \frac{1}{2}b_tJ_Y\zeta - \frac{1}{2}\alpha(v_t)\zeta + (\widetilde{\nabla_\eta J_\Sigma})w_t - \frac{K}{2}\omega_\Sigma(\eta, J_\Sigma w_t)R - \frac{K}{2}\omega_\Sigma(\eta,w_t)\partial_r \\
&= \widetilde{\dot{D}^\Sigma_w}\eta + K\omega_\Sigma(w_t,\eta)\partial_r - K\omega_\Sigma(w_s,\eta)R.
\end{aligned}$$

(Note that we use the fact that $\tilde{v}_s + J_Y\tilde{v}_t + h'(\mathrm{e}^b)\partial_r = 0$ in the cancellations.)

Writing $\zeta = (\zeta_a, \zeta_b)$ under the isomorphism $\tilde{v}^*T(\mathbb{R} \times Y) \cong (\mathbb{R} \oplus \mathbb{R}R) \oplus w^*T\Sigma$, we obtain the decomposition:

$$D_{\tilde{v}}(\zeta_a, \zeta_b)(\partial_s) = \begin{pmatrix} D_{aa} & D_{ab} \\ D_{ba} & D_{bb} \end{pmatrix} \begin{pmatrix} \zeta_a \\ \zeta_b \end{pmatrix} (\partial_s)$$

Our calculations now establish that $D_{aa} = D^{\mathbb{C}}_{\tilde{v}}$ and $D_{ba} = 0$, $D_{bb} = \dot{D}^{\Sigma}_{w}$, and $D_{ab}\zeta(\partial_s) = K\omega_\Sigma(w_t, \pi_\Sigma\zeta)\partial_r - K\omega_\Sigma(w_s, \pi_\Sigma\zeta)R$. Observe that in particular, D_{ab} is a pointwise linear map from $\tilde{v}^*\xi|_p$ to $\mathbb{R}\partial_r \oplus \mathbb{R}R$. The map is surjective except at critical points of the pseudoholomorphic map w, of which there are finitely many if w is non-constant. $\square$

Remark 6.13. Notice that for each puncture $z \in \{\pm\infty\} \cup \Gamma$, if $\gamma(t)$ denotes the corresponding asymptotic Hamiltonian or Reeb orbit, the previous result allows us to identify V_z with $T_{\gamma(0)}Y$ at a Hamiltonian orbit and with $\mathbb{R} \times T_{\gamma(0)}Y$ at a Reeb orbit.

Lemma 6.14. *Let $\tilde{v}\colon \mathbb{R} \times S^1 \backslash \Gamma \to \mathbb{R} \times Y$ be a finite hybrid energy Floer cylinder with punctures Γ.*

Then, the operator $D^{\mathbb{C}}_{\tilde{v}}$ defined in Equation (6.2) *is Fredholm for $\delta > 0$ sufficiently small.*

The restriction

$$D^{\mathbb{C}}_{\tilde{v}}|_{W^{1,p,\delta}}\colon W^{1,p,\delta}(\mathbb{R} \times S^1 \backslash \Gamma, \mathbb{C}) \to L^{p,\delta}(\mathrm{Hom}^{0,1}(T(\mathbb{R} \times S^1), \mathbb{C}))$$

has Fredholm index $-1 - 2\#\Gamma$ and is injective.

If $\tilde{v}$ converges at $-\infty$ to a closed Hamiltonian orbit,

$$D^{\mathbb{C}}_{\tilde{v}}\colon W^{1,p,\delta}_{\mathbf{V_0}}(\mathbb{R} \times S^1 \backslash \Gamma, \mathbb{C}) \to L^{p,\delta}(\mathrm{Hom}^{0,1}(T(\mathbb{R} \times S^1), \mathbb{C}))$$

has Fredholm index 1 *and is surjective.*

If, instead, $\tilde{v}$ converges at $-\infty$ to a closed Reeb orbit in $\{-\infty\} \times Y$, then $D^{\mathbb{C}}_{\tilde{v}}$ has Fredholm index 2 *and is surjective.*

In either of these cases, the kernel of $D^{\mathbb{C}}_{\tilde{v}}$ contains the constant section i, which can be identified with the Reeb vector field.

Proof. We will apply the punctured Riemann–Roch theorems A.6 and A.8.

If the Floer solution $\tilde{v}$ converges to a Hamiltonian orbit in $\{b_\pm\} \times Y$ as $s \to \pm\infty$, then the associated asymptotic operator associated to $D^{\mathbb{C}}_{\tilde{v}}$ at $\pm\infty$ is given by

$$A_\pm = -i\frac{d}{dt} - \begin{pmatrix} h''(\mathrm{e}^{b_\pm})\,\mathrm{e}^{b_\pm} & 0 \\ 0 & 0 \end{pmatrix}.$$

We recall that for $\delta > 0$, the space of functions $W^{1,p,\delta}(\mathbb{R} \times S^1 \backslash \Gamma, \mathbb{C})$ has exponentially fast decay to zero in cylindrical coordinates near the punctures in Γ and also approaching $\pm\infty \times S^1$.

By Theorem A.6, we need to compute the Conley–Zehnder indices of the appropriately perturbed asymptotic operators. We do so using Corollary A.4. Start with the case $\delta > 0$. The Conley–Zehnder index relevant at $+\infty$ is that of $A_+ + \delta$, which is 0. The Conley–Zehnder index relevant at $-\infty$ is that of $A_- - \delta$, which is 1. If we consider instead $\delta < 0$, then the Conley–Zehnder index of $A_+ + \delta$ is 1 and that of $A_- - \delta$ is 0.

Associated to a Reeb puncture at $-\infty$ or at $P \in \Gamma$, we have the asymptotic operator

$$-i\frac{d}{dt}.$$

Writing $\tilde{v} = (b, v)\colon \mathbb{R} \times S^1 \backslash \Gamma \to \mathbb{R} \times Y$, we have $b \to -\infty$ at both types of punctures. The same argument as above implies that if $\delta > 0$, these punctures have the Conley–Zehnder index of $-i\frac{d}{dt} - \delta$, which is 1. If $\delta < 0$, we have instead that the Conley–Zehnder index of $-i\frac{d}{dt} - \delta$ is -1.

The exponential weight, δ, must be smaller than the spectral gap for any of these punctures. In view of Lemma A.3, we take $|\delta| < \frac{-C_+ + \sqrt{C_+^2 + 16\pi^2}}{2} \leqslant 2\pi$, where $C_+ = h''(\mathrm{e}^{b_+})\,\mathrm{e}^{b_+}$. Notice that by Condition (v) of Definition 3.1, we have $h''(\mathrm{e}^{b_+})\,\mathrm{e}^{b_+} \geqslant h''(\mathrm{e}^{b_-})\,\mathrm{e}^{b_-}$ so this condition guarantees that $|\delta|$ is smaller than any spectral gap.

Applying now the punctured Riemann–Roch theorem A.6, and using the fact that the Euler characteristic of the punctured cylinder is $-\#\Gamma$, if $\delta > 0$, we obtain the Fredholm index

$$-\#\Gamma + 0 - 1 - \#\Gamma = -1 - 2\#\Gamma,$$

as claimed.

The injectivity of $D^{\mathbb{C}}_{\tilde{v}}$ restricted to $W^{1,p,\delta}$ follows from automatic transversality, applying [Wen10, Proposition 2.2]. Using the notation of Wendl [Wen10, Equation (2.5)], the fact that the curve has genus 0 and that the unique puncture with even Conley–Zehnder index is the positive puncture, we have

$$c_1(E, l, \mathbf{A}_\Gamma) = \frac{1}{2}(-1 - \#\Gamma - 2 + 1) = -1 - \frac{\#\Gamma}{2} < 0$$

as necessary.

Now, in order to consider the operator $D^{\mathbb{C}}_{\tilde{v}}\colon W^{1,p,\delta}_{\mathbf{V}_0} \to L^{p,\delta}$, it will be convenient to consider a related operator with the same formula, but on the much larger space of functions with exponential *growth*. By a slight abuse of notation, we will use the same name:

$$D^{\mathbb{C}}_{\tilde{v}}\colon W^{1,p,-\delta} \to L^{p,-\delta}$$
$$(D^{\mathbb{C}}_{\tilde{v}} F)(\partial_s) = F_s + iF_t + \begin{pmatrix} h''(\mathrm{e}^b)\,\mathrm{e}^b & 0 \\ 0 & 0 \end{pmatrix} F$$

(See also Lemma A.10). Then, the kernel and cokernel of the operator acting on the spaces of sections with with exponential growth will have kernel and cokernel that can be identified with the kernel and cokernel of the operator acting on $W^{1,p,\delta}_{\mathbf{V}_0}$.

Applying Theorem A.8, we compute the Fredholm index to be:

$$-\#\Gamma + 1 - (-\#\Gamma) - \begin{cases} 0 & \text{if } \tilde{v}(-\infty) \text{ converges to a Hamiltonian orbit} \\ -1 & \text{if } \tilde{v}(-\infty) \text{ converges to a Reeb orbit} \end{cases}$$
$$= 1 \text{ or } 2, \text{ depending on the negative end of } \tilde{v}.$$

Furthermore, the fact that the curve has genus 0 and one puncture with even Conley–Zehnder index precisely if $\lim_{s\to-\infty} \tilde{v}$ is a Hamiltonian orbit implies that

$$c_1(E, l, \mathbf{A}_\Gamma) = \begin{cases} \frac{1}{2}(1 - 2 + 1) = 0 < 1 & \text{if } \tilde{v}(-\infty) \text{ converges to a Hamiltonian orbit} \\ \frac{1}{2}(2 - 2) = 0 < 2 & \text{if } \tilde{v}(-\infty) \text{ converges to a Reeb orbit} \end{cases}$$

Therefore, $D^{\mathbb{C}}_{\tilde{v}}$ satisfies the automatic transversality criterion and is thus surjective, as wanted.

It follows immediately from the expression for $D^{\mathbb{C}}_{\tilde{v}}$ that the constant i is in the kernel. Recalling that $\mathbb{C} = \tilde{v}^*(\mathbb{R} \oplus \mathbb{R}R)$ in the splitting given by Lemma 6.12, we then may identify this with the Reeb vector field R.

□

To summarize then the results of this section, by Lemma 6.12, a punctured Floer cylinder in $\mathbb{R} \times S^1$ is regular if the operators $D^{\mathbb{C}}_{\tilde{v}}$ and $\dot{D}^{\Sigma}_{w}$ are surjective. Surjectivity of the latter is equivalent to surjectivity of D^{Σ}_{w}. Lemma 6.14 gives the surjectivity of $D^{\mathbb{C}}_{\tilde{v}}$. It thus remains to study transversality for D^{Σ}_{w}, specifically with respect to the evaluation maps that will allow us to define the moduli spaces of chains of pearls in Σ. Additionally, we need to consider transversality for moduli spaces of planes in W asymptotic to Reeb orbits in Y, or equivalently, the moduli spaces of spheres in X with an order of contact condition at Σ.

Remark 6.15. In Part **??**, we will also consider punctured J_Y-holomorphic cylinders $\tilde{u}\colon \mathbb{R} \times S^1 \backslash \Gamma \to \mathbb{R} \times Y$. They satisfy the J_Y-holomorphic curve equation, instead of its Hamiltonian-perturbed Floer counterpart (4.3). These are the types of curves contributing to the Morse–Bott contact homology of Y [Bou02]. The methods of this section can be adapted to show that the moduli spaces of such J_Y-holomorphic curves are also cut out transversely. The main difference is that the analogue of the vertical operator in (6.2) no longer has the term involving the Hamiltonian. Automatic transversality can again be used to adapt Lemma 6.14 to this situation.

6.3. **Transversality for chains of pearls in Σ.** In this section and the next, we show that for generic almost complex structure (in a sense to be made precise), the moduli spaces of chains of pearls and moduli spaces of chains of pearls with spheres in X (possibly augmented as well) are transverse. We begin with the definition of several moduli spaces that will be useful.

Definition 6.16. Let $J_\Sigma \in \mathcal{J}_\Sigma$ be an almost complex structure compatible with ω_Σ. Given $p, q \in \mathrm{Crit}(f_\Sigma)$ and a finite collection $A_1, \ldots, A_N \in H_2(\Sigma; \mathbb{Z})$, let

$$\mathcal{M}^*_{k,\Sigma}((A_1, \ldots, A_N); q, p; J_\Sigma)$$

denote the space of simple chains of pearls in Σ from p to q (see Definition 6.5), such that $(w_i)_*[\mathbb{C}P^1] = A_i$, with k marked points.

Let

$$\mathcal{M}^*_{k,\Sigma}((A_1, \ldots, A_N); J_\Sigma)$$

denote the moduli space of N parametrized holomorphic spheres in Σ, representing the classes $A_i, i = 1, \ldots, N$, with k marked points, also satisfying the simplicity criterion of Definition 6.5, i.e. so each sphere is either somewhere injective or constant, each constant sphere has at least one augmentation marked point, and no sphere has image contained in the image of another.

For $J_W \in \mathcal{J}_W$, let $J_\Sigma = P(J_W)$ be the corresponding almost complex structure in $\mathcal{J}_\Sigma$ and J_X the corresponding almost complex structure on X. Define

$$\mathcal{M}^*_{k,(X,\Sigma)}((B; A_1, \ldots, A_N); q, p, J_W)$$

to be the moduli space of simple chains of pearls in Σ with a sphere in X (as in Definitions 6.4 and 6.5), where q is a critical point of f_X and p is a critical point of f_Σ, and representing the spherical homology classes $[w_i] = A_i \in H_2(\Sigma;\mathbb{Z})$, $i = 1,\ldots,N$ and $[v] = B \in H_2(X;\mathbb{Z})\backslash 0$. In the following, we will write

$$l = B \bullet \Sigma = K\omega(B)$$

which is the order of contact of v with Σ.

Let

$$\mathcal{M}^*_{k,(X,\Sigma)}((B; A_1,\ldots,A_N); J_W)$$

denote the moduli space of N parametrized holomorphic spheres in Σ, representing the classes A_i, and of a holomorphic sphere in X representing the class B with order of contact $l = B \bullet \Sigma = K\omega(B)$, also satisfying the simplicity criterion of Definition 6.5, i.e. so each sphere in Σ is either somewhere injective or constant (if constant, it has at least one augmentation marked point), no image of a sphere is contained in the image of another and the image of the sphere in X is not contained in the tubular neighbourhood $\varphi(U)$ of Σ. Furthermore, the spheres in Σ have k marked points.

Let

$$\mathcal{M}^*_X((B_1, B_2,\ldots,B_k); J_W)$$

denote the moduli space of k parametrized holomorphic spheres in X, where each sphere is somewhere injective, no image of a sphere is contained in the image of another sphere, and so the image of each sphere is not contained in the tubular neighbourhood $\varphi(U)$ of Σ, and such that each sphere intersects Σ only at $\infty \in \mathbb{CP}^1$ with order of contact $B_i \bullet \Sigma$.

Finally, let

$$\mathcal{M}^a_{k,\Sigma}((A_1,\ldots,A_N),(B_1,\ldots,B_k); q, p; J_\Sigma, J_W)$$

denote the moduli space of simple augmented chains of pearls in Σ with k augmentation planes, and let

$$\mathcal{M}^a_{k,(X,\Sigma)}((B; A_1,\ldots,A_N); (B_1,\ldots,B_k); q, p; J_\Sigma, J_W)$$

denote the moduli space of simple augmented chains of pearls with a sphere in X. (See Definitions 6.4 and 6.5.)

In order to apply the Sard–Smale Theorem, we need to consider Banach spaces of almost complex structures, so we let $\mathcal{J}^r_\Sigma, \mathcal{J}^r_W$ be the space of C^r–regular almost complex structures otherwise satisfying the conditions of being in $\mathcal{J}_\Sigma$, $\mathcal{J}_W$. We impose $r \geqslant 2$ and in general will require r to be sufficiently large that the Sard–Smale theorem holds (this will depend on the Fredholm indices associated to the collection of homology classes and will also depend on the order of contact to Σ for the spheres in X).

For each of these moduli spaces, we also consider the corresponding universal moduli spaces as we vary the almost complex structure. For instance, we denote by $\mathcal{M}_{k,\Sigma}((A_1,\ldots,A_N),\mathcal{J}^r_\Sigma)$ the moduli space of pairs $((w_i)_{i=1}^N, J_\Sigma)$ with $J_\Sigma \in \mathcal{J}^r_\Sigma$ and $(w_i)_{i=1}^N \in \mathcal{M}_{k,\Sigma}((A_1,\ldots,A_N), J_\Sigma)$.

The main goal of this section and of the next is to prove that these moduli spaces of simple chains of pearls are transverse for generic J. This is analogous to [MS04, Theorem 6.2.6], and indeed, the transversality theorem of McDuff-Salamon will be an ingredient of our proof. We will furthermore require an extension of the results from [CM07] (see Section 6.4), and an additional technical transversality point required to be able to consider the lifted problem in $\mathbb{R} \times Y$.

Proposition 6.17. *There is a residual set $\mathcal{J}_W^{reg} \subset \mathcal{J}_W$ such that $\mathcal{J}_\Sigma^{reg} := P(\mathcal{J}_W^{reg})$ is a residual set in $\mathcal{J}_\Sigma$ and such that for all $J_\Sigma \in \mathcal{J}_\Sigma^{reg}$ and $J_W \in \mathcal{J}_W^{reg}$, $p \in \mathrm{Crit}(f_\Sigma)$, $q \in \mathrm{Crit}(f_\Sigma)$ or $q \in \mathrm{Crit}(f_W)$, the moduli spaces $\mathcal{M}^*_{k,\Sigma}((A_1,\ldots,A_N);q,p;J_\Sigma)$, $\mathcal{M}^*_{k,(X,\Sigma)}((B;A_1,\ldots,A_N);q,p,J_W)$, $\mathcal{M}^a_{k,\Sigma}((A_1,\ldots,A_N),(B_1,\ldots,B_k);q,p;J_\Sigma)$ and $\mathcal{M}^a_{k,(X,\Sigma)}((B;A_1,\ldots,A_N);(B_1,\ldots,B_k);q,p;J_\Sigma,J_W)$ are manifolds. Their dimensions are*

$$\dim \mathcal{M}^*_{k,\Sigma}((A_1,\ldots,A_N);q,p;J_\Sigma) = \ M(p) + \sum_{i=1}^N 2\langle c_1(T\Sigma),A_i\rangle - M(q) + N - 1 + 2k,$$

$$\begin{aligned}
&\dim \mathcal{M}^*_{k,(X,\Sigma)}((B;A_1,\ldots,A_N);q,p,J_W)\\
&\ = M(p) + \sum_{i=1}^N 2\langle c_1(T\Sigma),A_i\rangle + 2(\langle c_1(TX),B\rangle - B\bullet\Sigma) + M(q) - 2(n-1) + N - 1 + 2k\\
&\dim \mathcal{M}^a_{k,\Sigma}((A_1,\ldots,A_N),(B_1,\ldots,B_k);q,p;J_\Sigma,J_W)\\
&\ = M(p) + \sum_{i=1}^N 2\langle c_1(T\Sigma),A_i\rangle - M(q) + N - 1 + 4k + \sum_{i=1}^k \left(2\langle c_1(TX),B_i\rangle - 2B_i\bullet\Sigma\right),\\
&\dim \mathcal{M}^a_{k,(X,\Sigma)}((B;A_1,\ldots,A_N);(B_1,\ldots,B_k);q,p;J_\Sigma,J_W)\\
&\ = M(p) + \sum_{i=1}^N 2\langle c_1(T\Sigma),A_i\rangle + 2(\langle c_1(TX),B\rangle - B\bullet\Sigma) + M(q) - 2(n-1)+\\
&\qquad + N - 1 + 4k + \sum_{i=1}^k \left(2\langle c_1(TX),B_i\rangle - 2B_i\bullet\Sigma\right).
\end{aligned}$$

where $M(q)$ is the Morse index of the critical point q, for f_Σ if q is a critical point of f_Σ and for f_X if q is a critical point of f_X, and $M(p)$ is the Morse index of p for f_Σ.

Remark 6.18. Notice that we will eventually consider the augmentation planes modulo their 4 parameter family of automorphisms, thus effectively reducing the dimensions of each of $\mathcal{M}^a_\Sigma$ and $\mathcal{M}^a_{(X,\Sigma)}$ by $4k$.

Proposition 6.19 ([MS04, Proposition 6.2.7])**.** *$\mathcal{M}^*_{k,\Sigma}((A_1,\ldots,A_N);\mathcal{J}^r_\Sigma)$ is a Banach manifold.*

We will also make use of the following proposition, which we prove in the next section.

Definition 6.20. There is a *universal evaluation map*

$$\begin{aligned}
\mathrm{ev}_\Sigma \colon \mathcal{M}^*_{k,\Sigma}((A_1,\ldots,A_N);\mathcal{J}_\Sigma) &\to \Sigma^{2N}\\
(w_1,\ldots,w_N) &\mapsto (w_1(0),w_1(\infty),w_2(0),w_2(\infty),\ldots,w_N(\infty)).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\mathrm{ev}_{X,\Sigma} \colon \mathcal{M}^*_{k,(X,\Sigma)}((B;A_1,\ldots,A_N);\mathcal{J}_W) &\to X\times\Sigma^{2N+1}\\
(v,w_1,\ldots,w_N) &\mapsto (v(0),v(\infty),w_1(0),w_1(\infty),w_2(0),\ldots,w_N(\infty)),
\end{aligned}$$

where v is the holomorphic sphere in X and the w_i are the spheres in Σ.

We have an evaluation map coming from simple collections of spheres in X:

$$\begin{aligned}\mathrm{ev}^a_\Sigma \colon \mathcal{M}^*_X((B_1, B_2, \ldots, B_k); \mathcal{J}_W) &\to \Sigma^k \\ (v_1, \ldots, v_k) &\mapsto (v_1(\infty), v_2(\infty), \ldots, v_k(\infty)).\end{aligned}$$

For spheres in Σ, We also obtain evaluation maps at the augmentation punctures

$$\mathrm{ev}^a_\Sigma \colon \mathcal{M}^*_{k,\Sigma}((A_1, \ldots, A_N); \mathcal{J}_\Sigma) \to \Sigma^k$$

and

$$\mathrm{ev}^a_\Sigma \colon \mathcal{M}^*_{k,(X,\Sigma)}((B; A_1, \ldots, A_N); \mathcal{J}_W) \to \Sigma^k.$$

We refer to these three maps denoted ev^a as *augmentation evaluation maps.*

Proposition 6.21. *Let $B_0, \ldots, B_k$ be spherical classes in $H_2(X)$. Let*

$$r \geqslant \max B_i \bullet \Sigma + 2.$$

The universal moduli space $\mathcal{M}^*_X((B_1, \ldots, B_k); \mathcal{J}^r_W)$ is a Banach manifold and the evaluation maps

$$\begin{aligned}\mathrm{ev}^a_\Sigma \colon \mathcal{M}^*_X((B_1, \ldots, B_k); \mathcal{J}^r_W) &\to \Sigma^k : (f_1, f_2, \ldots, f_k) \mapsto (f_1(\infty), \ldots, f_k(\infty)) \\ \mathrm{ev}_{X,\Sigma} \colon \mathcal{M}^*_X((B_0); \mathcal{J}^r_W) &\to X \times \Sigma : f \mapsto (f(0), f(\infty))\end{aligned}$$

are submersions.

Recall that we have chosen a Morse function $f_\Sigma \colon \Sigma \to \mathbb{R}$ and a corresponding gradient-like vector field Z_Σ, such that (f_Σ, Z_Σ) is a Morse–Smale pair. The time-t flow of Z_Σ is denoted by $\varphi^t_{Z_\Sigma}$ and the stable (ascending) $W^s_\Sigma(q)$ and unstable (descending) manifolds $W^u_\Sigma(p)$ were defined in Equation (3.1).

Definition 6.22. The *flow diagonal* in $\Sigma \times \Sigma$ associated to the pair (f_Σ, Z_Σ) is

$$\Delta_{f_\Sigma} := \left\{ (x, y) \in (\Sigma \setminus \mathrm{Crit}(f_\Sigma))^2 \mid \exists t > 0 \text{ so } x = \varphi^t_{Z_\Sigma}(y) \right\}$$

where $\mathrm{Crit}(f_\Sigma)$ is the set of critical points of f_Σ.

We will now establish transversality of the evaluation maps to appropriate products of stable/unstable manifolds, critical points, diagonals and flow diagonals. By [MS04, Proposition 6.2.8], the key difficulty will be to deal with constant spheres. For this, we will need the following lemma about evaluation maps intersecting with the flow diagonals, essentially exploiting the composition property of the linearized flow of $\varphi^t_{Z_\Sigma}$, and interpreting a chain of pearls with a constant sphere in it as a chain of pearls without that constant sphere, but with an additional "stopping point" on one of the intermediate flow lines.

Lemma 6.23. *Suppose $\mathcal{M}$ is a manifold so that the map*

$$\mathrm{ev} = (\mathrm{ev}_-, \mathrm{ev}_+) \colon \mathcal{M} \to \Sigma \times \Sigma$$

is transverse to the flow diagonal Δ_{f_Σ}.

Then

$$\begin{aligned}\hat{\mathrm{ev}} \colon \mathcal{M} \times \Sigma &\to \Sigma^4 \\ (m, z) &\mapsto (\mathrm{ev}_-(m), z, z, \mathrm{ev}_+(m))\end{aligned}$$

is transverse to $\Delta_{f_\Sigma} \times \Delta_{f_\Sigma}$.

Proof. Suppose that $\hat{\text{ev}}(m,z) \in \Delta_{f_\Sigma} \times \Delta_{f_\Sigma}$. Let $m_- = \text{ev}_-(m)$, $m_+ = \text{ev}_+(m)$. Then, there exist times $t_1, t_2 > 0$ so that $\varphi^{t_1}_{Z_\Sigma}(m_-) = z$ and $\varphi^{t_2}_{Z_\Sigma}(z) = m_+$. In particular then, if $t = t_1 + t_2$, we have $\varphi^{t}_{Z_\Sigma}(\text{ev}_-(m)) = \text{ev}_+(m)$ so $\text{ev}(m) \in \Delta_{f_\Sigma}$.

If $(x,y) \in \Delta_{f_\Sigma}$, there exists $t > 0$ so that $\varphi^t_{Z_\Sigma}(x) = y$. Its tangent space is explicitly given by

$$T_{(x,y)}\Delta_{f_\Sigma} = \{(v, d\varphi^t_{Z_\Sigma}(x)v + cZ_\Sigma(y)) \,|\, v \in T_x\Sigma, c \in \mathbb{R}\} \subset T_x\Sigma \oplus T_y\Sigma.$$

Let $P_1 = d\varphi^{t_1}_{Z_\Sigma}(m_-)\colon T_{m_-}\Sigma \to T_z\Sigma$ and $P_2 = d\varphi^{t_2}_{Z_\Sigma}(z)\colon T_z\Sigma \to T_{m_+}\Sigma$. Then $P_2 \circ P_1 = d\varphi^{t}_{Z_\Sigma}(m_-)\colon T_{m_-} \to T_{m_+}$.

By hypothesis, we have

$$d\,\text{ev}_m \cdot T_m\mathcal{M} + T_{(m_-,m_+)}\Delta_{f_\Sigma} = T_{m_-}\Sigma \oplus T_{m_+}\Sigma.$$

Thus, for every $(w_-, w_+) \in T_{m_-}\Sigma \oplus T_{m_+}\Sigma$, there exist $(u_-, u_+) \in d\,\text{ev}_m \cdot T_m\mathcal{M}$, $c \in \mathbb{R}$, $v \in T_{m_-}\Sigma$ so that

$$\begin{aligned} u_- + v &= w_- \\ u_+ + P_2P_1v + cZ_\Sigma(m_-) &= w_+. \end{aligned}$$

We need to show that for every $(w_1, w_2, w_3, w_4) \in T_{(m_-,z,z,m_+)}\Sigma^4$, there exist $(u_-, u_+) \in d\,\text{ev}_m \cdot T_m\mathcal{M}$, $c_1, c_2 \in \mathbb{R}$, $v_1 \in T_{m_-}\Sigma$, $v_2, v_3 \in T_z\Sigma$ so that

$$\begin{aligned} u_- + v_1 &= w_1 \\ v_3 + P_1v_1 + c_1Z_\Sigma(z) &= w_2 \\ v_3 + v_2 &= w_3 \\ u_+ + P_2v_2 + c_2Z_\Sigma(m_+) &= w_4. \end{aligned}$$

From the hypothesis, there exist $(u_-, u_+) \in d\,\text{ev}_m \cdot T_m\mathcal{M}$, $c_2 \in \mathbb{R}$ and a $\hat{v} \in T_{m_-}\Sigma$ so that

$$\begin{aligned} u_- + \hat{v} &= w_1 - P_1^{-1}w_2 \\ u_+ + P_2P_1\hat{v} + c_2Z_\Sigma(m_-) &= w_4 - P_2w_3. \end{aligned}$$

Now, let $v_3 = -P_1\hat{v} \in T_z\Sigma$, $c_1 = 0$, $v_2 = w_3 - v_3$ and $v_1 = P_1^{-1}(w_2 - v_3) = P_1^{-1}w_2 + \hat{v}$. We now verify:

$$\begin{aligned} u_- + v_1 &= u_- + \hat{v} + P_1^{-1}w_2 = w_1 \\ v_3 + P_1v_1 &= v_3 + w_2 - v_3 = w_2 \\ v_3 + v_2 &= v_3 + w_3 - v_3 = w_3 \\ u_+ + P_2v_2 + c_2Z_\Sigma(m_+) &= u_+ + P_2w_3 - P_2v_3 + c_2Z_\Sigma(m_+) \\ &= u_+ + P_2P_1\hat{v} + P_2w_3 + c_2Z_\Sigma(m_+) \\ &= w_4 - P_2w_3 + P_2w_3 = w_4, \end{aligned}$$

as required. □

Lemma 6.24. *Let $N \geqslant 1$, and let $A_1, \ldots, A_N$ be spherical homology classes in Σ and let B be a spherical homology class in X.*

Suppose that $S \subset \Sigma^{2N-2}$ is obtained by taking the product of some number of copies of $\Delta_{f_\Sigma} \subset \Sigma^2$ and of the complementary number of copies of $\{(p,p) \,|\, p \in \text{Crit}(f_\Sigma)\} \subset \Sigma^2$, in arbitrary order. Let $\Delta \subset \Sigma^2$ denote the diagonal.

Then if $\sum_{i=1}^N A_i \neq 0$, the universal evaluation map

$$\text{ev}_\Sigma\colon \mathcal{M}^*_{k,\Sigma}((A_1, \ldots, A_N); \mathcal{J}_\Sigma) \to \Sigma^{2N}$$

is transverse to the submanifold $pt \times S \times pt$.

If $B \neq 0$, the universal evaluation map

$$\mathrm{ev}_{X,\Sigma} \colon \mathcal{M}^*_{k,(X,\Sigma)}((B; A_1, \ldots, A_N); \mathcal{J}_W) \to X \times \Sigma^{2N+1}$$

is transverse to the submanifold $pt \times \Delta \times S \times pt$.

Proof. We consider the case of $\mathcal{M}^*_{k,\Sigma}$ in detail, since the argument is essentially the same for $\mathcal{M}^*_{k,(X,\Sigma)}$, though notationally more cumbersome.

Suppose that $((v_1, \ldots, v_N), J) \in \mathcal{M}^*_{k,\Sigma}((A_1, \ldots, A_N); \mathcal{J}_\Sigma)$ is in the pre-image of $\mathrm{pt} \times S \times \mathrm{pt}$. Write $S = S_1 \times S_2 \times \cdots \times S_{N-1}$, where each $S_i \subset \Sigma^2$ is either the flow diagonal or the set of critical points.

Notice that the simplicity condition then requires that if some sphere v_i is constant, $1 < i < N$, we must have that S_{i-1} and S_i are flow diagonals. If v_1 is constant, then S_1 is a flow diagonal and if v_N is constant, S_{N-1} is a flow diagonal.

We will proceed by induction on N. The case $N = 1$ follows from [MS04, Proposition 3.4.2].

Now, for the inductive argument, we suppose the result holds for any $S \subset \Sigma^{2(N-1)-2}$ of the form specified, and for any $k \geqslant 0$, for any collection of $N-1$ spherical classes, not all of which are zero.

Let now $A_1, \ldots, A_N$ be spherical homology classes, not all of which are zero. Then, at least one of $A_1, \ldots, A_{N-1}$ or $A_2, \ldots, A_N$ is a collection of spheres satisfying the hypotheses. For simplicity of notation, let us assume that $A_1 + \cdots + A_{N-1} \neq 0$. Let $S_0 = S_1 \times S_2 \times \cdots \times S_{N-1}$. Let $k = k_0 + k_N$ where k_N is the number of marked points we consider on the last sphere. By the induction hypothesis, we have that

$$\mathrm{ev}_\Sigma \colon \mathcal{M}^*_{k_0,\Sigma}((A_1, \ldots, A_{N-1}); \mathcal{J}_\Sigma) \to \Sigma^{2(N-1)}$$

is transverse to $\mathrm{pt} \times S_0 \times \mathrm{pt}$. Denote this evaluation map by ev_0.

Notice that $\mathcal{M}^*_{k,\Sigma}((A_1, \ldots, A_N); \mathcal{J}_\Sigma) \subset \mathcal{M}^*_{k_0,\Sigma}((A_1, \ldots, A_{N-1}); \mathcal{J}_\Sigma) \times \mathcal{M}^*_{k_1,\Sigma}(A_N; \mathcal{J}_\Sigma)$. Let then $\mathrm{ev}_N \colon \mathcal{M}^*_{k,\Sigma}((A_1, \ldots, A_N); \mathcal{J}_\Sigma) \to \Sigma^2$ be the evaluation at 0 and ∞ in the Nth sphere. We therefore have

$$\mathrm{ev}_\Sigma \colon \mathcal{M}^*_{k,\Sigma}((A_1, \ldots, A_N); \mathcal{J}_\Sigma) \to \Sigma^{2N}$$

given by $\mathrm{ev}_\Sigma = (\mathrm{ev}_0, \mathrm{ev}_N)$.

If $A_N \neq 0$, the result follows again from [MS04, Proposition 6.2.8].

If, instead, $A_N = 0$, we have from above that $S_N = \Delta_{f_\Sigma}$. Notice that the evaluation map of constant spheres on Σ has image on the diagonal in $\Sigma \times \Sigma$. The result now follows by applying Lemma 6.23.

The case with a sphere in X follows a nearly identical induction argument, though the base case consists of a single sphere in X. The required submersion to $X \times \Sigma$ now follows from Proposition 6.21, and the induction proceeds as before. □

Proposition 6.25. *Let $N \geqslant 0$. Suppose that $S \subset \Sigma^{2N-2}$ is obtained by taking the product of some number of copies of $\Delta_{f_\Sigma} \subset \Sigma^2$ and of the complementary number of copies of $\{(p,p) \mid p \in \mathrm{Crit}(f_\Sigma)\} \subset \Sigma^2$, in arbitrary order.*

Let $\Delta \subset \Sigma \times \Sigma$ denote the diagonal and let Δ_k denote the diagonal Σ^k in $\Sigma^k \times \Sigma^k$.

Let p, q be critical points of f_Σ and let x be a critical point of f_W.

Then the universal evaluation maps together with augmentation evaluation maps

$$\mathrm{ev}_\Sigma \times \mathrm{ev}^a_\Sigma \times \mathrm{ev}^a_\Sigma \colon \mathcal{M}^*_{k,\Sigma}((A_1,\dots,A_N);\mathcal{J}_\Sigma) \times \mathcal{M}^*_0((B_1,\dots,B_k);\mathcal{J}_W) \to \Sigma^{2N} \times \Sigma^k \times \Sigma^k$$
$$\mathrm{ev}_{X,\Sigma} \times \mathrm{ev}^a_\Sigma \times \mathrm{ev}^a_\Sigma \colon \mathcal{M}^*_{k,(X,\Sigma)}((B;A_1,\dots,A_N);\mathcal{J}_W) \times \mathcal{M}^*_0((B_1,\dots,B_k);\mathcal{J}_W) \to X \times \Sigma^{2N+1} \times \Sigma^k \times \Sigma^k$$

are transverse to

$$W^s_\Sigma(q) \times S \times W^u_\Sigma(p) \times \Delta_k$$
$$W^u_X(q) \times \Delta \times S \times W^u_\Sigma(p) \times \Delta_k.$$

Proof. Notice first that by Proposition 6.21 the augmentation evaluation map

$$\mathrm{ev}^a \colon \mathcal{M}^*_0((B_1,\dots,B_k);\mathcal{J}_W) \to \Sigma^k$$

is a submersion. It suffices therefore to prove the result in the case $k = 0$.

The proposition follows immediately if at least one of the $A_i, i = 1,\dots,N$ is non-zero, or if we are considering the case of a chain of pearls with a sphere in X, by applying Lemma 6.24.

The only case then that must be examined is of a chain of pearls entirely in Σ with all spheres constant. In this case, the moduli space $\mathcal{M}^*_{0,\Sigma}((0,0,\dots,0),\mathcal{J}_W)$ can be identified with

$$\{(z_1,\dots,z_N) \in \Sigma^N \mid z_i = z_j \implies i = j\} \times \mathcal{J}_W.$$

Observe that in the case $N = 1$, transversality follows from the Morse–Smale condition on the gradient-like vector field Z_Σ. This gives that the intersection of $W^s_\Sigma(q)$ and $W^u_\Sigma(p)$ is transverse, and hence that the diagonal in $\Sigma \times \Sigma$ is transverse to $W^s_\Sigma(q) \times W^u_\Sigma(p)$. The case of $N \geqslant 2$ then follows from this together with Lemma 6.23.

□

Proposition 6.25 can be combined with standard Sard–Smale arguments, the fact that $P \colon \mathcal{J}_W \to \mathcal{J}_\Sigma$ is an open and surjective map and Taubes's method for passing to smooth almost complex structures (see for instance [MS04, Theorem 6.2.6]) to give the following proposition:

Proposition 6.26. *There exist residual sets of almost complex structures $\mathcal{J}^{reg}_W \subset \mathcal{J}_W$ and $\mathcal{J}^{reg}_\Sigma = P(\mathcal{J}^{reg}_W)$, so that for fixed $J_W \in \mathcal{J}^{reg}_W$ and $J_\Sigma = P(J_W)$, the restriction of the evaluation maps to $\mathcal{M}^*_{k,\Sigma}((A_1,\dots,A_N);J_\Sigma)$, $\mathcal{M}^*_{k,(X,\Sigma)}((B;A_1,\dots,A_N);J_W)$, and $\mathcal{M}^*_0((B_1,\dots,B_k);J_W)$ are transverse to the submanifolds of Proposition 6.25.*

The transversality statement of the main result of this section, Proposition 6.17 now follows. The dimension formulas follow from usual index arguments, combining Riemann–Roch with contributions from the constraints imposed by the evaluation maps.

6.4. **Transversality for spheres in X with order of contact constraints in Σ.** We will now consider transversality for a chain of pearls with a sphere in X. We will extend the results from Section 6 in [CM07]. In that paper, Cieliebak and Mohnke prove that the moduli space of simple curves not contained in Σ, with an order of contact condition, can be made transverse by a perturbation of the almost complex structure away from Σ. We will extend this result to show that additionally the evaluation map to Σ at the point of contact can be made transverse.

Recall that Σ is a symplectic divisor and $N\Sigma$ is its symplectic normal bundle equipped with a Hermitian structure. We have fixed a symplectic neighbourhood $\varphi\colon U' \to X$ where U, U' with $\overline{U} \subset U' \subset N\Sigma$ are open neighbourhoods of the zero section, as in Definition 2.1. From Definition 6.1, we require that all $J_X \in \mathcal{J}_W$ have that J_X is standard in the image $\varphi(U) \subset X$ of this neighbourhood.

Fix an almost complex structure $J_0 \in \mathcal{J}_W$. We may suppose that $P(J_0) \in \mathcal{J}_\Sigma$ is an almost complex structure in the residual set given by Proposition 6.17, though this isn't strictly speaking necessary.

Let $V := X \backslash \varphi(\overline{U})$. Following Cieliebak-Mohnke [CM07], let $\mathcal{J}(V)$ be the set of all almost complex structures compatible with ω that are equal to J_0 on $\varphi(U)$.

To define the order of contact, consider an almost complex structure $J_X \in \mathcal{J}_W$ and a J_X-holomorphic sphere $f\colon \mathbb{C}P^1 \to X$ with $f(0) \in \Sigma$, an isolated intersection. Choose coordinates $s+it = z \in \mathbb{C}$ on the domain and local coordinates near $f(0) \in \Sigma$ on the target, such that $f(0) \in \Sigma \subset X$ corresponds to $0 \in \mathbb{C}^{n-1} = \mathbb{C}^{n-1} \times \{0\} \subset \mathbb{C}^{n-1} \times \mathbb{C}$. Write $\pi_{\mathbb{C}}\colon \mathbb{C}^n \to \mathbb{C}$ for projection onto the last coordinate (which is to be thought of as normal to Σ). Assume also that $J_X(0) = i$. Then, f has order l tangency at 0 if the vector of all partial derivatives up to order l has trivial projection to $\mathbb{C}$. We define then the order of contact at an arbitrary point in $\mathbb{CP}^1$ by precomposing with a Möbius transformation.

Define the space of simple pseudoholomorphic maps into X that have order of tangency l at ∞ to a point in Σ to be

$$\begin{aligned}\mathcal{M}^*_{\infty,l,(X,\Sigma)}(\mathcal{J}_W;(\Sigma,l)) = \{(f,J_X) \in \mathcal{B}^{m,p} \times \mathcal{J}_W \mid \overline{\partial}_{J_X} f = 0, \\ f(\infty) \in \Sigma,\ d^l f(\infty) \in T_{f(\infty)}\Sigma, \\ f \text{ simple},\ f^{-1}(V) \neq \varnothing\}\end{aligned}$$

where $d^k\xi(0)$ is the vector of all partial derivatives of ξ of orders 1 through k of f at 0 and we require $m \geqslant l+2$. Note that our notation differs somewhat from the notation in [CM07].

In this section, we will prove Proposition 6.21, which was stated and used above:

Proposition 6.21. *Let $B_0, \ldots, B_k$ be spherical classes in $H_2(X)$. Let*

$$r \geqslant \max B_i \bullet \Sigma + 2.$$

The universal moduli space $\mathcal{M}^*_X((B_1,\ldots,B_k);\mathcal{J}^r_W)$ is a Banach manifold and the evaluation maps

$$\mathrm{ev}^a_\Sigma\colon \mathcal{M}^*_X((B_1,\ldots,B_k);\mathcal{J}^r_W) \to \Sigma^k : (f_1,f_2,\ldots,f_k) \mapsto (f_1(\infty),\ldots,f_k(\infty))$$
$$\mathrm{ev}_{X,\Sigma}\colon \mathcal{M}^*_X((B_0);\mathcal{J}^r_W) \to X \times \Sigma : f \mapsto (f(0), f(\infty))$$

are submersions.

Notice that this is a local question, so it suffices to show this for $(f, J_X) \in \mathcal{M}^*_{\infty,l,(X,\Sigma)}(\mathcal{J}^m_W;(\Sigma,l))$ specifically for $J_X \in \mathcal{J}(V)$ (and not for all $J_X \in \mathcal{J}_W$).

The proposition will follow by a modification of the proof given in [CM07, Section 6]. Instead of reproducing their proof, we indicate the necessary modifications. In order to be as consistent as possible with their notation, we consider the point of contact with Σ to be at 0.

Consider a J_X-holomorphic map $f : \mathbb{C}P^1 \to X$ such that $f(0) \in \Sigma$ with order of tangency l. In the notation of [CM07], we are interested in the case of only one component $Z = \Sigma$. We will obtain transversality of the evaluation map at 0 by varying J_X freely in V.

The linearized Cauchy–Riemann operator at f with respect to a torsion-free connection is

$$(D_f\xi)(z) = \nabla_s\xi(z) + J_X(f(z))\nabla_t\xi(z) + (\nabla_{\xi(z)}J_X(f(z)))\, f_t(z).$$

Specializing to the standard Euclidean connection in $\mathbb{R}^{2n} = \mathbb{C}^n$ (which preserves $\mathbb{C}^{n-1}$ along $\mathbb{C}^{n-1}$), we get

$$(D_f\xi)(z) = \xi_s(z) + J_X(f(z))\xi_t(z) + A(z)\xi(z)$$

where

$$A(z)\xi(z) = (D_{\xi(z)}J_X(f(z)))\, f_t(z)$$

(see page 317 in [CM07]).

We need the following adaptation of Corollary 6.2 in [CM07].

Lemma 6.27. *Suppose $(f, J_X) \in \mathcal{M}^*_{\infty,l,(X,\Sigma)}(\mathcal{J}^m_W; (\Sigma, l))$ with $J_X \in \mathcal{J}^m(V)$. Let $\xi\colon (D, 0) \to (\mathbb{C}^n, 0)$ be such that $D_f\xi = 0$, with J_X preserving $\mathbb{C}^{n-1}$. Given $0 < k \leqslant l$, if $\xi(0) \in \mathbb{C}^{n-1}$, $d^{k-1}\xi(0) \in \mathbb{C}^{n-1}$ and $\frac{\partial^k\xi}{\partial s^k}(0) \in \mathbb{C}^{n-1}$, then $d^k\xi(0) \in \mathbb{C}^{n-1}$.*

Proof. We need to show that $\frac{\partial^k\xi}{\partial s^{k-i}\partial t^i}(0) \in \mathbb{C}^{n-1}$ for all $0 \leqslant i \leqslant k$. It will be convenient to use multi-index notation for partial derivatives, and denote the previous expression by $D^{(k-i,i)}\xi(0)$. The case $i = 0$ is part of the hypotheses of the Lemma. For the induction step, note that $D_f\xi = 0$ combined with the product rule implies that

$$D^{(k-i,i)}\xi(z) = J_X(f(z))\Big(D^{(k-i+1,i-1)}\xi(z)+\sum_{\alpha,\beta} D^{\alpha}(J_X(f(z))D^{\beta}\xi(z)+\sum_{\alpha',\beta'} D^{\alpha'}A(z)D^{\beta'}\xi(z)\Big)$$

Here, α and β are multi-indices such that $\alpha = (a_1, a_2)$ for $0 \leqslant a_1 \leqslant k-i, 0 \leqslant a_2 \leqslant i-1, \alpha \neq (0,0)$ and $\alpha + \beta = (k-i, i)$. Similarly, α' and β' are multi-indices such that $\alpha' = (a_1', a_2')$ for $0 \leqslant a_1' \leqslant k-i, 0 \leqslant a_2' \leqslant i-1$ and $\alpha' + \beta' = (k-i, i-1)$. The hypotheses of the Lemma and the induction hypothesis implie that the derivatives of ξ on the right hand side take values in $T_{f(0)}\Sigma$. The fact that J_X and ∇ preserve $\mathbb{C}^{n-1}$ along $\mathbb{C}^{n-1}$, and that $d^l f(0) \in T_{f(0)}\Sigma$, implies the induction step. $\square$

Proposition 6.28. *For $m - 2/p > l$, the universal evaluation map*

$$\begin{aligned} \mathrm{ev}_{X,\Sigma}\colon \mathcal{M}^*_{\infty,l,(X,\Sigma)}(\mathcal{J}^m_W; (\Sigma, l)) &\to \Sigma \\ (f, J_X) &\mapsto f(0) \end{aligned}$$

is a submersion.

Proof. We show that for every $0 \leqslant k \leqslant l$ and $(f, J_X) \in \widetilde{\mathcal{M}}^*_z(\mathcal{J}(V); (\Sigma, k))$

$$\begin{aligned} (d\,\mathrm{ev}_{X,\Sigma})_{(f,J_X)} : T_{(f,J_X)}\widetilde{\mathcal{M}}^*_z(\mathcal{J}(V); (\Sigma, k)) &\to T_{f(0)}\Sigma \\ (\xi, Y) &\mapsto \xi(0) \end{aligned}$$

is surjective. By Lemma 6.5 in [CM07],

$$\begin{aligned} T_{(f,J_X)}\widetilde{\mathcal{M}}^*_z(\mathcal{J}(V); (\Sigma, k)) := \{(\xi, Y) \in T_f\mathcal{B}^{m,p}\times T_{J_X}\mathcal{J}(V) \mid D_f\xi + \frac{1}{2}Y(f)\circ df\circ j = 0, \\ \xi(0) \in T_{f(0)}\Sigma, d^k\xi(0) \in T_{f(0)}\Sigma\} \end{aligned}$$

We argue by induction on k. The case $k=0$ is a special case of Proposition 3.4.2 in [MS04]. We assume that the claim is true for $k-1$ and prove it for k.

Take any $v \in T_{f(0)}\Sigma$. By induction, there is $(\xi_1, Y_1) \in T_{(f,J_X)}\widetilde{\mathcal{M}}^*_z(\mathcal{J}(V);(\Sigma, k-1))$ such that $(d\,\mathrm{ev}_{X,\Sigma})_{(f,J_X)}(\xi_1,Y_1) = v$ and $d^{k-1}\xi_1(0) \in T_{f(0)}\Sigma$. Let now $\tilde{\xi} \in T_f\mathcal{B}^{m,p}$ be given by

$$\tilde{\xi}(z) = -\frac{z^k}{k!}\beta(z)\,\pi_{\mathbb{C}}\left(\frac{\partial^k}{\partial s^k}\xi_1\right)(0)$$

where $\beta : \mathbb{C} \to [0,1]$ is a smooth function that is identically 1 near 0 and has compact support contained in $\mathbb{C}\backslash f^{-1}(V)$. Writing

$$(D_f\xi)(z) = \xi_s(z) + i\xi_t(z) + (J_X(f(z)) - i)\xi_t(z) + A(z)\xi(z)$$

we have $(D_f\tilde{\xi})(0) = 0$ and $d^{k-1}(D_f\tilde{\xi})(0) = 0$ (this follows the fact that $\tilde{\xi}_s + i\tilde{\xi}_t \equiv 0$ near 0). By Lemma 6.6 in [CM07], there is $(\hat{\xi}, \hat{Y}) \in T_f\mathcal{B}^{m,p} \times T_{J_X}\mathcal{J}(V)$ such that $\hat{\xi}(0) = 0$, $d^k(\hat{\xi})(0) = 0$ and

$$D_f\hat{\xi} + \frac{1}{2}\hat{Y}(f) \circ df \circ j = -D_f\tilde{\xi}$$

Let now $\xi_2 = \xi_1 + \tilde{\xi} + \hat{\xi}$ and $Y_2 = Y_1 + \hat{Y}$. We have

$$D_f\xi_2 + \frac{1}{2}Y_2(f) \circ df \circ j = 0$$

as well as $\xi_2(0) = v$, $d^{k-1}(\xi_2)(0) \in T_{f(0)}\Sigma$ and $\pi_{\mathbb{C}}\left(\frac{\partial^k}{\partial s^k}\xi_2\right)(0) = 0$. Lemma 6.27 implies that $d^k(\xi_2)(0) \in T_{f(0)}\Sigma$, which completes the proof. □

Observe also that by combining this with standard arguments (see, for instance, [MS04, Proposition 3.4.2], which is also used in the proof of Proposition 6.17 above), we obtain the transversality for the evaluation at a point, taking values in X. This finishes the proof of Proposition 6.21.

6.5. **Proof of Proposition 6.9.** We are now ready to complete the proof of Proposition 6.9. To this end, we will show that the transversality problem for a cascade reduces to the already solved transversality problem for chains of pearls. The two key ingredients of this are the splitting of the linearized operator given by Lemma 6.12 and a careful study of the flow-diagonal in $Y \times Y$.

We denote by $\mathcal{J}_Y$ the set of cylindrical, Reeb-flow invariant almost complex structures on $\mathbb{R}\times Y$ obtained as lifts of the almost complex structures in $\mathcal{J}_\Sigma$. (Note that $d\pi_\Sigma \colon \mathcal{J}_Y \to \mathcal{J}_\Sigma$ gives a bijection between these sets.) Let $\mathcal{J}_Y^{reg}$ be the set of almost complex structures on $\mathbb{R}\times Y$ that are lifts of the almost complex structures in $\mathcal{J}_\Sigma^{reg}$. Recall from Section 2.2, if $J_W \in \mathcal{J}_W$ is an almost complex structure on W that is of the type we consider, it induces an almost complex structure $J_\Sigma \in \mathcal{J}_\Sigma$. The restriction of J_W to the cylindrical end of W, J_Y, is then a translation and Reeb-flow invariant almost complex structure on $\mathbb{R}\times Y$ that has $d\pi_\Sigma J_Y = J_\Sigma d\pi_\Sigma$.

Recall that the biholomorphism $\psi \colon W \to X\backslash\Sigma$ given in Lemma 2.8 allows us to identify holomorphic planes in W with holomorphic spheres in X. In the following, we will suppress the distinction when convenient.

Recall also that by the definition of an admissible Hamiltonian (Definition 3.1), for each non-negative integer m, there exists a unique b_m so that $h'(\mathrm{e}^{b_m}) = m$. Then $Y_m = \{b_m\}\times Y \subset \mathbb{R}\times Y$ is the corresponding Morse–Bott family of 1-periodic Hamiltonian orbits that wind m times around the fibre of $Y \to \Sigma$.

We now define the moduli spaces of Floer cylinders, from which we will extract the moduli spaces of cascades by imposing the gradient flow-line conditions. First, we define the moduli spaces relevant for the differential connecting two generators in $\mathbb{R} \times Y$. Then, we will define the moduli spaces relevant for the differential connecting to a critical point in W.

Definition 6.29. Let $N \geqslant 1$, let $A_1, \dots, A_N \in H_2(\Sigma; \mathbb{Z})$ be spherical homology classes. Let $J_Y \in \mathcal{J}_Y$.

Define $\widetilde{\mathcal{M}}^*_{H,k,\mathbb{R}\times Y;k_-,k_+}((A_1, \dots, A_N); J_Y)$ to be a set of tuples of punctured cylinders $(\widetilde{w}_1, \dots, \widetilde{w}_N)$ with the following properties:

(1) There is a partition of $\Gamma = \Gamma_1 \cup \dots \cup \Gamma_N$ of k augmentation marked points with
$$\widetilde{w}_i \colon \mathbb{R} \times S^1 \backslash \Gamma_i \to \mathbb{R} \times Y$$
so that $\widetilde{w}_i$ is a finite hybrid energy punctured Floer cylinder. For each $z_j \in \Gamma$, there is a positive integer multiplicity $k(z_j)$. Let w_i denote the projection to Y.

(2) There is an increasing list of $N+1$ multiplicities from k_- to k_+:
$$k_- = k_0 < k_1 < k_2 < \dots < k_N = k_+$$

(3) For each i, the cylinder $\widetilde{w}_i$ has multiplicities k_i and k_{i-1} at $\pm\infty$: $\tilde{w}_i(+\infty, \cdot) \in Y_{k_i}, \tilde{w}_i(-\infty, \cdot) \in Y_{k_{i-1}}$.

(4) The Floer cylinders $\widetilde{w}_i$ are simple in the sense that their projections to Σ are either somewhere injective or constant, if constant, they have at least one augmentation puncture, and their images are not contained one in the other.

(5) For each i, and for every puncture $z_j \in \Gamma_i$, the augmentation puncture has a limit whose multiplicity is given by $k(z_j)$; i.e. $\lim_{\rho \to -\infty} w_i(z_j + \mathrm{e}^{2\pi(\rho + i\theta)})$ is a Reeb orbit of multiplicity $k(z_j)$.

(6) The projections of the Floer cylinders to Σ represent the homology classes $A_i, i = 1, \dots, N$; i.e. $(\pi_\Sigma(\tilde{w}_i))_{i=1}^N \in \mathcal{M}^*_k((A_1, \dots, A_N), J_\Sigma)$.

Let $B \in H_2(X; \mathbb{Z})$ be a spherical homology class, $B \neq 0$. Let J_W be an almost complex structure on W as given by Lemma 2.8, matching J_Y on the cylindrical end.

Definition 6.30. Define the moduli space
$$\widetilde{\mathcal{M}}^*_{H,k,W;k_+}((B; A_1, \dots, A_N); J_W)$$
to consist of tuples
$$(v, \tilde{w}_1, \dots, \tilde{w}_N)$$
with the properties

(1) The map $v \colon \mathbb{R} \times S^1 \to W$ is a finite energy holomorphic cylinder with removable singularity at $-\infty$.

(2) There is a partition of $\Gamma = \Gamma_1 \cup \dots \cup \Gamma_N$ of k augmentation marked points with
$$\widetilde{w}_i \colon \mathbb{R} \times S^1 \backslash \Gamma_i \to \mathbb{R} \times Y$$
so that each $\widetilde{w}_i$ is a finite hybrid energy punctured Floer cylinder. For each $z_j \in \Gamma$, there is a positive integer multiplicity $k(z_j)$. Denote by w_i the projection of $\widetilde{w}_i$ to Y.

(3) There is an increasing list of $N+1$ multiplicities:
$$k_0 < k_1 < k_2 < \cdots < k_N = k_+$$
(4) For each i, and for every puncture $z_j \in \Gamma_i$, the augmentation puncture has a limit whose multiplicity is given by $k(z_j)$; i.e. $\lim_{\rho \to -\infty} w_i(z_j + e^{2\pi(\rho + i\theta)})$ is a Reeb orbit of multiplicity $k(z_j)$.
(5) The Floer cylinders $\tilde{w}_i$ are simple, in the strong sense that the projections to Σ are somewhere injective or constant, and have images not contained one in the other. The cylinder v is somewhere injective in W.
(6) The projections of the Floer cylinders to Σ represent the homology classes $A_i, i = 1, \ldots, N$; i.e. $\pi_\Sigma(\tilde{w}_i))_{i=1}^N \in \mathcal{M}_k^*((B; A_1, \ldots, A_N), J_W)$.
(7) After identifying v with a holomorphic sphere in X, v represents the homology class $B \in H_2(X; \mathbb{Z})$.
(8) The cylinder $\widetilde{w}_1$ has multiplicity k_1 at $+\infty$ and $\tilde{w}_1(+\infty, \cdot) \in Y_{k_1}$. At $-\infty$, $\widetilde{w}_1$ converges to a Reeb orbit in $\{-\infty\} \times Y$. This Reeb orbit has multiplicity k_0.
(9) For each $i \geqslant 2$, the cylinder $\widetilde{w}_i$ has multiplicities k_i and k_{i-1} at $\pm\infty$: $\tilde{w}_i(+\infty, \cdot) \in Y_{k_i}, \tilde{w}_i(-\infty, \cdot) \in Y_{k_{i-1}}$.
(10) The cylinder v converges at $+\infty$ to a Reeb orbit of multiplicity k_0.

Observe that these moduli spaces are non-empty only if for each $i = 1, \ldots, N$,
$$K\omega(A_i) = k_i - k_{i-1} - \sum_{z \in \Gamma_i} k(z).$$
Furthermore, for $\widetilde{\mathcal{M}}_{H,k,W}$, we must also have
$$B \bullet \Sigma = K\omega(B) = k_0.$$
Note also that these moduli spaces have a large number of connected components, where different components have different partitions of Γ or different intermediate multiplicities.

Identifying holomorphic spheres in X with finite energy J_W-planes in W, we consider also the moduli space of holomorphic planes $\mathcal{M}_0^*((B_1, \ldots, B_k); J_W)$ as in Definition 6.16.

The space $\widetilde{\mathcal{M}}^*_{H,k,\mathbb{R}\times Y}((A_1, \ldots, A_N); J_Y)$ consists of N-tuples of somewhere injective punctured Floer cylinders in $\mathbb{R} \times Y$. Similarly, $\widetilde{\mathcal{M}}^*_{H,k,W}$ consist of N-tuples of punctured Floer cylinders in $\mathbb{R} \times Y$ together with a holomorphic plane in W (which we can therefore also interpret as a holomorphic sphere in X). The cylinders and the eventual plane have asymptotics with matching multiplicities, but are otherwise unconstrained. These two moduli spaces, $\widetilde{\mathcal{M}}_{H,k,\mathbb{R}\times Y}$ and $\widetilde{\mathcal{M}}_{H,k,W}$ fail to be simple split Floer cascades (as in Definitions 4.1, 4.3) in two ways: they are missing the gradient trajectory constraints on their asymptotic evaluation maps, and they are missing their augmentation planes. In order to impose these conditions, we will need to study these evaluation maps and establish their transversality.

Proposition 6.31. *For $J_Y \in \mathcal{J}_Y^{reg}$, $\widetilde{\mathcal{M}}^*_{H,k,\mathbb{R}\times Y}((A_1, \ldots, A_N); J_Y)$ is a manifold of dimension*
$$N(2n-1) + \sum_{i=i}^{N} 2\langle c_1(T\Sigma), A_i\rangle + 2k$$

For $J_W \in \mathcal{J}_W^{reg}$, $\widetilde{\mathcal{M}}^*_{H,k,W}\left((B; A_1, \dots, A_N); J_W\right)$ *is a manifold of dimension*

$$N(2n-1) + 2n + 1 + \sum_{i=i}^{N} 2\left\langle c_1(T\Sigma), A_i\right\rangle + 2(\left\langle c_1(TX), B\right\rangle - B \bullet \Sigma) + 2k.$$

Proof. Consider first the case of cylinders in $\mathbb{R} \times Y$. Let

$$(\tilde{w}_1, \dots, \tilde{w}_N) \in \widetilde{\mathcal{M}}^*_{H,k,\mathbb{R}\times Y}\left((A_1, \dots, A_N); J_Y\right).$$

Recall from Lemma 6.17 that for $J_\Sigma \in \mathcal{J}_\Sigma^{reg}$, we have transversality for $D^\Sigma_{u_i}$ for each sphere $u_i = \pi_\Sigma(w_i)$.

Let $\delta > 0$ be sufficiently small. For each $i = 1, \dots, N$, by Lemma 6.14, $D^{\mathbb{C}}_{\tilde{w}_i}$ is surjective when considered on $W^{1,p,-\delta}$ (with exponential growth), and has Fredholm index 1. The operator considered instead on the space $W^{1,p,\delta}_{\mathbf{V}}$, with $V_{-\infty} = V_{+\infty} = i\mathbb{R}$ and $V_P = \mathbb{C}$ for any puncture P on the domain of $\tilde{w}_i$, has the same kernel and cokernel by Lemma A.10. Thus, the operator, acting on sections free to move in the Morse–Bott family of orbits is surjective and has index 1.

Since the operator $D_{\tilde{w}_i}$ is upper triangular from Lemma 6.12, and its diagonal components are both surjective, the operator is surjective. Since the Fredholm index is the sum of these, each component $\tilde{w}_i$ contributes an index of $1 + 2n - 2 + 2\left\langle c_1(T\Sigma), A_i\right\rangle + 2k_i = 2n - 1 + 2\left\langle c_1(T\Sigma), A_i\right\rangle + 2k_i$, where k_i is the number of punctures.

We now consider the case of a collection

$$(v, \tilde{w}_1, \dots, \tilde{w}_N) \in \widetilde{\mathcal{M}}^*_{H,k,W}\left((B; A_1, \dots, A_N); J_W\right).$$

The same consideration as previously gives that $\tilde{w}_2, \dots, \tilde{w}_N$ are transverse and each contributes an index of $2n - 1 + 2\left\langle c_1(T\Sigma), A_i\right\rangle + 2k_i$, where k_i is the number of punctures. For the component $\tilde{w}_1$, again applying Lemma 6.12 and applying Lemma 6.14 in the case where the $-\infty$ end of the cylinder converges to a Reeb orbit at $\{-\infty\} \times Y$, we obtain that the vertical Fredholm operator is surjective and has index 2. The linearized Floer operator at $\tilde{w}_1$ is then surjective and has index $2n + 2\left\langle c_1(T\Sigma), A_1\right\rangle + 2k_1$. By Lemma 2.8, the plane v can be identified with a sphere in X with an order of contact $l = B \bullet \Sigma$ with Σ. Its Fredholm index is $2n + 2(\left\langle c_1(TX), B\right\rangle - l)$. The total Fredholm index is therefore

$$(N-1)(2n-1) + 2n + 2n + \sum_{i=i}^{N} 2\left\langle c_1(T\Sigma), A_i\right\rangle + 2(\left\langle c_1(TX), B\right\rangle - B \bullet \Sigma) + 2k.$$

For both cases, the result now follows from the implicit function theorem. □

It now suffices to prove the transversality of evaluation maps to the products of stable/unstable manifolds and flow diagonals, and also transversality of the augmentation evaluation maps, in order to obtain the constraints coming from pseudo-gradient flow lines. Indeed, let $(\widetilde{w}_1, \dots, \widetilde{w}_N)$ be a collection of N cylinders in $\widetilde{\mathcal{M}}^*_{H,k,\mathbb{R}\times Y; k_-, k_+}\left((A_1, \dots, A_N); J_Y\right)$. Write each of the $\widetilde{w}_i \colon \mathbb{R} \times S^1 \to \mathbb{R} \times Y$ as a pair $\widetilde{w}_i = (b_i, w_i)$. We then have asymptotic evaluation maps

$$\begin{aligned}
&\widetilde{\mathrm{ev}}_Y : \widetilde{\mathcal{M}}^*_{H,k,\mathbb{R}\times Y;k_-,k_+}((A_1,\ldots,A_N);J_Y) \to Y^{2N} \\
&\quad (\widetilde{w}_1,\ldots,\widetilde{w}_N) \mapsto \left(\lim_{s\to-\infty} w_1(s,1), \lim_{s\to+\infty} w_1(s,1),\ldots, \lim_{s\to-\infty} w_N(s,1), \lim_{s\to+\infty} w_N(s,1)\right)
\end{aligned}$$

If $(v,\widetilde{w}_1,\ldots,\widetilde{w}_N) \in \widetilde{\mathcal{M}}^*_{H,k,W}((B;A_1,\ldots,A_N);J_W)$, we have

$$\begin{aligned}
&\widetilde{\mathrm{ev}}_{W,Y} : \widetilde{\mathcal{M}}^*_{H,k,W;k_+}((B;A_1,\ldots,A_N);J_W) \to W \times Y^{2N+1} \\
&\quad (v,\widetilde{w}_1,\ldots,\widetilde{w}_N) \mapsto \left(v(0), \lim_{r\to+\infty} \pi_Y v(r+i0), \lim_{s\to-\infty} w_1(s,1),\ldots, \lim_{s\to+\infty} w_N(s,1)\right)
\end{aligned}$$

This map is C^1 smooth, exploiting the asymptotic expansion of the Floer cylinder near its asymptotic limit, as described by [Sie08]. This is proved in [FS17].

We also have augmentation evaluation maps. For each puncture $z_0 \in \Gamma$, there exists an index $i \in \{1,\ldots,N\}$ so that the augmentation puncture z_0 is a puncture in the domain of w_i. For this augmentation puncture, we have the asymptotic evaluation map $w_i \mapsto \lim_{z\to z_0} \pi_\Sigma(w_i(z)) \in \Sigma$. Combining all of these evaluation maps over all punctures in Γ, we obtain

$$\begin{aligned}
\widetilde{\mathrm{ev}}^a_\Sigma &: \widetilde{\mathcal{M}}^*_{H,k,\mathbb{R}\times Y;k_-,k_+}((A_1,\ldots,A_N);J_Y) \to \Sigma^k \\
\widetilde{\mathrm{ev}}^a_\Sigma &: \widetilde{\mathcal{M}}^*_{H,k,W;k_+}((B;A_1,\ldots,A_N);J_W) \to \Sigma^k.
\end{aligned}$$

Note that this map is smooth by [Wen16b, Proposition 3.15].

Define the *flow diagonal* in $Y \times Y$ to be

$$\widetilde{\Delta}_{f_Y} := \left\{(x,y) \in (Y \setminus \mathrm{Crit}(f_Y))^2 \;:\; \exists t > 0 \text{ s.t. } \varphi^t_{Z_Y}(x) = y\right\}$$

where $\mathrm{Crit}(f_Y)$ is the set of critical points of f_Y.

Let $\tilde{p},\tilde{q} \in Y$ be critical points of f_Y and let $W^u_Y(\tilde{p})$, $W^s_Y(\tilde{q})$ be the unstable/stable manifolds of $\tilde{p},\tilde{q}$ with respect to the anti-gradient-like vector fields $-Z_Y$.

We may now describe the moduli space of simple split Floer cylinders from q_{k_-} to p_{k_+} as the unions of the fibre products of these moduli spaces under the asymptotic evaluation maps and augmentation evaluation maps. For notational convenience, we write

$$\begin{aligned}
\widetilde{\mathrm{ev}} : \widetilde{\mathcal{M}}^*_{H,k,\mathbb{R}\times Y;k_-,k_+}((A_1,\ldots,A_N);J_Y) \times \mathcal{M}^*_0((B_1,\ldots,B_k);J_W) &\to Y^{2N} \times \Sigma^k \times \Sigma^k \\
(\mathbf{w},\mathbf{v}) &\mapsto \left(\widetilde{\mathrm{ev}}_Y(\mathbf{w}), \widetilde{\mathrm{ev}}^a_\Sigma(\mathbf{w}), \mathrm{ev}^a_\Sigma(\mathbf{v})\right).
\end{aligned}$$

Write $\Delta_{\Sigma^k} \subset \Sigma^k \times \Sigma^k$ to denote the diagonal Σ^k. Then, define

$$\widetilde{\mathcal{M}}^*_H(\tilde{q}_{k_-},\tilde{p}_{k_+};(A_1,\ldots,A_N),(B_1,\ldots,B_k);J_W) = \widetilde{\mathrm{ev}}^{-1}\left(W^s_Y(\tilde{q}) \times \left(\widetilde{\Delta}_{f_Y}\right)^{N-1} \times W^u_Y(\tilde{p}) \times \Delta_{\Sigma^k}\right).$$

From this, we have

$$\begin{aligned}
&\widetilde{\mathcal{M}}^*_{H,N}(\tilde{q}_{k_-},\tilde{p}_{k_+};J_Y,J_W) = \\
&\quad \bigcup_{(A_1,\ldots,A_N)} \bigcup_{k\geqslant 0} \bigcup_{(B_1,\ldots,B_k)} \widetilde{\mathcal{M}}^*_H(\tilde{q}_{k_-},\tilde{p}_{k_+};(A_1,\ldots,A_N),(B_1,\ldots,B_k);J_W)
\end{aligned} \tag{6.5}$$

Similarly, if $x \in W$ is a critical point of f_W, and letting $W^u_W(x)$ be the descending manifold of x in W for the gradient-like vector field $-Z_W$, we similarly define

$$\widetilde{\mathrm{ev}}\colon \widetilde{\mathcal{M}}^*_{H,k,W;k_+}((B; A_1, \dots, A_N); J_W) \times \mathcal{M}^*_0((B_1, \dots, B_k); J_W) \to W \times Y^{2N+1} \times \Sigma^k \times \Sigma^k$$
$$((v, \tilde{\mathbf{w}}), \mathbf{v}) \mapsto (\widetilde{\mathrm{ev}}_{W,Y}(v, \mathbf{w}), \widetilde{\mathrm{ev}}^a_\Sigma(\mathbf{w}), \mathrm{ev}^a_\Sigma(\mathbf{v}))\,.$$

Then, define

$$\widetilde{\mathcal{M}}^*_H(x, \tilde{p}_{k_+}; (B; A_1, \dots, A_N), (B_1, \dots, B_k); J_W) =$$
$$\widetilde{\mathrm{ev}}^{-1}\left(W^u_W(x) \times \tilde{\Delta} \times \left(\tilde{\Delta}_{f_Y}\right)^{N-1} \times W^u_Y(\tilde{p}) \times \Delta_{\Sigma^k}\right).$$

Finally, we obtain

$$\begin{aligned}&\widetilde{\mathcal{M}}^*_{H,N}(x, \tilde{p}_{k_+}; J_W) = \\ &\bigcup_{(B;A_1,\dots,A_N)} \bigcup_{k \geqslant 0} \bigcup_{(B_1,\dots,B_k)} \widetilde{\mathcal{M}}^*_H(x, \tilde{p}_{k_+}; (B; A_1, \dots, A_N), (B_1, \dots, B_k); J_W)\end{aligned} \tag{6.6}$$

In order to establish transversality for our moduli spaces, it then becomes necessary to show transversality of the evaluation maps to these products of descending/ascending manifolds, diagonals and flow diagonals. Recall that the spaces of almost complex structures $J_Y \in \mathcal{J}^{reg}_Y$, $J_W \in J^{reg}_W$ are given from Proposition 6.26.

Proposition 6.32. *Let $J_W \in J^{reg}_W$ and let $J_Y \in \mathcal{J}^{reg}_Y$ be the induced almost complex structure on $\mathbb{R} \times Y$.*

Let $\tilde{q}, \tilde{p}$ denote critical points of f_Y, and let x be a critical point of f_W in W. Let k_+ and k_- be non-negative multiplicities, $k_+ > k_-$.

Let $A_1, \dots, A_N$ be spherical homology classes in Σ, let $B, B_1, \dots, B_k$ be spherical homology classes in X, $k \geqslant 0$.

Let $\tilde{\Delta} \subset Y \times Y$ and $\Delta_{\Sigma^k} \subset \Sigma^k \times \Sigma^k$ be the diagonals.

Then,

(1) *the evaluation map*

$$\widetilde{\mathrm{ev}}_Y \times \widetilde{\mathrm{ev}}^a_\Sigma \times \mathrm{ev}^a_\Sigma\colon : \widetilde{\mathcal{M}}^*_{H,k,\mathbb{R}\times Y;k_-,k_+}((A_1, \dots, A_N); J_Y) \times \mathcal{M}^*_0((B_1, \dots, B_k); J_W) \to Y^{2N} \times \Sigma^k \times \Sigma^k$$

is transverse to the submanifold

$$W^s_Y(\tilde{q}) \times \left(\tilde{\Delta}_{f_Y}\right)^{N-1} \times W^u_Y(\tilde{p}) \times \Delta_{\Sigma^k}$$

(2) *the evaluation map*

$$\widetilde{\mathrm{ev}}_{W,Y} \times \widetilde{\mathrm{ev}}^a_\Sigma \times \mathrm{ev}^a_\Sigma\colon \widetilde{\mathcal{M}}^*_{H,k,W;k_+}((B; A_1, \dots, A_N); J_W) \times \mathcal{M}^*_0((B_1, \dots, B_k); J_W) \to W \times Y^{2N+1} \times \Sigma^k \times \Sigma^k$$

is transverse to the submanifold

$$W^u_W(x) \times \tilde{\Delta} \times \left(\tilde{\Delta}_{f_Y}\right)^{N-1} \times W^u_Y(\tilde{p}) \times \Delta_{\Sigma^k}.$$

In order to prove this proposition, we will need a better description of the relationship between the moduli spaces of spheres in Σ, and the moduli spaces of Floer cylinders in $\mathbb{R} \times Y$ (or in W).

Lemma 6.33. *The maps*

$$\pi_\Sigma^{\mathcal{M}}\colon \widetilde{\mathcal{M}}^*_{H,k,\mathbb{R}\times Y;k_-,k_+}((A_1,\dots,A_N);J_Y) \to \mathcal{M}^*_{k,\Sigma}((A_1,\dots,A_N);J_\Sigma)$$
$$\pi_\Sigma^{\mathcal{M}}\colon \widetilde{\mathcal{M}}^*_{H,k,W;k_+}((B;A_1,\dots,A_N);J_W) \to \mathcal{M}^*_{k,X,\Sigma}((B;A_1,\dots,A_N);J_W)$$

induced by $\pi_\Sigma\colon \mathbb{R}\times Y \to \Sigma$ are submersions. The fibres have a locally free $(S^1)^N$ torus action by constant rotation by the action of the Reeb vector field.

Proof. We will study the case of

$$\pi_\Sigma^{\mathcal{M}}\colon \widetilde{\mathcal{M}}^*_{H,k,\mathbb{R}\times Y}((A_1,\dots,A_N);J_Y) \to \mathcal{M}^*_{k,\Sigma}((A_1,\dots,A_N);J_\Sigma)$$

in detail. The case with a sphere in X follows by the same argument with a small notational change. It also suffices to consider the case with $N = 1$, since moduli spaces with more spheres are open subsets of products of these.

Suppose $\pi_\Sigma(\tilde{w}) = w$ with $\tilde{w} \in \widetilde{\mathcal{M}}_{H,k,\mathbb{R}\times Y;k_-,k_+}(A;J_Y)$ and $w \in \mathcal{M}^*_{k,\Sigma}(A;J_\Sigma)$. Recall the splitting given in Lemma 6.12, which splits the linearized Floer operator at $\tilde{w}$ as

$$D_{\tilde{w}} = \begin{pmatrix} D^{\mathbb{C}}_{\tilde{w}} & M \\ 0 & D^\Sigma_w \end{pmatrix}$$

By definition, $T_w\mathcal{M}^*_{k,\Sigma}(A) = \ker D^\Sigma_w$ and $T_{\tilde{w}}\widetilde{\mathcal{M}}_{H,k,\mathbb{R}\times Y;k_-,k_+}(A;J_Y) = \ker D_{\tilde{w}}$. By Lemma 6.14, $D^{\mathbb{C}}_{\tilde{w}}$ is surjective. It follows then that any section ζ_0 of $w^*T\Sigma$ that is in the kernel of D^Σ_w can be lifted to a section (ζ_1,ζ_0) of $\tilde{w}^*T\Sigma \cong (\mathbb{R}\oplus\mathbb{R}R)\oplus w^*T\Sigma$ that is in the kernel of $D_{\tilde{w}}$.

Notice now that $d\pi_\Sigma(\zeta_1,\zeta_0) = \zeta_0$, establishing that the evaluation map is a submersion.

Also observe that S^1 acts on the curve $\tilde{w}$ by the Reeb flow. From the Reeb-invariance of J_Y and of the admissible Hamiltonian, this is then clearly in the fibre. If $A \neq 0$, this new solution is geometrically distinct from the previous solution, at least for small rotation. If $A = 0$, $\tilde{w}$ has image contained in a fibre $\mathbb{R}\times Y \to \Sigma$, and thus the Reeb rotation has the same image, but the parametrization is different. In this case also, we have then a different curve, at least for small rotation parameter. □

We now consider all Floer cylinders that project to the same *unparametrized* sphere in Σ. This will also be useful elsewhere in the paper. Let $\tilde{v} = (b,v)\colon \mathbb{R}\times S^1\backslash\Gamma \to \mathbb{R}\times Y$ be a finite hybrid energy solution to Floer's equation (4.3). There is an action of $S^1\times S^1$ on the space of such solutions, with each circle action by rotation on the domain and on the target, respectively:

$$(\theta_1,\theta_2).\tilde{v} = \tilde{v}_{(\theta_1,\theta_2)}$$

where

$$\tilde{v}_{(\theta_1,\theta_2)}(s,t) = (b(s,t+\theta_1), \phi_R^{\theta_2}\circ v(s,t+\theta_1)).$$

Here, ϕ^θ_R denotes the Reeb flow for time θ. If we identify the simple Reeb orbits underlying $x_\pm(t) = \lim_{s\to\pm\infty} v(s,t)$ with S^1, then the following map keeps track of the asymptotic effect of the action:

$$\begin{aligned} \mathrm{rot}_{\tilde{v}}\colon S^1\times S^1 &\to S^1\times S^1 \\ (\theta_1,\theta_2) &\mapsto \left(\lim_{s\to-\infty} v(s,\theta_1)+\theta_2, \lim_{s\to+\infty} v(s,\theta_1)+\theta_2\right). \end{aligned}$$

If $k_\pm$ are the multiplicities of the periodic orbits $x_\pm(t)$, then, $\mathrm{rot}_{\tilde v}$ is the linear map represented in matrix form as

$$\mathrm{rot}_{\tilde v} = \begin{pmatrix} k_- & 1 \\ k_+ & 1 \end{pmatrix}. \tag{6.7}$$

Suppose now that $w\colon \mathbb{R}\times S^1 \to W$ is a non-constant J_W-holomorphic cylinder with removable singularity at $-\infty$ (such as $\tilde v_2$ in Definition 4.14). There is an S^1-action on such curves, by rotation on the domain:

$$\theta.w(s,t) = w_\theta(s,t) = w(s,t+\theta).$$

Identifying the simple Reeb orbit underlying $\gamma(t) = \lim_{s\to\infty}\pi_Y(w(s,t))$ with S^1, we have

$$\begin{aligned} \mathrm{rot}_{\tilde v}\colon S^1 &\to S^1 \\ \theta &\mapsto \lim_{s\to+\infty} w(s,\theta). \end{aligned}$$

If the multiplicity of the periodic Reeb orbit $\gamma(t)$ is k, then rot_w is multiplication by k.

Lemma 6.34. *Let $A := [w] \in H_2(\Sigma;\mathbb{Z})$, where $w\colon \mathbb{C}P^1 \to \Sigma$ is the continuous extension of $\pi_\Sigma\circ\tilde v$. Assume that either $A\neq 0$ or $\Gamma\neq\varnothing$. Then, $k_+ > k_-$.*

Proof. Denote by w^*Y the pullback under w of the S^1-bundle $Y\to\Sigma$. The map $\tilde v$ gives a section s of w^*Y, defined in the complement of $\Gamma\cup\{0,\infty\}$. By [BT82, Theorem 11.16], the Euler number $\int_{\mathbb{C}P^1} e(w^*Y)$ (where e is the Euler class) is the sum of the local degrees of the section s at the points in $\Gamma\cup\{0,\infty\}$.

Denote the multiplicities of the periodic X_H-orbits $x_\pm(t) = \lim_{s\to\pm\infty} v(s,t)$ by $k_\pm$, respectively, and denote the multiplicities of the asymptotic Reeb orbits at the punctures $z_1,\dots,z_m\in\Gamma$ by $k_1,\dots,k_m$, respectively. The positive integers $k_\pm$ and k_i are the absolute values of the degrees of s at the respective points. Taking signs into account, we get

$$\int_{\mathbb{C}P^1} e(w^*Y) = k_+ - k_- - k_1 - \ldots - k_m.$$

But

$$\int_{\mathbb{C}P^1} e(w^*Y) = \int_{\mathbb{C}P^1} w^*e(Y\to\Sigma) = \int_{\mathbb{C}P^1} w^*e(N\Sigma)$$

where $N\Sigma$ is the normal bundle to Σ in Y. Now, $e(N\Sigma) = s^*\,\mathrm{Th}(N\Sigma)$, where $s\colon \Sigma\to N\Sigma$ is the zero section and $\mathrm{Th}(N\Sigma)$ is the Thom class of $N\Sigma$ [BT82, Proposition 6.41]. If $j\colon N\Sigma\to X$ is a tubular neighborhood, then $j_*\,\mathrm{Th}(N\Sigma) = PD([\Sigma]) = [K\omega]\in H^2(X;\mathbb{R})$ [BT82, Equation (6.23)]. So,

$$\begin{aligned} \int_{\mathbb{C}P^1} w^*e(N\Sigma) = \int_{\mathbb{C}P^1} w^*s^*\,\mathrm{Th}(N\Sigma) = \int_{\mathbb{C}P^1} w^*\iota^*j_*\,\mathrm{Th}(N\Sigma) = \\ = \int_{\mathbb{C}P^1} w^*\iota^*K\omega = K\omega(A) \geqslant 0 \end{aligned}$$

since $K>0$ and w is a J_Σ-holomorphic sphere. We conclude that

$$k_+ - k_- - k_1 - \ldots - k_m = K\omega(A) \geqslant 0.$$

If $A\neq 0$, we get a strict inequality. If $A=0$, we get an equality, but the assumptions of the Lemma imply that $\sum_{i=1}^m k_i > 0$. In either case, we get $k_+ > k_-$, as wanted. □

Recall that the gradient-like vector field Z_Y has the property that $d\pi_\Sigma Z_Y = Z_\Sigma$. Also recall that we may use the contact form α as a connection to lift vector fields from Σ to vector fields on Y, tangent to ξ. If V is a vector field on Σ, we write $\pi_\Sigma^* V := \widetilde{V}$ to be the vector field on Y uniquely determined by the conditions $\alpha(V) = 0$, $d\pi_\Sigma \widetilde{V} = V$. This extends as well to lifting vector fields on $\Sigma \times \Sigma$ to vector fields on $Y \times Y$.

Lemma 6.35. *The flow diagonal in Y satisfies*

$$\pi_\Sigma(\widetilde{\Delta}_{f_Y}) \subset \Delta_{f_\Sigma} \cup \{(p,p) \,|\, p \in \mathrm{Crit}(f_\Sigma)\}.$$

Let $(\tilde{x},\tilde{y}) \in \widetilde{\Delta}_{f_Y}$ and $x = \pi_\Sigma(\tilde{x})$, $y = \pi_\Sigma(\tilde{y})$. Let t so that $\tilde{y} = \varphi^t_{Z_Y}(\tilde{x})$. Then, if $x = y$, we have $x \in \mathrm{Crit}(f_\Sigma)$ and

$$T_{(\tilde{x},\tilde{y})}\widetilde{\Delta}_{f_Y} = \{(aR + v, bR + \pi_\Sigma^* d\varphi^t_{Z_\Sigma} d\pi_\Sigma v) \in TY \oplus TY \,|\, a, b \in \mathbb{R} \text{ and } \alpha(v) = 0\}. \tag{6.8}$$

If $x \neq y$, then $(x,y) \in \Delta_{f_\Sigma}$. Then, there exists a positive $g = g(\tilde{x},\tilde{y}) > 0$ so that

$$T_{(\tilde{x},\tilde{y})}\widetilde{\Delta}_{f_Y} = \mathbb{R}(R, gR) \oplus H \tag{6.9}$$

where $d\pi_\Sigma|_H \colon H \to T\Delta_{f_\Sigma}$ induces a linear isomorphism.

Proof. Observe first that if $x = \pi_\Sigma \tilde{x}$, we have

$$\pi_\Sigma \varphi^t_{Z_Y}(\tilde{x}) = \varphi^t_{Z_\Sigma}(x).$$

This gives $d\pi_\Sigma \, d\varphi^t_{Z_Y}(\tilde{x}) = d\varphi^t_{Z_\Sigma} d\pi_\Sigma(\tilde{x})$. From this, it follows that $d\varphi^t_{Z_Y}(\tilde{x})R$ is a multiple of the Reeb vector field. Observe also that $\varphi^t_{Z_\Sigma}$ and $\varphi^t_{Z_Y}$ are both orientation preserving diffeomorphisms for all t. We therefore obtain that if $y = \varphi^t_{Z_\Sigma}(x)$, $\varphi^t_{Z_Y}$ induces a diffeomorphism between the fibres $\pi_\Sigma^{-1}(x) \to \pi_\Sigma^{-1}(y)$. Additionally, we must have then that $d\varphi^t_{Z_Y}(\tilde{x})R$ is a positive multiple of the Reeb vector field. Let $g(\tilde{x},\tilde{y}) > 0$ such that $d\varphi^t_{Z_Y}(\tilde{x})R = g(\tilde{x},\tilde{y})R$.

In general, if $\tilde{y} = \varphi^t_{Z_Y}(\tilde{x})$, we have

$$T_{(\tilde{x},\tilde{y})}\widetilde{\Delta}_{f_Y} = \{(v, d\varphi^t_{Z_Y}(\tilde{x})v + cZ_Y(\tilde{y})) \,|\, v \in T_xY, c \in \mathbb{R}\}. \tag{6.10}$$

Consider first the case of $x = y$. Then, both $\tilde{x}$ and $\tilde{y}$ are in the same fibre of $Y \to \Sigma$. By definition of the flow diagonal, there exists $t > 0$ so that $\varphi^t_{Z_Y}(\tilde{x}) = \tilde{y}$, and hence Z_Y is vertical, $Z_\Sigma(x) = 0$. It follows $x \in \mathrm{Crit}_{f_\Sigma}$. From this, it now follows that $\pi_\Sigma(\widetilde{\Delta}_{f_Y}) \subset \Delta_{f_\Sigma} \cup \{(p,p) \,|\, p \in \mathrm{Crit}(f_\Sigma)\}$.

We now consider the consequences of Equation (6.10) in this case of $x = y$. Any $v \in T_xY$ may be written as $v_0 + aR$ where $\alpha(v_0) = 0$. Furthermore, since $x = y \in \mathrm{Crit}(f_\Sigma)$, and by definition, neither $\tilde{x}$ nor $\tilde{y}$ are critical points of f_Y, we obtain that $Z_Y(\tilde{y})$ is a non-zero multiple of the Reeb vector field. Equation (6.8) now follows from the fact that $d\pi_\Sigma \varphi^t_{Z_Y}(\tilde{x}) = d\varphi^t_{Z_\Sigma}(x) d\pi_\Sigma$.

We now consider when $x \neq y$. Let $H = \{(v, d\varphi^t_{Z_Y}(\tilde{x})v + cZ_Y(\tilde{y}) \,|\, \alpha(v) = 0\}$. Then,

$$d\phi_\Sigma(H) = \{(v, d\varphi^t_{Z_\Sigma}(x)v + cZ_\Sigma) \,|\, v \in T_x\Sigma\} = T\Delta_{f_\Sigma}.$$

By assumption, y is not a critical point of f_Σ, so $d\pi_\Sigma$ induces an isomorphism. The decomposition of $T\widetilde{\Delta}_{f_Y}$ now follows immediately from the definition of g and from Equation (6.10).

∎

Proof of Proposition 6.32. We consider first the case of

$$\widetilde{\mathrm{ev}}_Y : \widetilde{\mathcal{M}}^*_{H,k,\mathbb{R}\times Y;k_-,k_+}((A_1,\dots,A_N);J_Y) \to Y^{2N}$$

Suppose that $\tilde{\mathbf{v}} = (\tilde{v}_1,\dots,\tilde{v}_N) \in \widetilde{\mathcal{M}}^*_{H,k,\mathbb{R}\times Y;k_-,k_+}((A_1,\dots,A_N);J_Y)$. For each $i = 1,\dots N$, let $\tilde{y}_{i-1} = \tilde{v}_i(-\infty,0) \in Y$ and $\tilde{x}_i = \tilde{v}_i(+\infty,0) \in Y$, with

$$\begin{aligned}
&\tilde{y}_0 \in W^s_Y(\tilde{q}), \tilde{x}_N \in W^u_Y(\tilde{p})\\
&(\tilde{x}_i,\tilde{y}_i) \in \widetilde{\Delta}_{f_Y} \quad \text{for } 1 \leqslant i \leqslant N-1
\end{aligned}$$

Let $v_i = \pi_S igma(\tilde{v}_i)$ and $x_i = \pi_\Sigma(\tilde{x}_i)$, $y_i = \pi_\Sigma(\tilde{y}_i)$. Then, it follows that

$$\begin{aligned}
&y_0 \in W^s_\Sigma(q), x_N \in W^u_\Sigma(p)\\
&(x_i,y_i) \in \Delta_{f_\Sigma} \cup \{(p,p) \,|\, p \in \mathrm{Crit}(f_\Sigma)\} \quad \text{for } 1 \leqslant i \leqslant N-1.
\end{aligned}$$

Let $S \subset \Sigma^{2N-2}$ be the appropriate product of a number of copies of Δ_{f_Σ} and of $\{(p,p)\,|\,p \in \mathrm{Crit}(f_\Sigma)\}$. By Proposition 6.26, the evaluation map on $\mathcal{M}_\Sigma((A_1,\dots,A_N);J_Y)$ is transverse to S.

Then,

$$TS \subset d\,\mathrm{ev}_\Sigma\left(T_{\tilde{y}_0}W^s_Y(\tilde{q}) \times T_{(\tilde{x}_1,\tilde{y}_1)}\widetilde{\Delta}_{f_Y} \times \cdots \times T_{(\tilde{x}_{N-1},\tilde{y}_{N-1})}\widetilde{\Delta}_{f_Y} \times T_{\tilde{x}_N}W^u_Y(\tilde{p})\right).$$

It suffices therefore to obtain transversality in the vertical direction. Notice that by rotating by the action of the Reeb vector field on $\tilde{v}_i$, we obtain that the image of $d\widetilde{\mathrm{ev}}$ contains the subspace

$$\{(a_1R, a_1R, a_2R, a_2R, \dots, a_NR, a_NR) \,|\, (a_1,\dots,a_N) \in \mathbb{R}^N\} \subset (TY)^{2N}.$$

In the case of the chain of pearls in Σ, all of the spheres $v_i, i = 1,\dots,N$ must either be non-constant or have a non-trivial collection of augmentation punctures. Then, by Lemma 6.34, each punctured cylinder $\tilde{v}_i$ has different multiplicities k_i^+, k_i^- at $\pm\infty$, and thus the action of rotating the domain marker gives that the image of $d\widetilde{\mathrm{ev}}_Y(\tilde{v}_i)$ contains $(k_-R, k_+R) \in T_{\tilde{y}_{i-1}}Y \oplus T_{\tilde{x}_i}Y$. While this holds for each $i = 1,\dots,N$, we only require such a vector for one cylinder. Then, by taking this in the case of $i = 1$, we see that the following $N+1$ vertical vectors in $(\mathbb{R}R)^{2N} \subset TY^{2N}$ are in the image of the linearized evaluation map (the first two obtained by combining the two Reeb actions on $\tilde{v}_1$, the remainder by the Reeb action on $\tilde{v}_i$, $i \geqslant 2$):

$$\begin{aligned}
&(R,0,0,\dots,0),\\
&(0,R,0,\dots,0),\\
&(0,0,R,R,0,0,\dots,0),\\
&(0,0,0,0,R,R,0,\dots,0),\\
&\dots\\
&(0,0,\dots,0,R,R).
\end{aligned}$$

By Lemma 6.23, the tangent space $T_{(\tilde{x}_i,\tilde{y}_i)}\widetilde{\Delta}_{f_Y}$ contains at least the vertical vector (R, g_iR), where $g_i := g(\tilde{x}_i,\tilde{y}_i) > 0$, for each $1 \leqslant i \leqslant N-1$. In the vertical direction, this then contains the following $N-1$ vectors:

$$\begin{aligned}&(0, R, g_1 R, 0, \ldots, 0),\\&(0,0,0,R,g_2R,0,\ldots,0)\\&\ldots\\&(0,\ldots,0,R,g_{N-1}R,0).\end{aligned}$$

We now observe that this collection of $2N$ vectors spans $(\mathbb{R}R)^{2N}$. This now establishes the result in the case of $\widetilde{\mathcal{M}}^*_{H,k,\mathbb{R}\times Y;k_-,k_+}((A_1,\ldots,A_N);J_W)$.

We now consider the case of

$$\widetilde{\mathrm{ev}}_{W,Y} \colon \widetilde{\mathcal{M}}^*_{H,k,W;k_+}((B;A_1,\ldots,A_N);J_W) \to W \times Y^{2N+1}.$$

We will show this evaluation map is transverse to

$$\tilde{S} := W^u_W(x) \times \tilde{\Delta} \times \left(\tilde{\Delta}_{f_Y}\right)^{N-1} \times W^u_Y(\tilde{p}).$$

As before, it suffices to show transversality in a vertical direction, since, by Proposition 6.26, the projections to X, Σ are transverse. Let $S \subset W \times \Sigma \times \Sigma^{2N}$ be of the form $S = W^u_W(x) \times \Delta \times S' \times W^u_\Sigma(p)$, where $S' \subset \Sigma^{2N-2}$ is a product of some number of Δ_{f_Σ} and of $\{(p,p) \mid p \in \mathrm{Crit}(f_\Sigma)\}$ so that $TS \subset Td\pi_\Sigma(\tilde{S})$. Proposition 6.26 gives transversality to S.

Notice that the tangent space $T\tilde{S}$ contains at least the following vertical vectors (we put 0 in the first component since TW has no vertical direction):

$$\begin{aligned}&(0,R,R,0,0,\ldots,0)\\&(0,0,0,R,g_1R,0,\ldots,0)\\&\ldots\\&(0,\ldots,0,R,g_{N-1}R,0).\end{aligned}$$

Let $(w, \tilde{v}_1, \ldots, \tilde{v}_N) \in \widetilde{\mathcal{M}}^*_{H,k,W;k_+}$. The plane w converges to a Reeb orbit of multiplicity $l = B \bullet \Sigma$. Observe that domain rotation on the plane w then gives that $(0, lR, 0, \ldots, 0) \in TW \oplus TY \oplus TY^{2N}$ is in the image of $d\widetilde{\mathrm{ev}}_{W,Y}$.

As before, the Reeb rotation on each of the punctured cylinders $\tilde{v}_1, \ldots, \tilde{v}_N$ gives that the following vertical vectors are in the image of $d\widetilde{\mathrm{ev}}_{W,Y}$:

$$\begin{aligned}&(0,0,R,R,0,0,\ldots,0,0)\\&(0,0,0,0,R,R,\ldots,0,0)\\&(0,0,0,0,\ldots,0,R,R).\end{aligned}$$

We notice then that these vectors span $0 \oplus (\mathbb{R}R)^{2N-1}$, so it follows that the evaluation map is transverse to $\tilde{S}$.

Finally, the transversality of the evaluation maps at augmentation punctures comes from the fact that the augmentation evaluation maps

$$\begin{aligned}\widetilde{\mathrm{ev}}^a_\Sigma \times \mathrm{ev}^a_\Sigma \colon \widetilde{\mathcal{M}}^*_{H,k,\mathbb{R}\times Y;k_-,k_+}((A_1,\ldots,A_N);J_Y) \times \mathcal{M}^*_0((B_1,\ldots,B_k);J_W) \to \Sigma^k \times \Sigma^k\\ \widetilde{\mathrm{ev}}^a_\Sigma \times \mathrm{ev}^a_\Sigma \colon \widetilde{\mathcal{M}}^*_{H,k,W;k_+}((B;A_1,\ldots,A_N);J_W) \times \mathcal{M}^*_0((B_1,\ldots,B_k);J_W) \to \Sigma^k \times \Sigma^k\end{aligned}$$

factor through the evaluation maps $\mathrm{ev}^a \colon \mathcal{M}((A_1,\ldots,A_N);J_Y)) \to \Sigma^k$ and $\mathrm{ev}^a \colon \mathcal{M}((B;A_1,\ldots,A_N);J_W)$. The required transversality for these maps is given by Proposition 6.26. Furthermore, these evaluation maps are invariant under the domain rotation and Reeb rotations used to obtain transversality in the vertical directions, so the transversality follows immediately. ☐

Part 3. Index calculations and ruling out configurations

7. Fredholm index for Floer cylinders in the symplectization

7.1. **Conley–Zehnder indices.** In this section, we revisit the gradings of the generators of the Floer complex, as given in Equations (3.4) and (3.5), and relate this to the relevant Fredholm indices, as given by Proposition 6.9.

To this end, we compute the Conley–Zehnder indices of closed Reeb orbits in Y and of closed Hamiltonian orbits in $\mathbb{R} \times Y$ and in W.

We recall that the Conley–Zehnder index was originally defined for non-degenerate periodic orbits, and thus we need a modification to take into account the Morse–Bott nature of our situation. Furthermore, the Conley–Zehnder index depends on a choice of trivialization, either of the contact structure ξ over the orbit in the case of a Reeb orbit, or of the tangent bundle TW over the orbit in the Hamiltonian case. We refer the reader to Appendix A on Riemann–Roch for punctured holomorphic curves for a very brief summary of the relevant background. We also refer the reader to Abreu and Macarini's exposition of Conley–Zehnder and Robbin–Salamon indices in [AM16, Section 3], and also to Gutt [Gut14].

Suppose that we have two different trivializations of TW over a periodic orbit. The transition from one to the other gives a loop of matrices in $U(n)$. The difference in corresponding Conley–Zehnder indices is given by twice the winding number of the determinant of this loop. In particular, then, the Conley–Zehnder index associated to an orbit depends only on a trivialization of $\Lambda^n_{\mathbb{C}} TW$.

In the situation we consider, with Y obtained as a principal $U(1)$-bundle over Σ, there are three natural trivializations of $T(\mathbb{R} \times Y)$ that are useful to consider. The first is the *constant trivialization*, which uses the identification $T(\mathbb{R} \times Y) \cong \mathbb{R} \oplus \mathbb{R}R \oplus \xi$ and which is so that the linearized Reeb flow is the identity map on ξ. This has the advantage of corresponding to our construction of Floer solutions in terms of holomorphic spheres in Σ and of sections of the corresponding pull-back bundle. The second trivialization is closely related to the first, and involves seeing each Reeb orbit in Y as the boundary of the corresponding disk fibre in the normal bundle to Σ in X. This second version of the Conley–Zehnder index is most natural from the perspective of closed spheres in X. The final trivialization corresponds to the fractional SFT grading [EGH00] as used, for instance in [BO09b]. We use this point of view to define the grading of our complex. This fractional grading exists because our Reeb orbits are all torsion elements of $H_1(Y;\mathbb{Z})$. We will also relate our grading to the gradings introduced by Seidel in the case $c_1(TW) \in H_2(W;\mathbb{Z})$ vanishes and by McLean in the case $c_1(TW) \in H_2(W;\mathbb{Z})$ is torsion [Sei08, Section (3a); McL16, Section (4.1)]

Recall that the 1-periodic Hamiltonian orbits occur at levels $\{r\} \times Y$, where r verifies $h'(\mathrm{e}^r) \in \mathbb{Z}$.

7.1.1. *Constant trivializations along a fibre.* We recall that we denote the bundle map $\pi_\Sigma \colon Y \to \Sigma$. The contact structure ξ is a horizontal distribution, and the Reeb vector field generates the S^1 action on Y. Then, by definition, $TY = \mathbb{R}R \oplus \xi \cong \mathbb{R}R \oplus \pi_\Sigma^* T\Sigma$.

While each cover of a fibre of Y is a closed Reeb orbit, our chain complex additionally keeps track of the Morse function $f_\Sigma\colon \Sigma \to \mathbb{R}$. This Morse function represents infinitesimal perturbation data to perturb our contact form to a nearby non-degenerate one. (See [Bou02, BEH+03].) In our current calculation, we will not make any perturbations, but use this Morse function to define our moduli spaces of cylinders with cascades.

Let $p \in \Sigma$ be a point. Then, for a fixed trivialization of $T_p\Sigma$ and any point $q \in \pi_\Sigma^{-1}(p) \subset Y$ in the corresponding fibre, we obtain a trivialization of $\xi|_q \cong T_p\Sigma$. Furthermore, this trivialization is invariant under the linearized Reeb flow. We will refer to this trivialization as the *constant trivialization* over the orbit. In particular, the linearized Reeb flow, with respect to this trivialization, is the (constant) identity map.

The collection of unparametrized periodic Reeb orbits in Y is parametrized by a point p in Σ and an integer $k > 0$ indicating how many times the orbit covers the fiber of $\pi_\Sigma\colon Y \to \Sigma$ over p. Let $p \in \mathrm{Crit}(f_\Sigma)$ and take $k > 0$. We denote the corresponding unparametrized Reeb orbit by p_k. This has an associated Conley–Zehnder index, as in Definition A.9:

$$\mathrm{CZ}^0_\xi(p_k) = M(p) + 1 - n,$$

where $M(p)$ denotes the Morse index of p for the Morse function $f_\Sigma\colon \Sigma \to \mathbb{R}$. The superscript 0 indicates that the index is calculated with respect to the constant trivialization. Note that the contribution $1 - n$ comes from the last statement in Corollary A.4.

Hamiltonian orbits of period 1 in $\mathbb{R} \times Y$ are contained in level sets $\{r\} \times Y$. For fixed such r, the corresponding Hamiltonian orbits cover fibres of $\pi_\Sigma\colon Y \to \Sigma$ with multiplicity given by $h'(\mathrm{e}^r) = k \in \mathbb{Z}_+$. For each critical point p of $f_\Sigma\colon \Sigma \to \mathbb{R}$, there are two distinguished Hamiltonian orbits of multiplicity k. They correspond to the two critical points of the restriction of f_Y to the circle $\pi_\Sigma^{-1}(p)$, where $f_Y\colon Y \to \mathbb{R}$ is the fixed Morse function that lifts f_Σ. These two critical points are denoted by by $\widehat{p}$ and $\check{p}$ and the corresponding Hamiltonian orbits are denoted by $\widehat{p}_k$ and $\check{p}_k$. The latter are generators of the chain complex 3.2.

Given a periodic Reeb orbit, we extend the previously defined constant trivialization of ξ to a trivialization of $T(\mathbb{R} \times Y)$ over the corresponding Hamiltonian orbits, by noticing that $T(\mathbb{R} \times Y) = \mathbb{R} \oplus \mathbb{R}R \oplus \xi$ is obtained by stabilizing ξ by a trivialized complex line bundle.

With respect to this trivialization, the linearized Hamiltonian flow is no longer the identity map, but splits a non-trivial factor on $\mathbb{R} \oplus \mathbb{R}R$ and the identity on ξ. The non-trivial factor corresponds to the following linearized flow differential operator (asymptotic operator),

$$-i\frac{d}{dt} - \begin{pmatrix} h''(\mathrm{e}^r)\,\mathrm{e}^r & 0 \\ 0 & 0 \end{pmatrix}.$$

In the language of Appendix A, the Conley–Zehnder index of the perturbation of this asymptoptic operator in $\mathbb{R}^2$ by $\delta > 0$ is 0, by Corollary A.4. The Conley–Zehnder indices of a Reeb orbit and of the corresponding Hamiltonian orbits in this trivialization therefore agree.

Denote by $\widetilde{p}$ either $\check{p}$ or $\widehat{p}$ and write the Morse index of $\widetilde{p}$ for f_Y as $\tilde{M}(\widetilde{p}) = M(p) + i(\widetilde{p})$ (so $i(\widehat{p}) = 1$ and $i(\check{p}) = 0$). Using again Definition A.9, we obtain that with respect to the trivialization we fixed for $T(\mathbb{R} \times Y)$

$$\mathrm{CZ}^0_H(\widetilde{p}_k) = \mathrm{CZ}^0_\xi(p_k) + i(\widetilde{p}) = \tilde{M}(\widetilde{p}) + 1 - n.$$

Notice that with respect to this constant trivialization, the Conley–Zehnder index does not depend on the covering multiplicity of the orbit.

Finally, for a constant orbit $x \in \mathrm{Crit}(f_W)$, the constant trivialization gives

$$\mathrm{CZ}^0_H(x) = -n + (2n - M(x)) = n - M(x)$$

where $M(x)$ is the Morse index of x as a critical point of f_W.

7.1.2. *Orbits capped by the fibres of $N\Sigma$.* We take now a closely related trivialization: consider Y as a submanifold of X, arising as the concave boundary of a fixed r radius normal disk bundle $N_r\Sigma \subset N\Sigma \to \Sigma$. Observe that we then have $TX|_{N_r\Sigma} \cong N\Sigma \oplus \xi$ (where we extend ξ over $\Sigma \subset N_r\Sigma$ as $T\Sigma$). A simple Reeb orbit in Y bounds a fibre of this disk bundle $N_r\Sigma \to \Sigma$. The constant trivialization of the contact structure ξ extends across this disk fibre. The restriction $N\Sigma|_Y$ is a trivial complex line bundle. The computation of CZ^0 uses the trivialization of this line bundle given by the radial (Liouville) vector field and Reeb vector fields on Y. The trivialization of $N\Sigma|_Y$ that extends to all of $N_r\Sigma$, however, has winding 1 with respect to that one.

From this we obtain, for an orbit with multiplicity k, and considering the asymptotic boundary condition given by the descending manifold of $p \in \mathrm{Crit}(f_\Sigma)$, the following indices for Reeb and Hamiltonian orbits, with respect to the trivialization coming from the inclusion $Y \subset X$:

$$\mathrm{CZ}^X_\xi(p_k) = \mathrm{CZ}^0_\xi(p_k) = M(p) + 1 - n \tag{7.1}$$

and

$$\mathrm{CZ}^X_H(\widetilde{p}_k) = \mathrm{CZ}^0_H(\widetilde{p}_k) - 2k = \tilde{M}(\widetilde{p}) + 1 - n - 2k. \tag{7.2}$$

7.1.3. *The grading.* In order to construct the grading we defined in Equations (3.4) and (3.5), we consider a third trivialization of ξ and TW in the case of a Reeb orbit/Hamiltonian orbit that is contractible in W. We will later consider more general orbits.

As discussed above, in order to define a Conley–Zehnder index, it suffices to trivialize the line bundle

$$L \coloneqq \Lambda^n_{\mathbb{C}} TX$$

over the orbit. To compare two Conley–Zehnder indices coming from two different trivializations of L over an orbit, it suffices to understand the relative winding of these two trivializations.

Suppose that γ_l is a Reeb orbit of Y which is a l-fold cover of a simple orbit γ (i.e. of a fibre of $Y \to \Sigma$). Suppose that γ_l is contractible in W. Denote by $\dot{B}$ a disk in W whose boundary is γ_l. Note that γ_l is also the boundary of a l-fold cover of a fibre of the normal bundle to Σ, as described in the previous Section. This cover of a fibre can be concatenated with $\dot{B}$ to produce a spherical homology class $B \in H^S_2(X)$ such that $B \bullet \Sigma = l > 0$. Note that any $B \in H^S_2(X)$ such that $B \bullet \Sigma = l$ gives rise to a disk $\dot{B}$ bounding γ_l. The complex line bundle $L|_{\dot{B}}$ is trivial, since $\dot{B}$ is a disk. This induces a trivialization of L over γ_l, which can be identified with a trivialization of $L^{\otimes l}$ over γ.

The orbit γ bounds the fibre of the normal bundle to Σ, which induces a trivialization of L over γ. This is precisely the trivialization used in the previous section. The l-fold cover of the fibre of the normal bundle induces a trivialization over γ_l. Now, observe that the trivialization in the previous paragraph has a winding with respect to this second given by $\langle c_1(L), B\rangle$, since this represents the obstruction to extending the trivialization of L over $\dot{B}$ to all of B. Recall that $c_1(L) = c_1(TX)$.

Suppose now that γ is the Reeb orbit corresponding to $p \in \mathrm{Crit}(f_\Sigma)$. By (7.2), the Conley–Zehnder index with respect to the new trivialization is

$$\mathrm{CZ}^W_H(\widetilde{p}_l) = \tilde{M}(\widetilde{p}) + 1 - n - 2l + 2\langle c_1(TX), B\rangle \tag{7.3}$$

Using the fact that $K\omega(B) = l$ and the spherical monotonicity of X, we have:

$$\mathrm{CZ}^W_H(\widetilde{p}_l) = \tilde{M}(\widetilde{p}) + 1 - n + 2(\tau_X - K)\,\omega(B). \tag{7.4}$$

Now, for any $k > 0$, we define a fractional grading

$$\begin{aligned} |\widetilde{p}_k| &= \tilde{M}(\widetilde{p}) + 1 - n + 2(\tau_X - K)\,\omega(B)\frac{k}{l} \\ &= \tilde{M}(\widetilde{p}) + 1 - n + \frac{2(\tau_X - K)\,\omega(B)}{K\omega(B)}k \\ &= \tilde{M}(\widetilde{p}) + 1 - n + 2\left(\frac{\tau_X - K}{K}\right)k \end{aligned} \tag{7.5}$$

As before, for any critical point $x \in \mathrm{Crit}(f_W)$, we have the grading

$$|x| = n - M(x) \tag{7.6}$$

Finally, it will be convenient to introduce an index similar to the SFT grading for the Reeb orbits that arise as the limits of a Floer cylinder with punctures at $\{-\infty\} \times Y$. If γ is such a Reeb orbit, it is a k-fold cover a fibre of $Y \to \Sigma$ for some k. We then define its index to be:

$$|\gamma|_0 = n - 3 + 1 - n + 2\left(\frac{\tau_X - K}{K}\right)k = -2 + 2\left(\frac{\tau_X - K}{K}\right)k \tag{7.7}$$

Observe that if X is aspherical and thus vacuously spherically monotone, we no longer have capping disks $\dot{B}$ as above for any cover of a simple orbit γ. In that case, however, the differential will only consist of gradient trajectories. We therefore may always take the grading from Equations (3.4) and (3.5).

Notice that if Σ has a spherical homology class A with $l := \langle c_1(N\Sigma), A\rangle = K\omega(A) \neq 0$, then the l-fold cover of every Reeb orbit in Y is contractible in Y. In this case, using the corresponding cappings in Y, we obtain the fractional Conley–Zehnder index of [EGH00, Section 2.9.1].

Remark 7.1. Even though the idea of a fractional grading may seem unnatural at first, it can be thought of as a way of keeping track of some information about the homotopy classes of the Hamiltonian orbits. Indeed, two generators of our chain complex can only be homotopic Hamiltonian orbits (which is a necessary condition for the existence of a Floer cylinder connecting them) if the difference of their degrees is an integer. Alternatively, we could have introduced an additional parameter to keep track of the homotopy classes of Hamiltonian orbits, as done for instance in [BO09b].

Finally, we compare our gradings with those described by Seidel [Sei08] and generalized by McLean [McL16] (he considers Reeb orbits, but there is an analogous construction for Hamiltonian orbits). As before, assume that there exists a spherical class $B \in H_2(X;\mathbb{Z})$ such that $l = \langle c_1(TX), B\rangle > 0$. Suppose now that $c_1(TW) \in H_2(W;\mathbb{Z})$ is torsion, so $Nc_1(TW) = 0$ for a suitable choice of $N > 0$.

As explained earlier, let $\dot{B}$ be the disk associated to B that bounds γ_l. $L = \Lambda^n_{\mathbb{C}}(TW)$ is trivial over the disk $\dot{B}$, and induces a unique trivialization of TW over $\dot{B}$. This trivialization defines the Conley–Zehnder of γ_l.

Now, let us consider the associated trivialization of $(TW)^{\oplus N}$ over $\dot{B}$, which is also unique up to homotopy. This trivialization induces a Conley–Zehnder index on $(\gamma_l)^{\oplus N}$.

Now, since $Nc_1(TW) = 0$, it follows that $L^{\otimes N}$ is trivial. We fix a trivialization of this line bundle, which induces a trivialization of $(TW)^{\oplus N}$ over the 2-skeleton of W. The Seidel–McLean grading convention is to use this global trivialization to define the Conley–Zehnder index of $(\gamma_l)^{\oplus N}$.

Since $(TW)^{\oplus N}$ has a unique trivialization over $\dot{B}$, up to homotopy, the two Conley–Zehnder indices we obtained for $(\gamma_l)^{\oplus N}$ must agree. In our maximally Morse–Bott setting, the linearized return Reeb map is the identity. Thus, the Conley–Zehnder index of the cover of a fibre of Y is obtained as a constant term and a term linear in the covering multiplicity of the orbit. This linear term is the Robbin–Salamon index [RS93, Gut14] of the degenerate Reeb orbit. From this linearity, it follows then that our grading convention agrees for all covers with that of Seidel and McLean. It is interesting to note that our approach to fractional gradings uses the fact that in this Morse–Bott setting, the Conley–Zehnder indices are affine in the covering multiplicity and the linear part is given by the Robbin–Salamon index. The Seidel-McLean approach primarily uses the direct sum property, though equality of our two gradings also relies upon linearity.

7.2. **Index inequalities from monotonicity and transversality.** We now investigate the implications of our monotonicity assumptions (specifically that X is spherically monotone with $\langle c_1(TX), A\rangle = \tau_X\omega(A)$ for spherical classes A and that $\tau_X - K > 0$), specifically in combination with the transversality results from Section 6.

First, we consider the Fredholm index contributions of a plane in W that could appear as an augmentation plane to obtain some bounds on the possible indices.

Lemma 7.2. *If $v\colon \mathbb{C} \to W$ is a J_W holomorphic plane asymptotic to a given closed Reeb orbit γ in Y, the Fredholm index for the deformations of v (as an unparameterized curve) keeping γ fixed is non-negative. Furthermore, if v is multiply covered, this Fredholm index is at least* 2.

Proof. First, we observe that the Fredholm index (as an unparametrized curve), keeping the asymptotic limit fixed at γ is given by:

$$\mathrm{Ind}(v) = 2(\langle c_1(TX), B\rangle - m - 1) = 2(\tau_X\omega(B) - K\omega(B) - 1).$$

Since the plane is holomorphic, the induced spherical class B has $\omega(B) > 0$.

By our monotonicity assumptions, we have

$$\tau_X\omega(B) - K\omega(B) = (\tau_X - K)\,\omega(B) > 0.$$

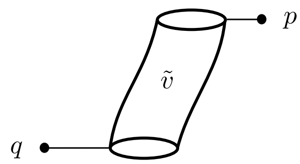

FIGURE 7.1. Option (1) in Proposition 7.4, call it Case 1

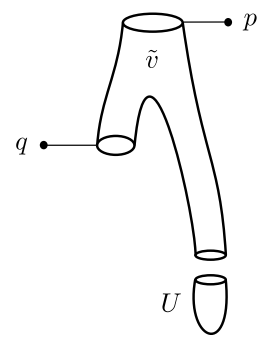

FIGURE 7.2. Option (2) in Proposition 7.4, call it Case 2

Finally, observe that $\tau_X\omega(B) = \langle c_1(TX), B\rangle \in \mathbb{Z}$ and $K\omega(B) = B \bullet \Sigma \in \mathbb{Z}$, so this quantity is an integer, and is thus at least 1..

It therefore follows that $\mathrm{Ind}(v) \geqslant 0$.

Suppose now that v is a k-fold cover of an underlying simple holomorphic plane v_0, representing classes $B = kB_0$ respectively. Then,

$$\mathrm{Ind}(v) + 2 = 2(\tau_X - K)\omega(kB_0) = k(\mathrm{Ind}(v_0) + 2).$$

Hence, $\mathrm{Ind}(v) \geqslant 2(k-1)$. □

Lemma 7.3. *Let $u\colon \mathbb{C} \to W$ be a holomorphic plane asymptotic to a multiplicity m Reeb orbit in Y corresponding to the critical point p, with marker condition going to $\widetilde{p} = \widehat{p}$ or $\widetilde{p} = \check{p}$, and with 0 mapping to the descending manifold in W of the critical point x. Denote by B the homology class $B \in H_2(X;\mathbb{Z})$ obtained by capping u with an m-fold cover of a fibre of E. The Fredholm index of u, as a parametrized curve, is given by*

$$2\langle c_1(TX), B\rangle - 2m + M(\widetilde{p}) + M(x) - 2n + 1$$

Proof. First, applying the Conley–Zehnder index formula from Equation (7.1), together with the standard index formula from SFT, we obtain that the index of an unparametrized plane satisfying the marked point conditions is:

$$n - 3 + 2\langle c_1(TX), B\rangle + (1-n) + M(p) - 2m + 2 - 2n + M(x).$$

We then obtain an additional two from the parametrization, but the marker condition reduces the index by 1. □

Proposition 7.4. *Any Floer cascade in $\mathbb{R} \times Y$ that can appear in the differential must be one of the following configurations:*

(0) *An index* 1 *gradient trajectory in* Y *without any (non-constant) holomorphic components and without any augmentation punctures.*
(1) *A smooth cylinder in* $\mathbb{R}\times Y$ *without any augmentation punctures and a non-trivial projection to* Σ*. The positive puncture converges to a* $\check{p}$ *orbit and the negative puncture converges to a* $\widehat{q}$ *orbit. The difference in multiplicites of the orbits is given by* $K\omega(A)$*, where* $A \in H_2(\Sigma;\mathbb{Z})$ *is the homology class represented by the projection of the cylinder to* Σ*. See Figure 7.1.*
(2) *A cylinder with one augmentation puncture and whose projection to* Σ *is trivial. The positive puncture converges to a* $\check{p}$ *orbit and the negative puncture converges to a* $\widehat{q}$ *orbit. The augmentation plane has index* 0*. If* $B \in H_2(X;\mathbb{Z})$ *is the class represented by the augmentation plane, then the difference in multiplicities is given by* $K\omega(B)$*. Furthermore,* $\check{p}$ *and* $\widehat{q}$ *are critical points of* f_Y *contained in the same fibre of* $Y \to \Sigma$*. See Figure 7.2.*

Proof. Consider a cascade with N levels and l augmentation planes appearing in the differential $d\widetilde{p}_{k_+} = \cdots + \widetilde{q}_{k_-} + \cdots$. Let $A_1,\dots,A_N \in H_2(\Sigma)$ denote the homology classes of the projections to Σ, let $B_1,\dots,B_k \in H_2(X)$ denote the homology classes corresponding to the augmentation planes. Let $\gamma_i, i = 1,\dots,k$ denote the limits at the augmentation punctures, and let k_i denote their multiplicites. Let $A = \sum_{i=1}^N A_i$.

We therefore have $k_+ - k_- \sum_{j=1}^k k_j = K\omega(A)$. We also have $k_j = B_j \bullet \Sigma = K\omega(B_j)$. Notice then that $|\gamma_i|_0 = 2\langle c_1(TX), B_j\rangle - 2B_j \bullet \Sigma - 2$.

We therefore have

$$\begin{aligned} 1 &= |\widetilde{p}_{k_+}| - |\widetilde{q}_{k_-}| \\ &= i(\widetilde{p}) + M(p) - i(\widetilde{q}) - M(q) + 2\frac{\tau_X - K}{K}(k_+ - k_- - k_0) + 2\frac{\tau_X - K}{K}k_0 \\ &= i(\widetilde{p}) + M(p) - i(\widetilde{q}) - M(q) + 2\langle c_1(T\Sigma), A\rangle + 2k + \sum_{j=1}^k |\gamma_j|_0 \end{aligned} \tag{7.8}$$

By Lemma 7.2, we have that for each $j = 1,\dots,k$, $2\langle c_1(TX), B_j\rangle - 2B_j\bullet\Sigma - 2 \geqslant 0$, so for each j, $|\gamma_j|_0 \geqslant 0$.

Consider the chain of pearls in Σ obtained by projecting the upper level of this split Floer trajectory to Σ. By Proposition 6.17, if this is a simple chain of pearls, it has Fredholm index

$$I_\Sigma := M(p) + 2\langle c_1(T\Sigma), A\rangle - M(q) + N - 1 + 2k.$$

If the chain of pearls is not simple, by monotonicity, we have that the index is at least as large as the index of the underlying simple chain of pearls.

Now let N_0 be the number of sub-levels that project to constant curves in Σ and let N_1 be the number of sub-levels that project to non-constant curves in Σ, $N = N_0 + N_1$. Note that by the stability condition, each cylinder that projects to a constant curve in Σ must have at least one augmentation puncture, so $N_0 \leqslant k$.

By transversality for simple chains of pearls (Proposition 6.17), we obtain the inequality

$$I_\Sigma \geqslant 2N_1 + 2k$$

by considering the 2-dimensional automorphism group for the N_1 non-constant spheres and by considering the $2l$ parameter family of moving augmentation marked points on the domains.

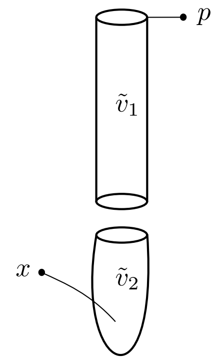

FIGURE 7.3. Configuration as in Proposition 7.5, call it Case 3

Combining with Equation (7.8), we obtain

$$1 = i(\widetilde{p}) - i(\widetilde{q}) + (I_\Sigma - N + 1) + \sum_{j=1}^{k} |\gamma_j|_0$$

$$1 \geqslant (i(\widetilde{p}) - i(\widetilde{q}) + 1) + 2N_1 + 2k - N + \sum_{j=1}^{k} |\gamma_j|_0$$

$$1 \geqslant (i(\widetilde{p}) - i(\widetilde{q}) + 1) + N_1 + k + (k - N_0) + \sum_{j=1}^{k} |\gamma_j|_0$$

Observe now that each term on the right-hand-side of the inequality is non-negative. In particular, there is at most one augmentation plane, and if there is one, it must have $|\gamma_1|_0 = 2\langle c_1(TX), B_1\rangle - 2B_1 \bullet \Sigma - 2 = 0$.

The inequality now reduces to

$$1 \geqslant (i(\widetilde{p}) - i(\widetilde{q}) + 1) + N_1 + k + (k - N_0).$$

Notice that $N_1 + 2k - N_0 \geqslant N$.

This inequality can be satisfied in one of the following ways:

0 $N = 0$. Then, either $i(\widetilde{p}) = i(\widetilde{q})$ or $\widetilde{p} = \check{p}$ and $\widetilde{q} = \hat{q}$. Since $N = 0$, this is a pure Morse differential term.

(1) $N_1 = 1$, $N_0 = k = 0$ and $\widetilde{p} = \check{p}$, $\widetilde{q} = \widehat{q}$. This case corresponds to a non-constant sphere in Σ without any augmentation punctures.

(2) $N_1 = 0$, $k = 1$, $N_0 = 1$, and $\widetilde{p} = \check{p}$, $\widetilde{q} = \widehat{q}$. In this case, the Floer cylinder has one augmentation puncture, but projects to a constant in Σ.

□

We now consider the possible terms in the differential that connect non-constant Hamiltonian trajectories in $\mathbb{R} \times Y$ to Morse critical points in X.

Proposition 7.5. *Any Floer cascade appearing in the differential, connecting a non-constant Hamiltonian orbit $\tilde{p}$ in $\mathbb{R} \times Y$ to a Morse critical point x in W, consists of two levels. The upper level, in $\mathbb{R} \times Y$, projects to a point in Σ and is a cylinder asymptotic at $+\infty$ to a $\check{p}$ orbit and at $-\infty$ to a Reeb orbit in $\{-\infty\} \times Y$.*

The lower level is a holomorphic plane in W with marker condition going to $\breve{p}$ and with 0 mapping to the descending manifold of the critical point x. As a parametrized curve, this has Fredholm index 1. See Figure 7.3.

Proof. Suppose such a cascade occurs in the differential, connecting the non-constant orbit $\widetilde{p}_{k_+}$ to the critical point q in the filling W.

Let N be the number of cylinders in $\mathbb{R} \times Y$ that appear in the split Floer cylinder. Let $A_i \in H_2(\Sigma), i = 1, \ldots, N$ denote the spherical classes represented by the projections of these cylinders to Σ. Let $A = \sum_{i=1}^N A_i$.

Let k be the number of augmentation planes, and let $B_j \in H_2(X), j = 1, \ldots, k$ be the corresponding spherical homology classes in X. Let $\gamma_j, j = 1, \ldots, k$ be the corresponding Reeb orbits with multiplicities $k_j = B_j \bullet \Sigma = K\omega(B_j)$.

Let $B \in H_2(X)$ be the spherical homology class in X represented by the lower level v in W connecting to the critical point x. Let $k_- = B \bullet \Sigma$ be the multiplicity of the orbit to which the plane v converges. As before, we have

$$k_+ - k_- - \sum_{j=1}^k k_j = K\omega(A).$$

We then have

$$\begin{aligned}
(7.9)& \\
1 &= |\widetilde{p}_{k_+}| - |x| \\
&= i(\widetilde{p}) + M(p) + 1 - 2n + M(x) + 2\frac{\tau_X - K}{K}\left(k_+ - k_- - \sum_{j=1}^k k_j + k_- + \sum_{j=1}^k k_j\right) \\
&= i(\widetilde{p}) + M(p) + 1 - 2n + M(x) + 2\langle c_1(T\Sigma), A\rangle + 2\langle c_1(TX), B\rangle - 2B \bullet \Sigma + 2k + \sum_{j=1}^k |\gamma_j|_0
\end{aligned}$$

Projecting to Σ, we obtain a chain of pearls with a sphere in X. Let N_0 be the number of constant spheres in Σ and let N_1 be the number of non-constant spheres in Σ, $N = N_0 + N_1$. Notice that each non-constant sphere in Σ has a 2-parameter family of automorphisms, and each augmentation marked point can be moved in a 2-parameter family. Furthermore, the holomorphic sphere v also has a 2-parameter family of automorphisms. By passing to a simple underlying chain of pearls as necessary, and applying monotonicity and Proposition 6.17, we obtain

$$\begin{aligned}
I_X &:= M(p) + 2\langle c_1(T\Sigma), A\rangle + 2\left(\langle c_1(TX), B\rangle - B \bullet \Sigma\right) + M(x) - 2n + 1 + N + 2k \\
&\geqslant 2N_1 + 2k + 2
\end{aligned}$$

We now combine the inequality with Equation (7.9):

$$1 = i(\widetilde{p}) + I_X - N + \sum_{j=1}^k |\gamma_j|_0$$

$$1 \geqslant i(\widetilde{p}) + 2N_1 + 2k + 2 - N_0 - N_1 + \sum_{j=1}^k |\gamma_j|_0$$

$$0 \geqslant i(\widetilde{p}) + N_1 + k + (k + 1 - N_0) + \sum_{j=1}^k |\gamma_j|_0.$$

Notice that we have $N_0 \leqslant k+1$ since the first sphere in the chain of pearls with a sphere in X is allowed to be constant without any marked points. This observation together with Lemma 7.2 gives that each term on the right-hand-side of the inequality is non-negative. It follows therefore that each term must vanish: $N_0 = 0$, $N_1 = 1$, $k = 0$ and $\widetilde{p} = \check{p}$. Notice that the Floer cylinder in $\mathbb{R} \times Y$ is contained in a single fibre of $\mathbb{R} \times Y \to \Sigma$, so the marker condition coming from $\check{p}$ can be interpreted as a marker condition on the holomorphic plane v. Without the marker condition, v has Fredholm index 2, and thus with the marker constraint, it has index 1.

□

Case (2) in Proposition 7.4 allows for the existence of augmented configurations contributing to the symplectic homology differential. We will now adapt an argument originally due to Biran and Khanevsky [BK13] to show that, if Σ is a symplectic hyperplane section in the sense of Definition 2.1 (so that the complement of a tubular neighborhood of Σ is a Weinstein domain) and has minimal Chern number at least 2, then there can only be rigid augmentation planes if the isotropic skeleton has codimension at most 2 (in particular, $\dim_{\mathbb{R}} X = 2n \leqslant 4$).

Lemma 7.6. *If W is a Weinstein domain with isotropic skeleton of real codimension at least 3, then X is symplectically aspherical if and only if Σ is.*

Furthermore, any symplectic sphere in X is in the image of the inclusion

$$\imath_* \pi_2(\Sigma) \to \pi_2(X).$$

Proof. The trivial direction is that if there exists a spherical class $A \in \pi_2(\Sigma)$ with $\omega(A) > 0$, then $\imath_* A \in \pi_2(X)$ and still has positive area.

We will now prove that any symplectic sphere in X is in the image of the inclusion. Let $C \subset W$ be the isotropic skeleton of W. Notice that by following the flow of the Liouville vector field on W, we obtain that $W \backslash C$ is symplectomorphic to a piece of the symplectization $(-\infty, a) \times Y$. Thus, we have that $X \backslash C$ is a subset of a symplectic disk bundle over Σ. We denote this bundle's projection map by $\pi \colon X \backslash C \to \Sigma$.

Suppose $A \in \pi_2(X)$ is a spherical class with $\omega(A) > 0$. By hypothesis, the skeleton C is of codimension at least 3. We may therefore perturb A in a neighbourhood of the skeleton so that it does not intersect the skeleton C. If $\iota \colon \Sigma \to X$ and $j \colon X \backslash C \to X$ are the inclusion maps, then $\omega_\Sigma = \iota^* \omega$ and $\iota \circ \pi$ is homotopic to j. This implies that $\omega_X(A) = \omega_\Sigma(\pi_* A)$, and the result follows.

□

Lemma 7.7. *Suppose W is a Weinstein domain with isotropic skeleton of real codimension at least 3 and Σ has minimal Chern number at least 2. Then, there do not exist any augmentation planes.*

Proof. Recall from Proposition 7.4 that an augmentation plane in the class B must have index 0, so $0 = 2(\langle c_1(TX), B\rangle - B \bullet \Sigma - 1)$. Now $\langle c_1(TX), B\rangle - B \bullet \Sigma = (\tau_X - K)\,\omega(B) \geqslant 1$. Thus, the augmentation plane can only exist if there is a spherical class B with $(\tau_X - K)\,\omega(B) = 1$.

Now, by applying Lemma 7.6, we have $B = \imath_* A$, where $A \in \pi_2(\Sigma)$ is a spherical class in Σ. It now follows from equation (1.1) that $1 = (\tau_X - K)\,\omega(A) = \tau_\Sigma\,\omega(A) = \langle c_1(T\Sigma), A\rangle$.

The minimal Chern number of Σ is at least 2 by hypothesis, so the augmentation plane cannot exist. □

Remark 7.8. The condition that the minimal Chern number of Σ is at least 2 is also used in upcoming joint work of the first author with D. Tonkonog, R. Vianna and W. Wu, studying the effect of the Biran circle bundle construction on superpotentials associated to Lagrangians [DTVW].

8. Orientations

In order to orient our moduli spaces, we will take the point of view of coherent orientations, and follow the method of orienting Morse–Bott moduli spaces as done in [Bou02,BO09b]. Some authors [Zap15,Sch16] have taken the point of view of canonical orientations. We find it more straightforward to use coherent orientations in our specific computational example, especially since there are very few choices involved.

We briefly recall the general method for obtaining signs in Floer homology, as first introduced in [FH93] and since generalized. Consider all Cauchy–Riemann operators on Hermitian vector bundles over punctured spheres, as described in Appendix A. Fix the asymptotic operators at each puncture, and suppose they are non-degenerate. The space of all Cauchy–Riemann operators with these asymptotic operators is then contractible and each such operator is Fredholm. Associated to this space of operators is the determinant line bundle, whose fibre at an operator D is given by

$$\det D = (\Lambda^{\max} \ker D) \otimes_{\mathbb{R}} (\Lambda^{\max} \operatorname{coker} D)^*.$$

An orientation corresponds to a nowhere vanishing continuous section of this determinant bundle over the space of Cauchy–Riemann operators (topologized in a way compatible with the discrete topology on the space of asymptotic operators).

An orientation is *coherent* if it respects the gluing operation on these Cauchy–Riemann operators. Indeed, given two such operators D and D' that have a matching asymptotic operator at a positive puncture for D and a negative puncture for D', we may form a glued operator $D\# D'$. This operator is not unique, but depends on a contractible family of choices, in particular on a gluing parameter. For sufficiently large gluing parameter, a section of $\det D$ and a section of $\det D'$ induce a section of $\det D\# D'$, well-defined up to a positive multiple. (See, for instance, [BM04].) Thus, an orientation of D and an orientation of D' induce an orientation of $D\# D'$.

We may also arrange for the coherent orientation to have the following two properties:

- the orientation of the direct sum of two operators is the tensor product of their orientations,
- the orientation of a complex linear operator is its canonical orientation.

Because of the Morse-Bott degeneracy in our situation, we must consider Cauchy-Riemann operators acting on exponentially weighted spaces in order to obtain Fredholm problems. Specifically, we consider a space of sections with exponential decay together with a (finite dimensional) space of infinitesimal movements in the corresponding Morse-Bott family of orbits. For a given degenerate asymptotic operator, we focus on two associated non-degenerate operators. The first corresponds to keeping the asymptotic orbit fixed. The second corresponds to letting the orbit to which the cylinder converges move in its Morse-Bott family. These are the δ-perturbed asymptotic operators of Definition A.5. Notice that the free/fixed asymptotic operator are perturbed by $\pm\delta$ differently depending on the sign of the puncture.

We now describe how to attach a sign to a Floer cylinder with cascades contributing to the differential. We will follow [BO09b] closely. By Propositions 7.4 and 7.5, there are four types of contributions to the differential, referred to as Cases 0 through 3. We will explain how to associate signs to each of these.

We begin with Case 1. Recall that such configurations consist of a Floer cylinder without augmentation punctures, together with two flow lines of $-Z_Y$ at the ends. Suppose that the Floer cylinder converges to orbits of multiplicity $k_\pm$ at $\pm\infty$. Such a cylinder is an element of the space $\widetilde{\mathcal{M}}^*_{H,1}(Y_{k_-}, Y_{k_+}; A; J_Y)$, for some $A \in H_2(\Sigma; \mathbb{Z})$.

In accordance with the prescription above for the non-degenerate case, we choose capping operators for the relevant asymptotic operators. By Lemma 6.12, the linearized operator associated to a Floer cylinder $\tilde{v}$ is a compact perturbation of a split operator $D^{\mathbb{C}}_{\tilde{v}} \oplus \dot{D}^{\Sigma}_{w}$, where $w = \pi_\Sigma \circ \tilde{v}$. There is also a corresponding splitting of the asymptotic operators at the asymptotic limits. In particular, $\dot{D}^{\Sigma}_{w}$ has complex linear asymptotic operators, and thus is a compact perturbation of a complex linear Cauchy–Riemann operator. Hence, its orientation is induced by the canonical one, and is independent of choice of trivialization or capping operator.

Remark 8.1. If it had been necessary to write down asymptotic operators associated to $\dot{D}^{\Sigma}_{w}$, we would have trivialized the contact distribution ξ over each stable or unstable manifold of the vector field Z_Y (even though it is not possible to trivialize ξ over all of Y, this can be done over the stable and unstable manifolds, since they are contractible).

It now remains to orient the operator $D^{\mathbb{C}}_{\tilde{v}}$. Write $\tilde{v} = (b, v) \colon \mathbb{R} \times S^1 \to \mathbb{R} \times Y$ and $b_\pm = \lim_{s\to\pm\infty} b(s,t)$. At $\pm\infty$, the δ-perturbed asymptotic operators (see Definition A.5) associated to $\tilde{v}$ are

$$A_\pm := -\left(J\frac{d}{dt} + \begin{pmatrix} h''(\mathrm{e}^{b_\pm})\,\mathrm{e}^{b_\pm} & 0 \\ 0 & 0 \end{pmatrix}\right) \pm \begin{pmatrix} \delta & 0 \\ 0 & \delta \end{pmatrix} \tag{8.1}$$

at $\pm\infty$, for a choice of $\delta > 0$ sufficiently small. Recall that these asymptotic operators are associated to converging to the descending manifold of the maximum of f_Y at $+\infty$, and to the ascending manifold of the minimum at $-\infty$, or equivalently, allowing the asymptotic S^1 marker to move in its Morse-Bott family at both punctures. We will choose capping operators for the $A_\pm$, which determines an orientation of $D^{\mathbb{C}}_{\tilde{v}}$ by the coherent orientation scheme.

By Lemma 6.14, $D^{\mathbb{C}}_{\tilde{v}}$ has Fredholm index 1, is surjective and its kernel contains an element that can be identified with the Reeb vector field.

Lemma 8.2. *There is a choice of capping operators for the asymptotic operators above, such that $D^{\mathbb{C}}_{\tilde{v}}$ is oriented in the Reeb direction.*

Proof. Recall that for each $b_k > 0$ that satisfies $h'(\mathrm{e}^{b_k}) = k \in \mathbb{Z}_+$, we have a Y-parametric family of 1-periodic Hamiltonian orbits. We can associate to each of these orbits two operators as in (8.1). We will define capping operators

$$\Phi_k^\pm \colon W^{1,p}(\mathbb{C}, \mathbb{C}) \to L^p(\mathrm{Hom}^{0,1}(T(\mathbb{C}), \mathbb{C}))$$

with these asymptotic operators.

We first define two families of auxiliary Fredholm operators. For each $k > 0$,

$$\Psi_k \colon W^{1,p}(\mathbb{R} \times S^1, \mathbb{C}) \to L^p(\mathrm{Hom}^{0,1}(T(\mathbb{R} \times S^1), \mathbb{C}))$$

is an operator given by

$$\Psi_k(F)(\partial_s) = F_s + iF_t + \begin{pmatrix} a(s)-\delta & 0 \\ 0 & -\delta \end{pmatrix} F$$

where $a\colon \mathbb{R} \to \mathbb{R}$ is such that $\lim_{s\to-\infty} a(s) = h''(e^{b_1})e^{b_1}$ and $\lim_{s\to+\infty} a(s) = h''(e^{b_k})e^{b_k}$.

$$\Xi_k\colon W^{1,p}(\mathbb{R}\times S^1, \mathbb{C}) \to L^p(\mathrm{Hom}^{0,1}(T(\mathbb{R}\times S^1), \mathbb{C}))$$

is an operator given by

$$\Xi_k(F)(\partial_s) = F_s + iF_t + \begin{pmatrix} h''(e^{b_k})e^{b_k} + \delta(s) & 0 \\ 0 & \delta(s) \end{pmatrix} F$$

where $\delta\colon \mathbb{R} \to \mathbb{R}$ is such that $\lim_{s\to-\infty} \delta(s) = -\delta < 0$ and $\lim_{s\to+\infty} \delta(s) = \delta$.

The operators Ψ_k are isomorphisms (in particular, they are canonically oriented). This follows from an argument analogous to the proof of Lemma 6.14. A version of the same argument implies that the operators Ξ_k are Fredholm of index 1 and surjective, and that their kernels contain elements that can be identified with the Reeb vector field.

Now, pick any capping operator Φ_1^-. Define Φ_k^- for $k > 1$ by gluing $\Phi_1^- \# \Psi_k$. Define Φ_k^+ for all $k > 0$ by gluing $\Phi_k^- \# \Xi_k$. For these choices of capping operators, $D^{\mathbb{C}}_{\tilde{v}}$ are oriented in the direction of the Reeb flow, as wanted. ☐

We have now oriented the operators $D^{\mathbb{C}}_{\tilde{v}}$ and $\dot{D}^{\Sigma}_{w}$. Since the linearized Floer operator is a compact perturbation of their direct sum, we get induced orientations on the spaces of Floer cylinders $\widetilde{\mathcal{M}}^*_{H,1}(Y_{k_-}, Y_{k_+}; A; J_Y)$.

Remark 8.3. Recall that in Symplectic Field Theory [EGH00], bad orbits arise when the Conley-Zehnder index of the even covers of a simple Reeb orbit have different parity from the odd covers (in that case, the bad orbits are the even covers). In our situation, all covers of orbits have the same parity (as in Equation (7.2)). This simplifies our discussion of orientations. See [BO09b] for a treatment of coherent orientations in a setting where bad orbits can exist.

Observe that cylinders with cascades in Case 1 that contribute to the differential are elements of spaces $\mathcal{M}^*_{H,1}(q,p;J_Y,J_W)$, for $p \neq q \in \mathrm{Crit}(f_Y)$. These spaces are unions of fiber products

$$W^s_{f_Y}(q) \times_{\mathrm{ev}} \widetilde{\mathcal{M}}^*_{H,1}(Y_{k_-}, Y_{k_+}; A; J_Y) \times_{\mathrm{ev}} W^u_{f_Y}(p) \tag{8.2}$$

defined with respect to the inclusion maps

$$W^s_{f_Y}(q), W^u_{f_Y}(p) \to Y$$

and the evaluation maps

$$\widetilde{\mathrm{ev}}_Y\colon \widetilde{\mathcal{M}}^*_{H,1}(Y_{k_-}, Y_{k_+}; A; J_Y) \to Y \times Y$$

for appropriate $A \in H_2(\Sigma;\mathbb{Z})$.

Let us now recall the convention in [BO09b] for how to orient a fiber sum (which agrees with [Joy12, Convention 7.2.(b)]).

Definition 8.4. Given linear maps between oriented vector spaces $f_i\colon V_i \to W$, $i = 1, 2$, such that $f_1 - f_2\colon V_1 \oplus V_2 \to W$ is surjective, the *fiber sum orientation* on $V_1 \oplus_{f_i} V_2 = \ker(f_1 - f_2)$ is such that

(1) $f_1 - f_2$ induces an isomorphism $(V_1 \oplus V_2)/\ker(f_1 - f_2) \to W$ which changes orientations by $(-1)^{\dim V_2 . \dim W}$,
(2) where a quotient U/V of oriented vector spaces is oriented in such a way that the isomorphism $V \oplus (U/V) \to U$ (associated to a section of the quotient short exact sequence) preserves orientations.

One key property of this orientation convention for fiber products is that it is associative. Pick now orientations for all unstable manifolds of (f_Σ, Z_Σ) (which, since Y is oriented, induces orientations also on the stable manifolds). We use the fiber sum convention to orient 8.2.

Observe now that if $\mathcal{M}^*_{H,1}(q, p; J_Y, J_W)$ is one-dimensional, then its tangent space at every point is generated by the infinitesimal translation in the s-direction on the domain of the Floer cylinder. This also induces an orientation on the spaces of Floer cylinders with cascades in Case 1. Comparing this orientation with the one defined above with the fiber sum rule, we get the sign of such a contribution to the differential.

Remark 8.5. In [BO09b], the authors describe the often studied process of adiabatic degeneration, in which one approximates a Morse–Bott Hamiltonian H by a small perturbation H_δ that is non-degenerate, and compares moduli spaces of Floer cylinders with cascades for H with moduli spaces of Floer cylinders for H_δ. In Proposition 3.9, they compare the signs prescribed by a coherent orientation scheme to rigid floer cylinders with cascades for H to the signs prescribed to the corresponding rigid cylinders for H_δ. A similar argument shows that the signs we obtain for cylinders with cascades in Case 1 agree with the signs we would get for the corresponding rigid Floer cylinders we would get from perturbing H to a non-degenerate Hamiltonian.

We adapt the argument above to associate a sign to a Case 2 contribution to the differential. The analogue of (8.2) is now

$$W^s_{f_Y}(\hat{q}) \times_{\mathrm{ev}} \big((\mathcal{M}_0(B; J_W)/\operatorname{Aut}(\mathbb{C})) \times_{\mathrm{ev}} \widetilde{\mathcal{M}}^*_{H,1}(Y_{k_-}, Y_{k_+}; 0; J_Y) \big) \times_{\mathrm{ev}} W^u_{f_Y}(\check{q}). \tag{8.3}$$

The manifold $(\mathcal{M}_0(B; J_W)/\operatorname{Aut}(\mathbb{C})) \times_{\mathrm{ev}} \widetilde{\mathcal{M}}^*_{H,1}(Y_{k_-}, Y_{k_+}; 0; J_Y)$ is diffeomorphic via gluing to the space of presplit Floer cylinders in W, denoted $\widetilde{\mathcal{M}}^*_{H,1}(Y_{k_-}, Y_{k_+}; B; \tilde{J}_W)$, for an almost complex structure $\tilde{J}_W$ close enough to being split. If we give $\mathcal{M}_0(B; J_W)$ and $\widetilde{\mathcal{M}}^*_{H,1}(Y_{k_-}, Y_{k_+}; 0; J_Y)$ their coherent orientations, then $(\mathcal{M}_0(B; J_W)/\operatorname{Aut}(\mathbb{C})) \times_{\mathrm{ev}}$ $\widetilde{\mathcal{M}}^*_{H,1}(Y_{k_-}, Y_{k_+}; 0; J_Y)$ can be given the fiber product orientation. If we also give $\widetilde{\mathcal{M}}^*_{H,1}(Y_{k_-}, Y_{k_+}; B; \tilde{J}_W)$ its coherent orientation, then the diffeomorphism above preserves orientations, by the properties of the fiber product orientation and coherent orientation.

We remark here that in Cases 2 and 3, we do not obtain an additional sign coming from the number of cylinders in the Floer cascade (in contrast to the situation in [BO09b, Proposition 3.9]) since the cylinders in question have matching asymptotics (whereas in [BO09b], there is a finite length gradient flow line).

Note that one might wonder if the fact that cascades in Case 2 have a holomorphic curve and a Floer cylinder would cause an additional sign to appear in the differential, as in [BO09b, Proposition 3.9] with $m = 2$. But that is not the case, because...

As for Case 3 Floer cylinders with cascades that contribute to the differential, they are elements of fiber products

$$W^u_{f_W}(x) \times_{\mathrm{ev}} \mathcal{M}^*_{H,1}(B; J_W) \times_{\mathrm{ev}} \mathcal{M}^*_{H,1}(Y_k, 0; J_Y) \times_{\mathrm{ev}} W^u_{f_Y}(\check{p}). \tag{8.4}$$

The fiber product $\mathcal{M}^*_{H,1}(B; J_W) \times_{\mathrm{ev}} \mathcal{M}^*_{H,1}(Y_k, 0; J_Y)$ has an action of $\mathbb{R}_1 \times \mathbb{R}_2$, where the 1-dimensional real vector space $\mathbb{R}_1$ acts by s-translation on an element of $\mathcal{M}^*_{H,1}(B; J_W)$ and $\mathbb{R}_2$ acts by s-translation on an element of $\mathcal{M}^*_{H,1}(Y_k, 0; J_Y)$. The fiber product can be oriented by using the coherent orientation on the spaces of cylinders, combined wih the fiber product rule for orientations. If $\tilde{J}_W$ is an almost complex structure close to split, then the moduli space of presplit cylinders $\mathcal{M}^*_{H,1}(B; \tilde{J}_W)$ has an $\mathbb{R}$-action, corresponding to s-translations on the domain. The moduli space of presplit Floer cylinders has a coherent orientation. The quotients $\Big(\mathcal{M}^*_{H,1}(B; J_W) \times_{\mathrm{ev}} \mathcal{M}^*_{H,1}(Y_k, 0; J_Y)\Big)/(\mathbb{R}_1 \times \mathbb{R}_2)$ and $\mathcal{M}^*_{H,1}(B; \tilde{J}_W)/\mathbb{R}$ are oriented diffeomorphic.

We might again be led to think that an analogue of [BO09b, Proposition 3.9] with $m = 2$ would cause a sign to appear in the differential, but that is not the case...

Part 4. Ansatz

We now show that Floer cylinders in $\mathbb{R} \times Y$ are equivalent punctured J_Y-holomorphic curves. We will first show how to obtain a pseudoholomorphic curve from a Floer cylinder and then explain how to go in the other direction.

9. From Floer cylinders to pseudoholomorphic curves

We show that every split Floer cylinder as in Definition 4.13 can be obtained from a J_Y-holomorphic curve and a solution to an auxiliary equation.

It is useful to first recall that for every integer $T \geqslant 1$, there is a Y-family of Reeb orbits of period T, and a bijective correspondence to the Y-family of 1-periodic Hamiltonian orbits contained in $\{b\} \times Y \subset \mathbb{R} \times Y$, where $h'(\mathrm{e}^b) = T$. A Reeb orbit γ *corresponds* to the Hamiltonian orbit $t \mapsto (b, \gamma(Tt))$. We say that two Reeb orbits of the same period are the *same as unparametrized orbits* if they differ by an additive constant on the domain.

Proposition 9.1. *Let $\Gamma \subset \mathbb{R} \times S^1$ be either the empty set or the singleton $\{P\}$. Suppose $\tilde{v} = (b, v)\colon \mathbb{R} \times S^1 \backslash \Gamma \to \mathbb{R} \times Y$ is the upper level of a split Floer cylinder, as in Definition 4.13. Then, there exists a pair $(\tilde{u}, \mathbf{e})$ consisting of a map $\tilde{u} = (a, u)\colon \mathbb{R} \times S^1 \backslash \Gamma \to \mathbb{R} \times Y$ and of a map $\mathbf{e} = (e_1, e_2)\colon \mathbb{R} \times S^1 \backslash \Gamma \to \mathbb{R} \times S^1$, with the following properties:*

1. *The map $\tilde{u}$ is a finite energy J_Y-holomorphic curve.*
2. *If $\tilde{v}$ converges to a Hamiltonian orbit at $\pm\infty$, then u converges to the corresponding parametrized Reeb orbit $\gamma_\pm$ at $\pm\infty$, i.e. $v(\pm\infty, t) = u(\pm\infty, t)$.*
3. *If $\Gamma = \{P\}$, then u is asymptotic to the same unparameterized Reeb orbit that v converges to at P, i.e. the asymptotic limit is the same orbit up to a phase shift.*
4. *The map $\mathbf{e}\colon \mathbb{R} \times S^1 \backslash \Gamma \to \mathbb{R} \times S^1$ satisfies the equation*

$$de_1 - de_2 \circ i + h'(\mathrm{e}^b) ds = 0. \tag{9.1}$$

(5) The original Floer solution is given by

$$(b, v) = (a + e_1, \phi_R^{e_2} \circ u)$$

where $\phi_R^t : Y \to Y$ denotes Reeb flow for time $t \in S^1$.

The pair $(\tilde{u}, \mathbf{e})$ is unique, up to replacing $\tilde{u} = (a, u)$ with $(a + c_1, u)$ and replacing $\mathbf{e} = (e_1, e_2)$ with $(e_1 - c_1, e_2)$, for some constant $c_1 \in \mathbb{R}$.

Suppose now that $\Gamma = \varnothing$ and that $\tilde{v}$ is the upper level of a split Floer cylinder, as in Definition 4.14. Then, there exists a pair $(\tilde{u}, \mathbf{e})$ as above, with the following difference:

(2') If $\tilde{v}$ converges to a Hamiltonian orbit at $+\infty$ and to a Reeb orbit at $-\infty$, then u converges to the corresponding Reeb orbit γ_+ at $+\infty$ and to the same Reeb orbit γ_- at $-\infty$.

The pair $(\tilde{u}, \mathbf{e})$ is again unique up to an $\mathbb{R}$-action.

In the following, we will construct the function $\mathbf{e}$ from Proposition 9.1 by studying the differential equation (9.1).

To this end, we need to construct function spaces of maps $\mathbf{e} : \mathbb{R} \times S^1 \backslash \Gamma \to \mathbb{R} \times S^1$. These will be subspaces of $W^{1,p}_{\text{loc}}(\mathbb{R} \times S^1 \backslash \Gamma, \mathbb{R} \times S^1)$ satisfying different types of asymptotic conditions. The maps $\mathbf{e}$ we consider will induce trivial maps of fundamental groups. It will be more convenient therefore to consider the lift to the universal cover $\tilde{\mathbf{e}} : \mathbb{R} \times S^1 \backslash \Gamma \to \mathbb{C}$, and study Equation (9.1) on this space.

As we will see below, all the punctures have the same asymptotic operator, namely

$$A := -i \frac{d}{dt} : W^{1,p}(S^1, \mathbb{C}) \to L^p(S^1, \mathbb{C}). \tag{9.2}$$

Note that A is self-adjoint, as a partially defined operator $L^2(S^1, \mathbb{C}) \to L^2(S^1, \mathbb{C})$. Lemma A.3 describes the spectrum of A. For the remainder of this section, we will fix $0 < \delta < 2\pi$, which is smaller than the absolute value of every non-zero eigenvalue of A. We will use this δ below when defining weighted Sobolev spaces.

We recall some notation first — this is discussed in more detail in Appendix A. For a fixed $z \in \mathbb{R} \times S^1$, or $z = \pm\infty$, we fix μ_z to be a bump function supported near z and identically 1 near z. The space of functions $W^{1,p,\delta}_{(0,\mathbb{C})}(\mathbb{R} \times S^1, \mathbb{C})$ consists of complex-valued functions exponentially decaying to 0 at $-\infty$ and exponentially decaying to an unspecified constant at $+\infty$. In the case of a punctured cylinder, $W^{1,p,\delta}_{(0,\mathbb{C};\mathbb{C})}(\mathbb{R} \times S^1 \backslash \{P\}, \mathbb{C})$ denotes the space of functions exponentially decaying to 0 at $-\infty$, decaying to a free constant at P and decaying to a free constant at $+\infty$.

In the case with $\Gamma = \{P\}$, we fix a parametrization

$$\begin{aligned} \varphi_P : (-\infty, -1] \times S^1 &\to \mathbb{R} \times S^1 \backslash \Gamma \\ (\rho, \eta) &\mapsto P + e^{2\pi(\rho + i\eta)} \end{aligned}$$

of a neighborhood of P, under the identification of $\mathbb{R} \times S^1$ with $\mathbb{C}/(z \sim z + i)$. Recall that the measure on $\mathbb{R} \times S^1 \backslash \Gamma$ and the norm on derivatives used for defining $W^{1,p}$ and L^p comes from these cylindrical coordinates near P. In particular, we make the following observation that will be used multiple times,

$$\varphi_P^* ds = 2\pi\, e^{2\pi\rho} \cos(2\pi\theta) d\rho - 2\pi\, e^{2\pi\rho} \sin(2\pi\theta) d\theta \tag{9.3}$$

In particular then, $|\varphi_P^* ds| = C\, e^{2\pi\rho}$.

We will consider solutions to equation (9.1) in three different spaces, one corresponding to each of the three possible Floer configurations appearing in Propositions 7.4 and 7.5. Cases 1 and 2 appear in Proposition 7.4, and are relevant for split Floer solutions in Definition 4.13 (connecting non-constant Hamiltonian orbits), represented by Figures 7.1 and 7.2, respectively. Case 3 is the one from Proposition 7.5, and is relevant for split Floer solutions in Definition 4.14 (connecting non-constant Hamiltonian orbits to constant ones), as in Figure 7.3.

Case 1: $\Gamma = \varnothing$ and both asymptotic limits of $\tilde{v}$ are Hamiltonian orbits.
We say $\tilde{\mathbf{e}} \in \tilde{\mathcal{X}}^1$ if

$$\tilde{\mathbf{e}} + (T_+ s + i0)\,\mu_{+\infty} + (T_- s + i0)\,\mu_{-\infty} \in W^{1,p,\delta}_{(\mathbb{C};\mathbb{C})}(\mathbb{R} \times S^1, \mathbb{C}).$$

(In particular then, $\mathbf{e}$ is of degree 0. The lift $\tilde{\mathbf{e}}$ is well-defined up to an additive constant.)

Case 2: $\Gamma = \{P\}$ and both asymptotic limits of $\tilde{v}$ are Hamiltonian orbits. We say $\tilde{\mathbf{e}} \in \tilde{\mathcal{X}}^2$ if

$$\tilde{\mathbf{e}} + (T_+ s + i0)\,\mu_{+\infty} + (T_- s + i0)\,\mu_{-\infty} \in W^{1,p,\delta}_{(\mathbb{C},\mathbb{C};\mathbb{C})}(\mathbb{R} \times S^1 \backslash \{P\}, \mathbb{C})$$

Case 3: $\Gamma = \varnothing$ and the $+\infty$ asymptotic limit of $\tilde{v}$ is a Hamiltonian orbit, whereas at $-\infty$, it converges to an orbit cylinder over a Reeb orbit in $\{-\infty\} \times Y$. We say $\tilde{\mathbf{e}} \in \tilde{\mathcal{X}}^3$ if

$$\tilde{\mathbf{e}} + (T_+ s + i0)\,\mu_{+\infty} \in W^{1,p\delta}_{(\mathbb{C};\mathbb{C})}(\mathbb{R} \times S^1, \mathbb{C}).$$

We then define $\mathcal{X}^1, \mathcal{X}^2, \mathcal{X}^3$ to be the set of functions obtained as compositions of functions in $\tilde{\mathcal{X}}^1, \tilde{\mathcal{X}}^2, \tilde{\mathcal{X}}^3$ with the projection map $\mathbb{C} \to \mathbb{R} \times S^1$.

Remark 9.2. The summands $(T_\pm s + i0)\,\mu_{\pm\infty}$ keep track of the fact that the Floer trajectory $\tilde{v} = (b, v)$ and the pseudoholomorphic curve $\tilde{u} = (a, u)$ in Proposition 9.1 have different asymptotic behavior at $\pm\infty$, and that $e_1 = b - a$.

Observe that in all three cases, we impose δ-decay to unspecified constants at $\pm\infty$ (and possibly P).

Denote the spaces of solutions to (9.1) in $\tilde{\mathcal{X}}^j$ by $\tilde{\mathcal{C}}^j(\tilde{v})$, where $j \in \{1, 2, 3\}$, and let $\mathcal{C}^j(\tilde{v})$ be the corresponding space of maps to $\mathbb{R} \times S^1$. We can write $\tilde{\mathcal{C}}^j(\tilde{v})$ as the preimage of zero under the operator

$$\begin{aligned} \tilde{\mathcal{X}}^j &\to L^{p,\delta}(T^*(\mathbb{R} \times S^1 \backslash \Gamma)) \\ \tilde{\mathbf{e}} = (e_1, e_2) &\mapsto de_1 - de_2 \circ i + h'(\mathrm{e}^b) ds. \end{aligned}$$

Observe that this map descends to a map $\mathcal{X}^j \to L^{p,\delta}(T^*(\mathbb{R} \times S^1 \backslash \Gamma))$. We then have that $\mathcal{C}^j(\tilde{v})$ is the preimage of zero under this quotient operator.

It will be useful to consider the corresponding linearized operators. Let $\mathbf{e} \in \mathcal{C}^j(\tilde{v})$. The corresponding linearized operator is

$$\begin{aligned} D^j \colon T_{\mathbf{e}}\mathcal{X}^j &\to L^{p,\delta}(T^*(\mathbb{R} \times S^1 \backslash \Gamma)) \\ E = E_1 + iE_2 &\mapsto dE_1 - dE_2 \circ i \end{aligned} \tag{9.4}$$

where in Cases (1) and (3), we have $T_{\mathbf{e}}\mathcal{X}^j = W^{1,p,\delta}_{(\mathbb{C};\mathbb{C})}(\mathbb{R} \times S^1, \mathbb{C})$ and and in Case (2), we have $T_{\mathbf{e}}\mathcal{X}^2 = W^{1,p,\delta}_{(\mathbb{C},\mathbb{C};\mathbb{C})}(\mathbb{R} \times S^1 \backslash \Gamma, \mathbb{C})$.

Evaluating (9.4) at ∂_s and ∂_t and rearranging terms, we get operators with values in $L^{p,\delta}(\mathbb{R} \times S^1 \backslash \Gamma, \mathbb{C})$ given by

$$\frac{\partial E}{\partial s} + i\frac{\partial E}{\partial t}. \tag{9.5}$$

By an abuse of notation, denote these operators also by D^j. Observe that the asymptotic operators to D^j at all punctures are given by Equation (9.2) above.

Lemma 9.3. *Fix* $\mathrm{e} \in \mathcal{C}^j(\tilde{v})$*, for* $j \in \{1, 2, 3\}$*. Then, the linearized operator* D^j *has index 2, is surjective and its kernel includes the span of the generator of the* S^1*-action on the target* $\mathbb{R} \times S^1$*. As a consequence, the space* $\mathcal{C}^j(\tilde{v})$ *has virtual dimension 2.*

Proof. The operator D^j in (9.4) is just the standard Cauchy–Riemann operator. This proof is perhaps more involved than necessary, but illustrates in a simple example the ideas used in the more technical setting of the next section. This proof is also similar to that of Lemma 6.14, the main difference being that we now apply the Morse–Bott Riemann–Roch Theorem A.8 instead of Theorem A.6.

We start with the Cases $j = 1$ or 3. Fix $\mathbf{e}$ as in the statement. Recall that the asymptotic operator is $A = -i\frac{d}{dt}$ at all punctures. We need to compute the Conley–Zehnder index of the perturbed asymptotic operator $A + \delta$, which is -1 by Corollary A.4. Therefore using the fact that all vector spaces in $\mathbf{V}$ are the kernels of the corresponding asymptotic operators, which we identified with $\mathbb{C}$, Theorem A.8 implies that

$$\operatorname{Ind} D^j = (-1 + 2) - (-1) = 2.$$

Following the notation of Wendl [Wen10, Equation (2.5)], we have

$$c_1(E, l, \mathbf{A}_\Gamma) = \frac{1}{2}(2 - 2) = 0 < 2 = \operatorname{Ind} \widehat{D}^j$$

and surjectivity and the statement about the kernel follow from [Wen10, Proposition 2.2] together with Lemma A.10.

Case 2 is slightly different, since there is an additional puncture at P. Its asymptotic operator is again A. The Euler characteristic of the domain is now -1. Therefore,

$$\operatorname{Ind} D^2 = -1 + \big((-1 + 2) - (-1) - (-1)\big) = 2,$$
$$c_1(E, l, \mathbf{A}_\Gamma) = \frac{1}{2}(2 - 2) = 0 < 2 = \operatorname{Ind} \widehat{D}^2$$

and we can once more apply [Wen10, Proposition 2.2] together with Lemma A.10. □

Lemma 9.4. *Let* $\mathbf{e} \in \mathcal{C}^j$*,* $j = 1, 2, 3$*. Then,* $\mathbf{e}$ *admits a smooth extension* $\mathbb{R} \times S^1 \to \mathbb{R} \times S^1$ *that satisfies Equation* (9.1)*.*

Proof. Let $\mathbf{e} \in \mathcal{C}^j$ be a solution. Then, in Cases (1) and (3), the result follows immediately from elliptic regularity and the fact that $h'(\mathrm{e}^b)$ is a smooth function on the cylinder.

For Case (2), observe that $h'(\mathrm{e}^r)$ vanishes for $r << 0$. Therefore, $h'(\mathrm{e}^b)$ vanishes in a neighbourhood of the puncture P, and thus extends as a smooth function on the cylinder. Furthermore, $\mathbf{e}$ is holomorphic in this same punctured neighbourhood, and the requirement it be in $W^{1,p,\delta}_{\mathbb{C}}(\mathbb{R}^- \times S^1)$ forces it to have a continuous and thus smooth extension across the puncture.

It follows therefore that $\mathbf{e}$ is smooth in Case (2) as well. □

Lemma 9.5. *Let* $\tilde{\mathbf{e}} = (e_1, \tilde{e}_2)\colon \mathbb{R} \times S^1 \to \mathbb{C}$ *be a lift of a solution* $\mathrm{e} = (e_1, e_2) \in \mathcal{C}^j$*. Then, for every* $t \in S^1$

$$\lim_{s\to-\infty} \tilde{e}_2(s,t) = \lim_{s\to\infty} \tilde{e}_2(s,t).$$

Proof. By Lemma 9.4, it follows that $\tilde{\mathbf{e}}$ is smooth and satisfies the equation

$$de_1 - d\tilde{e_2} \circ i + h'(\mathrm{e}^b)ds = 0.$$

For each $c >> 1$, let $S_c := [-c, c] \times S^1 \subset \mathbb{R} \times S^1$. Then,

$$\begin{aligned}
\int_{\partial S_c} \tilde{e}_2\, dt &= \int_{S_c} d\tilde{e}_2 \wedge dt = \int_{S_c} de_2 \wedge dt \\
&= \int_{S_c} de_2 \circ i \wedge dt \circ i \\
&= \int_{S_c} (de_1 + h'(\mathrm{e}^b)ds) \wedge ds \\
&= \int_{S_c} de_1 \wedge ds \\
&= \int_{\partial S_c} e_1\, ds = 0
\end{aligned}$$

since $ds|_{\partial S_c} = 0$. Hence, we obtain

$$\int_{\{c\}\times S^1} \tilde{e}_2(c,t)dt = \int_{\{-c\}\times S^1} \tilde{e}_2(-c,t)dt.$$

The result now follows from taking a limit as $c \to +\infty$. □

Lemma 9.6. *Equation* (9.1) *has a solution in* $\mathcal{X}^j$ *for* $j \in \{1, 2, 3\}$*, which is unique up to adding a constant in the target* $\mathbb{R} \times S^1$*.*

Proof. We will instead prove the result for the lifts $\tilde{\mathbf{e}} \in \tilde{\mathcal{X}}^j$, solving the same differential equation. Notice that any solution $\mathbf{e}$ admits a $\mathbb{Z}$ family of lifts. It suffices then to show that the solutions $\tilde{\mathbf{e}} = (e_1, \tilde{e}_2)$ are unique up to adding a constant in $\mathbb{C}$.

Write (9.1) as an inhomogeneous Cauchy–Riemann equation on $\mathbb{R} \times S^1\backslash\Gamma$:

$$d\tilde{e}_1 - d\tilde{e}_2 \circ i = -h'(\mathrm{e}^b)ds. \tag{9.6}$$

We will consider Cases 1, 2 and 3. Let $\nu_j : \mathbb{R} \to \mathbb{R}$ be smooth functions such that

$$\begin{cases} \nu_j(s) = T_- s \text{ for } s << 0 \text{ and } \nu_j(s) = T_+ s \text{ for } s >> 0 & j = 1, 2 \\ \nu_j(s) = 0 \text{ for } s << 0 \text{ and } \nu_j(s) = T_+ s \text{ for } s >> 0 & j = 3 \end{cases}$$

Instead of solving equation (9.6), we will consider the equivalent problem of finding $\mathbf{g} = (g_1, g_2) = (e_1 + \nu_j, \tilde{e}_2)$ such that

$$\mathbf{g} \in \begin{cases} W^{1,p,\delta}_{(\mathbb{C};\mathbb{C})}(\mathbb{R} \times S^1, \mathbb{C}), & \text{in cases } j = 1, 3 \\ W^{1,p,\delta}_{(\mathbb{C},\mathbb{C};\mathbb{C})}(\mathbb{R} \times S^1\backslash\Gamma, \mathbb{C}), & \text{in case } j = 2 \end{cases}$$

solves the equation

$$dg_1 - dg_2 \circ i = -h'(\mathrm{e}^b)ds + d\nu_j. \tag{9.7}$$

Notice that $\mathbf{g}$ is taken in the function space $T_{\mathbf{e}}X^j$.

Let $K = -h'(e^b)ds + d\nu_j$. In Cases (1) and (3), the fact that $K \in L^{p,\delta}(T^*(\mathbb{R}\times S^1))$ follows immediately from the exponential rate of convergence of b at $\pm\infty$ to $b_\pm$, and $T_\pm = h'(e^{b_\pm})$. In case (2), the fact $K \in L^{p,\delta}(T^*(\mathbb{R}\times S^1\backslash\{P\})$ follows from this and also from Equation 9.3 so that $\nu_2'(s)ds$ has exponential decay near P.

According to Lemma 9.3, the linearized operators (9.5) have index 2 and are surjective. Their kernels have dimension 2 and consist of constant functions. Let now $\mathbf{g} \in T_{\mathbf{e}}\mathcal{X}^j$, with $T_{\mathbf{e}}\mathcal{X}^j$ defined as in (9.4), be such that

$$dg_1 - dg_2 \circ i = K.$$

Any two solutions differ then by an element of the kernel of this linear operator, and thus differ by a constant. From this, we obtain the same statement for $\tilde{\mathbf{e}}$ and hence for $\mathbf{e}$. □

Proof of Proposition 9.1. Using the fact that $X_H = h'(e^r)R$, the Floer equation (4.3) satisfied by $\tilde{v} = (b, v)$ is equivalent to:

$$\begin{cases} db - v^*\alpha \circ i + h'(e^b)ds = 0 \\ \pi dv + J\pi dv \circ i = 0 \end{cases} \tag{9.8}$$

By Lemma 9.6, we have a solution $\mathbf{e}$ to (9.1). Now, $a := b - e_1$ and $u := \phi_R^{-e_2} \circ v$ are such that

$$\begin{aligned} da - u^*\alpha \circ i &= db - de_1 - v^*\alpha \circ i - de_2 \circ i \\ &= db - v^*\alpha \circ i + h'(e^b)ds \\ &= 0, \\ \pi du + J\pi du \circ i &= \pi dv + J\pi dv \circ i \\ &= 0. \end{aligned}$$

So, $\tilde{u} := (a, u)$ is J_Y-holomorphic. The asymptotic constraints on the spaces $\mathcal{X}^i$ are such that $\tilde{u}$ converges to Reeb orbits at $\pm\infty$ (and at P, if $\Gamma = \{P\}$). The uniqueness statement in Lemma 9.6 together with Lemma 9.5, imply that there is an $\mathbb{R}$-family of solutions $\mathbf{e} = (e_1, e_2) \in \mathcal{X}^j$, such that

$$\lim_{s\to\pm\infty} e_2(s,t) = 0 \in S^1,$$

where distinct solutions differ by adding a constant to e_1. This implies the existence statements in Proposition 9.1, and uniqueness up to an $\mathbb{R}$-action. □

10. From pseudoholomorphic curves to Floer cylinders

Proposition 9.1 implies that every split Floer cylinder $\tilde{v}\colon \mathbb{R}\times S^1\backslash\Gamma \to \mathbb{R}\times Y$ has an underlying J_Y-holomorphic curve $\tilde{u}\colon \mathbb{R}\times S^1\backslash\Gamma \to \mathbb{R}\times Y$. We will show that every J_Y-holomorphic curve $\tilde{u}\colon \mathbb{R}\times S^1\backslash\Gamma \to \mathbb{R}\times Y$ underlies a split Floer cylinder.

Proposition 10.1. *Let $\Gamma \subset \mathbb{R}\times S^1$ be either the empty set or the singleton $\{P\}$. Suppose $\tilde{u} = (a, u)\colon \mathbb{R}\times S^1\backslash\Gamma \to \mathbb{R}\times Y$ is a finite energy J_Y-holomorphic curve, asymptotic at P to a Reeb orbit at $-\infty$, if $\Gamma = \{P\}$. Then, there exists a pair $(\tilde{v}, \mathbf{e})$ consisting of a map $\tilde{v} = (b, v)\colon \mathbb{R}\times S^1\backslash\Gamma \to \mathbb{R}\times Y$ and of a map $\mathbf{e} = (e_1, e_2)\colon \mathbb{R}\times S^1 \to \mathbb{R}\times S^1$, with the following properties:*

1. *The map $\tilde{v}$ solves the Floer equation* (4.3).
2. *If u converges to a Reeb orbit $\gamma_\pm$ at $\pm\infty$, then $\tilde{v}$ converges to the Hamiltonian orbit corresponding to $\gamma_\pm$ at $\pm\infty$.*

(3) If $\Gamma = \{P\}$, then v is asymptotic to the same unparameterized Reeb orbit that u converges to at P.

(4) The map $\mathbf{e}\colon \mathbb{R}\times S^1\backslash\{P\} \to \mathbb{R}\times S^1$ satisfies the equation

$$de_1 - de_2\circ i + h'(\mathrm{e}^{a+e_1})ds = 0. \tag{10.1}$$

(5) The original J_Y-holomorphic curve is given by

$$(a,u) = (b - e_1, \phi_R^{-e_2}\circ v).$$

The pair $(\tilde v, \mathbf{e})$ is unique.

In the case that $\Gamma = \varnothing$ and $\tilde u(s,t) = (Ts + c, \gamma(Tt))$ is a trivial cylinder, for some Reeb orbit γ of period T, there also exists a pair $(\tilde v, \mathbf{e})$ as above, except for

(2') $\tilde v$ converges to a Hamiltonian orbit whose underlying Reeb orbit is γ at $+\infty$, and v converges to γ at $-\infty$.

The pair $(\tilde v, \mathbf{e})$ is now unique only up to replacing $b(s,t)$ with $\overline{b}(s,t) = b(s+s_0,t)$, for some constant $s_0\in\mathbb{R}$, and replacing e_1 with $\overline{e}_1 = \overline{b} - a$.

Equation (10.1) has a solution $\mathbf{e} = (e_1,e_2)$ iff there is a solution $\mathbf{f} = (f_1,f_2) = (e_1 + a, e_2)$ to the equation

$$\begin{aligned}\mathbf{f} =&(f_1,f_2)\colon \mathbb{R}\times S^1\backslash\Gamma \to \mathbb{R}\times S^1\\ &df_1 - df_2\circ i + h'(\mathrm{e}^{f_1})ds - da = 0,\end{aligned} \tag{10.2}$$

so we will study this equation instead. As we will see below equation (10.12), the linearized operator associated to (10.2) has asymptotic operators of the form

$$A_C = -i\frac{d}{dt} - \begin{pmatrix} C & 0\\ 0 & 0\end{pmatrix} : W^{1,p}(S^1,\mathbb{C}) \to L^p(S^1,\mathbb{C})$$

for some $C \geqslant 0$. Lemma A.3 implies that the kernel of A_C has real dimension 2 if $C = 0$ (given by constant functions $c_1 + ic_2 \in \mathbb{C}$) and 1 if $C \neq 0$ (given by constant functions $ic_2 \in i\mathbb{R}$).

Recall that we write $W^{1,p,\delta}_{(0,\mathbb{C};i\mathbb{R})}(\mathbb{R}\times S^1\backslash\{P\},\mathbb{C})$ to denote maps converging exponentially fast to 0 at $-\infty$, to an arbitrary complex constant at P and to an imaginary constant at $+\infty$, and write $W^{1,p,\delta}_{(0,i\mathbb{R})}(\mathbb{R}\times S^1,\mathbb{C})$ to denote maps converging exponentially fast to 0 at $-\infty$ and to an arbitrary imaginary constant at $+\infty$. Recall that we have chosen cylindrical coordinates near P, $\varphi\colon (-\infty,-1]\times S^1 \to \mathbb{R}\times S^1$ by $\varphi(\rho,\theta) = P + \mathrm{e}^{2\pi(\rho+i\theta)}$, and also a bump function μ_P that is identically 1 near P and supported in the image of φ.

Similarly to the previous section, we will study solutions to (10.2) in $W^{1,p}_{\mathrm{loc}}(\mathbb{R}\times S^1\backslash\Gamma, \mathbb{R}\times S^1)$ satisfying three types of asymptotic conditions.

Case 1: $\Gamma = \varnothing$. Define the space $\mathcal{Y}^1$ by the condition $\mathbf{f}\in\mathcal{Y}^1$ if it admits a lift $\tilde{\mathbf{f}}\in W^{1,p}_{\mathrm{loc}}(\mathbb{R}\times S^1,\mathbb{C})$ with

$$\tilde{\mathbf{f}} - (b_+ + i0)\,\mu_{+\infty} - (b_- + i0)\,\mu_{-\infty} \in W^{1,p,\delta}_{(0,i\mathbb{R})}(\mathbb{R}\times S^1,\mathbb{C})$$

Case 2: $\Gamma = \{P\}$. Define the space $\mathcal{Y}^2$ by the condition that $\mathbf{f}\in\mathcal{Y}^2$ if it admits a lift $\tilde{\mathbf{f}}\in W^{1,p}_{\mathrm{loc}}(\mathbb{R}\times S^1,\mathbb{C})$ with

$$\tilde{\mathbf{f}} - (b_+ + i0)\,\mu_{+\infty} - (b_- + i0)\,\mu_{-\infty} - \mu_P\, a \in W^{1,p,\delta}_{(0,\mathbb{C};i\mathbb{R})}(\mathbb{R}\times S^1\backslash\{P\},\mathbb{C}).$$

Case 3: $\Gamma = \varnothing$. Define the space $\mathcal{Y}^3$ by the condition that $\mathbf{f}$ admits a lift $\tilde{\mathbf{f}}$ with

$$\tilde{\mathbf{f}} - (b_+ + i0)\,\mu_{+\infty} - (b_- + i0)\,\mu_{-\infty} \in W^{1,p,\delta}_{(\mathbb{C};i\mathbb{R})}(\mathbb{R}\times S^1,\mathbb{C}).$$

Remark 10.2. Similarly to the previous section, the last part of the Proposition, where $\tilde{u}$ is assumed to be a trivial cylinder, is related to Case 3 (see Figure 7.3), and the first part of the Proposition is related to Cases 1 and 2 (see Figures 7.1 and 7.2, respectively).

It is important to observe that the asymptotic conditions imposed in this section are different from the ones that were imposed in the previous section. This will have an effect on the indices of the relevant Fredholm problems. Just to give an example, in Case 1 we are considering functions that converge at $+\infty$ to an unspecified constant in $\{b_+\} \times S^1 \subset \mathbb{R} \times S^1$, and that converge at $-\infty$ to $(b_-, 0) \in \mathbb{R} \times S^1$.

In this section, it will also be useful to consider auxiliary parametric versions of Equation (10.2). We will deform the solutions to (10.2) to solutions of a PDE for which we can explicitly describe the solution space. The deformation will give an identification of the solution spaces.

For Case 1, we will consider the map

$$\begin{aligned} \mathcal{F} \colon \mathcal{Y}^1 \times [0,1] &\to L^{p,\delta}(\mathbb{R} \times S^1, (\mathbb{R}^2)^*) \\ (\mathbf{f}, \tau) = ((f_1, f_2), \tau) &\mapsto df_1 - df_2 \circ i + h'(\mathrm{e}^{f_1})ds - \tau da - (1-\tau)d\nu \end{aligned} \tag{10.3}$$

where $\nu \colon \mathbb{R} \to \mathbb{R}$ is a smooth function such that $\nu(s) = T_- s$ for $s << 0$ and $\nu(s) = T_+ s$ for $s >> 0$ (just like ν_1 in the proof of Lemma 9.6). We will study solutions to equation

$$\mathcal{F}(\mathbf{f}, \tau) = 0, \tag{10.4}$$

which interpolates between (10.2) and the equation

$$df_1 - df_2 \circ i + h'(\mathrm{e}^{f_1})ds - d\nu = 0. \tag{10.5}$$

The latter has the advantage that $d\nu$ is independent of t. Denote the space of solutions to (10.4) in $\mathcal{Y}^1$ for fixed τ by $\mathcal{C}^{1,\tau}(\tilde{u})$.

In Case 2, it will be more convenient to study functions $\mathbf{g} = (g_1, g_2) = (f_1 - \mu_P\, a, f_2)$. We think of $\mathbf{g}$ as an element of the subspace $\hat{\mathcal{Y}}^2 \subset W^{1,p}_{\mathrm{loc}}(\mathbb{R} \times S^1 \backslash \Gamma, \mathbb{R} \times S^1)$, with $\mathbf{g} \in \hat{\mathcal{Y}}^2$ if it has a lift $\tilde{\mathbf{g}}$ with

$$\tilde{\mathbf{g}} - (b_+ + i0)\,\mu_{+\infty} - (b_- + i0)\,\mu_{-\infty} \in W^{1,p,\delta}_{(0,\mathbb{C};i\mathbb{R})}(\mathbb{R} \times S^1 \backslash \{P\}, \mathbb{C}).$$

We will consider the map

$$\begin{aligned} \mathcal{G} \colon \hat{\mathcal{Y}}^2 \times [0,1] &\to L^{p,\delta}(\mathbb{R} \times S^1 \backslash \Gamma, (\mathbb{R}^2)^*) \\ (\mathbf{g}, \tau) = ((g_1, g_2), \tau) &\mapsto dg_1 - dg_2 \circ i + h'(\mathrm{e}^{g_1 + \tau \mu_P a})ds - \tau d((1-\mu_P)a) - (1-\tau)d\nu \end{aligned} \tag{10.6}$$

where $\nu \colon \mathbb{R} \to \mathbb{R}$ is as above.

We will study solutions to

$$\mathcal{G}(\mathbf{g}, \tau) = 0. \tag{10.7}$$

This equation interpolates between the equivalent of equation (10.2) for $\mathbf{g}$:

$$dg_1 - dg_2 \circ i + h'(\mathrm{e}^{g_1 + \mu_P a})ds - d((1-\mu_P)a) = 0$$

and equation (10.5) applied to $\mathbf{g}$:

$$dg_1 - dg_2 \circ i + h'(\mathrm{e}^{g_1})ds - d\nu = 0,$$

The latter has again the advantage that $d\nu$ is independent of t. Denote the space of solutions to (10.7) in $\hat{\mathcal{Y}}^2$ for fixed τ by $\mathcal{C}^{2,\tau}(\tilde{u})$.

Also, denote the space of solutions to (10.2) in $\mathcal{Y}^3$ by $\mathcal{C}^3(\tilde{u})$. In Case 3, we will study solutions directly and do not require any deformation argument.

We need the following smoothness in order to apply the implicit function theorem, which we will need in the proof of Proposition 10.1.

Lemma 10.3. *The nonlinear operators $\mathcal{F}$ and $\mathcal{G}$ are C^1 in $\mathcal{Y}^1 \times [0,1] \to L^{p,\delta}(\mathbb{R} \times S^1, (\mathbb{R}^2)^*)$ and $\hat{\mathcal{Y}}^2 \times [0,1] \to L^{p,\delta}(\mathbb{R} \times S^1 \backslash \{P\}, (\mathbb{R}^2)^*)$, respectively.*

Proof. Notice that $\mathcal{F}(\mathbf{f}, \tau)$ is a Floer-type differential operator with an inhomogeneous term depending on τ. The claimed result holds for $\mathcal{F}$ since $-da + \nu'(s)\, ds$ is in $L^{p,\delta}$, which follows from the fact that $a(s,t)$ converges exponentially fast to $T_\pm s$ at $\pm\infty$.

In order to show the result for $\mathcal{G}(\mathbf{g}, \tau)$, it suffices to show that the following map is C^1, since the remaining terms in $\mathcal{G}$ are C^1 by standard arguments.

$$\begin{aligned} \mathcal{H} \colon W^{1,p,\delta}_{(0,\mathbb{C};i\mathbb{R})}(\mathbb{R} \times S^1 \backslash \Gamma, \mathbb{C}) &\to L^{p,\delta}(T^*(\mathbb{R} \times S^1 \backslash \Gamma)) \\ (g_1, g_2, \tau) &\mapsto h'(\mathrm{e}^{g_1 + \tau \mu_P a})\, ds. \end{aligned} \tag{10.8}$$

First, we restrict our attention to a $W^{1,p,\delta}_{(0,\mathbb{C};i\mathbb{R})}$-ball of radius R. We will show the map is C^1 on each such ball by showing its second derivative is bounded by a constant that depends on R. Exhausting the total space by such balls will prove the result.

Notice that the $W^{1,p,\delta}$ bound implies a uniform C^0 bound on g_1. Also notice that $\mu_P a$ is bounded above. Thus, $g_1 + \tau\mu_P a$ is bounded above (where the bound depends on R). Hence, there exists a C (depending on R) so that $h'(\mathrm{e}^{g_1+\tau\mu_P a}) \leqslant C$, $h''(\mathrm{e}^{g_1+\tau\mu_P a})\, \mathrm{e}^{g_1+\tau\mu_P a} \leqslant C$ and $h''(\mathrm{e}^{g_1+\tau\mu_P a})\, \mathrm{e}^{2(g_1+\tau\mu_P a)} + h'''(\mathrm{e}^{g_1+\tau\mu_P a})\, \mathrm{e}^{g_1+\tau\mu_P a} \leqslant C$.

The weak derivative of $\mathcal{H}$ is given by

$$d\mathcal{H}(\mathbf{g}, \tau)(\mathbf{G}, \tau') = G h''(\mathrm{e}^{g_1+\tau\mu_P a})\, \mathrm{e}^{g_1+\tau\mu_P a}\; ds + \tau' h''(\mathrm{e}^{g_1+\tau\mu_P a})\, \mathrm{e}^{g_1+\tau\mu_P a}\; \mu_P a\, ds$$

Notice that this expression is in $W^{1,p,\delta}$ away from P. Near P, in the cylindrical coordinates (ρ, θ), we note that this is bounded by $C(|G| + |\tau'|(T|\rho| + D))\, \mathrm{e}^{2\pi\rho}(|d\rho| + |d\theta|)$ for suitable constants C, D and where T is the period of the Reeb orbit to which (a, u) converges at P. From this, we obtain that $d\mathcal{H}$ is uniformly bounded on each ball of radius R, showing $\mathcal{H}$ is Lipschitz on each ball of radius R, and hence $\mathcal{H}$ is continuous.

By a similar argument, we estimate the second derivative to show C^1 smoothness of $\mathcal{H}$. The C^1-smoothness of $\mathcal{G}$ then follows. $\square$

We now discuss the linearized operators associated to $\mathcal{F}$ and $\mathcal{G}$, for fixed values of τ. In Case 1, writing a solution to (10.4) as $\mathbf{f} = f_1 + i f_2$, the linearization of the operator $\mathcal{F}^\tau$ is

$$\begin{aligned} D^{1,\tau} \colon T_{\mathbf{f}}\mathcal{Y}^1 &\to L^{p,\delta}(\mathbb{R} \times S^1, (\mathbb{R}^2)^*) \\ F = F_1 + iF_2 &\mapsto dF_1 - dF_2 \circ i + h''(\mathrm{e}^{f_1})\, \mathrm{e}^{f_1}\, F_1 ds. \end{aligned} \tag{10.9}$$

where $T_{\mathbf{f}}\mathcal{Y}^1 = W^{1,p,\delta}_{\mathbf{V}}(\mathbb{R} \times S^1, \mathbb{C})$ with $V_- = 0$ and $V_+ = i\mathbb{R}$.

In Case 2, writing $\mathbf{g} = g_1 + ig_2$ and again fixing τ, the linearized operator is

$$\begin{aligned} D^{2,\tau} \colon T_{\mathbf{g}}\hat{\mathcal{Y}}^2 &\to L^{p,\delta}(\mathbb{R} \times S^1 \backslash \Gamma, (\mathbb{R}^2)^*) \\ G = G_1 + iG_2 &\mapsto dG_1 - dG_2 \circ i + h''(\mathrm{e}^{g_1+\tau\mu a})\, \mathrm{e}^{g_1+\tau\mu a}\, G_1 ds. \end{aligned} \tag{10.10}$$

where $T_{\mathbf{g}}\hat{\mathcal{Y}}^2 = W^{1,p,\delta}_{\mathbf{V}}(\mathbb{R} \times S^1 \backslash \Gamma, \mathbb{C})$ with $V_- = 0$, $V_+ = i\mathbb{R}$ and $V_P = \mathbb{C}$.

In Case 3, the linearized operator

$$D^3 \colon T_{\mathbf{f}}\mathcal{Y}^3 \to L^{p,\delta}(\mathbb{R} \times S^1, (\mathbb{R}^2)^*),$$

where $T_{\mathbf{f}}\mathcal{Y}^3 = W^{1,p,\delta}_{\mathbf{V}}(\mathbb{R} \times S^1, \mathbb{C})$ with $V_- = \mathbb{C}$ and $V_+ = i\mathbb{R}$, is also given by (10.9)

Evaluating (10.9) and (10.10) at ∂_s and ∂_t and rearranging, we get operators with values in $L^{p,\delta}(\mathbb{R} \times S^1, \mathbb{C})$, given by

$$\frac{\partial F}{\partial s} + i\frac{\partial F}{\partial t} + h''(\mathrm{e}^{f_1})\,\mathrm{e}^{f_1}\,F_1 \tag{10.11}$$

and

$$\frac{\partial G}{\partial s} + i\frac{\partial G}{\partial t} + h''(\mathrm{e}^{g_1+\tau\mu a})\,\mathrm{e}^{g_1+\tau\mu a}\,G_1, \tag{10.12}$$

respectively.

The asymptotic operators at $\pm\infty$ in Cases 1 and 2, and at $+\infty$ in Case 3, are

$$A_\pm := -i\frac{d}{dt} - \begin{pmatrix} h''(\mathrm{e}^{b_\pm})\,\mathrm{e}^{b_\pm} & 0 \\ 0 & 0 \end{pmatrix} : W^{1,p}(S^1, \mathbb{C}) \to L^p(S^1, \mathbb{C})$$

In Case 2, there is also an asymptotic operator at the augmentation puncture:

$$A_P := -i\frac{d}{dt} : W^{1,p}(S^1, \mathbb{C}) \to L^p(S^1, \mathbb{C})$$

In Case 3, we have the asymptotic operator at $-\infty$ given by $A_- = -i\frac{d}{dt}$.

All these asymptotic operators are self-adjoint, as partially defined operators $L^2(S^1, \mathbb{C}) \to L^2(S^1, \mathbb{C})$. Their eigenvalues can be computed using Lemma A.3. Note that they all have eigenvalue 0. For the remainder of this section, we will fix $0 < \delta < 2\pi$ such that $\delta < C_\pm = h''(\mathrm{e}^{b_\pm})\,\mathrm{e}^{b_\pm} > 0$ and $\delta < \frac{1}{2}\left(-C_\pm + \sqrt{C_\pm^2 + 16\pi^2}\right)$ (see Remark 6.11). So, δ is smaller than the absolute value of every non-zero eigenvalue of $A_\pm$ and A_P. This δ will be used in the definitions of the relevant weighted Sobolev spaces.

Lemma 10.4. *Fix $\tau \in [0,1]$ and $\mathbf{f} \in \mathcal{C}^{1,\tau}(\tilde{u})$. Then, the linearized operator $D^{1,\tau}$ is Fredholm of index 0 and is an isomorphism.*

Fix $\tau \in [0,1]$ and $\mathbf{g} \in \mathcal{C}^{2,\tau}(\tilde{u})$. Then, the linearized operator $D^{2,\tau}$ is Fredholm of index 0 and is an isomorphism.

For fixed $\mathbf{f} \in \mathcal{C}^3(\tilde{u})$, D^3 is Fredholm of index 2, is surjective and its kernel includes the span of the generator of the S^1-action on the target $\mathbb{R} \times S^1$.

Proof. The proof has the same structure as that of Lemma 9.3. We start with Case 1. Fix τ and $\mathbf{f}$ as in the statement. Corollary A.4 implies that the Conley–Zehnder indices of the perturbed asymptotic operators $A_\pm + \delta$ are 0. Recall that the kernels of the asymptotic operators A are identified with $i\mathbb{R}$ and that the vector spaces in $\mathbf{V}$ are $V_- = 0$ and $V_+ = i\mathbb{R}$. Theorem A.8 now implies that

$$\operatorname{Ind} \widehat{D}^{1,\tau} = (0+1) - (0+1) = 0.$$

Following the notation of Wendl [Wen10, Equation (2.5)], we have

$$c_1(E, l, \mathbf{A}_\Gamma) = \frac{1}{2}(0-2) = -1 < 0 = \operatorname{Ind} \widehat{D}^{1,\tau}$$

and $\widehat{D}^{1,\tau}$ is an isomorphism by [Wen10, Proposition 2.2].

In Case 3, recall that the asymptotic operator at $-\infty$ is now $A_- = -i\frac{d}{dt}$. By Corollary A.4, the Conley–Zehnder index of the perturbed asymptotic operator

$A_- + \delta$ is -1. The kernel of the asymptotic operator A_- can be identified with $\mathbb{C}$, and the vector spaces in $\mathbf{V}$ are now $V_- = \mathbb{C}$ and $V_+ = i\mathbb{R}$. By Theorem A.8,

$$\operatorname{Ind}\widehat{D}^3 = (0+1) - (-1+0) = 2,$$

$$c_1(E, l, \mathbf{A}_\Gamma) = \frac{1}{2}(2-2) = 0 < 2 = \operatorname{Ind}\widehat{D}^3$$

and $\widehat{D}^3$ is again an isomorphism by [Wen10, Proposition 2.2].

In Case 2, we have the additional puncture P, with asymptotic operator $A_P = -i\frac{d}{dt}$, and the Euler charateristic of the domain is -1. We now have $V_- = 0$, $V_+ = i\mathbb{R}$ and $V_P = \mathbb{C}$, so Theorem A.8 implies

$$\operatorname{Ind}\widehat{D}^{2,\tau} = -1 + (0+1) - (0+1) - (-1+0) = 0,$$

$$c_1(E, l, \mathbf{A}_\Gamma) = \frac{1}{2}(0-2) = -1 < 0 = \operatorname{Ind}\widehat{D}^{2,\tau}$$

and we can apply once more [Wen10, Proposition 2.2]. □

We now prove a result analogous to Lemma 10.5.

Lemma 10.5. *Let $\tau \in [0,1]$.*

Suppose that $\mathbf{f} \in \mathcal{Y}^1$, $\mathbf{g} \in \hat{\mathcal{Y}}^2$ or $\mathbf{f} \in \mathcal{Y}^3$ satisfy

$$\mathcal{F}(\mathbf{f}, \tau) = 0, \quad \mathcal{G}(\mathbf{g}, \tau) = 0, \ \text{or } df_1 - df_2 \circ i + h'(\mathrm{e}^{f_1}) - da = 0,$$

respectively.

Then, the asymptotic constants satisfy $f_2(+\infty, \cdot) = f_2(-\infty, \cdot)$, $g_2(+\infty, \cdot) = g_2(-\infty, \cdot)$ and $f_2(+\infty, \cdot) = f_2(-\infty, \cdot)$.

In particular, in Cases (1) and (2), these asymptotic constants are 0.

Proof. The proof is very similar to the proof of Lemma 9.5.

Notice that if we have such a solution $\mathbf{f}$, $\mathbf{g}$, the resulting function is smooth on the punctured cylinder, by standard elliptic regularity arguments.

We will prove the result for functions that solve the equation

$$df_1 - df_2 \circ i + h'(\mathrm{e}^{f_1 + x})\, ds - \tau\, dy - (1-\tau)\, d\nu,$$

on $\mathbb{R} \times S^1 \backslash \Gamma$, where x, y satisfy the following:

Case 1: $x = 0, y = a$, $\Gamma = \varnothing$;
Case 2: $x = \tau\mu_P a$, $y = (1-\mu_P)a$, $\Gamma = \{P\}$;
Case 3: $\tau = 1$, $x = 0$, $y = a$, $\Gamma = \varnothing$.

We assume furthermore that f_2 converges exponentially fast to a constant at $\pm\infty$, and that f_1 converges exponentially fast to a constant at P in Case 2.

Notice in particular that if the cylinder has a puncture, y vanishes in a neighbourhood of the puncture.

For each $c >> 1$, let $S_c := [-c, c] \times S^1 \backslash D_P(1/c) \subset \mathbb{R} \times S^1 \backslash \Gamma$, where $D_P(\epsilon)$ denotes a disk of radius ϵ centered at P.

We then obtain (noticing that $d\nu = \nu'(s)\,ds$)

$$\begin{aligned}\int_{\partial S_c} \tilde{f}_2 dt &= \int_{S_c} d\tilde{f}_2 \wedge dt = \int_{S_c} df_2 \wedge dt \\ &= \int_{S_c} df_2 \circ i \wedge dt \circ i \\ &= \int_{S_c} (df_1 + h'(\mathrm{e}^{f_1+x})ds - \tau dy - (1-\tau)d\nu) \wedge ds \\ &= \int_{\partial S_c} (f_1 - \tau y) \wedge ds\end{aligned}$$

This vanishes in Cases (1) and (3), and we are done. In Case (2), ∂S_c also contains a small loop around P. In that case we continue to obtain:

$$\begin{aligned}&= \int_{\partial D_P(1/c)} (f_1 - \tau y)\,ds \\ &= \int_{\partial D_P(1/c)} f_1\,ds.\end{aligned}$$

Notice that $\int_{\partial D_P(1/c)} f_1\,ds$ decays like $\frac{1}{c^{2\pi\delta+1}}$.

The result now follows by taking the limit as $c \to +\infty$. $\square$

Lemma 10.6. *There is a unique solution to* (10.5) *in* $\mathcal{Y}^1$ *and in* $\hat{\mathcal{Y}}^2$.

If $a = Ts + c$, then there is an 2-parameter family of solutions to (10.2) *in* $\mathcal{Y}^3$, *parametrized by* $\mathbb{R} \times S^1$.

Proof. We begin with Case 1. Equation (10.5) can be rewritten in coordinates as

$$\begin{cases} \partial_s f_1 - \partial_t f_2 + h'(\mathrm{e}^{f_1}) - \nu' = 0 \\ \partial_t f_1 + \partial_s f_2 = 0 \end{cases} \tag{10.13}$$

Let us start by considering solutions where $f_2 \equiv k$ is constant, and $f_1(s,t) = f_1(s)$ is independent of t. Then, we get the s-dependent ODE

$$\frac{df_1}{ds} + h'(\mathrm{e}^{f_1}) - \nu' = 0.$$

Write it as

$$\frac{df_1}{ds} = F(f_1, s)$$

where $F(x,s) = -h'(\mathrm{e}^x) + \nu'(s)$.

Claim. For every $s_0 \in \mathbb{R}$ and every $c_1 \in \mathbb{R}$, there is a unique solution $f_1 : \mathbb{R} \to \mathbb{R}$ such that $f_1(s_0) = c_1$.

To prove the claim, recall the assumptions that $h'(\rho) \geqslant 0$ and $h'(\rho) = 0$ for $\rho \leqslant 2$. These and the asymptotic behavior of ν imply that there are constants C and $\tilde{C}$ such that:

- $F(x,s) \leqslant C$ for all (x,s);
- $F(x,s) \geqslant \tilde{C}$ for $x \leqslant \ln 2$, uniformly in s.

The claim now follows from standard existence and uniqueness theory for solutions to ODE's.

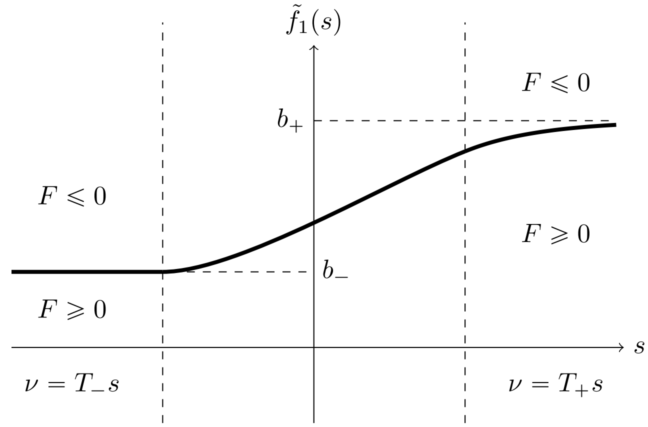

FIGURE 10.1. The function $\tilde f_1$

The facts that for $s << 0$ we have that b_- is the only zero of the function $x \mapsto F(x,s)$ and that $\partial_x F(x,s) \leqslant 0$ (a consequence of $h''(x) \geqslant 0$) imply that there is a unique solution $\tilde f_1$ of the ODE such that $\lim_{s\to-\infty} \tilde f_1(s) = b_-$, see Figure 10.1.

Now, $\mathbf{f}^0(s,t) = (\tilde f_1(s), 0)$ is a solution to (10.13) in $\mathcal{Y}^1$. The fact that it has the correct exponential convergence properties at $\pm\infty$ follow from $\tilde f_1$ solving the ODE above. We want to show that there are no other solutions to (10.13) in $\mathcal{Y}^1$. The proof will be an application of positivity of intersections of pseudoholomorphic curves in dimension 4.

Suppose that there is a solution $\mathbf{f}^1 = (f_1^1, f_2^1)$ of (10.13) in $\mathcal{Y}^1$ such that f_2^1 is not constant. By the definition of $\mathcal{Y}^1$ and by Lemma 10.5, $\lim_{s\to\pm\infty} f_2^1(s,t) = 0$. Since f_2^1 is not constant, there is $(s_0,t_0) \in \mathbb{R}\times S^1$ such that $f_2^1(s_0,t_0) \neq 0$. Let $f_1^2 : \mathbb{R} \to \mathbb{R}$ be the unique solution to the ODE above, for which $f_1^2(s_0) = f_1^1(s_0,t_0)$. Then, $\mathbf{f}^2(s,t) = (f_1^2(s), f_2^1(s_0,t_0))$ is another solution to (10.13) (although $(f_1^2, 0)$ does not have the correct asymptotic behavior to belong to $\mathcal{Y}^1$).

Now, observe that (10.13) is the Floer equation on the Kähler manifold $(\mathbb{R} \times S^1, dx \wedge dy, i)$, for the time-dependent Hamiltonian

$$\begin{aligned} H \colon \mathbb{R} \times (\mathbb{R}\times S^1) &\to \mathbb{R} \\ (s,x,y) &\mapsto \int_0^x h'(\mathrm{e}^z)dz - \nu'(s)x \end{aligned}$$

By Gromov's trick [Gro85], solutions to the Floer equation can be thought of as pseudoholomorphic curves $\mathbb{R}\times S^1 \to M := (\mathbb{R}\times S^1)\times(\mathbb{R}\times S^1)$, for some twisted almost complex structure on M. The projection of these curves to the first cylinder factor is the identity.

Since $\lim_{s\to-\infty} f_2^1 = \lim_{s\to-\infty} f_2^0$, Lemma 10.5 implies that they have lifts $\tilde f_2^1$ and $\tilde f_2^0$ to $\mathbb{R}$, with the same limits as $s \to \pm\infty$. By assumption, $\mathbf{f}^1$ and $\mathbf{f}^0$ are both in $\mathcal{Y}^1$, which implies that $\lim_{s\to\pm\infty} f_1^1(s,t) = \lim_{s\to\pm\infty} f_1^0(s,t) = b_\pm$. Therefore, there is a homotopy from $\mathbf{f}^1$ to $\mathbf{f}^0$ that is C^0-small on neighborhoods of $\pm\infty \times S^1$ in $\mathbb{R}\times S^1$. This gives a homotopy from the graph of $\mathbf{f}^1$ to the graph of $\mathbf{f}^0$ in M, and the intersections with the graph of $\mathbf{f}^2$ during the homotopy will remain in a compact region of M. So, we have an equality of signed intersection numbers

$$\#(Graph(\mathbf{f}^1) \cap Graph(\mathbf{f}^2)) = \#(Graph(\mathbf{f}^0) \cap Graph(\mathbf{f}^2)) = 0,$$

since the second components of $\mathbf{f}^0$ and $\mathbf{f}^2$ are different constants. But then positivity of intersections of pseudoholomorphic curves in dimension 4 [MS04, Exercise 2.6.1] implies that the graphs of $\mathbf{f}^1$ and $\mathbf{f}^2$ do not intersect, which, by the construction of $\mathbf{f}^2$, is a contradiction. Therefore, there is no such $\mathbf{f}^1$ to start with, which proves the first statement in the Lemma in Case 1.

Case 2 can be dealt with by the same argument, applied to $\mathbf{g} \in \hat{\mathcal{Y}}^2$. We again apply Lemma 10.5. The argument for Case 1 above produces a solution $\mathbf{g} = (g_1(s), 0) \in \mathcal{Y}^1$. We need to argue that such a solution is indeed in $\hat{\mathcal{Y}}^2$. This means that $(\mathbf{g} \circ \varphi_P) \in W^{1,p,\delta}_{\mathbb{C}_-}((-\infty, -1] \times S^1, \mathbb{R} \times S^1)$. This will be a consequence of the fact that g_1 is C^1, and of its asymptotics. Write $P = (P_s, P_t)$. Then,

$$(\mathbf{g} \circ \varphi_P)(\rho, \theta) = (g_1(\mathrm{e}^{2\pi\rho} \cos(2\pi\theta) + P_s), 0)$$

and for $\rho << -1$ the Mean Value Theorem gives

$$|g_1(\mathrm{e}^{2\pi\rho} \cos(2\pi\theta) + P_s) - g_1(P_s)| \leqslant ||g_1'||_{L^\infty}\ \mathrm{e}^{2\pi\rho}$$

where the sup norm is finite by the exponential convergence of g_1 at $\pm\infty$. Since $\delta < 2\pi$, we get the desired exponential convergence at P.

Let us now consider the case when $\tilde{u} = (a, u)$ is such that $a = Ts + c$, and look in $\mathcal{Y}^3$ for solutions to (10.2). This is equivalent to the system of equations

$$\begin{cases} \partial_s f_1 - \partial_t f_2 + h'(\mathrm{e}^{f_1}) - T = 0 \\ \partial_t f_1 + \partial_s f_2 = 0 \end{cases} \tag{10.14}$$

We again have solutions of the form $\mathbf{f}(s, t) = (f_1(s), f_2)$ with f_2 constant. There is an $\mathbb{R} \times S^1$ family of such solutions, parametrized by $\lim_{s\to-\infty}(f_1(s) - Ts, f_2) \in \mathbb{R} \times S^1$. The union of the graphs of these solutions foliates the region $(\mathbb{R} \times S^1) \times ((-\infty, b_+) \times S^1) \subset (\mathbb{R} \times S^1) \times (\mathbb{R} \times S^1)$. We can apply Lemma 10.5 to equation (10.2), since it is a special case of (10.4) when $\tau = 1$. The argument above using positivity of intersections can then be adapted to prove that there are no other solutions with the same asymptotics. To get different solutions, we can replace $(f_1(s), f_2)$ with $(\overline{f}_1(s), \overline{f}_2) = (f_1(s + c_1), f_2 + c_2)$, for a constant $(c_1, c_2) \in \mathbb{R} \times S^1$. □

Proof of Proposition 10.1. Most of the claims made in the statement follow from the existence of a solution $\mathbf{e}$ to (10.1) with the appropriate asymptotics. Indeed, given such $\mathbf{e} = (e_1, e_2)$, one can construct $\tilde{v} = (b, v)$ by taking

$$(b, v) = (a + e_1, \phi_R^{e_2} \circ u).$$

As we pointed out, the existence of such $\mathbf{e}$ is equivalent to the existence of a solution $\mathbf{f}$ to (10.2). Let us consider different cases separately.

In Case 1, note that Lemmas 10.4 and 10.3 and the Implicit Function Theorem imply that solutions to (10.4) vary smoothly in τ. Lemma 10.6 guarantees the existence and uniqueness of solutions to (10.5) in $\mathcal{Y}^1$ (which corresponds to $\tau = 0$). Therefore, we conclude that (10.2) (which corresponds to $\tau = 1$) also has a unique solution in $\mathcal{Y}^1$, which finishes the proof.

Case 2 works the same way, with solution $\mathbf{g}$ to (10.7) instead of solutions $\mathbf{f}$ to (10.4).

Case 3 corresponds to the final part of Proposition 10.1, when $\tilde{u}$ is assumed to be a trivial cylinder. Lemma 10.6 implies that there is an $\mathbb{R} \times S^1$-family of solutions to (10.2) in $\mathcal{Y}^3$. By Lemma 10.5, there is an $\mathbb{R}$-family of such solutions with $\lim_{s \to \pm\infty} f_2(s,t) = 0 \in S^1$. This finishes the proof. □

Remark 10.7. We now briefly compare the Fredholm indices of the Floer cylinders, the holomorphic cylinders in $\mathbb{R} \times Y$ and the cylinders satisfying equation (10.1).

Cases (1) and (2) are similar. We discuss Case (1) in detail. The Floer cylinder from $\widehat{q}_{k_-}$ to $\check{p}_{k_+}$ has Fredholm index $M(p) - M(q) - 1 + 2\frac{\tau_X - K}{K}(k_+ - k_-)$. Observe that if this projects to the class $A \in H_2(\Sigma;\mathbb{Z})$, we have $k_+ - k_- = K\omega(A)$, and thus this index is $M(p) - M(q) - 1 + 2\langle c_1(T\Sigma), A\rangle$. The corresponding holomorphic cylinder, without any marker constraints, has Fredholm index $M(p) - M(q) + 2\langle c_1(T\Sigma), A\rangle + 2$ and thus, with the two marker constraints, has index $M(p) - M(q) + 2\langle c_1(T\Sigma), A\rangle$. Equation (10.1) has Fredholm index 0. Notice that a holomorphic cylinder in $\mathbb{R} \times Y$ comes in a $\mathbb{R}$-parametric family obtained by translation in $\mathbb{R} \times Y$. Each translate gives rise to the same Floer solution, so we must consider the quotient by this action. Therefore, the index of the Floer equation is 1 less than the sum of the index of the holomorphic curve equation with that of (10.1).

Case (3) is different in that the split Floer cylinder goes from a critical point $x \in W$ to $\check{p}_k \in \mathbb{R} \times Y$. Notice that as a split Floer cylinder, this has $N = 2$, and thus will need to be quotiented by an $\mathbb{R}^2$ action, hence the Fredholm index must be 2. This will appear in the differential when the grading difference of x and of $\check{p}_k$ is 1: $1 = M(p) + 1 - 2n + M(x) + 2\frac{\tau_X - K}{K}k$. The lower level of this is a plane in W, which we interpret as a sphere in X, representing the spherical homology class B. We have $k = B \bullet \Sigma = K\omega(B)$. The Fredholm index of the plane W with asymptotic marker condition is given by $M(p) + M(x) + 1 - 2n + 2\langle c_1(TX), B\rangle - 2B \bullet \Sigma$. Notice that this is the grading difference between $\check{p}_k$ and x. The Fredholm index of Equation (10.1) is 2, with no marker condition. The marker condition cuts its dimension down by 1, giving us the index 2 required.

10.1. **Geometric interpretation.** Propositions 9.1 and 10.1 play a very important role in our description of the symplectic homology differential, so we will summarize their geometric meaning.

Recall that by Propositions 7.4 and 7.5, there are four types of contributions to the differential, referred to as Cases 0, 1, 2 and 3.

Case 0 configurations are gradient flow lines in Y, so they can be described by the Morse differential in Y.

Case 1 configurations are elements of 1-dimensional fiber products

$$W^s_{f_Y}(\widehat{q}) \times_{\mathrm{ev}} \widetilde{\mathcal{M}}^*_{H,1}(Y_k, Y_l; A; J_Y) \times_{\mathrm{ev}} W^u_{f_Y}(\check{p})$$

where $q, p \in \mathrm{Crit}(f_\Sigma)$ and $A \in H_2(\Sigma;\mathbb{Z})$. Recall that $\mathcal{M}^*_{H,1}(Y_k, Y_l; A; J_Y)$ is the space of parametrized unpunctured Floer cylinders $\tilde{v}$, going from orbits of multiplicity k (as $s \to -\infty$) to orbits of multiplicity l (as $s \to +\infty$), where $l > k$. The fiber product above has a free action of $\mathbb{R}$ by translations of $\tilde{v}$ in the s-direction, so it is a finite disjoint union of copies of $\mathbb{R}$. Applying Proposition 9.1 to such a Floer cylinder $\tilde{v}$, we get that $\tilde{v}$ can be expressed in terms of a pseudoholomorphic cylinder $\tilde{u}$ in $\mathbb{R} \times Y$ and a map $\mathbf{e}\colon \mathbb{R} \times S^1 \to \mathbb{R} \times S^1$ satisfying equation (9.1). If we require that the asymptotic limits of $\tilde{u}$ be the Reeb orbits underlying the asymptotic limits of $\tilde{v}$, we can say that the pair $(\tilde{u}, \mathbf{e})$ is unique up to an $\mathbb{R}$-shift on the domains of $\tilde{u}$ and $\mathbf{e}$.

If we now start with a pseudoholomorphic curve $\tilde{u}$ going from a Reeb orbit of multiplicity k to one of multiplicity l (like the one in the previous paragraph), Proposition 10.1 says that there is a Floer cylinder $\tilde{v}$ and a solution $\mathbf{e}$ of (10.1), such that $\tilde{u}$ can be expressed in terms of $\tilde{v}$ and $\mathbf{e}$. If we require that the Reeb orbits underlying the asymptotic limits of $\tilde{v}$ be the asymptotic limits of $\tilde{u}$, then the pair $(\tilde{v}, \mathbf{e})$ is unique. So, combining Propositions 9.1 and 10.1, we get a bijection between $\mathcal{M}^*_{H,1}(Y_k, Y_l; A; J_Y)$ and the space of J_Y-holomorphic cylinders in $\mathbb{R}\times Y$, going from Reeb orbits of multiplicity k to Reeb orbits of multiplicity l. In Section 11, we will relate such pseudoholomorphic cylinders with pseudoholomorphic spheres in Σ and meromorphic sections of line bundles over $\mathbb{C}P^1$.

Case 2 configurations are elements of 1-dimensional moduli spaces

$$W^s_{f_Y}(\hat{p}) \times_{\mathrm{ev}} \left(\mathcal{M}^*_0(B; J_W) \times_{\mathrm{ev}} \widetilde{\mathcal{M}}^*_{H,1}(Y_k, Y_l; 0; J_Y)\right) \times_{\mathrm{ev}} W^u_{f_Y}(\check{p})$$

where $p \in \mathrm{Crit}(f_\Sigma)$, $B \in H_2(X; \mathbb{Z})$. Recall that $\mathcal{M}^*_0(B; J_W)$ is a space of J_W-holomorphic planes U in W (which can be identified with a space of J_X-holomorphic spheres in X, by Lemma 2.8) and $\mathcal{M}^*_{H,1}(Y_k, Y_l; 0; J_Y)$ is a space of Floer cylinders $\tilde{v}\colon \mathbb{R}\times S^1\backslash\{P\} \to \mathbb{R}\times Y$ projecting to a point in Σ, asymptotic at $-\infty$ (resp. $+\infty$) to an orbit of multiplicity k (resp. l) and with a negative puncture at $P \in \mathbb{R}\times S^1$ that is asymptotic to a Reeb orbit of multiplicity $l-k$. The stable and unstable manifolds in the fiber product force $\tilde{v}$ to project to $p \in \Sigma$. Hence, U is asymptotic to a Reeb orbit of multiplicity $l-k = B \bullet \Sigma$ over p.

Applying Propositions 9.1 and 10.1 to $\tilde{v}$ now implies that $\mathcal{M}^*_{H,1}(Y_k, Y_l; 0; J_Y)$ can be identified with the space of J_Y-holomorphic cylinders in $\mathbb{R}\times Y$ that are multiple covers of a trivial cylinder, with negative punctures at $-\infty$ and P asymptotic to orbits of multiplicity k and $l-k$, respectively, and with a positive puncture at $+\infty$ asymptotic to an orbit of multiplicity l.

Finally, Case 3 configurations are elements of fiber products

$$W^u_{f_W}(x) \times_{\mathrm{ev}} \left(\mathcal{M}^*_{H,1}(B; J_W) \times_{\mathrm{ev}} \mathcal{M}^*_{H,1}(0; J_Y, J_W)\right) \times_{\mathrm{ev}} W^u_{f_Y}(\check{p})$$

where $p \in \mathrm{Crit}(f_\Sigma)$, $x \in \mathrm{Crit}(f_W)$ and $B \in H_2(X; \mathbb{Z})$. Here, $\mathcal{M}^*_{H,1}(B; J_W)$ is a space of J_W-holomorphic planes U in W that are asymptotic to a Reeb orbit of multiplicity $k = B\bullet\Sigma$ (recall that $H = 0$ in W after neck-stretching, so we can think of J_W-holomorphic planes as Floer cylinders that asymptote to constants at $-\infty$). $\mathcal{M}^*_{H,1}(0; J_Y, J_W)$ is a space of Floer cylinders $\tilde{v}$ in $\mathbb{R}\times Y$ that project to a point in Σ, asymptote at $+\infty$ to a Hamiltonian orbit of multiplicity k, and asymptote at $-\infty$ to a Reeb orbit of multiplicity k. The unstable manifold at the right of the fiber product forces $\tilde{v}$ to project to p in Σ. The orbit U asymptotes to must also project to p in Σ.

The last parts of Propositions 9.1 and 10.1 applied to $\tilde{v}$ now imply that $\mathcal{M}^*_{H,1}(0; J_Y, J_W)$ can be identified with the space of trivial J_Y-holomorphic spheres in $\mathbb{R}\times Y$, over Reeb orbits of multiplicity k.

11. Pseudoholomorphic spheres in Σ and meromorphic sections

The previous sections explain that split Floer cylinders are in some sense equivalent to J_Y-holomorphic curves. We now want to describe J_Y-holomorphic curves in a manner that is more suitable for computing the Floer differential.

Let $\tilde{u} = (a, u) : \mathbb{R} \times S^1 \backslash \Gamma \to \mathbb{R} \times Y$ be a J_Y-holomorphic curve, where Γ is either empty or the singleton $\{P\}$. Since J_Y is $\mathbb{R} \times S^1$-equivariant, it is a lift of an almost complex structure J_Σ in Σ, and one can project $\tilde{u}$ to obtain a J_Σ-holomorphic map $w : \mathbb{R} \times S^1 \backslash \Gamma \to \Sigma$. Since punctures of finite energy pseudoholomorphic curves in $\mathbb{R} \times Y$ are asymptotic to Reeb orbits in Y, and these are multiple covers of the fibres of $Y \to \Sigma$, w extends to a J_Σ-holomorphic map $\mathbb{C}P^1 \to \Sigma$ [MS04, Theorem 4.1.2]. We are using the standard identification of $\mathbb{R} \times S^1$ with $\mathbb{C}^* \subset \mathbb{C}P^1$.

The symplectization $\mathbb{R} \times Y$ can be identified with the complement of the zero section in a complex line bundle $E \to \Sigma$ (the dual to the normal bundle to Σ in X). We have the following commutative diagram

$$
\begin{tikzcd}
 & w^*(E\backslash 0) \arrow[r] \arrow[d] & \mathbb{R} \times Y = E\backslash 0 \arrow[d] \\
\mathbb{R} \times S^1\backslash\Gamma \arrow[r, hook] \arrow[urr, bend left, "{(a,u)}"] & \mathbb{C}P^1 \arrow[r, "w"] \arrow[u, dashed, bend left, "s"] & \Sigma
\end{tikzcd}
$$

where s is a section of $w^*E \to \mathbb{C}P^1$ with a zero at 0 (and at P, if $\Gamma = \{P\}$) and a pole at ∞. The line bundle w^*E has an induced complex linear Cauchy–Riemann operator, in the sense of [MS04, Section C.1]. Complex linearity is a consequence of the Reeb-invariance of J_Y. Such a Cauchy–Riemann operator corresponds to a unique holomorphic structure on w^*E [Kob87, Proposition 1.3.7]. The section s is in the kernel of this operator and is therefore meromorphic. This proves the following result.

Lemma 11.1. *Every pseudoholomorphic curve $\tilde{u} = (a, u) : \mathbb{R} \times S^1 \backslash \Gamma \to \mathbb{R} \times Y$ defines a J_Σ-holomorphic map $w : \mathbb{C}P^1 \to \Sigma$ and a meromorphic section of $w^*E \to \mathbb{C}P^1$, with zero at* 0 *(and at P, if $\Gamma = \{P\}$) and pole at ∞.*

We can now reduce the question of counting punctured J_Y-holomorphic maps $\tilde{u}$ to that of counting J_Σ-holomorphic curves $w : \mathbb{C}P^1 \to \Sigma$, together with meromorphic sections s of w^*E, with prescribed zeros and poles. The count of maps w is related with the computation of genus zero Gromov–Witten numbers of Σ [MS04]. We can also give a complete description of the relevant meromorphic sections s. Given a divisor of points D in $\mathbb{C}P^1$ and a holomorphic line bundle L over $\mathbb{C}P^1$, where both D and L have degree d, there is a $\mathbb{C}^*$-family of meromorphic sections of L, such that the divisor associated with each section is D. One can justify this fact by reducing it to the simplest case of trivial L: use a trivialization of L over $\mathbb{C} \subset \mathbb{C}P^1$ to identify meromorphic sections of L with meromorphic functions on $\mathbb{C}P^1$ [Mir95, pp 342–345].

Remark 11.2. Ignoring transversality issues, the fact that meromorphic sections of line bundles over $\mathbb{C}P^1$ come in $\mathbb{C}^*$-families implies that the contact homology differential (without point constraints) should vanish for prequantization bundles (see

[EGH00]), because the moduli spaces of non-constant pseudoholomorphic curves with cylindrical ends have an S^1-action without fixed points. This is not the case in our setting, though, because we impose marker conditions on our asymptotic limits (as in *non-equivariant* contact homology, see Section 3.2 of [BO09a]).

Fix a J_Σ-holomorphic map $w : \mathbb{C}P^1 \to \Sigma$ and let $\tilde{u} = (a, u) \colon \mathbb{R} \times S^1 \backslash \Gamma \to \mathbb{R} \times Y$ be a J_Y-holomorphic lift. Given $(\theta_1, \theta_2) \in S^1 \times S^1$, we can produce a new J_Y-holomorphic curve

$$(\theta_1, \theta_2).\tilde{u} = \tilde{u}_{(\theta_1,\theta_2)}$$

such that

$$\tilde{u}_{(\theta_1,\theta_2)}(s,t) = (a(s, t + \theta_1), \phi^{\theta_2} \circ u(s, t + \theta_1)).$$

Lemma 11.3. *The number of such lifts that have prescribed asymptotic markers at* $\pm\infty$ *is* $k_+ - k_- = -\langle c_1(E \to \Sigma), w_*[\mathbb{C}P^1]\rangle + \sum_{P\in\Gamma} k_P > 0$.

Proof. This result follows immediately from the observation that the discussion of the linear maps $\mathrm{rot}_{\tilde{v}}$ before Lemma 6.34 can be adapted to also to J_Y-holomorphic curves (instead of just Floer cylinders). The number of lifts we want is the absolute value of the degree of the appropriate linear map $T^2 \to T^2$, which is the absolute value of the determinant of the matrix in equation (6.7), i.e. $k_+ - k_- > 0$ (where the inequality follows from Lemma 6.34). The alternative formula for $k_+ - k_-$ in the statement follows from the consideration in the proof of Lemma 6.34. □

11.1. **Back to orientations.** We wish to compare the signs that appear in the differential to the signs involved in counts of pseudoholomorphic spheres (as in Gromov–Witten theory). It is useful to begin with some general properties of the fiber sum orientation, introduced in Definition 8.4. The first result is Proposition 7.5.(a) in [Joy12] (which has the same fiber sum orientation convention as Definition 8.4). We include a proof here because the structure of its argument will be the model for other proofs in this section.

Lemma 11.4. *Let* $f_i \colon V_1 \to W$ *be as in Definition 8.4. Consider the map* $r \colon V_1 \oplus V_2 \to V_2 \oplus V_1$*, such that* $r(v_1, v_2) = (v_2, v_1)$*. Then,* r *induces an identification*

$$\ker\big((f_1 - f_2) \colon V_1 \oplus V_2 \to W\big) \to \ker\big((f_2 - f_1) \colon V_2 \oplus V_1 \to W\big),$$

which changes the fiber sum orientations by $(-1)^{(\dim V_1+\dim W)(\dim V_2+\dim W)}$.

Proof. Call $f := (f_1 - f_2) \colon V_1 \oplus V_2 \to W$ and $g := (f_2 - f_1) \colon V_2 \oplus V_1 \to W$. Denoting also by f and g the appropriate isomorphisms in Part (1) of Definition 8.4, we have an isomorphism

$$g \circ r \circ f^{-1} \colon W \to W,$$

which is $-\mathrm{Id}$, hence it changes orientation by $(-1)^{\dim W}$.

Denote also by r the induced map $(V_1 \oplus V_2)/\ker f \to (V_2 \oplus V_1)/\ker g$. A simple calculation yields the following.

Claim. Let $s_{12} \colon (V_1 \oplus V_2)/\ker f \to V_1 \oplus V_2$ be a right-inverse for the projection $\pi_{12} \colon V_1 \oplus V_2 \to (V_1 \oplus V_2)/\ker f$. Then, $s_{21} := r \circ s_{12} \circ r^{-1} \colon (V_2 \oplus V_1)/\ker g \to V_2 \oplus V_1$ is a right-inverse for $\pi_{21} \colon V_2 \oplus V_1 \to (V_2 \oplus V_1)/\ker g$.

We can now justify the sign in the statement of the Lemma. By the Claim, the following diagram commutes:

$$\begin{tikzcd}
\ker f \bigoplus \big((V_1 \oplus V_2)/\ker f\big) \arrow[r, "i_{12}+s_{12}"] \arrow[d, "r\oplus r"] & V_1 \oplus V_2 \arrow[d, "r"] \\
\ker g \bigoplus \big((V_2 \oplus V_1)/\ker g\big) \arrow[r, "i_{21}+s_{21}"] & V_2 \oplus V_1
\end{tikzcd}$$

where i_{12} and i_{21} are inclusion maps.

By Part (1) in Definition 8.4 and the argument above the Claim, $r\colon (V_1 \oplus V_2)/\ker f \to (V_2 \oplus V_1)/\ker g$ changes orientations by $(-1)^{(\dim V_1 + \dim V_2 + 1)\dim W}$. By Part (2) in Definition 8.4 and the fact that $r\colon V_1 \oplus V_2 \to V_2 \oplus V_1$ changes orientations by $(-1)^{\dim V_1 \dim V_2}$, we get the remaining contribution to the sign in the Lemma. □

Definition 11.5. Let f_1, f_2 be as in Definition 8.4. Denote by $\Delta \subset W \oplus W$ the diagonal, and orient it as the image of the map $w \mapsto (w, w)$. The *preimage orientation* on the fiber sum $V_1\, {}_{f_1}\!\oplus_{f_2} V_2 = (f_1, f_2)^{-1}(\Delta) \subset V_1 \oplus V_2$ is so that, if $H \subset V_1 \oplus V_2$ is a complementary subspace to $(f_1, f_2)^{-1}(\Delta)$, the following isomorphisms preserve orientations

(1) $H \oplus (f_1, f_2)^{-1}(\Delta) \to (-1)^{\dim V_2(\dim V_1 + \dim W)} V_1 \oplus V_2$;
(2) $(f_1, f_2)(H) \oplus \Delta \to W \oplus W$.

Remark 11.6. If $g_1\colon X \to Y$ and $g_2\colon Z \to Y$ are transverse maps of smooth oriented manifolds, then the previous definition can be applied to maps of tangent spaces to orient the fiber product $X\, {}_{g_1}\!\times_{g_2} Z$. In the special case when g_2 is the inclusion of a submanifold, this agrees with the preimage orientation defined in [GP74]. This is the reason for the name used for this orientation convention, and for the complicated sign in Part (1) above.

Lemma 11.7. *The identification of* $\ker f$ *with the fiber sum orientation (as in Definition 8.4) and* $(f_1, f_2)^{-1}(\Delta)$ *with preimage orientation (as in Definition 11.5) changes orientations by* $(-1)^{(\dim V_1 + \dim W)(\dim V_2 + \dim W)}$.

Proof. For convenience, denote Parts (1) and (2) in Definition 11.5 by (P1) and (P2), respectively, and similarly denote by (F1) and (F2) the Parts in Definition 8.4. Note that to write (F2) in this case, we need a section $s\colon \frac{V_1 \oplus V_2}{\ker f} \to V_1 \oplus V_2$. We will take $H = s(\frac{V_1 \oplus V_2}{\ker f})$ in (P1).

The result follows from a sequence of oriented isomorphisms:

$$
\begin{aligned}
W \oplus W \oplus \ker f &\cong (-1)^{(\dim V_1+\dim V_2-\dim W)\dim W} W \oplus \ker f \oplus W \cong \\
&\overset{(F1)}{\cong} (-1)^{\dim V_1 \dim W + \dim W} W \oplus \ker f \oplus \frac{V_1 \oplus V_2}{\ker f} \cong \\
&\overset{(F2)}{\cong} (-1)^{\dim V_1 \dim W + \dim W} W \oplus V_1 \oplus V_2 \cong \\
&\overset{(P1)}{\cong} (-1)^{\dim V_1 \dim W + \dim V_1 \dim V_2 + \dim V_2 \dim W + \dim W} \cdot \\
&\qquad\qquad\qquad\qquad . W \oplus H \oplus (f_1, f_2)^{-1}(\Delta) \cong \\
&\cong (-1)^{\dim V_1 \dim W + \dim V_1 \dim V_2 + \dim V_2 \dim W + \dim W} \cdot \\
&\qquad\qquad\qquad\qquad . \Delta \oplus (f_1, f_2)(H) \oplus (f_1, f_2)^{-1}(\Delta) \cong \\
&\cong (-1)^{\dim V_1 \dim W + \dim V_1 \dim V_2 + \dim V_2 \dim W} \cdot \\
&\qquad\qquad\qquad\qquad . (f_1, f_2)(H) \oplus \Delta \oplus (f_1, f_2)^{-1}(\Delta) \cong \\
&\overset{(P2)}{\cong} (-1)^{\dim V_1 \dim W + \dim V_1 \dim V_2 + \dim V_2 \dim W} W \oplus W \oplus (f_1, f_2)^{-1}(\Delta) \cong \\
&\overset{\varphi}{\cong} (-1)^{\dim V_1 \dim W + \dim V_1 \dim V_2 + \dim V_2 \dim W + \dim W} W \oplus W \oplus (f_1, f_2)^{-1}(\Delta)
\end{aligned}
$$

where $\varphi = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \oplus \mathrm{id} \colon W \oplus W \oplus (f_1, f_2)^{-1}(\Delta) \to W \oplus W \oplus (f_1, f_2)^{-1}(\Delta)$. Note that φ changes orientations by $(-1)^{\dim W}$. The composition of the chain of isomorphisms above is of the form

$$
\begin{pmatrix} 2 & * \\ 0 & 1 \end{pmatrix} \oplus \mathrm{id} \colon W \oplus W \oplus \ker f \to W \oplus W \oplus (f_1, f_2)^{-1}(\Delta)
$$

so it preserves orientations iff the identity $\ker f \to (f_1, f_2)^{-1}(\Delta)$ does. The result now follows. □

Remark 11.8. Since the signs appearing in Lemmas 11.4 and 11.7 are the same, the order-reversing map r in Lemma 11.4 induces an orientation-preserving isomorphism between $\ker(f_1 - f_2)$ with the fiber sum orientation and $(f_2, f_1)^{-1}(\Delta)$ with the preimage orientation (note the order of the f_i).

Let us now study the signs that appear in the split symplectic homology differential. We start with Case 1 configurations. Let $p, q \in \mathrm{Crit}(f_\Sigma)$ and consider a Floer cylinder with one cascade giving a contribution of $\hat{q}_k$ to the differential of $\check{p}_l$. As we saw in (8.2), these cascades form spaces

$$
\mathcal{M}(\hat{q}_k, \check{p}_l; A) := W^s_{f_Y}(\hat{q}) \times_{\mathrm{ev}} \widetilde{\mathcal{M}}^*_{H,1}(Y_k, Y_l; A; J_Y) \times_{\mathrm{ev}} W^u_{f_Y}(\check{p}) \tag{11.1}
$$

for suitable $A \in H_2(\Sigma; \mathbb{Z})$. Each such cascade projects to a chain of pearls in Σ from q to p, with exactly one sphere, contained in

$$
\mathcal{M}(q, p; A) := W^s_{f_\Sigma}(q) \times_{\mathrm{ev}} \mathcal{M}^*_2(A; J_\Sigma) \times_{\mathrm{ev}} W^u_{f_\Sigma}(p). \tag{11.2}
$$

Note that $\mathcal{M}(q, p; A)$ can be identified with $\mathrm{ev}^{-1}\left(W^s_{f_\Sigma}(q) \times W^u_{f_\Sigma}(p)\right)$, for

$$
\begin{aligned}
\mathrm{ev} \colon \mathcal{M}^*_2(A; J_\Sigma) &\to \Sigma \times \Sigma \\
w &\mapsto (w(0), w(\infty))
\end{aligned}
$$

Definition 11.9. The *Gromov–Witten orientation* on $\mathcal{M}(q,p;A)$ is the one induced by the identification with $\mathrm{ev}^{-1}\left(W^s_{f_\Sigma}(q) \times W^u_{f_\Sigma}(p)\right)$, where the latter is equipped with the preimage orientation.

Remark 11.10. To understand the reason for this terminology, suppose that these stable and unstable manifolds defined cycles in $H_*(\Sigma)$. An element in $\mathcal{M}(q,p;A)$ would then contribute to a two-point Gromov–Witten invariant of Σ, with insertions given by the stable and unstable manifolds. The sign of this contribution would be the the one prescribed by the Gromov–Witten orientation (see [MS04, Exercise 7.1.2]).

Our goal is to compare

- the sign of the contribution of a Floer cylinder with cascades in (11.1) to the symplectic homology differential
- to the sign induced on the projected chain of pearls in (11.2) by the Gromov–Witten orientation.

This will be the content of Proposition 11.15. To compare the two signs, it will be useful to also consider the fiber products

$$\mathcal{M}(\check{q}_k,\widehat{p}_l;A) := W^s_{f_Y}(\check{q}) \times_{\mathrm{ev}} \widetilde{\mathcal{M}}^*_{H,1}(Y_k,Y_l;A;J_Y) \times_{\mathrm{ev}} W^u_{f_Y}(\widehat{p}) \tag{11.3}$$

and

$$\overline{\mathcal{M}(\check{q}_k,\widehat{p}_l;A)} := \pi_\Sigma^{-1}\left(W^s_{f_\Sigma}(q)\right) \times_{\mathrm{ev}} \widetilde{\mathcal{M}}^*_{H,1}(Y_k,Y_l;A;J_Y) \times_{\mathrm{ev}} \pi_\Sigma^{-1}\left(W^u_{f_\Sigma}(p)\right), \tag{11.4}$$

which contains $\mathcal{M}(\check{q}_k,\widehat{p}_l;A)$ as an open dense subset (which justifies the notation).

$W^s_{f_Y}(\check{q})$ is an open subset of $\pi_\Sigma^{-1}\left(W^s_{f_\Sigma}(q)\right)$, which is an S^1-bundle over $W^s_{f_\Sigma}(q)$. Note also that $W^s_{f_Y}(\widehat{q})$ specifies a section of this bundle, which we use to trivialize $\pi_\Sigma^{-1}\left(W^s_{f_\Sigma}(q)\right) \cong W^s_{f_\Sigma}(q) \times S^1$ (identifying $W^s_{f_Y}(\widehat{q})$ with $W^s_{f_\Sigma}(q) \times \{1\}$). We can now write $W^s_{f_Y}(\widehat{q}) = \pi_\Sigma^{-1}\left(W^s_{f_\Sigma}(q)\right) {}_{\check{s}}\times_i \{1\}$, where the maps in the fiber product are $\check{s}\colon \pi_\Sigma^{-1}\left(W^s_{f_\Sigma}(q)\right) \to S^1$ and $i\colon \{1\} \hookrightarrow S^1$. Similarly, we can write $W^u_{f_Y}(\check{p}) = \pi_\Sigma^{-1}\left(W^u_{f_\Sigma}(p)\right) {}_{\hat{u}}\times_i \{1\}$ for the analogous map $\hat{u}\colon \pi_\Sigma^{-1}\left(W^u_{f_\Sigma}(p)\right) \to S^1$.

Note that $\mathcal{M}(\widehat{q}_k,\check{p}_l;A)$ is a codimension 2 submanifold of $\overline{\mathcal{M}(\check{q}_k,\widehat{p}_l;A)}$. Given an element $(x,u,y) \in \mathcal{M}(\widehat{q}_k,\check{p}_l;A)$, there is a splitting

$$T_{(x,u,y)}\overline{\mathcal{M}(\check{q}_k,\widehat{p}_l;A)} \cong \left(T_{(x,u,y)}\mathcal{M}(\widehat{q}_k,\check{p}_l;A)\right) \oplus \mathbb{R}_{\mathrm{domain}} \oplus \mathbb{R}_{\mathrm{target}} \tag{11.5}$$

where $\mathbb{R}_{\mathrm{domain}}$ is the direction spanned by an infinitesimal rotation of the Floer cylinder u on the domain (in the t-direction) and $\mathbb{R}_{\mathrm{target}}$ is spanned by the infinitesimal rotation of u on the target (in the Reeb direction).

Lemma 11.11. *If we take the fiber product orientation on* (11.1) *and* (11.4)*, then the splitting* (11.5) *preserves orientations....*

Proof. Using the associativity of the fiber product orientation and Lemma 11.4 on the third identity, we have the oriented diffeomorphisms (dropping A and J_Y from the notation)

$$
\begin{aligned}
\mathcal{M}(\widehat{q}_k,\check{p}_l) &= W^s_{f_Y}(\widehat{q}) \times_{\mathrm{ev}} \widetilde{\mathcal{M}}^*_{H,1}(Y_k,Y_l) \times_{\mathrm{ev}} W^u_{f_Y}(\check{p}) = \\
&= \left(\pi_\Sigma^{-1}\left(W^s_{f_\Sigma}(q)\right) {}_{\check{s}}\times_i \{1\}\right) \times_{\mathrm{ev}} \widetilde{\mathcal{M}}^*_{H,1}(Y_k,Y_l) \times_{\mathrm{ev}} \left(\pi_\Sigma^{-1}\left(W^u_{f_\Sigma}(p)\right) {}_{\hat{u}}\times_i \{1\}\right) = \\
&= (-1)^{\dim \pi_\Sigma^{-1}\left(W^s_{f_\Sigma}(q)\right)+1}\cdot \\
&\qquad \cdot \{1\} {}_i\times_{\check{s}} \left(\pi_\Sigma^{-1}\left(W^s_{f_\Sigma}(q)\right) \times_{\mathrm{ev}} \widetilde{\mathcal{M}}^*_{H,1}(Y_k,Y_l) \times_{\mathrm{ev}} \pi_\Sigma^{-1}\left(W^u_{f_\Sigma}(p)\right)\right) {}_{\hat{u}}\times_i \{1\} = \\
&= (-1)^{\mathrm{ind}_{f_\Sigma}(q)} \{1\} {}_i\times_{\check{s}} \overline{\mathcal{M}(\check{q}_k,\widehat{p}_l)} {}_{\hat{u}}\times_i \{1\}
\end{aligned}
$$

and we get an oriented isomorphism of vector spaces

$$
T_{(x,u,y)}\mathcal{M}(\widehat{q}_k,\check{p}_l;A) \cong (-1)^{\mathrm{ind}_{f_\Sigma}(q)} 0 {}_{di}\oplus_{d\check{s}} \left(T_{(x,u,y)}\overline{\mathcal{M}(\check{q}_k,\widehat{p}_l;A)}\right) {}_{d\hat{u}}\oplus_{di} 0
$$

(where we wrote 0 instead of $T_{\{1\}}\{1\}$).

On the other hand, abbreviating $\mathbb{R}^2 := \mathbb{R}_{\mathrm{domain}} \oplus \mathbb{R}_{\mathrm{target}}$ and thinking of the direct sum as a fiber product (of maps to the zero vector space), we get

$$
\begin{aligned}
0 {}_{di}\oplus_{d\check{s}} & \left(\left(T_{(x,u,y)}\mathcal{M}(\widehat{q}_k,\check{p}_l;A)\right) \oplus \mathbb{R}^2\right) {}_{d\hat{u}}\oplus_{di} 0 = \\
&= 0 {}_{di}\oplus_{d\check{s}} \left(T_{(x,u,y)}\mathcal{M}(\widehat{q}_k,\check{p}_l;A) \oplus (\mathbb{R}^2 {}_{d\hat{u}}\oplus_{di} 0)\right) = \\
&= (-1)^{\dim \mathcal{M}(\widehat{q}_k,\check{p}_l;A)} 0 {}_{di}\oplus_{d\check{s}} \left((\mathbb{R}^2 {}_{d\hat{u}}\oplus_{di} 0) \oplus T_{(x,u,y)}\mathcal{M}(\widehat{q}_k,\check{p}_l;A)\right) = \\
&= -\left(0 {}_{di}\oplus_{d\check{s}} \mathbb{R}^2 {}_{d\hat{u}}\oplus_{di} 0\right) \oplus T_{(x,u,y)}\mathcal{M}(\widehat{q}_k,\check{p}_l;A) = \\
&= T_{(x,u,y)}\mathcal{M}(\widehat{q}_k,\check{p}_l;A).
\end{aligned}
$$

The first identity above uses the associativity of the fiber product, the second uses Lemma 11.4, the third uses the fact that $\dim \mathcal{M}(\widehat{q}_k,\check{p}_l;A) = 1$ (since we are studying contributions to the differential) and again the associativity property. The last identity follows from the fact that the fiber sum orientation on the zero dimensional vector space $0 {}_{di}\oplus_{d\check{s}} \mathbb{R}^2 {}_{d\hat{u}}\oplus_{di} 0$ is negative, which is a simple calculation.

The Lemma follows from combining the previous two computations. □

In what follows, it is useful to work from a slightly more abstract point of view. Let $\pi_A\colon \tilde{A} \to A$ and $\pi_B\colon \tilde{B} \to B$ be S^1-bundles. Suppose that there are S^1-equivariant maps $\tilde{f}_1\colon \tilde{A} \to Y$ and $\tilde{f}_2\colon \tilde{B} \to Y$, and denote their projections by $f_1\colon A \to \Sigma$ and $f_2\colon B \to \Sigma$. Then, the fiber product $\tilde{A} {}_{\tilde{f}_1}\times_{\tilde{f}_2} \tilde{B}$ is an S^1-bundle over $A {}_{f_1}\times_{f_2} B$ (with 'diagonal' S^1-action on $\tilde{A} {}_{\tilde{f}_1}\times_{\tilde{f}_2} \tilde{B}$).

Recall that the contact form on Y induces an Ehresmann connection on $\pi_\Sigma\colon Y \to \Sigma$. The S^1-equivariance of $\tilde{f}_1$ implies that there is a unique Ehresmann connection on $\tilde{A}$ inducing the top arrow in the commutative diagram

$$
\begin{tikzcd}
T_a A \oplus \mathbb{R} \arrow[r] \arrow[d, "df_1 \oplus \mathrm{id}"'] & T_{\tilde{a}}\tilde{A} \arrow[d, "d\tilde{f}_1"] \\
T_{f_1(a)}\Sigma \oplus \mathbb{R} \arrow[r] & T_{\tilde{f}_1(\tilde{a})} Y
\end{tikzcd}
\tag{11.6}
$$

where $\tilde{a} \in \tilde{A}$ and $a = \pi_A(\tilde{a})$. Do the same for $\tilde{B}$.

These connections induce a connection on the S^1-bundle $\tilde{A} {}_{\tilde{f}_1}\times_{\tilde{f}_2} \tilde{B}$ as follows. Given $(\tilde{a},\tilde{b}) \in \tilde{A} {}_{\tilde{f}_1}\times_{\tilde{f}_2} \tilde{B}$, we can take the horizontal subspace $H_{\tilde{a}} {}_{d(\pi_\Sigma\circ\tilde{f}_1)}\oplus_{d(\pi_\Sigma\circ\tilde{f}_2)} H_{\tilde{b}}$.

This can be identified with $T_{(a,b)}(A\,{}_{f_1}\!\times_{f_2} B)$, via the restriction of (π_A, π_B) to the fiber sum. This gives an identification

$$T_{(\tilde{a},\tilde{b})}(\tilde{A}\,{}_{\tilde{f}_1}\!\times_{\tilde{f}_2} \tilde{B}) \cong \big(T_{(a,b)}(A\,{}_{f_1}\!\times_{f_2} B)\big) \oplus \mathbb{R}, \tag{11.7}$$

where $\mathbb{R}$ stands for the vector space generated by the infinitesimal generator of the S^1-action on $\tilde{A}\,{}_{\tilde{f}_1}\!\times_{\tilde{f}_2} \tilde{B}$. We rewrite (11.7) as an isomorphism

$$\chi\colon \ker(df_1 - df_2)_{(a,b)} \oplus \mathbb{R} \to \ker(d\tilde{f}_1 - d\tilde{f}_2)_{(\tilde{a},\tilde{b})}.$$

Pick an orientation on A and orient $\tilde{A}$ so that the splitting $T_{\tilde{a}}\tilde{A} \cong T_a A \oplus \mathbb{R}$ induced by the connection on $\tilde{A}$ preserves orientations. Pick an orientation on B and orient $\tilde{B}$ in an analogous manner.

Lemma 11.12. *The identification* (11.7) *preserves orientations, if we use the fiber product orientation on both fiber products, and orient* $\mathbb{R}$ *in the direction of the infinitesimal* S^1*-action on* $\tilde{A}\,{}_{\tilde{f}_1}\!\times_{\tilde{f}_2} \tilde{B}$.

Proof. The structure of the argument is parallel to that of Lemma 11.4.

Fix $(\tilde{a},\tilde{b}) \in \tilde{A}\,{}_{\tilde{f}_1}\!\times_{\tilde{f}_2} \tilde{B}$, so that $\pi_A(\tilde{a}) = a$ and $\pi_B(\tilde{b}) = b$. Write $d\tilde{f} := (d\tilde{f}_1 - d\tilde{f}_2)\colon T\tilde{A} \oplus T\tilde{B} \to TY$ and $df := (df_1 - df_2)\colon TA \oplus TB \to T\Sigma$, where we omited subscripts $a, b, \tilde{a}, \tilde{b}$ to make the notation somewhat lighter. We will do this throughout the proof. Denote by $\xi\colon T\Sigma \oplus \mathbb{R} \to TY$ the isomorphism induced by the contact structure on Y. The commutative diagram

$$\begin{tikzcd} \dfrac{TA \oplus TB}{\ker(df)} \oplus \mathbb{R} \arrow[r, "{[df] \oplus \mathrm{id}}"] \arrow[d, "\varphi"] & T\Sigma \oplus \mathbb{R} \arrow[d, "\xi"] \\ \dfrac{T\tilde{A} \oplus T\tilde{B}}{\ker(d\tilde{f})} \arrow[r, "{[d\tilde{f}]}"] & TY \end{tikzcd}$$

defines the isomorphism φ.

Denote by ψ the composition $TA \oplus TB \oplus \mathbb{R} \oplus \mathbb{R} \to TA \oplus \mathbb{R} \oplus TB \oplus \mathbb{R} \to T\tilde{A} \oplus T\tilde{B}$, where the first isomorphism permutes the second and third factors and the second isomorphism is induced by the connections on $\tilde{A}$ and $\tilde{B}$.

Claim. Let $s\colon \frac{TA \oplus TB}{\ker(df)} \to TA \oplus TB$ be a right-inverse for the projection $\pi\colon TA \oplus TB \to \frac{TA \oplus TB}{\ker(df)}$. Then, $\tilde{s} := \psi \circ (s \oplus (1,0)) \circ \varphi^{-1}\colon \frac{T\tilde{A} \oplus T\tilde{B}}{\ker(d\tilde{f})} \to T\tilde{A} \oplus T\tilde{B}$ is a right-inverse for $\tilde{\pi}\colon T\tilde{A} \oplus T\tilde{B} \to \frac{T\tilde{A} \oplus T\tilde{B}}{\ker(d\tilde{f})}$.

The claim follows from the fact that the following diagram commutes, which is a simple calculation (using (11.6) and its analogue for $\tilde{B}$).

$$\begin{tikzcd} TA \oplus TB \oplus \mathbb{R} \oplus \mathbb{R} \arrow[r, "{\pi \oplus \begin{pmatrix} 1 & -1 \end{pmatrix}}"] \arrow[d, "\psi"] & \dfrac{TA \oplus TB}{\ker(df)} \oplus \mathbb{R} \arrow[d, "\varphi"] \\ T\tilde{A} \oplus T\tilde{B} \arrow[r, "\tilde{\pi}"] & \dfrac{T\tilde{A} \oplus T\tilde{B}}{\ker(d\tilde{f})} \end{tikzcd}$$

We can now justify the sign in the statement of the Lemma. The Claim can be used to show that the following diagram commutes:

$$\begin{tikzcd}
\big(\ker(df)\oplus\mathbb{R}\big)\bigoplus\left(\dfrac{TA\oplus TB}{\ker(df)}\oplus\mathbb{R}\right) \arrow[r, "i+\alpha+s+\beta"] \arrow[d, "\chi\oplus\varphi"] & TA\oplus TB\oplus\mathbb{R}\oplus\mathbb{R} \arrow[d, "\psi"] \\
\ker(d\tilde{f})\bigoplus\dfrac{T\tilde{A}\oplus T\tilde{B}}{\ker(d\tilde{f})} \arrow[r, "\tilde{i}+\tilde{s}"] & T\tilde{A}\oplus T\tilde{B}
\end{tikzcd}$$

where $\alpha\oplus\beta\colon\mathbb{R}\oplus\mathbb{R}\to\mathbb{R}^2$ is the linear map represented by $\begin{pmatrix}1&1\\1&0\end{pmatrix}$.

By Part (1) in Definition 8.4, φ changes orientations by $(-1)^{\dim B\dim\Sigma+\dim\tilde{B}\dim Y}=(-1)^{\dim B+1}$. The map ψ changes orientations by $(-1)^{\dim B}$ (the sign comes from the permutation, since the orientations on $\tilde{A}$ and $\tilde{B}$ behave well with the connections). Note that to get the map $\alpha\oplus\beta$ (which has negative determinant), we need to permute $\mathbb{R}$ with $\frac{TA\oplus TB}{\ker(df)}$ at the top of the diagram. The permutation picks up no sign, since $\dim\frac{TA\oplus TB}{\ker(df)}=\dim\Sigma$ is even. By Part (2) in Definition 8.4 and the commutativity of the diagram, we conclude that the isomorphism χ changes orientations by $(-1)^{\dim B+1+\dim B+1}=1$, as wanted. □

The space (11.4) is an S^1-bundle over (11.2). With respect to the connections discussed above (see (11.6)), we have a decomposition

$$T_{(x,u,y)}\overline{\mathcal{M}(\check{q}_k,\widehat{p}_l;A)}\cong\big(T_{(\pi_\Sigma(x),\pi_\Sigma\circ u,\pi_\Sigma(y))}\mathcal{M}(q,p;A)\big)\oplus\mathbb{R}\tag{11.8}$$

Lemma 11.13. *The isomorphism* (11.8) *preserves orientations, if the fiber products on the left and right are given their fiber product orientations.*

Proof. This follows from applying the Lemma 11.12 twice, to the S^1-bundles $\pi_\Sigma^{-1}\left(W^s_{f_\Sigma}(q)\right)\to W^s_{f_\Sigma}(q)$, $\widetilde{\mathcal{M}}^*_{H,1}(Y_k,Y_l;A;J_Y)\to\mathcal{M}^*_2(A;J_\Sigma)$ and $\pi_\Sigma^{-1}\left(W^u_{f_\Sigma}(p)\right)\to W^u_{f_\Sigma}(p)$. Note that Lemma 8.2 implies that the coherent orientation on $\tilde{B}=\widetilde{\mathcal{M}}^*_{H,1}(Y_k,Y_l;A;J_Y)$ and the usual complex orientation on $B=\mathcal{M}^*_2(A;J_\Sigma)$ are such that the identification $T_{\tilde{b}}\tilde{B}\cong T_bB\oplus\mathbb{R}$ is orientation-preserving, as was assumed in Lemma 11.12 (see the paragraph above the Lemma). □

Lemma 11.14. *The fiber product orientation on* (11.2) *differs from the preimage orientation on* (11.2) *by* $(-1)^{\dim W^s_{f_\Sigma}(q)\dim W^u_{f_\Sigma}(p)}$.

Proof. This follows from applying Lemma 11.7 twice. Writing (11.2) as

$$(W^s_{f_\Sigma}(q)\times_{\mathrm{ev}}\mathcal{M}^*_2(A;J_\Sigma))\times_{\mathrm{ev}}W^u_{f_\Sigma}(p)$$

we apply Lemma 11.7 to $W^s_{f_\Sigma}(q)\hookrightarrow\Sigma$ and $\mathrm{ev}_1\colon\mathcal{M}^*_2(A;J_\Sigma)\to\Sigma$ to get a sign

$$(-1)^{(\dim W^s_{f_\Sigma}(q)+\dim\Sigma)(\dim\mathcal{M}^*_2(A;J_\Sigma)+\dim\Sigma)}=1$$

since $\mathcal{M}^*_2(A;J_\Sigma)$ and Σ are even dimensional. We then apply Lemma 11.7 to $\mathrm{ev}_2\colon W^s_{f_\Sigma}(q)\times_{\mathrm{ev}}\mathcal{M}^*_2(A;J_\Sigma)\to\Sigma$ and $W^u_{f_\Sigma}(p)\hookrightarrow\Sigma$ to get the sign

$$\begin{aligned}&(-1)^{\dim(W^s_{f_\Sigma}(q)\times_{\mathrm{ev}}\mathcal{M}^*_2(A;J_\Sigma)+\dim\Sigma)(\dim W^u_{f_\Sigma}(p)+\dim\Sigma)}\\&\qquad=(-1)^{\dim W^s_{f_\Sigma}(q)\dim W^u_{f_\Sigma}(p)}.\end{aligned}$$

□

Combining these results, we get the following.

Proposition 11.15. *The sign with which an element of* (11.1) *contributes to the symplectic homology differential agrees with the sign with which the projection of this element (to an element in* (11.2)*) would contribute to a Gromov–Witten invariant in* Σ.

Proof. Combining Lemmas 11.11 and 11.13, we can relate the fiber product orientations on (11.1) and (11.2):

$$\begin{aligned}\big(T_{(x,u,y)}\mathcal{M}(\widehat{q}_k, \check{p}_l; A)\big) \oplus \mathbb{R}_{\text{domain}} \oplus \mathbb{R}_{\text{target}} &\cong T_{(x,u,y)}\overline{\mathcal{M}(\check{q}_k, \widehat{p}_l; A)} \cong \\ &\cong \big(T_{(\pi_\Sigma(x), \pi_\Sigma \circ u, \pi_\Sigma(y))}\mathcal{M}(q,p;A)\big) \oplus \mathbb{R}\end{aligned}$$

The factors $\mathbb{R}_{target}$ on the left and $\mathbb{R}$ on the right can be identified, yielding an orientation-preserving isomorphism

$$\big(T_{(x,u,y)}\mathcal{M}(\widehat{q}_k, \check{p}_l; A)\big) \oplus \mathbb{R}_{\text{domain}} \cong T_{(\pi_\Sigma(x), \pi_\Sigma \circ u, \pi_\Sigma(y))}\mathcal{M}(q,p;A).$$

Observe that $\mathcal{M}(q,p;A)$ in (11.2) can be identified with

$$\mathcal{M}_2^*(A; J_\Sigma) \times_{\text{ev}} \big(W^s_{f_\Sigma}(q) \times W^u_{f_\Sigma}(p)\big). \tag{11.9}$$

The sign with which an element $(x,u,y) \in \mathcal{M}(\widehat{q}_k, \check{p}_l; A)$ contributes to the symplectic homology differential is the sign of the zero-dimensional vector space obtained by taking the quotient of $T_{(x,u,y)}\mathcal{M}(\widehat{q}_k, \check{p}_l; A)$ by translation in the s-variable in the domain of the Floer cylinder u. This oriented zero-dimensional vector space coincides with the quotient of $\big(T_{(x,u,y)}\mathcal{M}(\widehat{q}_k, \check{p}_l; A)\big) \oplus \mathbb{R}_{\text{domain}}$ by the infinitesimal $\mathbb{C}^*$-action on the domain of u. On the other hand, the sign with which $(\pi_\Sigma(x), \pi_\Sigma \circ u, \pi_\Sigma(y)) \in \mathcal{M}(q,p;A)$ would contribute to a Gromov–Witten invariant is the sign of the zero-dimensional vector space obtained as the quotient of $T_{(\pi_\Sigma(x), \pi_\Sigma \circ u, \pi_\Sigma(y))}\mathcal{M}(q,p;A)$ by the $\mathbb{C}^*$-action, where $\mathcal{M}(q,p;A)$ is identified with (11.9) with the preimage orientation. The result now follows from

Claim. If we equip (11.2) with the fiber product orientation and (11.9) with the preimage orientation, then the natural identification of the two spaces is orientation-preserving.

By Lemma 11.14, the fiber product and preimage orientations on (11.2) differ by $(-1)^{\dim W^s_{f_\Sigma}(q) \dim W^u_{f_\Sigma}(p)}$. But this is also the difference between the preimage orientations on (11.2) and on (11.9). The Claim now follows. □

Remark 11.16. The Claim in the previous proof is also a consequence of the following argument. If we equip (11.2) and (11.9) with their fiber product orientations, then their natural identification preserves orientations, by [Joy12, Proposition 7.5.(c)]. On the other hand, the fiber product and preimage orientations on (11.9) are the same, by Lemma 11.7.

There is an analogous argument for Case 2 contributions to the symplectic homology differential. Recall that they come from elements of fiber products

$$W^s_{f_Y}(\widehat{p}) \times_{\text{ev}} \big(\mathcal{M}_0(B; J_W) \times_{\text{ev}} \widetilde{\mathcal{M}}^*_{H,1}(Y_{k_-}, Y_{k_+}; 0; J_Y)\big) \times_{\text{ev}} W^u_{f_Y}(\check{p}), \tag{11.10}$$

which are given the fiber product orientation. An element in this fiber product has an underlying plane in $\mathcal{M}_0(B; J_W)$, which can be thought of as a sphere in $\mathcal{M}_0(B; J_X)$ that is asymptotic to $p \in \Sigma$. Denote by $\text{ev}\colon \mathcal{M}_0(B; J_X) \to \Sigma$ the evaluation map.

Definition 11.17. Given a generic point $pt \in \Sigma$, the preimage orientation on $\mathrm{ev}^{-1}(pt)$ is called its *Gromov–Witten orientation.*

Analogously to what was pointed out in Remark 11.10, this orientation determines the sign with which an element in $\mathrm{ev}^{-1}(p)$ contributes to a relative Gromov–Witten invariant of the pair (X, Σ), with a single insertion of the point p (chosen to be generic) in Σ, with order of contact $B \bullet \Sigma$.

Proposition 11.18. *The sign with which an element of* (11.10) *contributes to the symplectic homology differential agrees with the sign with which the underlying J_W-holomorphic plane in W contributes to a relative Gromov–Witten in (X, Σ), with one point constraint of order of contact $k_+ - k_- = B \bullet \Sigma$ at $p \in \Sigma$.*

Proof. The proof is analogous to that of Propostion 11.15. We have oriented isomorphisms

$$\begin{aligned}
&\big[W^s_{f_Y}(\widehat{p}) \times_{\mathrm{ev}} \big(\mathcal{M}_0(B; J_W) \times_{\mathrm{ev}} \widetilde{\mathcal{M}}^*_{H,1}(Y_{k_-}, Y_{k_+}; 0; J_Y)\big)\big] \times_{\mathrm{ev}} W^u_{f_Y}(\check{p}) \\
&\quad\cong \big[\big(\mathcal{M}_0(B; J_W) \times_{\mathrm{ev}} \widetilde{\mathcal{M}}^*_{H,1}(Y_{k_-}, Y_{k_+}; 0; J_Y)\big) \times_{ev} W^s_{f_Y}(\widehat{p})\big] \times_{\mathrm{ev}} W^u_{f_Y}(\check{p}) \\
&\quad\cong \big[\mathcal{M}_0(B; J_W) \times_{\mathrm{ev}} \big(\widetilde{\mathcal{M}}^*_{H,1}(Y_{k_-}, Y_{k_+}; 0; J_Y) \times_{ev} W^s_{f_Y}(\widehat{p})\big)\big] \times_{\mathrm{ev}} W^u_{f_Y}(\check{p}) \\
&\quad\cong \mathcal{M}_0(B; J_W) \times_{\mathrm{ev}} \big[\big(\widetilde{\mathcal{M}}^*_{H,1}(Y_{k_-}, Y_{k_+}; 0; J_Y) \times_{ev} W^s_{f_Y}(\widehat{p})\big) \times_{\mathrm{ev}} W^u_{f_Y}(\check{p})\big] \\
&\quad\cong (-1)^{1+M(p)} \mathcal{M}_0(B; J_W) \times_{\mathrm{ev}} \big[\widetilde{\mathcal{M}}^*_{H,1}(Y_{k_-}, Y_{k_+}; 0; J_Y) \times_{ev} \big(W^s_{f_Y}(\widehat{p}) \times W^u_{f_Y}(\check{p})\big)\big] \\
&\quad\cong (-1)^{M(p)} \big(\Delta_\Sigma \cap (W^s_{f_\Sigma}(p) \times W^u_{f_\Sigma}(p))\big) \mathcal{M}_0(B; J_W) \times_{\mathrm{ev}} \mathbb{R} \\
&\quad\cong (W^s_{f_\Sigma}(p) \cap W^u_{f_\Sigma}(p)) |\,\mathrm{ev}^{-1}(p)|\, \mathbb{R},
\end{aligned}$$

where we identified the space of Floer cylinders from $\widehat{p}$ to $\check{p}$ with $\mathbb{R}$ in the last two lines, with the orientation induced by s-translation on the domain (any two such cylinders differ by translation by a constant $s_0 \in \mathbb{R}$). The first isomorphism preserves orientations, by Lemma 11.4 ($\dim \big(\mathcal{M}_0(B; J_W) \times_{\mathrm{ev}} \widetilde{\mathcal{M}}^*_{H,1}(Y_{k_-}, Y_{k_+}; 0; J_Y)\big) + \dim Y$ is even). The second and third isomorphisms are orientation-preserving, by the associativity of the fiber product orientation. The sign in the fourth isomorphism comes from [Joy12, Proposition 7.5.(c)]. Note that Lemma 11.7, the fiber sum and preimage orientations on $\big[\widetilde{\mathcal{M}}^*_{H,1}(Y_{k_-}, Y_{k_+}; 0; J_Y) \times_{ev} \big(W^s_{f_Y}(\widehat{p}) \times W^u_{f_Y}(\check{p})\big)\big]$ agree. In the penultimate line, $\big(\Delta_\Sigma \cap (W^s_{f_\Sigma}(p) \times W^u_{f_\Sigma}(p))\big)$ stands for the sign of the intersection, which is the same as the sign of $(-1)^{M(p)}(W^s_{f_\Sigma}(p) \cap W^u_{f_\Sigma}(p))$, by Remark 11.6. In the last line, $|\,\mathrm{ev}^{-1}(p)|$ is the absolute value of the signed count of points in $\mathrm{ev}^{-1}(p)$. □

Finally, Case 3 contributions to the symplectic homology differential come from

$$W^u_{f_W}(x) \times_{\mathrm{ev}} \mathcal{M}^*_{H,1}(B; J_W) \times_{\mathrm{ev}} \mathcal{M}^*_{H,1}(Y_k, 0; J_Y) \times_{\mathrm{ev}} W^u_{f_Y}(\check{p}) \tag{11.11}$$

with the fiber product orientation. An element in the fiber product contains a pseudoholomorphic plane in $\mathcal{M}^*_2(B; J_W)$ (recall that $H = 0$ in W after stretching the neck), which can be thought of as a pseudoholomorphic sphere in $\mathcal{M}^*_2(B; J_W)$ with one constraint at $\psi\big(W^u_{f_W}(x)\big) \subset X$ and one constraint at $W^u_{f_\Sigma}(p) \subset \Sigma$ with order of contact $B \bullet \Sigma$. Recall from Lemma 2.8 that the map $\psi \colon W \to X$ is a diffeomorphism onto its image $X \backslash \Sigma$. Denote by $\mathrm{ev} \colon \mathcal{M}_2(B; J_X) \to X \times \Sigma$ the evaluation map.

Definition 11.19. The preimage orientation on $\mathrm{ev}^{-1}\Big(\psi\big(W^u_{f_W}(x)\big)\times W^u_{f_\Sigma}(p)\Big)$ is called its *Gromov–Witten orientation.*

If $W^u_{f_W}(x)\subset X$ and $W^u_{f_\Sigma}(p)\subset\Sigma$ represent homology classes, then the Gromov–Witten orientation determines the signs with which elements in $\mathrm{ev}^{-1}\Big(\psi\big(W^u_{f_W}(x)\big)\times W^u_{f_\Sigma}(p)\Big)$ contribute to a relative Gromov–Witten invariant.

Proposition 11.20. *The sign with which an element of* (11.11) *contributes to the symplectic homology differential agrees with the sign with which the underlying J_W-holomorphic plane in W contributes to a relative Gromov–Witten in (X,Σ), with one point constraint at $W^u_{f_W}(x)\subset X$ and one point constraint of order of contact $k=B\bullet\Sigma$ at $W^u_{f_\Sigma}(p)\subset\Sigma$.*

Proof. We have the oriented diffeomorphisms

$$\begin{aligned}
&W^u_{f_W}(x)\times_{\mathrm{ev}}\mathcal{M}^*_{H,1}(B;J_W)\times_{\mathrm{ev}}\mathcal{M}^*_{H,1}(0;J_Y,J_W)\times_{\mathrm{ev}}W^u_{f_Y}(\breve{p})\\
&\quad\cong(-1)^{\dim W^u_{f_Y}(\breve{p})}\big(W^u_{f_W}(x)\times_{\mathrm{ev}}\mathcal{M}^*_{H,1}(B;J_W)\times_{\mathrm{ev}}W^u_{f_Y}(\breve{p})\big)\times\mathbb{R}_2\\
&\quad\cong(-1)^{\dim W^u_{f_Y}(\breve{p})+\dim W^u_{f_W}(x)}\big(\mathcal{M}^*_{H,1}(B;J_W)\times_{\mathrm{ev}}\big(W^u_{f_W}(x)\times W^u_{f_Y}(\breve{p})\big)\big)\times\mathbb{R}_2
\end{aligned}$$

where we denoted by $\mathbb{R}_2$ the domain of the s-coordinate of a Floer cylinder in $\mathcal{M}^*_{H,1}(0;J_Y,J_W)$. We used the oriented diffeomorphism $\mathcal{M}^*_{H,1}(0;J_Y,J_W)\cong\mathbb{R}_2\times Y$, the fact that W and $\mathcal{M}^*_{H,1}(B;J_W)$ are even dimensional and [Joy12, Proposition 7.5.(a)&(c)]. Now, if we identify $\mathcal{M}^*_{H,1}(B;J_W)$ with $\mathcal{M}^*_2(B;J_W)$ and denote $\mathcal{M}^*_2(B;J_W)/\mathbb{C}^*$ by $\widetilde{\mathcal{M}}^*_2(B;J_W)$, then we also have the oriented diffeomorphisms

$$\begin{aligned}
&\big(\mathcal{M}^*_{H,1}(B;J_W)\times_{\mathrm{ev}}\big(W^u_{f_W}(x)\times W^u_{f_Y}(\breve{p})\big)\big)\\
&\quad\cong(-1)^{\dim W^u_{f_W}(x)+\dim W^u_{f_\Sigma}(p)}\Big(\widetilde{\mathcal{M}}^*_2(B;J_W)\times_{\mathrm{ev}}\big(W^u_{f_W}(x)\times W^u_{f_\Sigma}(p)\big)\Big)\times\mathbb{R}_1
\end{aligned}$$

where $\mathbb{R}_1$ is the domain of the s-coordinate of a Floer cylinder in $\mathcal{M}^*_{H,1}(B;J_W)$. By Lemma 11.7, the fiber sum and preimage orientations agree on $\widetilde{\mathcal{M}}^*_2(B;J_W)\times_{\mathrm{ev}}\Big(W^u_{f_W}(x)\times W^u_{f_\Sigma}(p)\Big)$. Combining the two calculations, we get an oriented diffeomorphism between

$$W^u_{f_W}(x)\times_{\mathrm{ev}}\mathcal{M}^*_{H,1}(B;J_W)\times_{\mathrm{ev}}\mathcal{M}^*_{H,1}(0;J_Y,J_W)\times_{\mathrm{ev}}W^u_{f_Y}(\breve{p})$$

with the fiber product orientation (which gives rigid contributions to the symplectic homology differential when we mod out by $\mathbb{R}_1\times\mathbb{R}_2$) and

$$\Big(\widetilde{\mathcal{M}}^*_2(B;J_W)\times_{\mathrm{ev}}\big(W^u_{f_W}(x)\times W^u_{f_\Sigma}(p)\big)\Big)\times\mathbb{R}_1\times\mathbb{R}_2$$

with the preimage orientation (which contributes to a relative Gromov–Witten invariant when we mod out by $\mathbb{R}_1\times\mathbb{R}_2$, according to Definition 11.19). The result now follows. $\square$

We now define some curve counts that will be useful when we write a formula for the symplectic homology differential.

Definition 11.21. Let $q,p\in\mathrm{Crit}(f_\Sigma)$ and $A\in H_2(\Sigma;\mathbb{Z})$. Define

$$n_A(q,p):=\#\mathcal{M}_A(q,p),$$

where

$$\mathcal{M}_A(q,p):=\mathcal{M}(q,p;A)/\mathbb{C}^*.$$

Here, $\mathcal{M}(q,p;A)$ is given the Gromov–Witten orientation (see Definition 11.9).

This makes sense when $\dim \mathcal{M}_A(p,q) = 2\langle c_1(T\Sigma), A\rangle + M(p) - M(q) = 0$ (see Lemma **??**).

Definition 11.22. Let $B \in H_2(X;\mathbb{Z})$ and define

$$n_B := \#\mathcal{M}_B,$$

where

$$\mathcal{M}_B := \mathrm{ev}^{-1}(pt)/\operatorname{Aut}(\mathbb{C}P^1, 0).$$

Here, $\mathrm{ev}^{-1}(pt)$ is as in Definition 11.17, and is given its Gromov–Witten orientation. [2]

This makes sense when $\dim \mathcal{M}_B = 2\langle c_1(TX), B\rangle - 2(B\bullet\Sigma) - 2 = 0$ (see Lemma **??**).

Definition 11.23. Let $p \in \operatorname{Crit}(f_\Sigma)$, $x \in \operatorname{Crit}(f_W)$ and $B \in H_2(X;\mathbb{Z})$. Define

$$n_B(x,p) := \#\mathcal{M}_B(x,p)$$

where

$$\mathcal{M}_B(x,p) := \mathrm{ev}^{-1}\left(\psi\big(W^u_{f_W}(x)\big) \times W^u_{f_\Sigma}(p)\right)/\mathbb{C}^*.$$

The formula makes sense when $\dim \mathcal{M}_B(p,x) = 2\langle c_1(TX), B\rangle - 2(B\bullet\Sigma) + M(p) + M(x) - 2n = 0$ (see proof of Lemma 7.3).

Remark 11.24. The coefficients $n_A(q,p)$ above can only be non-trivial if

$$\langle c_1(T\Sigma), A\rangle = (\tau_X - K)\,\omega(A) \leqslant n.$$

by the dimension formula for $\mathcal{M}_A(q,p)$. Similarly, the coefficients n_B can only be non-trivial if

$$\langle c_1(TX), B\rangle - B\bullet\Sigma = (1 - K/\tau_X)\langle c_1(TX), B\rangle = (\tau_X - K)\,\omega(B) = 1.$$

The coefficients $n_B(x,p)$ can only be non-zero if

$$\langle c_1(TX), B\rangle - B\bullet\Sigma = (1 - K/\tau_X)\langle c_1(TX), B\rangle = (\tau_X - K)\,\omega(B) \leqslant n.$$

These inequalities are useful when computing the symplectic homology differential, as we will see in an example below.

12. A formula for the symplectic homology differential

Recall that the symplectic chain complex (3.2) is

$$SC_*(W,H) = \left(\bigoplus_{k>0}\ \bigoplus_{p_k\in\operatorname{Crit}(f_\Sigma)} \mathbb{Z}\langle \check{p}_k, \hat{p}_k\rangle\right) \oplus \left(\bigoplus_{x\in\operatorname{Crit}(f_W)} \mathbb{Z}\langle x\rangle\right)$$

as an abelian group, where $f_\Sigma : \Sigma \to \mathbb{R}$ and $f_W : W \to \mathbb{R}$ are Morse functions, with associated Morse–Smale gradient-like vector fields Z_Σ and Z_W, respectively. Given $p \in \operatorname{Crit}(f_\Sigma)$, its stable and unstable manifolds ($W^s_\Sigma(p)$ and $W^u_\Sigma(p)$) were defined in (3.1). We use the analogous conventions to define critical manifolds for f_W and f_Y (recall that we also have a Morse–Smale pair (f_Y, Z_Y) in Y).

[2] We thank Viktor Fromm for observing that the correct constraint is a generic point in Σ, rather than a submanifold of possibly higher dimension.

Recall that, given $\tilde{p} \in \mathrm{Crit}(f_Y)$, its Morse differential is

$$\partial_{f_Y}(\tilde{p}) = \sum_{\substack{\tilde{q}\in\mathrm{Crit}(f_Y)\\ \mathrm{ind}_Y(\tilde{p})-\mathrm{ind}(\tilde{q})=1}} \#\mathcal{M}(\tilde{q},\tilde{p})\,\tilde{q}$$

where the coefficients are signed counts of elements in

$$\mathcal{M}(\tilde{q},\tilde{p}) = \big(W_Y^s(\tilde{q}) \cap W_Y^u(\tilde{p})\big)/\mathbb{R}.$$

We are now ready to write the differential on $SC_*(W,H)$.

Theorem 12.1. *The differential on* (3.2)*, denoted ∂, is as follows. Given $p \neq q \in \mathrm{Crit}(f_\Sigma)$, the coefficient of $\widehat{q}_l$ in $\partial\check{p}_k$ is*

$$\langle \partial(\check{p}_k), \widehat{q}_l\rangle = (k-l) \sum_{A\in H_2(\Sigma;\mathbb{Z})} \delta_{k-l,\langle c_1(N\Sigma),A\rangle}\, n_A(q,p) + \delta_{k,l}\,\langle \partial_{f_Y}(\check{p}_k), \widehat{q}_l\rangle$$

where $2\langle c_1(T\Sigma),A\rangle + M(p) - M(q) = 0$, ∂_{f_Y} is the Morse differential in Y and the $\delta_{*,*}$ are Kronecker deltas.

If $p \in \mathrm{Crit}(f_\Sigma)$, then

$$\langle \partial(\check{p}_k), \widehat{p}_l\rangle = (k-l) \sum_{B\in H_2(X;\mathbb{Z})} \delta_{k-l,(B\bullet\Sigma)}\, n_B$$

where $(1-K/\tau_X)\langle c_1(TX),B\rangle - 1 = 0$. If W is Weinstein, this term is trivial if $n \geqslant 3$ and the minimal Chern number of spheres in Σ is at least 2.

Given $p \in \mathrm{Crit}(f_\Sigma)$ and $x \in \mathrm{Crit}(f_W)$,

$$\langle \partial(\check{p}_k), x\rangle = k \sum_{B\in H_2(X;\mathbb{Z})} \delta_{k,(B\bullet\Sigma)}\, n_B(x,p)$$

where $2(1-K/\tau_X)\langle c_1(TX),B\rangle + M(p) + M(x) - 2n = 0$.

Given $x \in \mathrm{Crit}(f_W)$,

$$\partial(x) = \partial_{-f_W}(x),$$

where ∂_{-f_W} is the Morse differential in W with respect to the function $-f_W$.

Proof. The split Floer differential on (3.2) was introduced at the end of Section 4.2. Propositions 7.4 and 7.5 describe those split Floer cylinders with cascades that can contribute to the differential.

The two types of contributions in $\langle \partial(\check{p}_k), \widehat{q}_l\rangle$ are (1) and (0), respectively, in Proposition 7.4. The former are given by Floer cylinders in $\mathbb{R}\times Y$ and correspond to Case 1 in Sections 9 and 10. According to Propositions 9.1 and 10.1, counts of Floer cylinders in $\mathbb{R}\times Y$ are equivalent to counts of pseudoholomorphic cylinders in $\mathbb{R}\times Y$. By Lemma 11.1 and the discussion that follows, the latter are equivalent to counts of rigid pseudoholomorphic spheres in Σ (given by the $n_A(q,p)$), with additional factors $k-l$ in accordance with Lemma 11.3.

The contributions in $\langle \partial(\check{p}_k), \widehat{p}_l\rangle$ correspond to (2) in Proposition 7.4, namely split Floer cylinders with one augmentation. They correspond to Case 2 in Sections 9 and 10. We can appeal again to Propositions 9.1 and 10.1, to relate punctured Floer cylinders in $\mathbb{R}\times Y$ to counts of punctured pseudoholomorphic cylinders in $\mathbb{R}\times Y$. The latter are branched covers of trivial cylinders in this case, and project to constant spheres in Σ. The number of lifts of these constant spheres is $k-l$, again by Lemma 11.3 (in the notation of that Lemma, $k^- + k_P^- = k^+$ in this case, since the spheres in Σ are constant). The counts of augmentation planes in W, capping the punctures in these cylinders, are given by the numbers n_B, by Lemma

2.8. The comment about W Weinstein follows from Lemma 7.7 (and, by Lemma 12.5 below, can be interpreted as saying that a certain relative Gromov–Witten invariant of the pair (X, Σ) vanishes).

The contributions in $\langle \partial(\check{p}_k), x \rangle$ are the ones in Proposition 7.5, and consist of a split Floer cylinder in $\mathbb{R} \times Y$ with one end asymptotic to a Reeb orbit in Y, matching the asymptotics of a pseudohomorphic plane in W, which is in turn connected to a critical point of f_W by a flow line of Z_W. They correspond to Case 3 in Sections 9 and 10. We can appeal to Propositions 9.1 and 10.1, to relate the Floer cylinders in $\mathbb{R} \times Y$ to trivial pseudoholomorphic cylinders in $\mathbb{R} \times Y$. The component in W is a pseudoholomorphic cylinder $\tilde{v}_2 : \mathbb{R} \times S^1 \to W$ with a removable singularity at $-\infty$. At $+\infty$, the cylinder converges to a Reeb orbit of multiplicity k. Therefore, precomposing $\tilde{v}_2$ with constant rotations on the domain (as in the discussion following Lemma 6.34), we get k many maps with the correct asymptotic marker condition at $+\infty$. By Lemma 2.8, such $\tilde{v}_2$ correspond to pseudoholomorphic spheres in X with appropriate tangency condition to Σ. The contribution of these configurations to the differential will then be $k\, n_B(x, p)$, as wanted.

The only contributions to the differential of $x \in \mathrm{Crit}(f_W)$ are from the Morse differential, because W is exact and hence has no non-constant pseudoholomorphic spheres. □

Remark 12.2. For every $p \in \mathrm{Crit}(f_\Sigma)$ and $k > 0$, we have $\partial_{f_Y}(\hat{p}_k) = 0$. There are two flow lines of $-Z_Y$ from $\hat{p}_k$, which cancel one another.

Remark 12.3. At least if there were no contributions $\langle \partial(\check{p}_k), x \rangle$, it would be immediate from Theorem 12.1 that $\partial^2 = 0$.

12.1. **Relation to Gromov–Witten invariants.** All the coefficients contributing to the differential in Theorem 12.1 are either combinatorial or topological, except for the $n_A(q, p)$, the n_B and the $n_B(x, p)$. These are harder to determine, but can sometimes be related to Gromov–Witten invariants, absolute or relative.

We denote by $\mathrm{GW}^{\Sigma}_{0,k,A}(C_1, \ldots, C_k)$ the genus 0 Gromov–Witten invariant of (Σ, ω_Σ), counting pseudohomolorphic spheres in Σ in class $A \in H_2(\Sigma; \mathbb{Z})$ with k marked points constrained to go through representatives of the classes $C_i \in H_*(\Sigma; \mathbb{Z})$. To see a definition of these invariants when (Σ, ω_Σ) is monotone, see [MS04]. We also denote denote by $\mathrm{GW}^{(X,\Sigma)}_{0,k,(s_1,\ldots,s_l),B}(C_1, \ldots, C_k; D_1, \ldots, D_l)$ the genus 0 relative Gromov–Witten invariant of (X, Σ, ω), counting pseudohomolorphic spheres in X class $B \in H_2(X; \mathbb{Z})$ with k marked points constrained to go through representatives of the classes $C_i \in H_*(X; \mathbb{Z})$, and l additional marked points constrained to be tangent to Σ with order of contact s_j and go through representatives of the classes $D_j \in H_*(\Sigma; \mathbb{Z})$. For details on how to define these invariants, see [IP03, LR01, TZ14] (for an algebro-geometric approach, see [Li01]). Note that our assumptions that (X, ω) is monotone with monotonicity constant τ_X, that Σ is Poincaré-dual to $K\omega$ and that $\tau_X - K > 0$ imply that the tuple (X, Σ, ω) is *strongly semi-positive*, as in [TZ14, Definition 4.7].

Recall that a Morse function is *perfect* if the corresponding Morse differential vanishes. Furthermore, it is *lacunary* if it does not have critical points of consecutive Morse index.

Lemma 12.4. *If the Morse function f_Σ is perfect, then $W^s_\Sigma(p)$ and $W^u_\Sigma(p)$ represent classes in $H_*(\Sigma;\mathbb{Z})$ and*

$$n_A(q,p) = \mathrm{GW}^\Sigma_{0,2,A}\big([W^s_\Sigma(q)],[W^u_\Sigma(p)]\big).$$

If the Morse function f_W is also perfect, then $\psi\big(W^u_W(x)\big)$ represents a class in the image of the map $H_*(X\backslash\Sigma;\mathbb{Z}) \to H_*(X;\mathbb{Z})$ and

$$n_B(x,p) = \mathrm{GW}^{X,\Sigma}_{0,1,(B\bullet\Sigma),B}\big([\psi(W^u_W(x))];[W^u_\Sigma(p)]\big).$$

If f_Σ is lacunary, then

$$\langle \partial_{f_Y}(\check{p}_k), \widehat{q}_k\rangle = \big\langle c_1(N\Sigma), \big(W^s_\Sigma(q) \cap W^u_\Sigma(p)\big)\big\rangle$$

where $N\Sigma$ is the normal bundle to Σ in X.

Proof. For the first statement, the fact that the stable and unstable submanifolds of f_Σ define homology classes in Σ implies that the $n_A(q,p)$ are the Gromov–Witten invariants in the statement, and not merely numbers defined at the (Morse) chain level. The orientation conventions in Definition 11.21 are precisely the ones used in Gromov–Witten theory (recall Remark 11.10). The second statement is proven along similar lines. Note that the $W^u_W(x)$ define homology classes in W, but that the $W^s_W(x)$ (which we do not consider) would define relative classes in $H_*(W,Y;\mathbb{Z})$ instead. On a technical note, an admissible J_X as in Definition 2.4 satisfies the vanishing normal Nijenhuis tensor condition [TZ14, (4.6)], and is hence suitable for defining relative Gromov–Witten invariants.

We now discuss the last statement. For index reasons, the coefficients $\langle \partial_{f_Y}(\check{p}_k), \widehat{q}_k\rangle$ can only be non-zero if $\mathrm{ind}_{f_\Sigma}(p) - \mathrm{ind}_{f_\Sigma}(q) = 2$. Since f_Σ is lacunary, $W^s_\Sigma(q) \cap W^u_\Sigma(p)$, together with p and q, form a 2-sphere S. The restriction of $Y \to \Sigma$ to S is an S^1-bundle E over S, and (f_Y, Z_Y) restricts to a Morse–Smale pair (f_E, Z_E) on E. By the description of the Gysin sequence in Morse homology [Oan03, (3.26)], $\langle \partial_{f_E}(\check{p}), \widehat{q}\rangle$ is the Euler class of $E \to S$ integrated over S, which is exactly what we wanted to show. □

The following result does not need any hypotheses on the auxiliary Morse functions.

Lemma 12.5. *The count of augmentation planes*

$$n_B = \mathrm{GW}^{X,\Sigma}_{0,0,(B\bullet\Sigma),B}(\varnothing;[pt])$$

is the genus 0 relative Gromov–Witten invariant of (X,Σ), in class $B \in H_2(X,\mathbb{Z})$, with no point constraints in X and order of contact $B\bullet\Sigma$ to a fixed (generic) point in Σ.

The proof is as in the second case of Lemma 12.4.

Remark 12.6. When f_Σ is not a perfect Morse function, the stable and unstable manifolds of its critical points may not represent classes in $H_*(\Sigma;\mathbb{Z})$. In such cases, the $n_A(q,p)$ can still be thought of as *chain level Gromov–Witten numbers* of Σ. These are not invariants, and may thus be harder to compute. Similarly, if f_W is not perfect, the $n_B(x,p)$ are only chain level relative Gromov–Witten numbers.

We will see below an example in which we compute Gromov–Witten invariants of $(\mathbb{C}P^1\times\mathbb{C}P^1, \Delta, \omega_{FS}\oplus\omega_{FS})$ using the integral J, even though it is not 'cylindrical near Σ', as we require of admissible J_X in Definition 2.4. That is not a problem, though, since the numbers we are computing are invariants.

Remark 12.7. Computing Gromov–Witten invariants can be a very hard problem, both in the absolute and relative case, but it has been done for a large class of important examples, especially using tools from algebraic geometry. For absolute invariants, see for instance [Bea95, Zin14]. Gathmann wrote a computer program called `GROWI` to compute absolute and relative Gromov–Witten invariants of hypersurfaces in projective spaces [Gat].

Relative invariants are often harder to compute than absolute ones. Nevertheless, under certain assumptions, a relative invariant where all intersections with Σ are transverse (i.e. have order of tangency 1) can be related to an absolute invariant where one forgets Σ (in the spirit of the divisor axiom for absolute Gromov–Witten invariants) [McD08, HR13, TZ16].

It is also worth pointing out that the relative invariants of a tuple (X, Σ, ω) can in principle be obtained from the absolute invariants of X and of Σ (incluing more complicated insertions like ψ-classes, in general) [MP06]. Nevertheless, it can be computationally quite challenging to use this fact to compute relative invariants of (X, Σ, ω).

Remark 12.8. Since explicit computations of Gromov–Witten invariants are often easier in algebraic geometry than in symplectic geometry, we implicitly assume that the two definitions agree in geometric settings where they both make sense. For more details about the relation between symplectic and algebraic absolute Gromov–Witten invariants, see [Sie99, LT97].

Remark 12.9. Since the computation of relative Gromov–Witten invariants can be more difficult than that of absolute invariants, it is worth pointing out that the symplectic homology differential does not have contributions n_A when $n \geqslant 3$, the minimal Chern number of Σ is at least 2 and W is Weinstein, as we saw in Theorem 12.1. Also, if one were only interested in the *positive* (or *large-action*) symplectic homology of W [BO09a], then one would ignore the contributions $\langle \partial(\check{p}_k), x\rangle$, and hence would not need to determine the coefficients $n_B(x, p)$.

Remark 12.10. The following was pointed out to us by D. Pomerleano. We also thank R. Leclercq and M. Sandon for helpful conversations about their related work in progress. Theorem 12.1 may seem unsatisfactory since it relates the differential ∂, which is a chain level object depending on the model we use to compute symplectic homology, to counts of pseudoholomorphic spheres which, at least under the assumption of Lemma 12.4, are given by Gromov–Witten invariants, which are homological. The chain complex SC_* in (3.2) has a quotient complex, referred to in [BO09a] as SC_*^+, obtained by modding out generators coming from critical points of f_W. We can write SC_*^+ as a complex of the form

$$C_*(\Sigma)[t]t \oplus \big(C_*(\Sigma)[t]t\big)[1] \tag{12.1}$$

where $C_*(\Sigma)$ denotes the Morse complex of f_Σ and t^k encodes generators associated to Reeb orbits of multiplicity k. The variable t has degree $2\frac{\tau_X - K}{K}$, by (3.4) (which should also force us to take a global shift of $C_*(\Sigma)$ by $1-n$; we don't do this to avoid making the notation even heavier). The first summand contains the orbits decorated with $\check{}$ and the second summand, with the degree shift up by 1, contains the orbits decorated with $\hat{}$. Theorem 12.1 implies that the differential in (12.1) is given by a matrix

$$\partial^+ = \begin{pmatrix} d_\Sigma & \Delta \\ 0 & d_\Sigma \end{pmatrix}$$

where d_Σ is the Morse differential in Σ and $\Delta\colon C_*(\Sigma)[t]t \to C_*(\Sigma)[t]t$ is a chain map counting: flow lines in Σ connecting critical points of index difference 2, pseudohomolorphic spheres in Σ (the terms involving $n_A(p,q)$) and in X (the terms involving n_B). This means that SC_*^+ is a cone complex on Δ. There is an induced long exact sequence

$$\ldots \to H_*(\Sigma)[t]t[1] \to SH_*^+(W,H) \to H_*(\Sigma)[t]t \xrightarrow{[\Delta]} H_*(\Sigma)[t]t[1] \to \ldots$$

with connecting homomorphism $[\Delta]$ (we are not being careful keeping track of the degrees $*$ in the sequence). This can be thought of as the long exact sequence in [BO09a], in the special case when the Liouville manifold is the completion of $X\backslash\Sigma$. It is also related to the spectral sequence in [Sei08, (3.2)].

We can think of this exact sequence as a more invariant consequence of Theorem 12.1. Note that if we were interested in computing $SH_*^+(W)$ with field coefficients, then the exact sequence suggests that it should be enough to compute Gromov–Witten invariants, instead of chain level counts of pseudoholomorphic spheres, even if one does not start with a perfect Morse function in Σ. Observe that $SC_*(W)$ is also a cone complex on a chain map $SC_*^+(W) \to C_*(W)$, counting pseudoholomorphic spheres in X with tangency conditions in Σ (the terms involving $n_B(p,x)$). The same argument as above should enable us to compute $SH_*(W)$ with the additional information of relative Gromov–Witten invariants of the pair (X,Σ), instead of the non-invariant counts $n_B(p,x)$, at least over field coefficients.

Part 5. **Example:** T^*S^2

We now illustrate the results in this paper with the computation of the symplectic homology of the completion W of $X\backslash\Sigma$, where $(X,\Sigma,\omega) = (\mathbb{C}P^1 \times \mathbb{C}P^1, \Delta, \omega_{FS} \oplus \omega_{FS})$, where Δ is the diagonal and ω restricts to the Fubini-Study form of area 1 in each factor. The manifold W is symplectomorphic to T^*S^2 (see Exercise 6.20 in [MS98]), so its symplectic homology is isomorphic to the homology of the based loop space of S^2. We will see that our computation recovers this result. X and Σ are both monotone, with $\tau_X = 2$ and $\tau_\Sigma = 1$. Also, $[\Sigma]$ is Poincaré-dual to $[\omega]$, so $K = 1$. The S^1-bundle $Y \to \Sigma$ is $\mathbb{R}P^3$.

13. The coefficients in the differential

The manifold $\Sigma = \mathbb{C}P^1$ admits a lacunary Morse function $f_\Sigma : \Sigma \to \mathbb{R}$, with one zero of index 0, denoted by m (for *minimum*), and one zero of index 2, denoted by M (for *maximum*). Lemma 12.4 implies that all the numbers $n_A(q,p)$ (giving the Case 1 contributions to the differential) are absolute Gromov–Witten invariants of $\Sigma = \mathbb{C}P^1$. Definition 11.21 and Remark 11.24 indicate that the only potentially non-trivial invariant is

$$n_L(M,m) = \mathrm{GW}_{L,2}^{\mathbb{C}P^1}([pt],[pt]) = 1$$

where $L \in H_2(\mathbb{C}P^1;\mathbb{Z})$ is the class of a line. This invariant counts the number of lines in $\mathbb{C}P^1$ through two generic points, which is 1 (the integral complex structure is regular in $\mathbb{C}P^1$).

The manifold $W = T^*S^2$ also admits a lacunary Morse function $f_W : W \to \mathbb{R}$ growing at infinity, with one zero of index 0, denoted by e, and one zero of index 2, denoted by c. The manifold $W^u(c)$ represents the zero section in T^*S^2. We orient it as $L_1 - L_2$, where $L_1, L_2 \in H_2(\mathbb{C}P^1 \times \mathbb{C}P^1;\mathbb{Z})$ are homology classes representing the factors $\mathbb{C}P^1 \times \{pt\}$ and $\{pt\} \times \mathbb{C}P^1$, respectively. Definition 11.21 and Remark 11.24 again indicate that the only potentially non-trivial coefficients in Cases 2 and 3 are

$$n_{L_i}, \quad n_{L_i}(e, M), \quad n_{L_i}(c, m), \quad n_{2L_i}(e, m) \quad \text{and} \quad n_{L_1+L_2}(e, m).$$

Lemmas 12.4 and 12.5 imply that these numbers are relative Gromov–Witten invariants of $(\mathbb{C}P^1 \times \mathbb{C}P^1, \Delta, \omega)$. We will now compute these invariants.

Proposition 13.1. *(1)* $n_{L_i} = \mathrm{GW}^{\mathbb{C}P^1\times\mathbb{C}P^1,\Delta}_{L_i,0,(1)}(\varnothing; pt) = 1$, *for* $i = 1, 2$ *(the point constraint is in* Δ*, not in* $\mathbb{C}P^1 \times \mathbb{C}P^1$*);*

(2) $n_{L_i}(e, M) = \mathrm{GW}^{\mathbb{C}P^1\times\mathbb{C}P^1,\Delta}_{L_i,1,(1)}([pt]; [\Delta]) = 1$*;*

(3) $n_{L_i}(c, m) = \mathrm{GW}^{\mathbb{C}P^1\times\mathbb{C}P^1,\Delta}_{L_i,1,(1)}([S^2]; [pt]) = (-1)^i$ *where* $S^2 \subset T^*S^2$ *is the zero section, oriented so as to represent the homology class* $L_1 - L_2$*;*

(4) $n_{2L_i}(e, m) = \mathrm{GW}^{\mathbb{C}P^1\times CP^1,\Delta}_{2L_i,1,(2)}([pt]; [pt]) = 0$*;*

(5) $n_{L_1+L_2}(e, m) = \mathrm{GW}^{\mathbb{C}P^1\times CP^1,\Delta}_{L_1+L_2,1,(2)}([pt]; [pt]) = 1$*.*

We need an auxiliary result.

Lemma 13.2. *The product complex structure on* $X = \mathbb{C}P^1 \times \mathbb{C}P^1$ *is regular for all immersed holomorphic spheres* $U : \mathbb{C}P^1 \to \mathbb{C}P^1 \times \mathbb{C}P^1$ *such that* $\int_{\mathbb{C}P^1} U^*\omega \geqslant 1$*. In particular, this is true when* $U_*[\mathbb{C}P^1] \in \{L_1, L_2, L_1 + L_2\}$*.*

Proof. By [MS04, Lemma 3.3.3], we just need to make sure that $\langle c_1(TX), U_*[\mathbb{C}P^1]\rangle \geqslant 1$, but $\langle c_1(TX), U_*[\mathbb{C}P^1]\rangle = 2\int_{\mathbb{C}P^1} U^*\omega \geqslant 2$. □

For the proof of cases (1) through (3) of Proposition 13.1, one could appeal to the relation between relative and absolute Gromov–Witten invariants, when all the tangencies to Σ are of order 1, as mentioned in Remark 12.7. This is not necessary, though, since Lemma 13.2 allows us to do the computations explicitly using the integral complex structure in $\mathbb{C}P^1 \times \mathbb{C}P^1$. This satisfies the vanishing normal Nijenhuis tensor condition [TZ14, (4.6)], and can thus be used for computing relative Gromov–Witten invariants.

Proof of Proposition 13.1. (1) Given $i = 1, 2$,

$$\mathrm{GW}^{\mathbb{C}P^1\times\mathbb{C}P^1,\Delta}_{L_i,0,(1)}(\varnothing; [pt]) = 1$$

is the fact that if one fixes any point $p \in \Delta$, then there is exactly one holomorphic sphere in class L_i that goes through p.

(2) Given $i = 1, 2$,

$$\mathrm{GW}^{\mathbb{C}P^1\times\mathbb{C}P^1,\Delta}_{L_i,1,(1)}([pt]; [\Delta]) = 1$$

expresses the fact that, if we fix any point $p \in \mathbb{C}P^1 \times \mathbb{C}P^1 \backslash \Delta$, there is a unique holomorphic sphere in class L_i that goes through p and intersects Δ.

(3) Given $i, j = 1, 2$,

$$\mathrm{GW}^{\mathbb{C}P^1\times\mathbb{C}P^1,\Delta}_{L_i,1,(1)}(L_j; [pt]) = 1 - \delta_{i,j}.$$

This means, for instance, that there is a unique vertical sphere in $\mathbb{C}P^1 \times \mathbb{C}P^1$ intersecting a horizontal sphere and a generic point $p \in \mathbb{C}P^1 \times \mathbb{C}P^1$, but there is no horizontal sphere intersecting another horizontal sphere and a generic p. This is because two different horizontal spheres do not intersect. Since $S^2 = L_1 - L_2$,

$$\mathrm{GW}^{\mathbb{C}P^1\times\mathbb{C}P^1,\Delta}_{L_i,1,(1)}([S^2];[pt]) = \mathrm{GW}^{\mathbb{C}P^1\times\mathbb{C}P^1,\Delta}_{L_i,1,(1)}(L_1;[pt]) - \mathrm{GW}^{\mathbb{C}P^1\times\mathbb{C}P^1,\Delta}_{L_i,1,(1)}(L_2;[pt]) = (-1)^i$$

(4) Given $i = 1,2$,

$$\mathrm{GW}^{\mathbb{C}P^1\times\mathbb{C}P^1,\Delta}_{2L_i,1,(2)}([pt];[pt]) = 0$$

encodes the fact that holomorphic curves in classes $2L_i \in H_2(\mathbb{C}P^1\times\mathbb{C}P^1;\mathbb{Z})$ are covers of either vertical or horizontal spheres, and therefore cannot go through one generic point in $\mathbb{C}P^1\times\mathbb{C}P^1$ and another generic point in Δ.

(5) To prove that

$$\mathrm{GW}^{\mathbb{C}P^1\times CP^1,\Delta}_{L_1+L_2,1,(2)}(pt;pt) = 1$$

we show that, up to domain automorphism, there is a unique holomorphic map $U : \mathbb{C}P^1 \to \mathbb{C}P^1\times\mathbb{C}P^1$, such that $U_*[\mathbb{C}P^1] = L_1 + L_2 \in H_2(\mathbb{C}P^1 \times \mathbb{C}P^1;\mathbb{Z})$, $U(\infty) = (\infty,1)$, $U(0) = (0,0)$, and such that U intersects the diagonal $\Delta \subset \mathbb{C}P^1\times\mathbb{C}P^1$ in a non-transverse way. Since

$$U(\infty) = U\big([1;0]\big) = (\infty,1) = \big([1;0],[1;1]\big),$$

we can write in homogeneous coordinates

$$U\big([z;1]\big) = \big([az+b;1],[z+c;z+d]\big)$$

which we will abbreviate as

$$U(z) = \left(az+b, \frac{z+c}{z+d}\right).$$

Since $U(0) = (0,0)$, we get $b = c = 0$, and so $U(z) = \big(az, z/(z+d)\big)$. Now, for the tangency condition, note that

$$U'(z) = \left(a, \frac{d}{(z+d)^2}\right)$$

and so $U'(0) = \big(a, 1/d\big)$. So U is tangent to the diagonal at $(0,0)$ precisely when $a = 1/d$. Therefore, the space of maps U can be identified with the space of $a \in \mathbb{C}^*$. Taking a quotient by the group $\mathbb{C}^*$ of automorphisms of the domain $(\mathbb{C}P^1,\{0,\infty\})$, we get the uniqueness of U.

□

14. The group $SH_*(T^*S^2)$

We can now compute the symplectic homology chain complex for T^*S^2. Using the functions f_Σ and f_W introduced above, we have

$$SC_*(T^*S^2) = \mathbb{Z}\langle e,c\rangle \oplus \bigoplus_{k>0} \mathbb{Z}\left\langle \check{m}_k, \hat{m}_k, \widecheck{M}_k, \widehat{M}_k \right\rangle$$

We can use Theorem 12.1, Lemma 12.4 and Proposition 13.1 to compute the differential. Given $k \geqslant 2$,

$$\begin{aligned}\partial\check{m}_{k+1} &= 2\,n_L(M,m)\,\widehat{M}_{k-1} + \big(n_{L_1} + n_{L_2}\big)\,\hat{m}_k = \\ &= 2\times 1\,\widehat{M}_{k-1} + \big(1+1\big)\,\hat{m}_k = 2\,\widehat{M}_{k-1} + 2\,\hat{m}_k\end{aligned}$$

and

$$\partial \widecheck{M}_k = \big(n_{L_1} + n_{L_2}\big)\, \widehat{M}_{k-1} + \langle c_1(\mathbb{R}P^3 \to \mathbb{C}P^1), \mathbb{C}P^1 \rangle\, \widehat{m}_k = 2\, \widehat{M}_{k-1} + 2\, \widehat{m}_k$$

The remaining differentials are

$$\partial \check{m}_2 = \big(n_{L_1} + n_{L_2}\big)\, \widehat{m}_1 + 2\, n_{L_1+L_2}(e,m)\, e = 2\, \widehat{m}_1 + 2\, e,$$

$$\begin{aligned}\partial \widecheck{M}_1 &= \big(n_{L_1}(e,M) + n_{L_2}(e,M)\big)\, e + \langle c_1(\mathbb{R}P^3 \to \mathbb{C}P^1), \mathbb{C}P^1 \rangle\, \widehat{m}_1 = \\ &= 2\, e + 2\, \widehat{m}_1\end{aligned}$$

and

$$\partial \check{m}_1 = \big(n_{L_1}(c,m) + n_{L_2}(c,m)\big)\, c = (-1+1)\, c = 0.$$

Summing up, the non-trivial contributions to the differential are

$$\begin{cases} \partial \check{m}_{k+1} = 2\, \widehat{m}_k + 2\, \widehat{M}_{k-1} \\ \partial \widecheck{M}_k = 2\widehat{m}_k + 2\, \widehat{M}_{k-1} \\ \partial \check{m}_2 = 2\, \widehat{m}_1 + 2\, e \\ \partial \widecheck{M}_1 = 2\, \widehat{m}_1 + 2\, e \end{cases}$$

for $k \geqslant 2$. We can conclude the following.

Proposition 14.1.

$$SH_*(T^*S^2;\mathbb{Z}) = \mathbb{Z}\left\langle c, \check{m}_1, e, \widecheck{M}_k - \check{m}_{k+1}, \widehat{M}_k \right\rangle \oplus \mathbb{Z}/2\left\langle e + \widehat{m}_1, \widehat{M}_k + \widehat{m}_{k+1} \right\rangle$$

where we take all $k \geqslant 1$.

We can compare these results with earlier computations of the symplectic homology of T^*S^2. By Viterbo's Theorem, it is isomorphic as a ring to the homology of the free loop space of S^2 [AS12]. As a graded abelian group, this was computed by McCleary [McC90]. The a ring, it was computed in [CJY04] to be

$$H_*(LS^2;\mathbb{Z}) \cong (\Lambda(b) \otimes \mathbb{Z}[a,v])/(a^2, ab, 2av) \tag{14.1}$$

for some $a \in H_0(LS^2;\mathbb{Z})$, $b \in H_1(LS^2;\mathbb{Z})$ and $v \in H_4(LS^2;\mathbb{Z})$. The following tables show how our computations match these, at least as graded abelian groups. For the free part, we get

$SH_d(T^*S^2)$	c	$-\check{m}_1$	e	$\widecheck{M}_k - \check{m}_{k+1}$	$\widehat{M}_k$
$H_d(LS^2)$	a	b	1	bv^k	v^k
d	0	1	2	$2k+1$	$2k+2$

for $k \geqslant 1$. For the $\mathbb{Z}/2$-torsion part:

$SH_d(T^*S^2)$	$e + \widehat{m}_1$	$\widehat{M}_k + \widehat{m}_{k+1}$
$H_d(LS^2)$	av	av^{k+1}
d	2	$2k+2$

for $k \geqslant 1$. One can also adapt the arguments presented above to describe the ring structure on symplectic homology and show that it matches 14.1, but that is beyond the scope of this paper.

Appendix A. Morse–Bott Riemann–Roch

In this section, we collect some facts about Cauchy–Riemann-type operators on Hermitian vector bundles over punctured Riemann surfaces. These facts are well-established in parts of the literature, but we collect them here for the convenience of the reader. The main reference for these results is [Sch95]. Additional references include [HWZ99; Sch95; Wen10; ACH05, Sections 2.1–2.3; BM04].

We begin by introducing some Sobolev spaces of sections of appropriate bundles. Let $\Gamma \subset \mathbb{R} \times S^1$ be a finite set of punctures and denote $\mathbb{R} \times S^1 \backslash \Gamma$ by $\dot{S}$. Write $\Gamma_+ = \{+\infty\}$ and $\Gamma_- = \{-\infty\} \cup \Gamma$. Consider, for each puncture $z \in \Gamma$, exponential cylindrical polar coordinates of the form $(-\infty, -1] \times S^1 \to \mathbb{R} \times S^1 \backslash \Gamma \colon \rho + i\eta \mapsto z_0 + \epsilon\, \mathrm{e}^{2\pi(\rho + i\eta)}$. Choose $\epsilon > 0$ sufficiently small that the image of these embeddings for any two different punctures are disjoint.

Let $E \to \dot{S}$ be a (complex) rank n Hermitian vector bundle over $\dot{S}$ together with a preferred set of trivializations in a small neighbourhood of $\Gamma \cup \{\pm\infty\}$. While the bundle E over $\dot{S}$ is trivial if there is at least one puncture, this is no longer the case once we specify these preferred trivializations near $\Gamma \cup \{\pm\infty\}$. We therefore associate a first Chern number to this bundle relative to the asymptotic trivializations. There are several equivalent definitions. One approach is to consider the complex determinant bundle $\Lambda^n E$. The trivialization of E at infinity gives a trivialization of this determinant bundle at infinity, and we can now count zeros of a generic section of $\Lambda^n E$ that is constant (with respect to the prescribed trivializations) near the punctures. We denote this Chern number by $c_1(E)$, but emphasize that it depends on the choice of these trivializations near the punctures.

Since we cannot specify where an augmentation puncture appears when we stretch the neck on a Floer cylinder, we should have the punctures in Γ free to move on the domain $\mathbb{R} \times S^1$. This creates a problem when we try to linearize the Floer operator in a family of domains where the positions of the punctures are not fixed. We will instead consider a $2\#\Gamma$ parameter family of almost complex structures on $\mathbb{R} \times S^1$, but fix the location of the punctures, as follows. We specify a fixed collection Γ of punctures on $\mathbb{R} \times S^1$ and, for any other collection of augmentation punctures, choose an isotopy with compact support from the new punctures to the fixed ones. We take the push-forward of the standard complex structure in $\mathbb{R} \times S^1$ by the final map of the isotopy, to produce a family of complex structures on $\mathbb{R} \times S^1$, which can be assumed standard near Γ and outside of a compact set.

For each $z \in \Gamma$, let $\beta_z \colon \dot{S} \to [0, +\infty)$ be a function supported in a small neighbourhood of z, with $\beta_z(\rho, \eta) = -\rho$ near the puncture (where (ρ, η) are cylindrical polar coordinates near z, as above). Similarly, let $\beta_+ \colon \mathbb{R} \times S^1 \to [0, +\infty)$ be supported in a region where s is sufficiently large and $\beta_+(s,t) = s$ for s large enough. Let $\beta_- \colon \mathbb{R} \times S^1 \to [0, +\infty)$ have support near $-\infty$, and $\beta_-(s,t) = -s$ for s sufficiently small.

In many situations, it will be convenient to consider the function

$$\beta := \sum_{z \in \Gamma} \beta_z + \beta_- + \beta_+. \tag{A.1}$$

Given a vector of weights $\delta \colon \Gamma \cup \{\pm\infty\} \to \mathbb{R}$, we define $W^{1,p,\delta}(\dot{S}, E)$ to be the space of sections u of E for which $u\, \mathrm{e}^{\sum \delta_z \beta_z + \delta_- \beta_- + \delta_+ \beta_+} \in W^{1,p}(\dot{S}, E)$ (with respect to a measure on $\dot{S}$ that equals $ds\, dt$ for $|s|$ sufficiently large, and $d\rho\, d\eta$ in cylindrical polar coordinates near each puncture in Γ). Note that these sections decay exponentially fast at the punctures where $\delta > 0$. We can similarly define $L^{p,\delta}(\dot{S}, E)$. While these definitions involve making various choices, the resulting metrics are strongly equivalent.

We will say that a differential operator $D\colon \Gamma(E) \to \Lambda^{0,1}T^*\dot{S} \otimes E$ is a Cauchy–Riemann operator if it is a real linear Cauchy–Riemann operator [MS04, Definition C.1.5] such that, near $\pm\infty$, it takes the form:

$$(D\sigma)\frac{\partial}{\partial s} = \frac{\partial}{\partial s}\sigma + J(s,t)\frac{\partial}{\partial t}\sigma + A(s,t)\sigma \tag{A.2}$$

where $J(s,t)$ is a smooth function on $\mathbb{R}^{\pm} \times S^1$ with values in almost complex structures on $\mathbb{C}^n$ compatible with the standard symplectic form, and $A(s,t)$ takes values in real matrices on $\mathbb{R}^{2n} \cong \mathbb{C}^n$. We further impose that these functions converge as $s \to \pm\infty$, $J(s,t) \to J_z(t)$ and $A(s,t) \to A_z(t)$, where $A_z(t)$ is a loop of self-adjoint matrices. We impose the same conditions near punctures $z \in \Gamma$, using the local coordinates (ρ, η) instead of (s,t) in (A.2).

Associated to such a Cauchy–Riemann operator D, we obtain *asymptotic operators* at each puncture in $\Gamma \cup \{\pm\infty\}$ by $\mathbf{A}_z := -J_z(t)\frac{d}{dt} - A_z(t)$. This is a densely defined unbounded self-adjoint operator on $L^2(S^1, \mathbb{R}^{2n})$. Let $\sigma(\mathbf{A}_z) \subset \mathbb{R}$ denote its spectrum. This will consist of a discrete set of eigenvalues. If an asymptotic operator $\mathbf{A}_z$ does not have 0 in its spectrum, we say the asymptotic operator is non-degenerate. If all the asymptotic operators are non-degenerate, we say D itself is non-degenerate.

Note that we obtain a path of symplectic matrices associated to the asymptotic operator $\mathbf{A}_z$ by finding the fundamental matrix Φ to the ODE $\frac{d}{dt}x = J_z(t)A_z(t)x$. The asymptotic operator is non-degenerate if and only if the time-1 flow of the ODE does not have 1 in the spectrum. We will consider a description of the Conley–Zehnder index in terms of properties of the asymptotic operator itself [HWZ95, Lemmas 3.4, 3.5, 3.6, 3.9].

Remark A.1. In this paper, we also want to associate a Conley–Zehnder index to a periodic orbit of a Hamiltonian vector field, given a trivialization of the tangent bundle along the orbit. In order to do so, we take the linearized flow map, which defines a path $\Phi\colon [0,1] \to \mathrm{Sp}(2n)$ with respect to the fixed trivialization. If we fix a path of almost complex structures, this path of symplectic matrices satisfies an ODE as in the previous paragraph, which in turn specifes an asymptotic operator. The Conley–Zehnder index of the Hamiltonian orbit is by definition the Conley-Zehnder index of this asymptotic operator.

Proposition A.2. *Suppose $\mathbf{A}_z$ is non-degenerate and E is a rank 1 vector bundle. Then, each eigenfunction $u\colon S^1 \to \mathbb{C}$ of $\mathbf{A}_z$ has a winding number that depends only on its eigenvalue λ, $w(\lambda)$. The function $w\colon \sigma(\mathbf{A}_z) \to \mathbb{Z}$ is non-decreasing in λ and is surjective. If $\lambda_\pm$ are eigenvalues so that $\lambda_- < 0 < \lambda_+$ and there are no eigenvalues in the interval (λ_-, λ_+), then*

$$CZ(\mathbf{A}_z) = w(\lambda_-) + w(\lambda_+).$$

This formulation will be the most useful for our calculations. Furthermore, in the case of a higher rank bundle, we use the axiomatic description, see for instance [HWZ95, Theorem 3.1] to observe that $\mathrm{CZ}(\mathbf{A}_z)$ is invariant under deformations for which 0 is never in the spectrum, and that if the operator can be decomposed as the direct sum of operators, then the CZ-index is additive.

The following statement is useful at several points in the paper. It can often be combined with Proposition A.2 to compute Conley–Zehnder indices of interest.

Lemma A.3. *Given a constant $C \geqslant 0$, the spectrum $\sigma(A_C)$ of the operator*

$$A_C := -i\frac{d}{dt} - \begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix} : W^{1,p}(S^1, \mathbb{C}) \to L^p(S^1, \mathbb{C})$$

is the set

$$\left\{\frac{1}{2}\left(-C - \sqrt{C^2 + 16\pi^2 k^2}\right) \mid k \in \mathbb{Z}\right\} \cup \left\{\frac{1}{2}\left(-C + \sqrt{C^2 + 16\pi^2 k^2}\right) \mid k \in \mathbb{Z}\right\}.$$

If λ is an eigenvalue associated to $k \in \mathbb{Z}$, then the winding number of the corresponding eigenvector is $|k|$ if $\lambda \geqslant 0$ and $-|k|$ if $\lambda \leqslant 0$. If $C = 0$, then all eigenvalues have multiplicity 2 (see Table 1). If $C > 0$, then the same is true except for the eigenvalues $-C$ and 0, corresponding to $k = 0$ above, both of which have multiplicity 1 (see Table 2).

In particular, the $\sigma(A_0) = 2\pi\mathbb{Z}$ and the winding number of $2\pi k$ is k.

Proof. An eigenvector $v : S^1 \to \mathbb{C}$ of A_C with eigenvalue λ solves the equation

$$-\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\dot{v} - \begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix} v = \lambda v \Leftrightarrow \dot{v} = \begin{pmatrix} 0 & -\lambda \\ C + \lambda & 0 \end{pmatrix} v.$$

Computing the eigenvalues of the matrix on the right, and requiring that they be of the form $2\pi i k$, $k \in \mathbb{Z}$ (since $v(t+1) = v(t)$), yields the result. □

eigenvalues	...	-4π	-2π	0	2π	4π	...
multiplicities	...	2	2	2	2	2	...
winding numbers	...	-2	-1	0	1	2	...

TABLE 1. Eigenvalues of A_0

eigenvalues	...	$\frac{1}{2}\left(-C - \sqrt{C^2 + 16\pi^2}\right)$	$-C$	0	$\frac{1}{2}\left(-C + \sqrt{C^2 + 16\pi^2}\right)$	...
multiplicities	...	2	1	1	2	...
winding #s	...	-1	0	0	1	...

TABLE 2. Eigenvalues of A_C in increasing order, if $C > 0$

Corollary A.4. *Take $C \geqslant 0$ and $\delta > 0$ such that $[-\delta, \delta] \cap \sigma(A_C) = \{0\}$. Then*

$$CZ(A_C + \delta) = \begin{cases} 0 & \text{if } C > 0 \\ -1 & \text{if } C = 0 \end{cases} \qquad \text{and} \qquad CZ(A_C - \delta) = 1.$$

For any $n \geqslant 0$, taking

$$-i\frac{d}{dt} : W^{1,p}(S^1, \mathbb{C}^n) \to L^p(S^1, \mathbb{C}^n),$$

we have

$$CZ\left(-i\frac{d}{dt} \pm \delta\right) = \mp n.$$

Proof. The case $n = 1$ follows from Proposition A.2 and Lemma A.3. The case of general n uses the additivity of CZ under direct sums. □

Definition A.5. A key observation for our computations of Fredholm indices (as noted, for instance, in [HWZ99]) is that a Cauchy–Riemann operator

$$D\colon W^{1,p,\delta}(\dot{S},E)\to L^{p,\delta}(\dot{S},\Lambda^{0,1}T^*\dot{S}\otimes E)$$

with asymptotic operators $\mathbf{A}_z$ is conjugate to the Cauchy–Riemann operator

$$D^\delta\colon W^{1,p}(\dot{S},E)\to L^{p}(\dot{S},\Lambda^{0,1}T^*\dot{S}\otimes E)$$
$$D^\delta = \mathrm{e}^{\sum\delta_z\beta_z(s)}\,D\,\mathrm{e}^{-\sum\delta_z\beta_z(s)}.$$

This has asymptotic operators $\mathbf{A}_z^\delta = \mathbf{A}_z \pm \delta_z$ (where the sign is positive at positive punctures and negative at negative punctures). We refer to these as the *δ-perturbed asymptotic operators.*

This observation about the conjugation of the weighted operator to the non-degenerate case, combined with Riemann–Roch for punctured domains (see for instance, [Sch95, Theorem 3.3.11; HWZ99, Theorem 2.8; Wen16a, Theorem 5.4]) gives the following.

Theorem A.6. *Let $\delta\colon\Gamma\to\mathbb{R}$ such that $\mp\delta_z\notin\sigma(\mathbf{A}_z)$. Then, the Cauchy–Riemann operator*

$$D\colon W^{1,p,\delta}(\dot{S},E)\to L^{p,\delta}(\dot{S},\Lambda^{0,1}T^*\dot{S}\otimes E)$$

with asymptotic operators $\mathbf{A}_z$, $z\in\Gamma$ is Fredholm and its index is given by

$$\operatorname{Ind}(D,\delta) = n\chi_{\dot{S}} + 2c_1(E) + \sum_{z\in\Gamma_+} CZ(\mathbf{A}_z+\delta_z) - \sum_{z\in\Gamma_-} CZ(\mathbf{A}_z-\delta_z).$$

Now, a useful fact for us is a description of how the Conley–Zehnder index changes as a weight crosses the spectrum of the operator:

Lemma A.7. *Suppose that $[-\delta,+\delta]\cap\sigma(\mathbf{A}_z)=\{0\}$. Then,*

$$CZ(\mathbf{A}_z-\delta) - CZ(\mathbf{A}_z+\delta) = \dim(\ker\mathbf{A}_z)$$

For a proof, see for instance [Wen05, Proposition 4.5.22].

To obtain a result that is useful for our moduli spaces of cascades asymptotic to Morse–Bott families of orbits, we consider the following modification of our function spaces.

To each puncture, we associate a subspace of the kernel of the corresponding asymptotic operator, which we denote by $V_z, z\in\Gamma$, V_-, V_+ and write $\mathbf{V}$ for this collection. Then, for each puncture $z\in\Gamma$ and also $\pm\infty$, we associate a smooth bump function μ_z, $\mu_\pm$, supported near and identically 1 even nearer to its puncture. We then define

$$\begin{aligned} W^{1,p,\delta}_{\mathbf{V}}(\dot{S},E) = \{&u\in W^{1,p}_{\mathrm{loc}}(\dot{S},E)\mid \exists\, c_z\in V_z, z\in\Gamma, c_-\in V_-, c_+\in V_+ \\ &\text{such that } u-\sum c_z\mu_z - c_-\mu_- - c_+\mu_+\in W^{1,p,\delta}(\dot{S},E)\}.\end{aligned} \tag{A.3}$$

We remark that we are using the asymptotic cylindrical coordinates near Γ and the asymptotic trivialization of E in order to define the local sections $c_z\mu_z$.

In this paper, we are primarily concerned with Cauchy-Riemann operators defined on $\dot{S}=\mathbb{R}\times S^1$ and on $\dot{S}=\mathbb{R}\times S^1\backslash\{P\}$ (a cylinder with one additional negative puncture). In the case of $\mathbb{R}\times S^1$, we will write $\mathbf{V}=(V_-;V_+)$, and in the case of $\mathbb{R}\times S^1\backslash\{P\}$, we will write $\mathbf{V}=(V_-,V_P;V_+)$. (The negative punctures are enumerated first, and separated from the positive puncture by a semicolon.)

Observe that since the vector spaces V_z are in the kernel of the corresponding asymptotic operators, for any choice of $\mathbf{V}$ and any vector of weights δ, we have that the Cauchy–Riemann-type operator D can be extended to

$$D\colon W^{1,p,\delta}_{\mathbf{V}}(\dot{S},E) \to L^{p,\delta}(\dot{S},\Lambda^{0,1}T^*\dot{S}\otimes E).$$

Let $\dim V_z$ denote the dimension of the vector space V_z and let $\operatorname{codim} V_z = \dim(\ker \mathbf{A}_z/V_z)$. Combining Theorem A.6 with Lemma A.7, we have:

Theorem A.8. *Let $\delta > 0$ be sufficiently small that for $z \in \Gamma_+$, $[-\delta,0)\cap\sigma(\mathbf{A}_z) = \varnothing$ and such that for $z \in \Gamma_-$, $(0,\delta]\cap\sigma(\mathbf{A}_z) = \varnothing$.*

For each $z \in \Gamma$, fix the subspace $V_z \subset \ker \mathbf{A}_z$.

Then,

$$D\colon W^{1,p,\delta}_{\mathbf{V}}(\dot{S},E) \to L^{p,\delta}(\dot{S},\Lambda^{0,1}T^*\dot{S}\otimes E).$$

is Fredholm, and its Fredholm index is given by

$$\operatorname{Ind}(D) = n\chi_{\dot{S}}+2c_1(E)+\sum_{z\in\Gamma_+}\big(CZ(\mathbf{A}_z+\delta)+\dim(V_z)\big)-\sum_{z\in\Gamma_-}\big(CZ(\mathbf{A}_z+\delta)+\operatorname{codim}(V_z)\big).$$

In applications where there are Morse–Bott manifolds of orbits, we will sometimes use V_z to be the tangent space to the descending manifold of a critical point p_z on the manifold of orbits at a positive puncture, and V_z will be the tangent space to ascending manifold of a critical point p_z at a negative puncture. In either case, the contribution to $\operatorname{Ind}(D)$ of $\dim V_z$ or of $\operatorname{codim} V_z$ will be the Morse index of the appropriate critical point. This motivates the following definition.

Definition A.9. Let $\delta > 0$ be sufficiently small. If p_z is a critical point of an auxiliary Morse function on the manifold of orbits associated to z, then the Conley–Zehnder index of the pair $(\mathbf{A}_z, p_z)$ is

$$\mathrm{CZ}(\mathbf{A}_z,p_z) = \mathrm{CZ}(\mathbf{A}_z+\delta) + M(p_z),$$

where $M(p_z)$ is the Morse index of p_z.

In this case, we can write the Fredholm index as

$$\operatorname{Ind}(D) = n\chi_{\dot{S}} + 2c_1(E) + \sum_{z\in\Gamma_+}\mathrm{CZ}(\mathbf{A}_z,p_z) - \sum_{z\in\Gamma_-}\mathrm{CZ}(\mathbf{A}_z,p_z).$$

We conclude with a lemma that is particularly useful when applying the automatic transversality result [Wen10, Proposition 4.22]. The lemma states that the Fredholm index of an operator with a small negative weight at a puncture is the same as that of the corresponding operator with a small positive weight at that puncture, if the puncture is decorated with the kernel of the corresponding asymptotic operator. The former indices are used in [Wen10, Proposition 4.22], whereas the latter can be computed using Theorem A.8.

We first learned this result from Wendl [Wen05]. We give a proof of this formulation since it is slightly stronger than what we found in the literature (and is still not as strong as can be proved.)

Lemma A.10. *Let D be a Cauchy–Riemann-type operator. Fix a puncture $z_0 \in \Gamma\cup\{\pm\infty\}$.*

Let δ and δ' be vectors of sufficiently small weights so that the differential operator induces a Fredholm operator on $W^{1,p,\delta}$ and on $W^{1,p,\delta'}$, and for which there is a

$z_0 \in \Gamma \cup \{\pm\infty\}$ with $\delta_{z_0} > 0$ and $\delta'_{z_0} < 0$ and so that for each $z \in \Gamma \cup \{\pm\infty\}$ with $z \neq z_0$, the weights $\delta_z = \delta'_z$.

Let $\mathbf{V}$ be the trivial vector space at each puncture other than z_0 and let V_{z_0} be the kernel of the asymptotic operator at z_0.

Then, the induced operators

$$D_\delta \colon W^{1,p,\delta}_{\mathbf{V}}(\dot{S}, E) \to L^{p,\delta}(\dot{S}, \Lambda^{0,1}T^*\dot{S} \otimes E)$$
$$D_{\delta'} \colon W^{1,p,\delta'}(\dot{S}, E) \to L^{p,\delta'}(\dot{S}, \Lambda^{0,1}T^*\dot{S} \otimes E)$$

have the same Fredholm index and their kernels and cokernels are isomorphic.

Proof. The main idea of the lemma is contained in [Wen05, Proposition 4.5.22], which contains a proof of the equality of Fredholm indices. See also the very closely related [Wen16b, Proposition 3.15].

Note that $W^{1,p,\delta}_{\mathbf{V}}(\dot{S}, E)$ is a subspace of $W^{1,p,\delta'}(\dot{S}, E)$, and thus the kernel of D_δ is contained in the kernel of $D_{\delta'}$.

Now, by a linear version of the analysis done in [HWZ99, Sie08], any element of the kernel of $D_{\delta'}$ converges exponentially fast at z_0 to an eigenfunction of the asymptotic operator, with exponential rate governed by the eigenvalue (in this case 0). Therefore, any element of the kernel of $D_{\delta'}$ must converge exponentially fast to an element of the kernel of the asymptotic operator at z_0. Hence, the kernel of $D_{\delta'}$ is contained in the kernel of D_δ.

We conclude that the kernels of the two operators may be identified. Since their Fredholm indices are the same, their cokernels are also isomorphic. □

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