MONOTONE LAGRANGIANS IN THE COTANGENT BUNDLE OF THE 3-SPHERE

MOHAMMED ABOUZAID AND LUÍS DIOGO

We study the compact monotone Fukaya category of T^*S^3 and show that it is split-generated by two classes of objects: the zero-section S^3 (equipped with bounding cochains) and a 1-parameter family of monotone Lagrangian tori T^3_τ, $\tau > 0$ (equipped with unitary local systems of rank 1). As a consequence, any closed orientable spin monotone Lagrangian with non-trivial Floer homology (possibly equipped with a bounding cochain and/or a unitary local system of rank 1) is non-displaceable from either S^3 or one of the T^3_τ. The Lagrangians T^3_τ can be obtained by surgery on two copies of S^3, and can be thought of as abstract analogues of flux deformations of two copies of S^3 along a generator of $H^3(S^3)$.

Contents

1. Introduction 1
2. Monotone Lagrangians in T^*S^3 4

Part 1. Wrapped Floer homology of Lagrangians in T^*S^3 6
3. Wrapped Fukaya categories 6
3.1. Coefficients 6
3.2. Morse–Bott Floer cohomology for clean intersections 7
3.3. Wrapped Fukaya categories 8
4. Computations in T^*S^3 10
4.1. The Lagrangians N_w 10
4.2. Floer cohomology of F_w with S^3 12
4.3. Floer cohomology of N_w with T^3_τ 13
4.4. Module actions on the Fukaya category 13

Part 2. Algebras and modules 14
5. Hochschild cohomology and formality of $CW^*(F,F)$ 14
6. Ext groups and formality of A_∞-modules 15
References 18

1. Introduction

The study of Lagrangian submanifolds in cotangent bundles is a central problem in symplectic topology. Recall that an embedded Lagrangian L in $(T^*N, d(pdq))$, is *exact* if $pdq|_L = df$ for some function $f : N \to \mathbb{R}$. Arnold's nearby Lagrangian conjecture predicts that if N and L are closed, then L is Hamiltonian-isotopic to

Date: February 22, 2017.

the zero-section $N \subset T^*N$. This result is currently only known to hold for a limited list of examples, like $N = S^2$ [12] and T^2 [8]. Work by many people has also led to a proof that the composition $L \to T^*N \to N$ (where the last map is the projection) is a simple homotopy equivalence [2].

Very little is know if one drops the requirement that L be exact. A natural next case to consider is when L is *monotone*. This means that there is a constant $\tau \geq 0$ such that, for every map $u : (D^2, \partial D^2) \to (T^*N, L)$,

$$\int_{D^2} u^*\omega = \tau\mu(u)$$

where $\mu(u)$ is the *Maslov index* of u, which measures the rotation of the path of Lagrangian subspaces $T_{u(e^{2\pi i\theta})}L \subset T_{u(e^{2\pi i\theta})}(T^*N)$, for $\theta \in [0, 1]$. An exact Lagrangian L is monotone, with $\tau = 0$. Little is known about monotone Lagrangians in cotangent bundles, but see for instance [11].

The goal of this paper is to study closed monotone Lagrangians in cotangent bundles, from the point of view of Floer theory, more specifically using *wrapped Floer cohomology*. Given two (potentially equal) Lagrangians L, L' (possibly equipped with additional data like bounding cochains or unitary local systems of rank 1) in a symplectic manifold, one can sometimes define their Floer cohomology $HF^*(L, L')$, which is invariant under Hamiltonian perturbations of either L or L'. If $HF^*(L, L') \neq 0$, then L is not Hamiltonian-displaceable from L' (which means that $\varphi(L) \cap L' \neq \emptyset$ for every Hamiltonian diffeomorphism φ of T^*N) [9].

We will focus on the case $N = S^3$ in this paper, even though analogous results are expected to hold for cotangent bundles of other manifolds. In [7], the authors construct a 1-parameter family of disjoint monotone Lagrangian tori T^3_τ, one for each monotonicity constant $\tau > 0$, which can be equipped with local systems with respect to which $HF^*(T^3_\tau, T^3_{\tau'}) \neq 0$ iff $\tau = \tau'$. We will review the construction of [7] below.

We will prove the following result.

Theorem 1.1. *Let $L \subset T^*S^3$ be a closed orientable spin monotone Lagrangian, possibly equipped with a bounding cochain and/or a unitary local system of rank 1 for which $HF^*(L, L) \neq 0$. Then, either $HF^*(L, S^3) \neq 0$ (where the zero-section S^3 is equipped with a certain bounding cochain) or there is a $\tau > 0$ for which $HF(L, T^3_\tau) \neq 0$ (where T^3_τ is equipped with a certain unitary local system of rank 1). In particular, L is non-displaceable from either S^3 or from T^3_τ for some $\tau > 0$.*

For a specific application of this result, T^*S^3 contains a 1-parameter family of disjoint monotone Lagrangians $(S^1 \times S^2)_\tau$, one for each monotonicity constant $\tau > 0$. They can be constructed, for instance, with the help of the standard Lefschetz fibration with two singularities on the affine complex quadric of dimension 3, which is symplectomorphic to T^*S^3. We will give more details below.

Corollary 1.2. *For every $\tau > 0$, the Lagrangians $(S^1 \times S^2)_\tau$ can be equipped with local systems, such that $HF^*((S^1 \times S^2)_\tau, T^3_\tau) \neq 0$. In particular, $(S^1 \times S^2)_\tau$ is not Hamiltonian-displaceable from T^3_τ.*

Proof. □

We now describe the structure of the proof of Theorem 1.1. The Lagrangians L in the statement are elements of a monotone wrapped Fukaya category of T^*S^3, which also includes a cotangent fiber $F = T^*_qS^3$ (for some $q \in S^3$). This is an A_∞-category (with only a $\mathbb{Z}/2\mathbb{Z}$-grading, since we allow monotone Lagrangians), which we denote temporarily by $\mathcal{W}$ (and will refer to as $\mathcal{W}^{\mathbb{Z}/2\mathbb{Z}}_{\text{mon}}(Y;\Lambda^{\mathbb{C}})$ in Section 3.3). Morphism spaces in $\mathcal{W}$ between two Lagrangians L and L' are given by wrapped Floer cochain complexes $CW^*(L,L')$ (which can be thought of as the usual Lagrangian Floer cochain complexes $CF^*(L,L')$ when at least one of L or L' is closed), and the higher A_∞-operations are defined in terms of counts of pseudoholomorphic polygons with boundary edges mapping to Lagrangians.

The category $\mathcal{W}$ is generated (in a sense which we review below) by the cotangent fiber F. This is an adaptation of a result in [4] (which in its original form was about a version of the wrapped Fukaya category containing only exact Lagrangians). This generation result can be restated in a more algebraic manner. Let $A := HW^*(F,F)$ be the wrapped Floer cohomology algebra of F. There is a *Yoneda functor*

$$\begin{aligned} Y : \mathcal{W} &\to \text{mod}(A) \\ L &\mapsto HF^*(F,L) \end{aligned}$$

where $\text{mod}(A)$ is the category of ($\mathbb{Z}/2\mathbb{Z}$-graded) right A_∞-modules over A. The morphism space between two objects M, M' in $\text{mod}(A)$ is $\text{Ext}^*(M,M')$.

The generation criterion mentioned above, together with formality results for A_∞-modules that we prove in Sections 5 and 6, imply that Y is a cohomologically fully faithful functor, in the sense that it induces an isomorphism on cohomology

$$HW^*(L,L') \cong \text{Ext}^*(Y(L),Y(L'))$$

for any pair of objects L, L'. Since we are interested in compact Lagrangians, we take the subcategory $\mathcal{F} \subset \mathcal{W}$ (denoted as $\mathcal{F}^{\mathbb{Z}/2\mathbb{Z}}_{\text{mon}}(Y;\mathbb{Z}^{\mathbb{C}})$ in Section 3.3) that does not include the object F, but only compact Lagrangians. Given such an L, $HW^*(F,L) = HF^*(F,L)$ is a finite dimensional vector space, so Y restricts to a cohomologically full and faithful embedding

$$Y_c : \mathcal{F} \to \text{mod}_{pr}(A)$$

where $\text{mod}_{pr}(A) \subset \text{mod}(A)$ is the subcategory of proper A-modules M, whose cohomology is finite dimensional. With this in mind, the approach of this paper is to study the category $\mathcal{F}$ by analyzing the algebraic category $\text{mod}_{pr}(A)$.

Theorem 4.10 below is a classification result for $\text{mod}_{pr}(A)$, according to which it is split-generated by modules coming from the zero section S^3 (with auxiliary bounding cochains) and T^3_τ (with auxiliary unitary locals system of rank 1). It follows from the more algebraic Corollary 6.10. Theorem 1.1 is a consequence of the following result (which appears as Corollary 4.11 below).

Theorem 1.3. *The Fukaya category $\mathcal{F}$ of monotone Lagrangians in T^*S^3 (possibily equipped with bounding cochains and/or unitary local systems of rank 1) is split-generated by the uncountable collection of objects consisting of S^3 (equipped with local systems) and the T^3_τ (equipped with unitary local systems of rank 1).*

Proof of Theorem 1.1. Given L such that $HF^*(L,L) \neq 0$, L is a non-trivial object in $\mathcal{F}$. Theorem 1.3 then implies that either $HF^*(L,S^3)$ or $HF^*(L,T^3_\tau)$ are non-trivial, as wanted. □

Remark 1.4 (Relation to mirror symmetry). As mentioned above, the tori T^3_τ were studied in [7]. They are fibers of an SYZ fibration in the complement of a symplectic hypersurface H in T^*S^3. The divisor H is *anticanonical*, in the sense that the Lagrangian tori in the SYZ fibration have vanishing Maslov class.

The authors compute the superpotentials associated to fibers by studying wall-crossing of pseudoholomorphic disks. This information is used to construct an explicit mirror to T^*S^3, and the critical locus of the superpotential corresponds precisely to the monotone Lagrangians T^3_τ (equipped with appropriate local systems). This suggests that the tori T^3_τ, together with the zero-section S^3, should in some sense contain all the relevant Floer-theoretic information in T^*S^3. One can think of Theorems 1.1 and 1.3 as giving substance to this intuition.

Remark 1.5 (Relation to abstract flux). The monotone Lagrangian tori T^3_τ can be obtained geometrically as follows. Let $f : S^3 \to \mathbb{R}$ be a Morse–Bott function whose critical locus consists of a Hopf link. The graph of df intersects the zero section of T^*S^3 cleanly along that Hopf link, and one can perform surgery on this clean intersection (see [6, 14] for details) to produce the family T^3_τ.

Recall that given a compact manifold N and a class $\alpha \in H^1(N;\mathbb{R})$, one can take the *flux deformation* of the zero-section of T^*N in the direction of α, by flowing N along a symplectic vector field X such that $[\omega(.,X)] = i^*\alpha$ (where $i : N \to T^*N$ is the inclusion). Using the Weinstein tubular neighborhood theorem, one can similarly deform a compact Lagrangian L in a symplectic manifold (M,ω) along a class $\alpha \in H^1(L;\mathbb{R})$. Motivated by [19], one can think of the family of tori T^3_τ as an *abstract flux deformation* of two copies of the zero section S^3 in the direction of a generator of $H^*(S^3)$. More precisely, each T^3_τ has an idempotent summand S_τ in the split-closure of $\mathcal{W}$, and the 1-parameter family of *abstract objects* S_τ is an abstract deformation of the zero-section S^3, in the direction of a generator of $H^3(S^3;\mathbb{R})$. The endomorphism algebra of S_τ is isomorphic to that of S^3 equipped with a certain bounding cochain related to that generator of $H^3(S^3;\mathbb{R})$.

This paper is organized as follows. In Section 2, we present the construction of the monotone Lagrangians T^3_τ and $(S^1 \times S^2)_\tau$ in T^*S^3. In Part 1, we study the monotone wrapped Fukaya category of T^*S^3. We review the definition of wrapped Floer cohomology, including the case of Lagrangians with clean intersections. We compute Floer cohomology algebras and modules associated to Lagrangians in T^*S^3, with a view towards proving Theorem 1.3. In Part 2, we study A_∞-algebras and categories of A_∞-modules. We establish formality and generation results for a category of modules associated to a cotangent fiber of T^*S^3.

2. Monotone Lagrangians in T^*S^3

We borrow some notation from [7]. Write

$$Y = \{(z,u_1,v_1,u_2,v_2) \in \mathbb{C}^5 \,|\, u_1v_1 = z - b_1, u_2v_2 = z - b_2\}$$

where $b_1 \neq b_2$ and $b_i \neq 0$. Y is a complex affine 3-quadric, and can be equipped with a symplectic structure that makes it symplectomorphic to T^*S^3 [15, Exercise 6.20]. We will mostly consider the case $b_1 = -1$ and $b_2 = 1$, but will leave the b_i unspecified as much as possible. Consider the Lefschetz fibrations

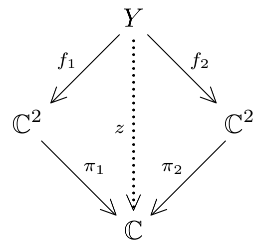
$$\pi_i : \mathbb{C}^2 \to \mathbb{C}$$
$$(u_i, v_i) \mapsto u_iv_i + b_i.$$

π_i has a unique critical value at b_i and, given $p \in \mathbb{C} \setminus \{b_i\}$, the vanishing circle in $\pi_i^{-1}(p)$ is

$$V_{i,p} := \{(u_i, v_i) \in \mathbb{C}^2 \mid \pi_i(u_i, v_i) = p, |u_i| = |v_i|\}.$$

For more details, see [7] and [18, Example 16.5].

Y is the fiber product of these two fibrations:



The map $z : Y \to \mathbb{C}$ is not a Lefschetz fibration, but can be thought of as a Morse–Bott analogue, whose critical values are the b_i and such that the critical points mapping to each b_i form a copy of $\mathbb{C}^*$.

Definition 2.1. Given a simple closed curve $C \subset \mathbb{C} \setminus \{b_1, b_2\}$, let

$$T_C := \{(z, u_1, v_1, u_2, v_2) \in Y \mid (u_i, v_i) \in V_{i,z}, \text{ for } i = 1, 2\} = \bigcup_{z \in C} V_{1,z} \times V_{2,z}.$$

Definition 2.2. Given an embedding $\gamma : [0, \infty) \to \mathbb{C}$ such that

- $\gamma(0) = b_2$,
- $\gamma(0, \infty) \subset \mathbb{C} \setminus \{b_1, b_2\}$ and
- $\gamma(t) = at + b$ for some $a \in \mathbb{C}^*$, $b \in \mathbb{C}$ and t large enough

let

$$N_\gamma := \{(z, u_1, v_1, u_2, v_2) \in Y \mid (u_i, v_i) \in V_{i,\gamma(t)}, \text{ for } i = 1, 2 \text{ and } t \in [0, \infty)\} = \\ = \bigcup_{t \geq 0} V_{1,\gamma(t)} \times V_{2,\gamma(t)}.$$

Recall the definitions of exact and monotone Lagrangian from the Introduction.

Lemma 2.3. *T_C and N_γ are Lagrangian submanifolds of Y. T_C is diffeomorphic to T^3 and monotone, with monotonicity constant $\tau = A/2$, where A is the area of the region bounded by C in $\mathbb{C}$. N_γ is diffeomorphic to $S^1 \times \mathbb{R}^2$ and exact.*

Proof. □

The N_γ are actually Hamiltonian isotopic to the conormal bundle of the unknot in S^3.

Fix $\tau > 0$ and a smooth simple closed curve $C \subset \mathbb{C} \setminus \{b_1, b_2\}$ that winds around b_1 and b_2 and bounds area 2τ in $\mathbb{C}$. Denote by T^3_τ the corresponding Lagrangian torus T_C. Fix also a family of embeddings γ_w as in Definition 2.2, for $w \in \mathbb{Z}_{\geq 0}$, each of which intersecting C transversely in a single point p_w (where $p_w \neq p_{w'}$ if $w \neq w'$). We require that for all w and for all T large enough, the concatenation of $\gamma_w|_{[0,T]}$ with a line segment from $\gamma_w(T)$ to b_2 has winding number w around b_1. In addition, for all $w > w'$, γ_w intersects $\gamma_{w'}$ transversely in exactly $w - w' + 1$ points (b_2 being one of them). See Figure 1. We denote the Lagrangian N_{γ_w} by N_w.

Recall that two Lagrangians $L_0, L_1 \in M$ *intersect cleanly* if $K := L_0 \cap L_1$ is a manifold and for every $x \in K$ we have $T_xK = T_xL_0 \cap T_xL_1 \subset T_xM$.

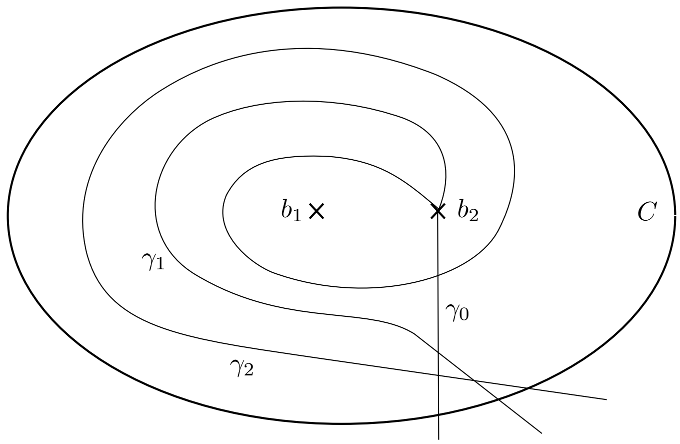


FIGURE 1. Some curves γ_w

Lemma 2.4. *For every $w \geq 0$, N_w and T^3_τ intersect cleanly. For every $w, w' \geq 0$, N_w and $N_{w'}$ intersect cleanly (this is trivial if $w = w'$).*

Proof. This follows from the fact that the Lagrangians project to transverse curves in the base of the map $z : Y \to \mathbb{C}$. □

Part 1. Wrapped Floer homology of Lagrangians in T^*S^3

3. Wrapped Fukaya categories

The wrapped Fukaya category of a Liouville domain M was introduced in [3]. In the original definition, the objects are exact Lagrangians in the completed Liouville manifold $\widehat{M}$. The Lagrangians are either compact or agree outside of a compact set with the product of $\mathbb{R}$ with a Legendrian submanifold of the contact manifold ∂M. We want to consider different versions of wrapped Fukaya category, possibly allowing for closed monotone Lagrangians, as in [16]. Given Lagrangians intersecting cleanly, we will use a Morse–Bott formalism similar to that of [19] to compute the associated A_∞ maps μ^i. Because of the formality results we will establish in Part 2, we will actually only need to compute differentials μ^1 and products μ^2.

3.1. **Coefficients.** Some of the Floer cohomology groups we will study are defined with coefficients in $\mathbb{Z}$, and some over the Novikov field

$$\Lambda^{\mathbb{C}} := \left\{ \sum_{i=0}^{\infty} a_i T^{\lambda_i} \,|\, a_i \in \mathbb{C}, \lambda_i \in \mathbb{R}, \lambda_i < \lambda_{i+1}, \lim_{i \to \infty} \lambda_i = \infty \right\}.$$

We could have replaced $\mathbb{C}$ with any algebraically closed field of characteristic zero, so that the Novikov field is algebraically closed. See Section 6 for more on this point.

There is a *valuation* map

$$\mathrm{val}\colon \Lambda^{\mathbb{C}} \to (-\infty, \infty]$$
$$\sum_{i=0}^{\infty} a_i T^{\lambda_i} \mapsto \min\{\lambda_i \,|\, a_i \neq 0\}$$

where $\mathrm{val}(\alpha) = \infty$ iff $\alpha = 0$. Say that $\alpha \in \Lambda^{\mathbb{C}}$ is *unitary* if $\mathrm{val}(\alpha) = 0$. Denote by

$$U_{\Lambda^{\mathbb{C}}} := \left\{\alpha \in \Lambda^{\mathbb{C}} \mid \mathrm{val}(\alpha) = 0\right\}$$

the group of unitary elements in $\Lambda^{\mathbb{C}}$, and by

$$\Lambda_0^{\mathbb{C}} := \left\{\alpha \in \Lambda^{\mathbb{C}} \mid \mathrm{val}(\alpha) \geq 0\right\}$$

the *Novikov ring.*

3.2. **Morse–Bott Floer cohomology for clean intersections.** We will work in the setting of [19, Section 3.2]. Let L_0, L_1 be two oriented spin Lagrangians such that each L_i is equipped with:

- a Pin structure (as in [18, Section 12a]).
- a unitary local system $\nabla_i \in H^1(L_i; U_{\Lambda^{\mathbb{C}}})$ on a rank 1 $\Lambda^{\mathbb{C}}$-bundle E_i over L_i;
- a bounding cochain $b_i \in H^{2k+1}(L_i; \Lambda_0^{\mathbb{C}})$, where $3 \leq 2k+1 \leq \dim L$.

The categories of exact Lagrangians we will consider are $\mathbb{Z}$-graded, so we will need additional choices of gradings for the L_i (as in [18, Section 12a]). The categories of monotone Lagrangians will only be $\mathbb{Z}/2\mathbb{Z}$-graded, so we will not need the L_i to be graded in that case.

Write $\mathcal{L}_i$ for $(L_i, E_i, \nabla_i, b_i)$. Assume that the Lagrangians intersect cleanly and let $f : L_0 \cap L_1 \to \mathbb{R}$ be a Morse function. Define the chain complex

$$CF^k(\mathcal{L}_0, \mathcal{L}_1) = \bigoplus_{C \subset L_0 \cap L_1} \bigoplus_{\substack{p \in \mathrm{Crit}(f|_C) \\ |p| = k}} \mathrm{Hom}_{\Lambda^{\mathbb{C}}}((E_0)_p, (E_1)_p)$$

The operations μ^1 and μ^2 on these chain complexes will be defined by counts of *pearly configurations*, which we will now describe.

3.2.1. *Differential.* To define the differential on $CF^*(\mathcal{L}_0, \mathcal{L}_1)$, we need the choice of a Morse–Smale vector field Z_f on $L_0 \cap L_1$ that is gradient-like for f. In what follows, the operator $\overline{\partial}$ is defined with respect to the usual integrable complex structures on $\mathbb{R} \times [0,1]$ and Y.

Given $p, q \in \mathrm{Crit}(f)$, a *pearly chain* from p to q consists of the following data:

- a finite set of real numbers $S_2, \ldots, S_k > 0$ ($k \in \mathbb{Z}_{>0}$). Write $S_1 = 0$;
- a tuple $\underline{\gamma} = (\gamma_1, \ldots, \gamma_{k+1})$ of maps $\gamma_1 : (-\infty, 0] \to L_0 \cap L_1$, $\gamma_i : [0, S_i] \to L_0 \cap L_1$ for $2 \leq i \leq k$ and $\gamma_{k+1} : [0, \infty) \to L_0 \cap L_1$, such that $\dot{\gamma}_i(s) = Z_f(\gamma_i(s))$ for all $1 \leq i \leq k+1$, $\lim_{s \to -\infty} \gamma_1(s) = q$ and $\lim_{s \to \infty} \gamma_{k+1}(s) = p$;
- a tuple $\underline{u} = (u_1, \ldots, u_k)$ of maps $u_i : \mathbb{R} \times [0,1] \to Y$ for $1 \leq i \leq k$ such that $\overline{\partial} u_i = 0$, $u(\mathbb{R} \times \{0\}) \subset L_0$, $u(\mathbb{R} \times \{1\}) \subset L_1$, $\lim_{s \to -\infty} u_i(s,t) = \gamma_i(S_i)$ and $\lim_{s \to \infty} u_i(s,t) = \gamma_{i+1}(0)$.

We also allow for pearly chains with $k = 0$, given by $\gamma_1 : \mathbb{R} \to M$ such that $\dot{\gamma}_1(s) = Z(\gamma_1(s))$, $\lim_{s \to -\infty} \gamma_1(s) = q$ and $\lim_{s \to \infty} \gamma_1(s) = p$.

We denote such a pearly chain by $(\underline{\gamma}, \underline{u})$. It has an index given by... One can associate to $(\underline{\gamma}, \underline{u})$ a continuous path $c_0 : \mathbb{R} \to L_0$, obtained by successively concatenating the γ_i with $u_i \circ \phi|_{[-1,1] \times \{0\}}$, where $\phi : [-1,1] \times [0,1] \to [-\infty, \infty] \times [0,1]$ is an orientation-preserving diffeomorphism such that $\phi(\pm 1, t) = (\pm\infty, t)$. Similarly, we can construct a path $c_1 : \mathbb{R} \to L_1$, by concatenating the γ_i with the $(u_1 \circ \phi)|_{[-1,1] \times \{1\}}$.

Let $\mathcal{M}^0(p,q)$ be the space of pearly chains from p to q of index 0. The differential in $\mu^1 \colon CF^*(\mathcal{L}_0,\mathcal{L}_1) \to CF^*(\mathcal{L}_0,\mathcal{L}_1)$ is given by

$$\mu^1(\psi_p) = \sum_{q\in \mathrm{Crit}(f)} \sum_{(\underline{\gamma},\underline{u})\in\mathcal{M}^0(p,q)} \pm T^{\underline{u}}\,(P^1_{c_1})^{-1}\circ\psi_p\circ P^0_{c_0}$$

for every $\psi_p \in \mathrm{Hom}((E_0)_p,(E_1)_p)$, where $P^i_{c_i} : (E_i)_q \to (E_i)_p$ is parallel transport along c_i, with respect to ∇_i. Denote by $HF^*(\mathcal{L}_0,\mathcal{L}_1)$ the homology of $CF^*(\mathcal{L}_0,\mathcal{L}_1)$ with this differential.

Note that if L_0 is an exact Lagrangian, the differential in $CF^*(\mathcal{L}_0,\mathcal{L}_0)$ is the Morse differential on L_0.

3.2.2. *Product.* The product given by μ^2 in the Fukaya category can also be defined in terms of pearly configurations. Let $\mathcal{L}_i = (L_i, E_i, \nabla_i, b_i)$, $i \in \{0,1,2\}$, be three Lagrangians equipped with unitary local systems and bounding cochains. Assume that L_i intersects L_j cleanly for all i and j, and choose Morse functions $f_{ij} : L_i \cap L_j \to \mathbb{R}$ for all $0 \le i \le j \le 2$. Choose also Morse–Smale vector fields Z_{ij} that are grandient-like for the f_{ij}. Given $p \in \mathrm{Crit}(f_{12})$, $q \in \mathrm{Crit}(f_{01})$ and $r \in \mathrm{Crit}(f_{02})$, a *pearly triangle* from p and q to r consists of:

- a continuous map $u : D^2 \to Y$ such that $\overline{\partial}(u|_{\mathring{D}^2}) = 0$, the points in ∂D^2 with argument in $[-\pi,-\pi/3]$ map to L_0, those with argument in $[-\pi/3,\pi/3]$ map to L_1 and those with argument in $[\pi/3,\pi]$ map to L_2;
- a finite sequence of gradient flow segments for Z_{12}, alternating with pseudoholomorphic strips with boundaries in L_1 and L_2 as in the definition of a Floer trajectory above, but with the first (half-infinite) flow line starting at p and the last (finite) flow segment ending at $u(e^{i\pi/3})$;
- a similar sequence of flow segments for Z_{01} and pseudoholomorphic strips with boundaries in L_0 and L_1, starting at q and ending at $u(e^{i\pi/3})$;
- a similar sequence of flow segments for Z_{02} and pseudoholomorphic strips with boundaries in L_0 and L_2, but this time ending (with a half-infinite segment) at r and starting (with a finite segment) at $u(-1)$.

Denoting by $\mathcal{M}^0(p,q;r)$ the space of pearly triangles from p and q to r or index 0, the product

$$\mu^2 \colon CF^*(\mathcal{L}_1,\mathcal{L}_2)\otimes CF^*(\mathcal{L}_0,\mathcal{L}_1) \to CF^*(\mathcal{L}_0,\mathcal{L}_2)$$

is given by

$$\mu^2(\psi_p\otimes\psi_q) = \sum_{r\in\mathrm{Crit}(f_{02})}\sum_{\mathcal{M}^0(p,q;r)} \pm T^{\underline{u}}\,(P^1_{c_2})^{-1}\circ\psi_p\circ(P^1_{c_1})^{-1}\circ\psi_q\circ P^0_{c_0}$$

for every $\psi_p \in \mathrm{Hom}((E_0)_p,(E_1)_p)$ and $\psi_q \in \mathrm{Hom}((E_0)_q,(E_1)_q)$, where $P^i_{c_i} : (E_i)_q \to (E_i)_p$ is parallel transport along c_i, with respect to ∇_i. The product μ^2 descends to Floer cohomology.

3.3. **Wrapped Fukaya categories.** It will be useful to consider different versions of the Fukaya A_∞-category of T^*S^3.

- $\mathcal{W}^{\mathbb{Z}}(T^*S^3;\mathbb{Z})$ is a category whose objects are of two types: either N_0 or (S^3,b) for some bounding cochain $b \in H^3(S^3;\mathbb{Z})$. Both N_0 and S^3 are exact Lagrangians, and we equip them with $\mathbb{Z}$-gradings and Pin structures. The morphism spaces are wrapped Floer cochain complexes with coefficients in

$\mathbb{Z}$. The higher A_∞ operations are given by counts of pseudo-holomorphic polygons.

- $\mathcal{W}^{\mathbb{Z}}(T^*S^3;\Lambda^{\mathbb{C}})$ also has objects associated to either N_0 or S^3, equipped with $\mathbb{Z}$-gradings and Pin structures. S^3 can now be equipped with a bounding cochain $b \in H^3(S^3;\Lambda_0^{\mathbb{C}})$. The other difference with respect to $\mathcal{W}^{\mathbb{Z}}(T^*S^3;\mathbb{Z})$ is that the morphism spaces are now wrapped Floer cochain complexes with coefficients in $\Lambda^{\mathbb{C}}$, keeping track of the symplectic area of pseudoholomorphic polygons.
- $\mathcal{W}^{\mathbb{Z}/2\mathbb{Z}}_{\mathrm{mon}}(T^*S^3;\Lambda^{\mathbb{C}})$ has the same objects as $\mathcal{W}^{\mathbb{Z}}(T^*S^3;\Lambda^{\mathbb{C}})$, in addition to all closed orientable spin monotone Lagrangians. The objects are equipped with orientations and Pin structures, and are only $\mathbb{Z}/2\mathbb{Z}$-graded. The monotone Lagrangians also carry rank 1 unitary local systems and bounding cochains. As in $\mathcal{W}^{\mathbb{Z}}(T^*S^3;\Lambda^{\mathbb{C}})$, the morphism spaces are defined with coefficients in $\Lambda^{\mathbb{C}}$.
- $\mathcal{F}^{\mathbb{Z}/2\mathbb{Z}}_{\mathrm{mon}}(T^*S^3;\Lambda^{\mathbb{C}})$ is the full subcategory of $\mathcal{W}^{\mathbb{Z}/2\mathbb{Z}}_{\mathrm{mon}}(T^*S^3;\Lambda^{\mathbb{C}})$ containing only those objects whose underlyin Lagrangians are compact.

Note that Hamiltonian isotopies induce isomorphic objects in all these categories. We can define functors between these categories:

- $\mathcal{W}^{\mathbb{Z}}(T^*S^3;\mathbb{Z}) \to \mathcal{W}^{\mathbb{Z}}(T^*S^3;\Lambda^{\mathbb{C}})$ is the identity on objects. Let u be a pseudoholomorphic disk with boundary punctures, contributing to an A_∞ operation μ^k, $k \geq 1$, in $\mathcal{W}^{\mathbb{Z}}(T^*S^3;\mathbb{Z})$. Then, the contribution of u in $\mathcal{W}^{\mathbb{Z}}(T^*S^3;\Lambda^{\mathbb{C}})$ is weighted by a factor $T^{-\int_{D^2} u^*\omega}$.
- $\mathcal{W}^{\mathbb{Z}}(T^*S^3;\Lambda^{\mathbb{C}}) \to \mathcal{W}^{\mathbb{Z}/2\mathbb{Z}}_{\mathrm{mon}}(T^*S^3;\Lambda^{\mathbb{C}})$ is the inclusion of objects, only remembering the parity of the $\mathbb{Z}$-grading.

Remark 3.1. In principle, we could have allowed arbitrary bounding cochains $b \in H^3(S^3;\Lambda^{\mathbb{C}})$ in $\mathcal{W}^{\mathbb{Z}}(T^*S^3;\Lambda^{\mathbb{C}})$. The reason we did not do so is that such bounding cohains would not define objects in $\mathcal{W}^{\mathbb{Z}/2\mathbb{Z}}(T^*S^3;\Lambda^{\mathbb{C}})$. Indeed, the Floer differentials between S^3 with such bounding cochains and monotone Lagrangians might not converge over $\Lambda^{\mathbb{C}}$.

Lemma 3.2. *Let $L \subset T^*S^3$ be a compact monotone Lagrangian with a unitary rank 1 local system (E,∇) and bounding cochain b. Write $\mathcal{L} = (L,E,\nabla,b)$ If $HF^*(\mathcal{L},\mathcal{L}) \neq 0$, then $\mu^0_{\mathcal{L}} = 0$.*

Proof. □

3.3.1. *Yoneda functors.* Let $\mathcal{A} := CW^*(F,F;\Lambda^{\mathbb{C}})$ be the A_∞-algebra of a cotangent fiber. There is a Yoneda functor

$$\mathcal{Y} : \mathcal{W}^{\mathbb{Z}/2\mathbb{Z}}_{\mathrm{mon}}(T^*S^3;\Lambda^{\mathbb{C}}) \to \mathrm{mod}(\mathcal{A})$$
$$\mathcal{L} \mapsto CW^*(F,\mathcal{L})$$

where $\mathrm{mod}(\mathcal{A})$ is the differential $\mathbb{Z}/2\mathbb{Z}$-graded category of right A_∞-modules over $\mathcal{A}$. Given two objects $\mathcal{M},\mathcal{M}'$ in $\mathrm{mod}(\mathcal{A})$, the morphism space $\hom_{\mathrm{mod}(\mathcal{A})}(\mathcal{M},\mathcal{M}')$ is a chain complex computing $\mathrm{Ext}^*(\mathcal{M},\mathcal{M}')$.

The functor $\mathcal{Y}$ restricts to

$$\mathcal{Y}_c : \mathcal{F}^{\mathbb{Z}/2\mathbb{Z}}_{\mathrm{mon}}(T^*S^3;\Lambda^{\mathbb{C}}) \to \mathrm{mod}_{pr}(\mathcal{A})$$

where $\mathrm{mod}_{pr}(\mathcal{A}) \subset \mathrm{mod}(\mathcal{A})$ is the subcategory of *proper modules* $\mathcal{M}$, such that $H^*(\mathcal{M})$ is finite dimensional (the subscript in $\mathcal{Y}_c$ stands for *compact*).

Now, let $A = H^*(\mathcal{A})$ be the cohomology algebra of $\mathcal{A}$. There is a functor

$$\begin{aligned} H : \mathrm{mod}(\mathcal{A}) &\to \mathrm{mod}(A) \\ \mathcal{M} &\mapsto H^*(\mathcal{M}) \end{aligned}$$

where $\mathrm{mod}(A)$ is the $\mathbb{Z}/2\mathbb{Z}$-graded category of right A_∞-modules over A. This is *not* simply the category of modules over the algebra A, since morphism spaces are Ext^* groups and not simply module homomorphisms.

This functor restricts to

$$H_c : \mathrm{mod}_{pr}(\mathcal{A}) \to \mathrm{mod}_{pr}(A)$$

where $\mathrm{mod}_{pr}(A) \subset \mathrm{mod}(A)$ is the subcategory of $\mathbb{Z}/2\mathbb{Z}$-graded A_∞-modules over A with finite dimensional cohomology.

The proof of Proposition 4.1 implies that the functor $\mathcal{Y}$ (hence also $\mathcal{Y}_c$) is cohomologically fully faithful. According to Corollary 6.7 below, H (hence also H_c) is an equivalence.

Corollary 3.3. *The composition*

$$\begin{aligned} Y := H \circ \mathcal{Y} : \mathcal{W}^{\mathbb{Z}/2\mathbb{Z}}_{\mathrm{mon}}(T^*S^3; \Lambda^{\mathbb{C}}) &\to \mathrm{mod}(A) \\ \mathcal{L} &\mapsto HW^*(F, \mathcal{L}) \end{aligned}$$

and its restriction

$$Y_c := H_c \circ \mathcal{Y}_c : \mathcal{F}^{\mathbb{Z}/2\mathbb{Z}}_{\mathrm{mon}}(T^*S^3; \Lambda^{\mathbb{C}}) \to \mathrm{mod}_{pr}(A)$$

are cohomologically fully faithful embeddings.

4. Computations in T^*S^3

4.1. The Lagrangians N_w.

Proposition 4.1. *The Lagrangian N_0 split-generates $\mathcal{W}^{\mathbb{Z}}(T^*S^3; \mathbb{Z})$, $\mathcal{W}^{\mathbb{Z}}(T^*S^3; \Lambda)$ and $\mathcal{W}^{\mathbb{Z}/2\mathbb{Z}}_{\mathrm{mon}}(T^*S^3; \Lambda)$.*

Proof. Let mod-$C_{-*}(\Omega_p S^3)$ be the category of right A_∞-modules over chains in the based loop space of S^3 (note the cohomological grading). According to [5], there is a functor

$$\mathcal{W}^{\mathbb{Z}}(T^*S^3; \mathbb{Z}) \to \text{mod-}C_{-*}(\Omega_p S^3).$$

By [4], a cotangent fiber F generates the wrapped Fukaya category of T^*S^3, which means that...

For $\mathcal{W}^{\mathbb{Z}/2\mathbb{Z}}_{\mathrm{mon}}(T^*S^3; \Lambda)$, use Viterbo functoriality to compare with a deformation of λ to pdq. Quote [16] and [20].

□

Proposition 4.2. *For every $w \geq 0$, N_w is quasi-isomorphic to $F \oplus F[1]$ in $\mathcal{W}^{\mathbb{Z}}(T^*S^3; \mathbb{Z})$. In particular, $HW^*(N_w, N_w; \mathbb{Z})$ is isomorphic to the graded matrix algebra*

$$B := \begin{pmatrix} \mathbb{Z}[u] & \mathbb{Z}[u][1] \\ \mathbb{Z}[u][-1] & \mathbb{Z}[u] \end{pmatrix}$$

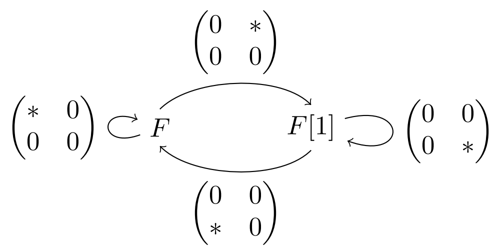
over $\mathbb{Z}[u]$, where u is of degree -2.

Proof. This follows from the fact that N_w intersects the zero section cleanly along a copy of S^1...

To see that $HW^*(N_w, N_w; \mathbb{Z})$ is isomorphic to the matrix algebra in the statement, recall that

$$HW^*(F, F; \mathbb{Z}) \cong H_*(\Omega S^3; \mathbb{Z}) \cong \mathbb{Z}[u]$$

where u is of degree -2 [1, 4]. The endomorphisms of the object $F \oplus F[1]$ can be represented as follows [18, Sections 3b and 3d].



□

Define

$$e_1 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \; e_2 := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \; e_{21} := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \; e_{12} := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Note that

$$|e_1| = 0 = |e_2|, \; |e_{21}| = 1, \; |e_{12}| = -1.$$

If we think of B as a graded free abelian group, it has generators in low degrees given by

degree	1	0	-1	-2	-3
generator	e_{21}	e_1, e_2	e_{12}, ue_{21}	ue_1, ue_2	ue_{12}, u^2e_{21}

The following simple observations will be useful.

Lemma 4.3. *(1) The unit in B is $e := e_1 + e_2$.*

(2) $e_i e_{jk} = \delta_{ij} e_{jk}$ and $e_{ij} e_k = \delta_{jk} e_{ij}$.

(3) ue and $-ue$ are the only elements $y \in B$ of degree -2 such that
- *y is part of a $\mathbb{Z}$-basis for the degree -2 summand of B;*
- *$ye_{21} = e_{21}y$.*

Proof. Only (3) is not immediate. To prove this, observe that any element $y \in B$ of degree -2 can be written as

$$y = aue_1 + bue_2$$

where $a, b \in \mathbb{Z}$.

Now, $ye_{21} = bue_{21}$ and $e_{21}y = aue_{21}$, so $ye_{21} = e_{21}y$ implies $a = b$. Since y is part of a $\mathbb{Z}$-basis for the degree -2 summand of B, together with an element

$$z = cue_1 + due_2$$

such that $c, d \in \mathbb{Z}$, we have

$$\det \begin{pmatrix} a & a \\ c & d \end{pmatrix} = a(d - c) = \pm 1.$$

This implies that $a = \pm 1$, as wanted. □

Given $w > 0$, the intersection $N_{w+1} \cap N_w$ is clean, and diffeomorphic to a disjoint union of S^1 (over b_2) with T^2 (over some point in $\mathbb{C} \setminus \{b_1, b_2\}$). Denote by $e_w \in HF^0(N_{w+1}, N_w)$ the class that corresponds to the fundamental class of S^1. The triangle product gives a map

$$\kappa_w \colon HF^*(N_w, N_0; \mathbb{Z}) \to HF^*(N_{w+1}, N_0; \mathbb{Z})$$
$$x \mapsto \mu^2(x, e_w)$$

We can now define the *wrapped Floer homology* of N_0 to be the direct limit

$$HW^*(N_0, N_0; \mathbb{Z}) = \lim_{w\to\infty} HF^*(N_w, N_0; \mathbb{Z})$$

with respect to the maps κ_w [?AbouzaidSeidelFuture].

Lemma 4.4. *The maps κ_w are injective.*

Proof. □

This implies that the induced maps $HF^*(N_1, N_0) \to HW^*(N_0, N_0)$ are graded inclusions, and we can use $HF^*(N_1, N_0)$ for module computations.

We now study the group $HF^*(N_1, N_0; \mathbb{Z})$ from the point of view of Morse–Bott Floer cohomology, with respect to a Morse function $f_{10} : N_1 \cap N_0 \to \mathbb{R}$ and an associated Morse–Smale vector field Z_{10} whose differential vanishes. Note that the clean intersection $N_1 \cap N_0$ is diffeomorphic to the disjoint union $S^1 \cup T^2$. Since the N_w bound no non-constant pseudoholomorphic disks (they are exact) and the Morse differential of Z_{10} vanishes, we get an isomorphism

$$HF^*(N_1, N_0; \mathbb{Z}) \cong H^*(S^1 \cup T^2; \mathbb{Z}). \tag{4.1}$$

As we saw in Proposition 4.2 and the discussion that followed, the free abelian group $HW^*(N_0, N_0; \mathbb{Z})$ has two generators in degree -2, denoted by ue_1 and ue_2.

Lemma 4.5. *(1) The class $e_{21} \in HW^*(N_0, N_0; \mathbb{Z})$ is represented by the class of a point in S^1, under* (4.1).

(2) The class of the unit $e = e_1 + e_2 \in HW^*(N_0, N_0, \mathbb{Z})$ is represented by the fundamental class of S^1, under (4.1).

(3) The class $ue_1 + ue_2 = ue \in HW^*(N_0, N_0; \mathbb{Z})$ is represented by the fundamental class of T^2, under (4.1).

Proof. For (1), note that the summand of A in homogeneous degree 1 is spanned by e_{21}. For (2) ... For (3), we use Lemma 4.3(3). If $y \in HF^*(N_1, N_0; \mathbb{Z})$ represents the fundamental class of T^2, as in the statement, we show that

$$\mu^2(y, e_{21}) = \mu^2(e_{21}, y).$$

□

4.2. Floer cohomology of F_w with S^3.

Proposition 4.6. *There is an isomorphism*

$$HF^*(F_w, (S^3, T^\lambda[pt]); \Lambda^{\mathbb{C}}) \cong \Lambda^{\mathbb{C}}$$

possibly with a degree shift.

4.3. **Floer cohomology of N_w with T^3_τ.** Fix a local system (E,∇) on T^3_τ such that ...

Denote the triple (T^3_τ, E, ∇) by $\mathcal{T}_\tau$. We want to apply the framework of Morse–Bott Floer cohomology to compute $HF^*(N_w, \mathcal{T}_\tau)$, for $w \geq 0$. We are implicitly equipping the Lagrangian N_w with the trivial rank 1 local system. The clean intersection $N_w \cap T_\tau$ is diffeomorphic to T^2 and we define $CF^*(N_w, \mathcal{T}_\tau)$ with respect to a Morse function $f_w : N_w \cap T^3_\tau \to \mathbb{R}$ and an associated Morse–Smale vector field Z_w whose associated Morse differential vanishes.

Proposition 4.7. *There is an isomorphism*

$$HF^*(N_w, \mathcal{T}_\tau; \Lambda^{\mathbb{C}}) \cong H^*(T^2; \Lambda),$$

possibly with a degree shift.

Proof. We wish to prove that the differential in $CF^*(N_w, \mathcal{T}_\tau)$ vanishes. The proof follows an idea that can be found in [17, 18]. First, note that contributions to the differential consisting solely of gradient flow lines all cancel each other, since the Morse differential of Z_w is assumed to vanish. We will show that any contributions to the differential containing pseudoholomorphic curves cancel in pairs, by showing that the moduli space of such configurations is cobordant to another moduli space, whose signed count of points is zero.

□

4.4. **Module actions on the Fukaya category.** Recall that $\alpha \in \Lambda^{\mathbb{C}}_0$ and a generator $[pt] \in H^3(S^3;\mathbb{Z})$ define a bounding cochain $\alpha[pt]$ in S^3. We want to relate F_0 and $(S^3, \alpha[pt])$ in $\mathcal{W}^{\mathbb{Z}}(T^*S^3; \Lambda^{\mathbb{C}})$.

Given $x \in HF^*(F_1, F_0; \Lambda^{\mathbb{C}})$, define a map

$$\psi_x \colon HF^*(F_0, (S^3, \alpha[pt]); \Lambda^{\mathbb{C}}) \to HF^*(F_1, (S^3, \alpha[pt]); \Lambda^{\mathbb{C}})$$
$$y \mapsto \mu^2(y, x)$$

Proposition 4.8. $\psi_{ue} = \alpha\, \psi_e$.

Proof. □

We now want to relate N_0 and $\mathcal{T}_\tau$ in $\mathcal{W}^{\mathbb{Z}/2\mathbb{Z}}_{\mathrm{mon}}(T^*S^3; \Lambda^{\mathbb{C}})$.

Given $x \in HF^*(N_1, N_0; \Lambda^{\mathbb{C}})$, define a map

$$\phi_x \colon HF^*(N_0, \mathcal{T}_\tau; \Lambda) \to HF^*(N_1, \mathcal{T}_\tau; \Lambda^{\mathbb{C}})$$
$$y \mapsto \mu^2(y, x)$$

Proposition 4.9. $\phi_{ue} = T^{-2\tau}\, \phi_e$.

Proof. □

Recall that $A = \Lambda^{\mathbb{C}}[u]$.

Theorem 4.10. *The collection of A-modules*

$$\{HF^*(N_0, (S^3, \alpha[pt]); \Lambda^{\mathbb{C}})\}_{\mathrm{val}(\alpha) \geq 0} \cup \{HF^*(N_0, \mathcal{T}_\tau; \Lambda^{\mathbb{C}})\}_{\tau > 0}$$

split-generates the category $\mathrm{mod}_{pr}(A)$.

Proof. By Propositions 4.6 and 4.8,

$$HF^*(F_0, (S^3, \alpha[pt]); \Lambda^{\mathbb{C}}) \cong S_\alpha$$

as right A-modules, where S_α is the 1-dimensional (over $\Lambda^{\mathbb{C}}$) right A-module in which $u \in A$ acts as multiplication by α (as in Lemma 6.9 below).

By Propositions 4.7 and 4.9,

$$HF^*(N_0, \mathcal{T}_\tau; \Lambda^{\mathbb{C}}) \cong M \oplus M[1]$$

as right A-modules, where $M = S_\lambda \oplus S_\lambda[1]$, $\lambda := T^{-2\tau} \in \Lambda^{\mathbb{C}}$ and S_λ is the 1-dimensional (over $\Lambda^{\mathbb{C}}$) right A-module in which $u \in A$ acts as multiplication by λ.

By Proposition 4.2, we have

$$HF^*(N_0, \mathcal{T}_\tau; \Lambda^{\mathbb{C}}) \cong HF^*(F, \mathcal{T}_\tau; \Lambda^{\mathbb{C}}) \oplus HF^*(F, \mathcal{T}_\tau; \Lambda^{\mathbb{C}})[1]$$

so $HF^*(F, \mathcal{T}_\tau; \Lambda^{\mathbb{C}}) \cong M = S_\lambda \oplus S_\lambda[1]$ as a right A-module.

Proposition 6.10 now implies the result. □

Corollary 4.11. *The category $\mathcal{F}^{\mathbb{Z}/2\mathbb{Z}}_{mon}(T^*S^3; \mathbb{Z})$ is split-generated by the collection of objects $\{(S^3, \alpha[pt])\}_{\mathrm{val}(\alpha)\geq 0} \cup \{\mathcal{T}_\tau\}_{\tau>0}$.*

Proof. This follows from Theorem 4.10 and Corollary 3.3. □

Part 2. **Algebras and modules**

5. Hochschild cohomology and formality of $CW^*(F,F)$

Let S be a simply connected smooth manifold, with an orientation, spin structure and $\mathbb{Z}$-grading. As shown in [4] for every $p \in S$, there is a quasi-isomorphism of A_∞-algebras

$$CW^*(T^*_pS, T^*_pS) \cong C_{-*}(\Omega_p S).$$

Recall that an A_∞-algebra $\mathcal{A}$ is *formal* if it is quasi-isomorphic to its cohomology algebra $A = H^*(\mathcal{A})$. By the homological perturbation lemma [18, Section 1i], $\mathcal{A}$ is quasi-isomorphic to an A_∞-structure on the graded algebra A. The latter is *intrinsically formal* (and hence $\mathcal{A}$ is formal) if all A_∞-structures on A are quasi-isomorphic to A. It is known that A is intrinsically formal if $HH^2(A, A) = 0$ [?KadeishviliHH].

The cohomology algebra of $C_{-*}(\Omega_p S^n)$ is the polynomial algebra $A = \Lambda^{\mathbb{C}}[u]$ over the Novikov field $\Lambda^{\mathbb{C}}$, where u has degree $1 - n$.

Remark 5.1. This notation may be confusing, but $\Lambda^{\mathbb{C}}[u]$ is *not* an exterior algebra.

Lemma 5.2. *A is intrinsically formal when $n \geq 2$.*

Proof. This is a consequence of the Hochschild–Kostant–Rosenberg theorem [13], according to which

$$HH^*(A, A) \cong \Lambda^{\mathbb{C}}[[u]] \otimes_{\Lambda^{\mathbb{C}}} A$$

where $\Lambda^{\mathbb{C}}[[u]]$ is the algebra of formal power series in the variable u. By degree reasons, we then have $HH^2(A, A) = 0$, which implies the result. □

This lemma and the results cited above imply the following. The last statement will be useful in the following section.

Corollary 5.3. *$\mathcal{A} = CW^*(T_p^*S^n, T_p^*S^n)$ is formal when $n \geq 2$. Hence, it is quasi-isomorphic to its cohomology A, which is the polynomial algebra $\Lambda^{\mathbb{C}}[u]$ in a variable u of degree $1-n$. Therefore, $\mathcal{A}$ and A are also quasi-isomorphic as $\mathbb{Z}/2\mathbb{Z}$-graded A_∞-algebras.*

6. Ext groups and formality of A_∞-modules

Let A be a $\mathbb{Z}/2\mathbb{Z}$-graded algebra and let M, N be right $\mathbb{Z}/2\mathbb{Z}$-graded A-modules. We will adopt the following notation for certain graded homomorphism spaces:

$$\mathrm{ext}^r(M,N)^0 := \mathrm{Hom}^0_\Lambda(M \otimes_\Lambda A^{\otimes r}, N)$$

and

$$\mathrm{ext}^{r+1}(M,N)^1 := \mathrm{Hom}^1_\Lambda(M \otimes_\Lambda A^{\otimes r}, N)$$

Denote also by $\partial_{r,s} : \mathrm{ext}^{r+s}(M,N)^s \to \mathrm{ext}^{r+s+1}(M,N)^s$ the operator

$$\begin{aligned}(\partial_{r,s}\phi)(m, a_{r+1}, \dots, a_1) =& (-1)^*\phi(ma_{r+1}, a_r, \dots, a_2)a_1 + \\ &+ \sum_{n=1}^{r}(-1)^*\phi(m, a_{r+1}, \dots, a_{n+1}a_n, \dots, a_1) + \\ &+ (-1)^*\phi(m, a_{r+1}, \dots, a_2)a_1 +\end{aligned}$$

Lemma 6.1. $\partial_{r+1,s} \circ \partial_{r,s} = 0$.

Proof. Standard. □

Denote the associated cohomology groups by $\mathrm{Ext}^{r+s}(M,N)^s$, where $r \in \mathbb{N}_0$ and $s \in \{0,1\}$. The reason for this notation will be given below.

Proposition 6.2. *If $\mathrm{Ext}^{r+1}(M,M)^1 = 0$ for all $r \geq 2$ even and $\mathrm{Ext}^r(M,M)^0 = 0$ for all $r \geq 2$ odd, then any A_∞-structure on M whose cohomology module is M is quasi-isomorphic to M.*

Proof. Adapt Kadeishvili. Note that labeling of inputs differs from Seidel's book p 19 by 1. □

Call a module M satisfying the conclusion of the Proposition *intrinsically formal*. We now turn to computing the groups introduced in this section. Recall that M is a right A-module, and observe that the same is true of $M \otimes_\Lambda A^{\otimes r}$, for every $r \geq 1$. The A-action is given by

$$(m, a_k, \dots, a_1).a = (m, a_k, \dots, a_1 a)$$

Given right A-modules M and N, an A-module homomorphism splits as a sum of its degree 0 and degree 1 parts. In particular, for any $r \geq 0$, we have a splitting

$$\mathrm{Hom}_A(M \otimes_\Lambda A^{\otimes r+1}, N) \cong \mathrm{Hom}^0_A(M \otimes_\Lambda A^{\otimes r+1}, N) \oplus \mathrm{Hom}^1_A(M \otimes_\Lambda A^{\otimes r+1}, N)$$

Lemma 6.3. *Suppose that A is supported in degree 0. For $s = 0$ or 1, $\mathrm{ext}^{r+s}(M,N)^s$ is isomorphic to $\mathrm{Hom}^s_A(M \otimes_\Lambda A^{\otimes r+1}, N)$, as a Λ-vector space.*

Proof. This is similar to [`?GinzburgNotes`, page 22]. Recall that $\mathrm{ext}^{r+s}(M,N)^s = \mathrm{Hom}^s_\Lambda(M \otimes_\Lambda A^{\otimes r}, N)$. The isomorphism is given by a map

$$\mathrm{Hom}^s_A(M \otimes_\Lambda A^{\otimes r+1}, N) \to \mathrm{Hom}^s_\Lambda(M \otimes_\Lambda A^{\otimes r}, N)$$

taking $\varphi \in \mathrm{Hom}^s_A(M \otimes_\Lambda A^{\otimes r+1}, N)$ to $\overline{\varphi} \in \mathrm{Hom}^s_\Lambda(M \otimes_\Lambda A^{\otimes r}, N)$ such that

$$\overline{\varphi}(m, a_k, \dots, a_1) = \varphi(m, a_k, \dots, a_1, 1_A).$$

Note that $\overline{\varphi}$ has the same s-degree as φ since A is supported in degree 0. □

Lemma 6.4. *Any right A-module M admits a projective resolution by right A-modules*

$$\dots \xrightarrow{f_2} M \otimes_\Lambda A^{\otimes 2} \xrightarrow{f_1} M \otimes_\Lambda A \xrightarrow{f_0} M \to 0$$

Proof. The maps $f_k : M \otimes_\Lambda A^{\otimes k+1} \to M \otimes_\Lambda A^{\otimes k}$ are given by
(6.1)

$$f_k(m, a_{k+1}, \dots, a_1) = (-1)^*(m a_{k+1}, \dots, a_1) + \sum_{n=1}^{k} (-1)^*(m, a_{k+1}, \dots, a_{n+1}a_n, \dots, a_1)$$

□

We are now in a position to justify the notation adopted in this section. The following result will also be instrumental for computing the groups $\mathrm{Ext}^{r+s}(M, N)^s$.

Corollary 6.5. *If M, N are right A-modules, then*

$$\mathrm{Ext}^r(M, N)^0 \oplus \mathrm{Ext}^{r+1}(M, N)^1 \cong \mathrm{Ext}^r_A(M, N).$$

Proof. Recall that $\mathrm{Ext}^*_A(M, N)$ is the homology of a chain complex

$$0 \to \mathrm{Hom}_A(P_0, N) \to \mathrm{Hom}_A(P_1, N) \to \mathrm{Hom}_A(P_2, N) \to \dots$$

obtained from a projective resolution of M

$$\dots \to P_2 \to P_1 \to P_0 \to M \to 0$$

By Lemma 6.4, $\mathrm{Ext}^r_A(M, N)$ is the homology of the chain complex

$$0 \to \mathrm{Hom}_A(M \otimes_\Lambda A, N) \xrightarrow{f_1^*} \mathrm{Hom}_A(M \otimes_\Lambda A^{\otimes 2}, N) \xrightarrow{f_2^*} \mathrm{Hom}_A(M \otimes_\Lambda A^{\otimes 3}, N) \xrightarrow{f_3^*} \dots$$

which, by Lemma 6.3, is isomorphic to

$$0 \to \mathrm{ext}^0(M, N)^0 \oplus \mathrm{ext}^1(M, N)^1 \to \mathrm{ext}^1(M, N)^0 \oplus \mathrm{ext}^2(M, N)^1 \to \mathrm{ext}^2(M, N)^0 \oplus \mathrm{ext}^3(M, N)^1 \to \dots$$

One can check that the differential induced by the map f_k^* agrees with $\partial_{k-1,0} \oplus \partial_{k-1,1}$, and the result follows. □

In what follows, we will restrict our attention the case $A = \Lambda[u]$, where u has degree 0.

Proposition 6.6. *Let M and N be right A-modules. Then,* $\mathrm{Ext}^{r+1}(M, N)^1 = 0$ *and* $\mathrm{Ext}^r(M, N)^0 = 0$ *if $r \geq 2$. In particular, all right A-modules are intrinsically formal.*

Proof. By Corollary 6.5, it is enough to show that $\mathrm{Ext}^r_A(M, N) = 0$ for all $r \geq 2$, which we compute using a convenient projective resolution of M. Indeed, consider the Koszul resulution of the A-bimodule A:

$$0 \to A \otimes_\Lambda A \xrightarrow{g_2} A \otimes_\Lambda A \xrightarrow{g_1} A \to 0$$

where

$$g_2(x_1 \otimes x_2) = (ux_1) \otimes x_2 - x_1 \otimes (ux_2), \qquad g_1(x_1 \otimes x_2) = x_1 x_2.$$

We can now get a projective resolution of any right A-module M by tensoring it (over A, not Λ) with the resolution of A:

$$0 \to M \otimes_\Lambda A \to M \otimes_\Lambda A \to M \to 0$$

This immediately implies that for any right A-module N and any $r \geq 2$,

$$\mathrm{Ext}^r_A(M, N) = 0.$$

□

Corollary 6.7. *The functor*

$$\begin{aligned} H : \mathrm{mod}(\mathcal{A}) &\to \mathrm{mod}(A) \\ \mathcal{M} &\mapsto H^*(\mathcal{M}) \end{aligned}$$

is an equivalence. Note that morphisms in $\mathrm{mod}(A)$ *are* Ext *groups.*

Proof. □

Definition 6.8. Let $\mathrm{mod}_{pr}(A)$ be the category of $\mathbb{Z}/2\mathbb{Z}$-graded right A_∞-modules over A, with finite dimensional cohomology (the subscript stands for *proper*).

The fact that $\mathbb{C}$ is algebraically closed of characteristic zero implies that $\Lambda^{\mathbb{C}}$ is also algebraically closed [10, Appendix A].[1] This will enable us to study the category $\mathrm{mod}_{pr}(A)$ using Jordan normal forms.

Recall that $A = \Lambda^{\mathbb{C}}[u]$. Take an object $M \oplus N$ of $\mathrm{mod}_{pr}(A)$, where M is in degree 0, N is in degree 1 and they are both finite dimensional $\Lambda^{\mathbb{C}}$-vector spaces. By Proposition 6.6, we can assume that M and N are cohomological modules (i.e. $M = H^*(M)$ and $N = H^*(N)$). Since $\Lambda^{\mathbb{C}}$ is algebraically closed, M has a splitting

$$M = \bigoplus_{i=1}^{m} M^{k_i}_{\lambda_i}$$

where $\lambda_i \in \Lambda^{\mathbb{C}}$, $k_i \in \mathbb{Z}_+$ and M^k_λ is the right A-module isomorphic as a vector space to $(\Lambda^{\mathbb{C}})^k$, on which u acts on the right as the $k \times k$ transposed Jordan block

$$\begin{pmatrix} \lambda & & & & \\ 1 & \lambda & & & \\ & \ddots & \ddots & & \\ & & 1 & \lambda & \\ & & & 1 & \lambda \end{pmatrix}$$

N also has a spliting

$$N = \bigoplus_{j=1}^{n} M^{k^j}_{\lambda^j}[1]$$

where $\lambda^j \in \Lambda^{\mathbb{C}}$ and $k^j \in \mathbb{Z}_+$.

Denote the 1-dimensional module M^1_λ by S_λ.

Lemma 6.9. M^k_λ *is in the triangulated closure of* S_λ.

[1]In characteristic $p > 0$, the polynomial $x^p - x - T^{-1}$ does not have roots in Λ. See [?Kedlaya] for a discussion of the algebraic closure of the Novikov field in prime characteristic.

Proof. Observe that there are A-module homomorphisms

$$\varphi_\lambda^k \colon M_\lambda^k \to S_\lambda$$

obtained by projecting onto the last coordinate. Recall that $\operatorname{Cone}(\varphi_\lambda^k)$ is the right A_∞-module over A [18, Section (3e)] given by the chain complex

$$(M_\lambda^k[1] \oplus S_\lambda, \mu^1 = (0, \varphi)),$$

with $\mu^2 = (\mu^2_{M_\lambda^k[1]}, \mu^2_{S_\lambda})$ and trivial higher A_∞ maps. We have that $H\operatorname{Cone}(\varphi_\lambda^k) \cong M_\lambda^{k-1}[1]$ and so $\operatorname{Cone}(\varphi)$ is quasi-isomorphic to $M_\lambda^{k-1}[1]$.

We can argue by induction on k to prove the statement in the Lemma. Since there is a distinguished triangle

$$M_\lambda^k \to S_\lambda \to M_\lambda^{k-1}[1] \to M_\lambda^k[1],$$

axiom TR2 of triangulated categories [21, 22] implies that there is also a distinguished triangle

$$S_\lambda \to M_\lambda^{k-1}[1] \to M_\lambda^k[1] \to S_\lambda[1]$$

and by induction on k we get that M_λ^k is in the triangulated closure of S_λ for all $k \geq 1$. □

We can now conclude the following.

Corollary 6.10. *The collection of modules* $\{S_\lambda\}_{\lambda \in \Lambda}$ *generates the category* $\operatorname{mod}_{pr}(A)$.

Remark 6.11. If use used a Novikov field that is not algebraically closed (for instance, if we worked in prime characteristic), then the argument above would not imply that the S_λ generate $\operatorname{mod}_{pr}(A)$. This was used to show that S^3 equipped with bounding cochains, together with the tori T^3_τ equipped with rank 1 unitary local systems, split-generate the category of compact monotone Lagrangians in T^*S^3 (Corollary 4.11). One might be able to consider fields of characteristic $p > 0$, possibly at the expense of allowing higher rank local systems on the T^3_τ.

References

[1] Alberto Abbondandolo and Matthias Schwarz, *On the Floer homology of cotangent bundles*, Comm. Pure Appl. Math. **59** (2006), no. 2, 254–316.

[2] M. Abouzaid and T. Kragh, *Simple Homotopy Equivalence of Nearby Lagrangians*, 2016. arXiv:1603.05431.

[3] M. Abouzaid and P. Seidel, *An open string analogue of Viterbo functoriality*, Geom. Topol. **14** (2010), no. 2, 627–718.

[4] Mohammed Abouzaid, *A cotangent fibre generates the Fukaya category*, Adv. Math. **228** (2011), no. 2, 894–939.

[5] ______, *On the wrapped Fukaya category and based loops*, J. Symplectic Geom. **10** (2012), no. 1, 27–79.

[6] Miguel Abreu and Agnès Gadbled, *Toric constructions of monotone Lagrangian submanifolds in* $\mathbb{C}P^2$ *and* $\mathbb{C}P^1 \times \mathbb{C}P^1$, 2014. arXiv:1411.6564.

[7] Kwokwai Chan, Daniel Pomerleano, and Kazushi Ueda, *Lagrangian torus fibrations and homological mirror symmetry for the conifold*, Comm. Math. Phys. **341** (2016), no. 1, 135–178.

[8] Georgios Dimitroglou Rizell, Elizabeth Goodman, and Alexander Ivrii, *Lagrangian isotopy of tori in* $S^2 \times S^2$ *and* $\mathbb{C}P^2$, 2016. arXiv:1602.08821.

[9] Andreas Floer, *Morse theory for Lagrangian intersections*, J. Differential Geom. **28** (1988), no. 3, 513–547.

[10] Kenji Fukaya, Yong-Geun Oh, Hiroshi Ohta, and Kaoru Ono, *Lagrangian Floer theory on compact toric manifolds. I*, Duke Math. J. **151** (2010), no. 1, 23–174.

[11] Agnès Gadbled, *Obstructions to the existence of monotone Lagrangian embeddings into cotangent bundles of manifolds fibered over the circle*, Ann. Inst. Fourier (Grenoble) **59** (2009), no. 3, 1135–1175.

[12] Richard Hind, *Lagrangian isotopies in Stein manifolds*, 2003. arXiv:math/0311093.

[13] G. Hochschild, Bertram Kostant, and Alex Rosenberg, *Differential forms on regular affine algebras*, Trans. Amer. Math. Soc. **102** (1962), 383–408.

[14] Cheuk Yu Mak and Weiwei Wu, *Dehn twists exact sequences through Lagrangian cobordism*, 2015. arXiv:1509.08028.

[15] D. McDuff and D. Salamon, *Introduction to symplectic topology*, Second, Oxford Mathematical Monographs, Oxford University Press, 1998.

[16] Alexander F. Ritter and Ivan Smith, *The monotone wrapped Fukaya category and the open-closed string map*, Selecta Math. (N.S.) **23** (2017), no. 1, 533–642.

[17] Paul Seidel, *A long exact sequence for symplectic Floer cohomology*, Topology **42** (2003), no. 5, 1003–1063.

[18] ______, *Fukaya categories and Picard-Lefschetz theory*, Zurich Lectures in Advanced Mathematics, European Mathematical Society (EMS), Zürich, 2008.

[19] ______, *Abstract analogues of flux as symplectic invariants*, Mém. Soc. Math. Fr. (N.S.) **137** (2014), 135.

[20] Nick Sheridan, *On the Fukaya category of a Fano hypersurface in projective space*, Publ. Math. Inst. Hautes Études Sci. **124** (2016), 165–317.

[21] Jean-Louis Verdier, *Des catégories dérivées des catégories abéliennes*, Astérisque **239** (1996), xii+253 pp. (1997). With a preface by Luc Illusie, Edited and with a note by Georges Maltsiniotis.

[22] Charles A. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, Cambridge, 1994.