Categorifying Quantum Groups

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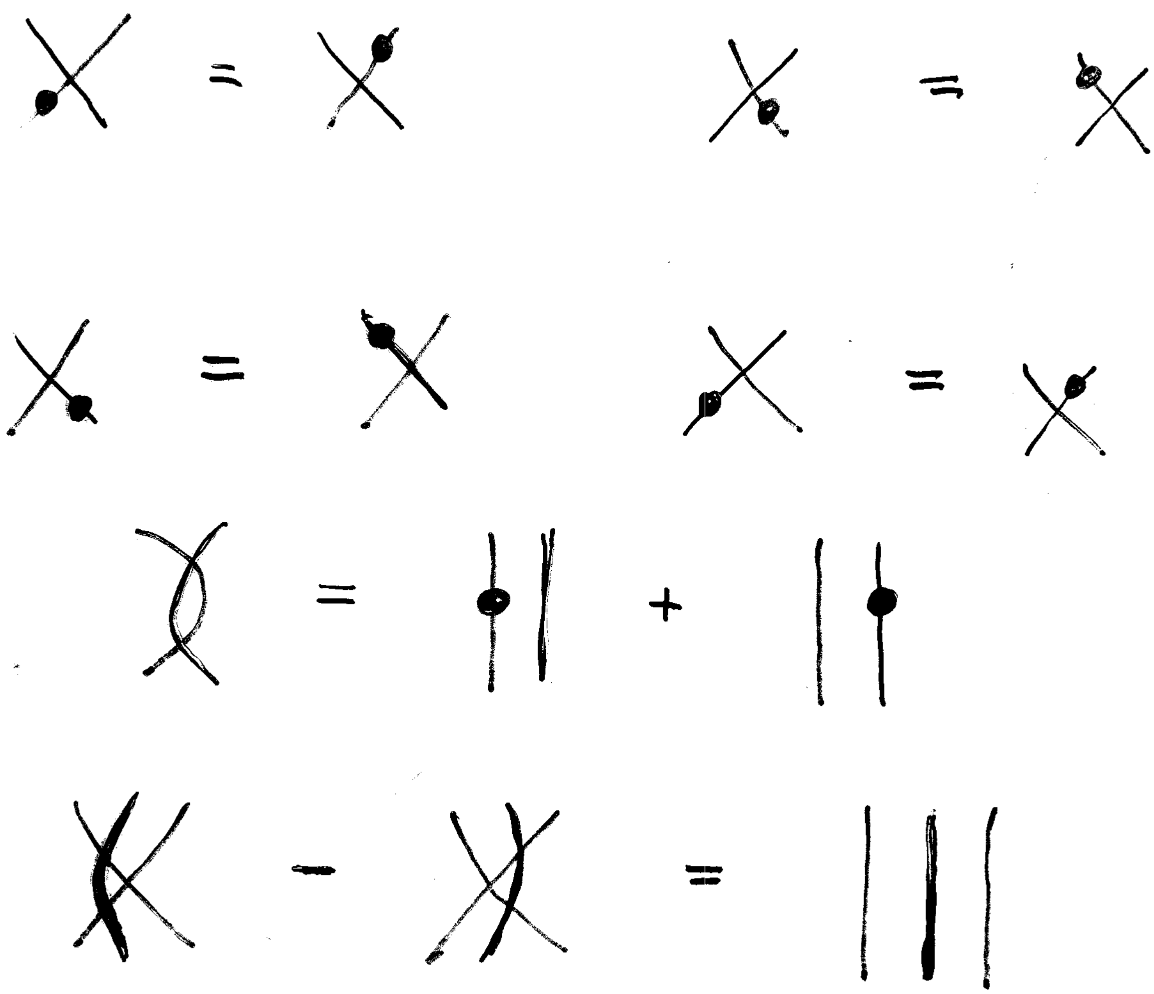
Joint work with M. Khovanov



CRANE-FRENKEL CONJECTURE

GOAL: COMBINATORIAL 4D-TQFT

Idea:

Hopf Algebra $\longleftrightarrow$ 3D-TQFT

QUANTUM GROUP
$U_q(\mathfrak{g})$

Rich class of 3D-TQFTs related to Chern-Simons-Witte[n] Theory

CATEGORIFY

(replace algebraic structure by similar categorical algebraic structure.)

Hopf Category $\longleftrightarrow$ 4D TQFTs

CATEGORIFIED QUANTUM GROUP

Resulting combinatorial 4D-TQFTs should be sensitive to smooth structure in 4D.

EXAMPLES OF CATEGORIFICATION

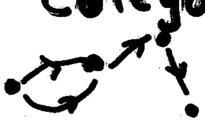
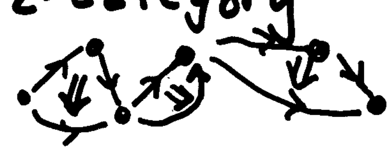
$\xrightarrow{\text{CATEGORIFICATION}}$

$\xleftarrow[\text{DECATEGORIFICATION}]{}$

number n	$\longrightarrow$ $\xleftarrow[\text{dim}]{}$	vector space V of dimension n
(POSITIVE COEFFICIENT) Laurent Polynomial $q^2 + 1 + q^{-2}$	$\longrightarrow$ $\xleftarrow[\text{gdim}]{}$	Graded vector space $V = V_2 \oplus V_0 \oplus V_{-2}$
Set 	$\longrightarrow$ $\longleftarrow$	category 
Algebra	$\longrightarrow$ $\xleftarrow[\text{Grothendieck Ring}]{}$	Monoidal (additive) Category

Grothendieck Ring

K_0, G_0, split G_0 $\quad A \oplus B \rightsquigarrow [A] + [B]$

category 	$\longrightarrow$ $\longleftarrow$	2-category 

$U_q(\mathfrak{sl}_2)$

Generators: $E\ ,\ F\ ,\ K^{\pm 1}$

Relations: $KK^{-1} = 1 = K^{-1}K$

$$KE = q^2 EK \qquad KF = q^{-2} FK \qquad EF - FE = \frac{K - K^{-1}}{q - q^{-1}}$$

Basis: $\{E^a F^b K^c\}_{a,b \in \mathbb{Z}_+,\ c \in \mathbb{Z}}$

What would it mean to categorify $U_q(\mathfrak{sl}_2)$

$$\begin{tikzcd} U_q(\mathfrak{sl}_2) \arrow[r, shift left, "\text{CATEGORIFICATION}"] & \mathcal{U}_q(\mathfrak{sl}_2) \arrow[l, shift left, "\text{Grothendieck Ring}"] \end{tikzcd}$$

(algebra) (category + extra structure ?)

X (algebra basis element) object of $\mathcal{U}_q(\mathfrak{sl}_2)$

$q^{\alpha} X$ $\Rightarrow$ objects of $\mathcal{U}_q(\mathfrak{sl}_2)$ should be graded $X\{\alpha\}$

$$X \cdot y = \sum_{\text{basis}} \underbrace{m_{xy}^{z}}_{\text{structure constants in } \mathbb{Q}(q)} z$$

- Grothendieck ring is a $\mathbb{Z}[q, q^{-1}]$-module

Any finite-dimensional representation of $U_q(sl_2)$ has a weight decomposition

$$\begin{tikzcd} V(n+2) \arrow[d, bend left, "F"] \\ V(n) \arrow[u, bend left, "E"] \arrow[d, bend left, "F"] \\ V(n-2) \arrow[u, bend left, "E"] \end{tikzcd}$$

$$V = \bigoplus_{n \in \mathbb{Z}} V(n)$$

$$v \in V(n) \iff Kv = q^n v$$

Add projection operators 1_n to $U_q(sl_2)$ projecting onto $V(n)$

$$U_q(sl_2) \longrightarrow \dot{U}(sl_2)$$

$1 \qquad \qquad \{1_n\}_{n \in \mathbb{Z}}$

(mutually-orthogonal idempotents)

$$K 1_n = q^n 1_n \quad \text{— No } K \text{ in } \dot{U}$$

$$E 1_n = 1_{n+2} E = 1_{n+2} E 1_n$$

$$F 1_n = 1_{n-2} F = 1_{n-2} F 1_n$$

$$(EF - FE) 1_n = [n] 1_n \qquad [n] = \frac{q^n - q^{-n}}{q - q^{-1}}$$

$\dot{U}$ has an integral form ${}_{\mathcal{A}}\dot{U}$ spanned by

$$E^{(a)} := \frac{E^a}{[a]!} \qquad F^{(b)} := \frac{E^b}{[b]!}$$

$\dot{U}$ has canonical basis $\dot{B} = \begin{cases} E^{(a)}F^{(b)}1_n & n \leq b-a \\ F^{(b)}E^{(a)}1_n & n \geq b-a \end{cases}$

STRUCTURE CONSTANTS IN BASIS $\dot{B}$ are in $\mathbb{N}[q, q^{-1}]$

$\dot{U}(\mathfrak{sl}_2)$, $\dot{U}(\mathfrak{sl}_n)$ Beilinson - Lusztig - MacPherson

$\dot{U}(\mathfrak{g})$ Lusztig

Conjecturally structure constants are in $\mathbb{N}[q, q^{-1}]$ when Cartan datum is symmetric.

$\Rightarrow$ ${}_{\mathcal{A}}\dot{U}$ is a natural candidate for categorification (Idea goes back to Igor Frenkel!)

Algebra without unit and a collection of orthogonal idempotents $\longleftrightarrow$ pre-additive category

$\Rightarrow$ $\dot{U}$ is a pre-additive category

objects: $n \in \mathbb{Z}$

morphisms: $1_m \dot{U} 1_n$ (abelian group)

identities: 1_n

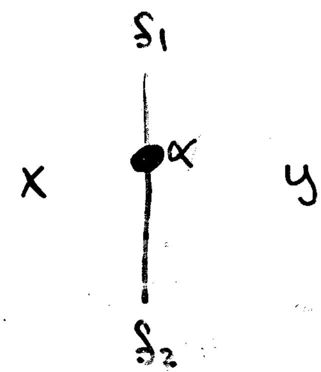
composition: $1_{m'}\dot{U}1_{n'} \times 1_m \dot{U} 1_n \xrightarrow[\text{multiplication}]{} 1_{m'}\dot{U}$

2-Categories

Normally category theorist like to draw

- Objects: x

- Morphisms: $x \xrightarrow{f} y$



- 2-Morphisms:

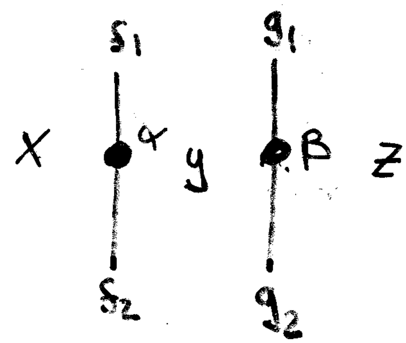
$$\begin{tikzcd} x \arrow[r, bend left=50, "f_1"] \arrow[r, bend right=50, "f_2"'] \arrow[r, phantom, "\alpha\Downarrow"] & y \end{tikzcd}$$

- Composition of morphisms:

$$x \xrightarrow{f} y \xrightarrow{g} z$$

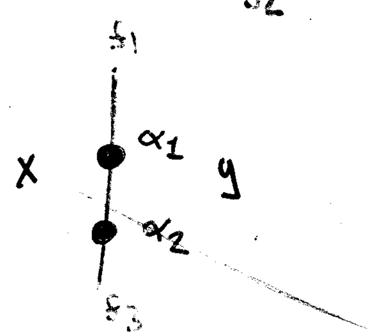
- Horizontal composition of morphisms:

$$\begin{tikzcd} x \arrow[r, bend left=50, "f_1"] \arrow[r, bend right=50, "f_2"'] \arrow[r, phantom, "\alpha\Downarrow"] & y \arrow[r, bend left=50, "g_1"] \arrow[r, bend right=50, "g_2"'] \arrow[r, phantom, "\beta\Downarrow"] & z \end{tikzcd}$$



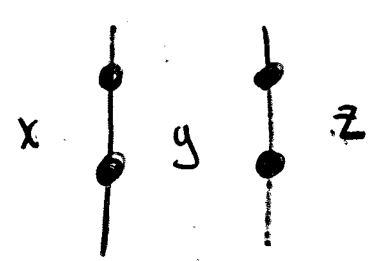
- Vertical composition of morphisms:

$$\begin{tikzcd} x \arrow[r, bend left=60, "f_1"] \arrow[r] \arrow[r, bend right=60, "f_3"'] \arrow[r, phantom, shift left=2.5ex, "\alpha_1\Downarrow"] \arrow[r, phantom, shift right=2.5ex, "\alpha_2\Downarrow"] & y \end{tikzcd}$$



- The interchange law:

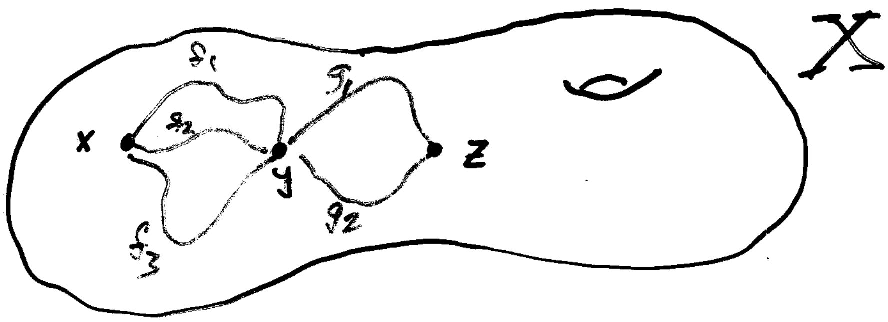
$$\begin{tikzcd} x \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift left=2.5ex, "\Downarrow"] \arrow[r, phantom, shift right=2.5ex, "\Downarrow"] & y \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift left=2.5ex, "\Downarrow"] \arrow[r, phantom, shift right=2.5ex, "\Downarrow"] & z \end{tikzcd}$$



EXAMPLES

1) X topological space

$$\underline{\underline{\Pi_2(X)}} = \begin{cases} \text{objects: points of } X \\ \text{morphisms: paths in } X \\ \text{2-morphisms: homotopies between paths} \end{cases}$$

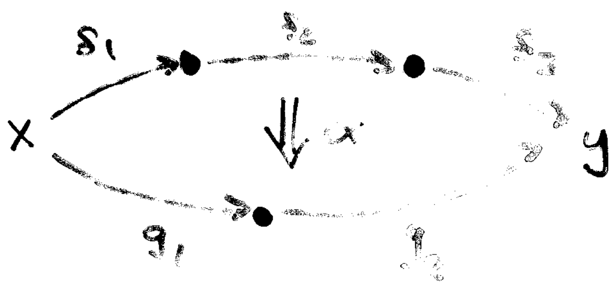


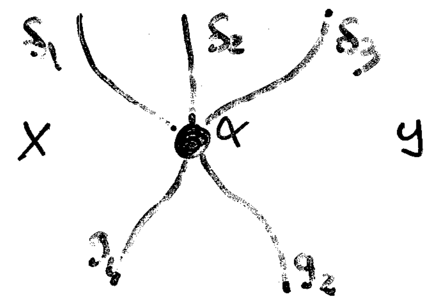
2) $$\underline{\underline{\text{BiMod}}} = \begin{cases} \text{objects: rings } A, B, \ldots \\ \text{morphisms: } A \xrightarrow{{}_A M_B} B \quad (A,B)\text{-bimodules} \\ \text{2-morphisms: Bimodule maps} \end{cases}$$

$$A \xrightarrow{{}_A M_B} B \xrightarrow{{}_B M'_C} C = {}_A M_B \otimes_B {}_B M'_C$$

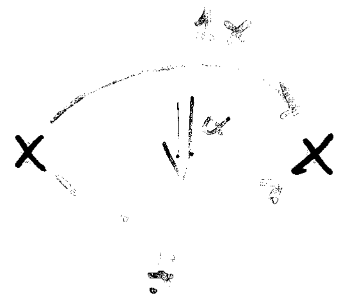
3) $$\underline{\underline{\text{Cat}}} = \begin{cases} \text{objects: categories} \\ \text{morphisms: functors} \\ \text{2-morphisms: natural transformations} \end{cases}$$

More string diagrams!

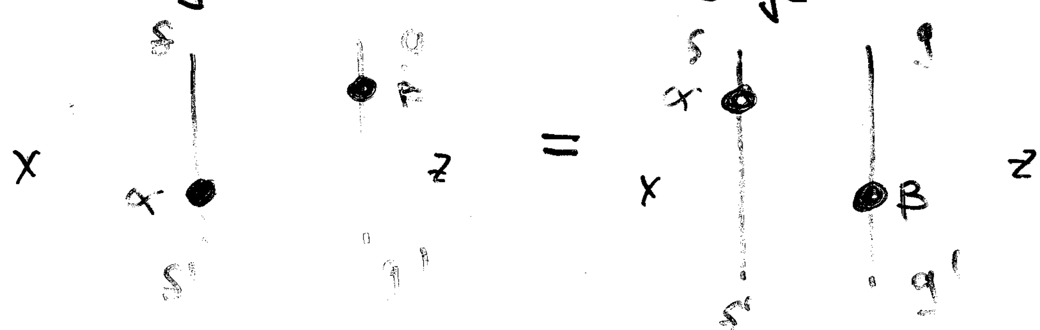


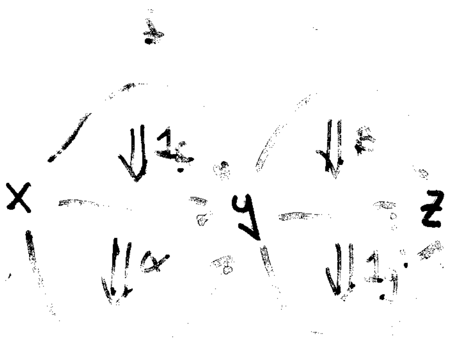


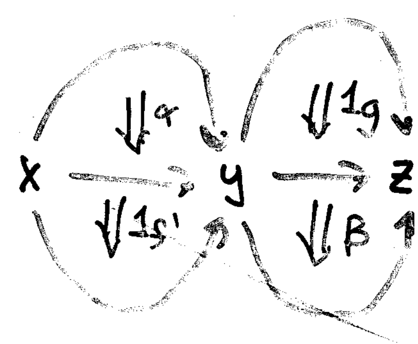
Do not draw identity 1-morphisms!



Identity axioms + Interchange ⇒



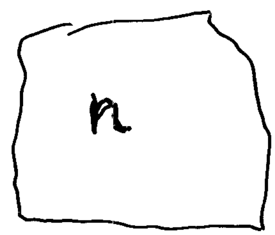




$$\dot{U} \underset{\longleftarrow}{\xrightarrow{\text{CATEGORIFICATION}}} \dot{\mathcal{U}}$$

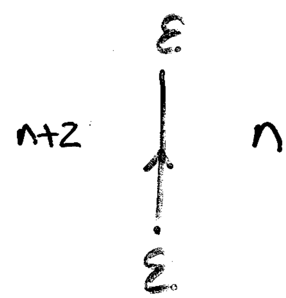
(category) (2-category)

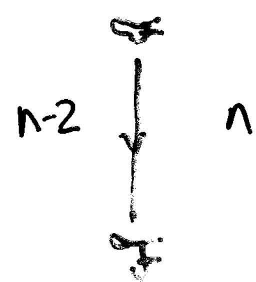
objects:

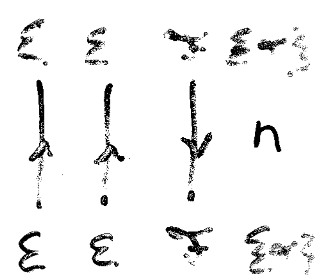
$n \longmapsto$ 

region labelled n

morphisms:

$1_{n+2} E 1_n \longmapsto 1_{n+2} \mathcal{E} 1_n :=$ 

$1_{n-2} F 1_n \longmapsto 1_{n-2} \mathcal{F} 1_n :=$ 

$q^{\alpha} E^2 F 1_n \longmapsto \mathcal{E}^2 \mathcal{F} 1_n \{\alpha\} =$ 

2-morphisms ???

LOOK FOR REMNANTS OF CATEGORIFICATION

$$(,) : \dot{U} \times \dot{U} \longrightarrow \mathbb{Z}[q, q^{-1}] \xleftarrow{\text{gdim}} \dot{\mathcal{U}}(\mathcal{E}^{\alpha}\mathcal{F}^{\beta}1_n, \mathcal{E}^{\gamma}\mathcal{F}^{\delta}1_n)$$

some form on $\dot{U}$

(graded vector space)

If $\quad x \xrightarrow{f} y \qquad \deg(f) = r$

$$\Rightarrow \quad x\{s\} \xrightarrow{f} y \qquad\qquad x \xrightarrow{f} y\{s\}$$

$$\deg = r - s \qquad\qquad \deg = r + s$$

$$\Rightarrow \quad \operatorname{qdim} \dot{\mathcal{U}}(x\{s\}, y) = q^{-s} \operatorname{qdim} \dot{\mathcal{U}}(x, y)$$

$$\operatorname{qdim} \dot{\mathcal{U}}(x, y\{s\}) = q^{s} \operatorname{qdim} \dot{\mathcal{U}}(x, y)$$

$\Rightarrow \quad (\ ,\ ) : {}_{\mathcal{A}}\dot{U} \times {}_{\mathcal{A}}\dot{U} \longrightarrow \mathbb{Z}[q, q^{-1}]$ is semilinear

(antilinear in first slot, linear in second

Thm (Lusztig) $\exists$ a unique form $(\ ,\ ) : {}_{\mathcal{A}}\dot{U} \times {}_{\mathcal{A}}\dot{U} \longrightarrow \mathbb{Z}[q, q^{-1}]$

1) Form is semilinear $(fx, y) = \bar{f}(x, y)$ $\quad f \in \mathbb{Z}[q, q^{-1}]$, $x, y \in \dot{U}$

2) $(1_{m'} x 1_{n'}, 1_m y 1_n) = 0$ unless $m' = m$, $n' = n$

3) $(ux, y) = (x, \tau(u) y)$ $\qquad \tau(1_{n+2} E 1_n) = q^{-1-n} 1_n F 1_{n+2}$

$\qquad \tau(1_n F 1_{n+2}) = q^{1+n} 1_{n+2} E 1_n$

4) $$(E^{(a)} 1_n, E^{(a)} 1_n) = \prod_{j=1}^{a} \frac{1}{1 - q^{2j}}$$

Idea: Use the semilinear form to guess

- Generating 2-morphisms $\dot{\mathcal{U}}(x1_n, y1_n)$
- Degrees of generating 2-morphisms
- Relations on 2-morphisms

Example

$$\mathrm{gdim}\left(\dot{\mathcal{U}}(\mathcal{E}1_n,\mathcal{E}1_n)\right) = (E1_n, E1_n) = \frac{1}{1-q^2} = 1 + q^2 + q^4 + \cdots$$

$q^0 =$ degree of identity map $\mathcal{E}1_n \Rightarrow \mathcal{E}1_n$

$\Rightarrow$ Add a new generator  of degree 2.

No relations

$$\Rightarrow \quad \mathrm{gdim}\left(\dot{\mathcal{U}}(\mathcal{E}1_n,\mathcal{E}1_n)\right) = \deg\Big($$  $$\Big) + \deg\Big($$  $$\Big) + \deg\Big($$  $$\Big) + \cdots$$

$$= 1 + q^2 + q^4 + \cdots$$

write  ^{a} $=$ $\Big($  $\Big)^a$ $\quad \deg = 2a$

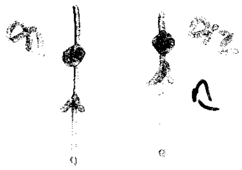
vertically compose new generator with itself a times.

$$\mathrm{gdim}\left(\dot{\mathcal{U}}(\mathcal{E}^2 1_n, \mathcal{E}^2 1_n)\right) = (E^2 1_n, E^2 1_n) = (1 + q^{-2}) \frac{1}{1-q^2} \frac{1}{1-q^2}$$

degree zero generator

degree -2 generator

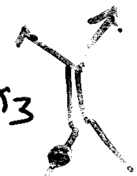
Two places to put dots

$$\sum_{a_1, a_2 \geq 0} \deg$$  $$= \underline{\underline{1}} \cdot \frac{1}{1-q^2} \frac{1}{1-q^2}$$

deg -2 ,

$$\sum_{a_1, a_2 \geq 0}$$  $$= \underline{\underline{q^{-2}}} \frac{1}{1-q^2} \cdot \frac{1}{1-q^2}$$

Only 3 linearly independent generators in degree 0

$\Rightarrow$ a_1  $+ a_2$  $+ a_3$  $+ a_4$  $+ a_5$  $= 0$

$\Rightarrow$ SOME RELATIONS

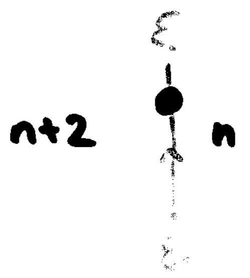
$$\mathrm{gdim}\left(\dot{\mathcal{U}}(1_n, \mathcal{E}\mathcal{F} 1_n)\right) = (1_n, \mathcal{E}\mathcal{F} 1_n) = q^{1-n} \frac{1}{1-q^2}$$

one generator

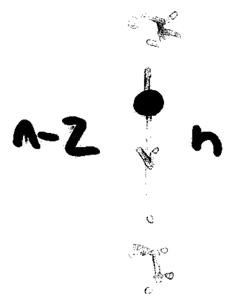
one place to put dots

deg () $= 1-n$

2-morphisms of $_{A}\mathcal{U}$:

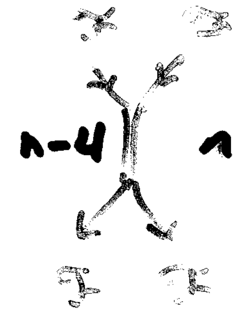


deg 2



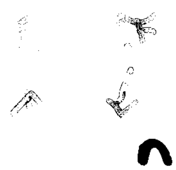
deg 2

deg -2



deg -2

deg 1+n

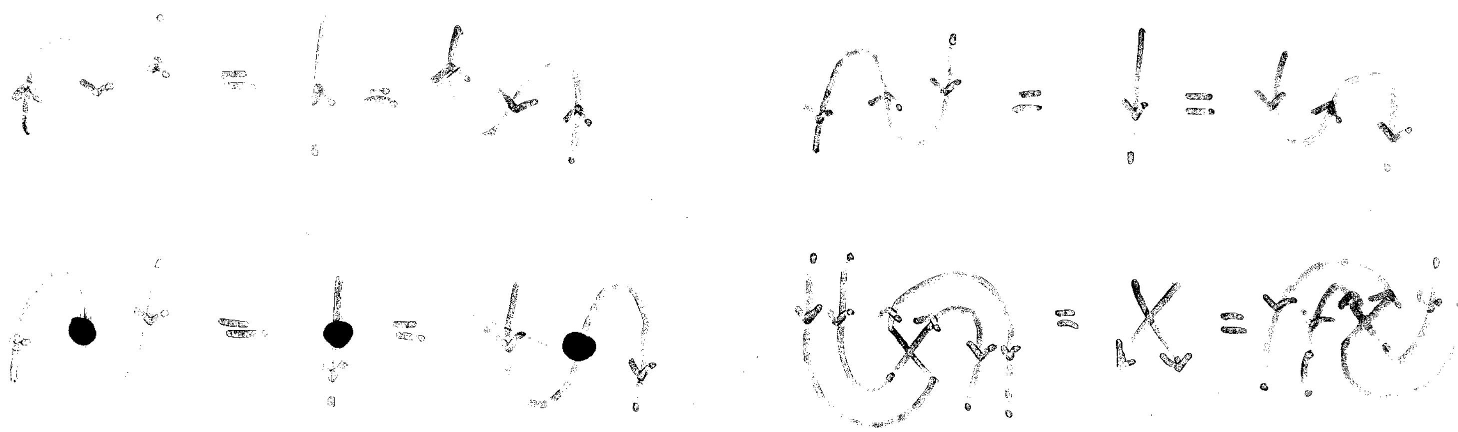


deg 1-n

deg 1+n

deg 1-n

Modulo relations:



$\Rightarrow$ All isotopies of diagrams represent the same 2-morphism

Closed diagrams of negative degree $= 0$

 $^{\underline{n}}$  $^{\underline{n}}$

$\deg = 2(n+1)$ $\deg = 2(-n+1)$

Examples:

 $^{-2}$ $\deg = 2(-\underline{\underline{2}}+1) = -2 \Rightarrow$  $^{-2} = 0$

It is convenient to write

 $^{\underline{n}}$  $^{\underline{n}}$

$(-n-1)+\alpha$ $(n-1)+\alpha$

$\deg = 2\alpha$ $\deg = 2\alpha$

sometimes negative numbers

Fake bubbles  $_a$ $\deg \geq 0, \quad a < 0$

$$\Big(\sum_{a \geq 0}$$  $$_{(-n-1)+a}\, t^a\Big)\Big(\sum_{b \geq 0}$$  $$_{(n-1)+b}\, t^b\Big) = 1$$

Analog to defining relations in $H^*(Gr(\infty,\infty))$

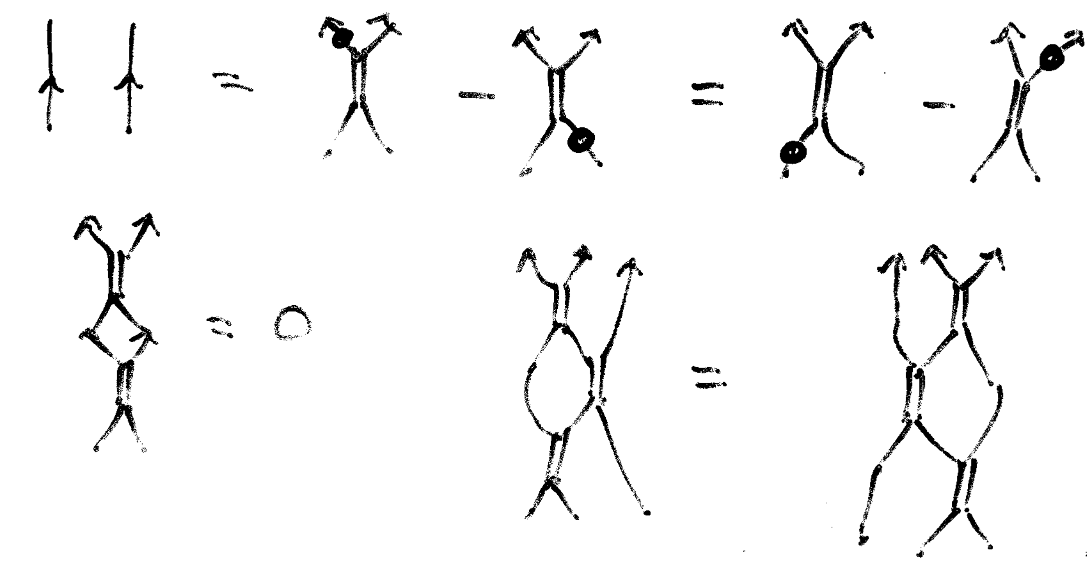
$\lim_{m\to\infty} Gr(m, 2m)$

$\mathbb{C}^m \subset \mathbb{C}^{2m}$

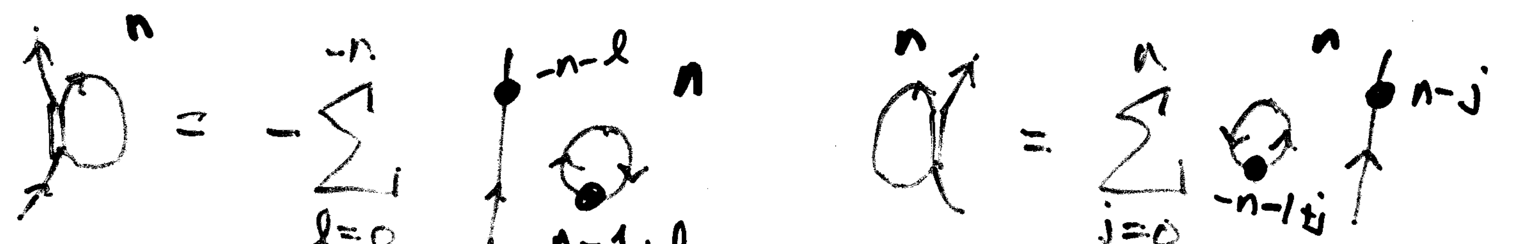
$$(1 + x_1 t + x_2 t^2 + \dots)(1 + y_1 t + \dots) = 1$$

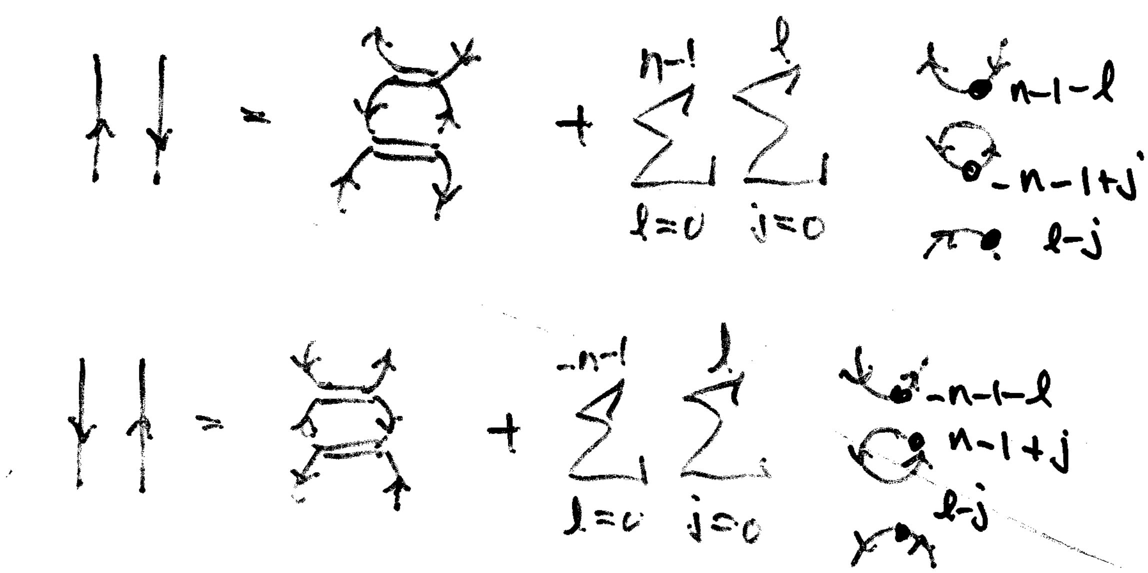
$$\Rightarrow x_1 + x_2 = 0, \quad x_2 + x_1 y_1 + y_2 = 0$$

Nil Hecke Algebra:

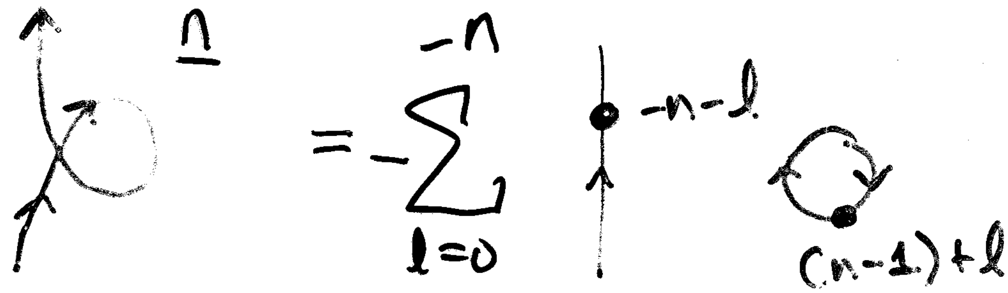


other relations:





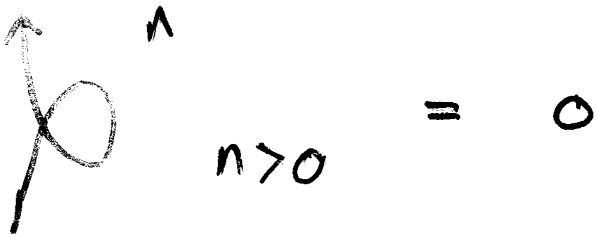
All bubbles are fake!

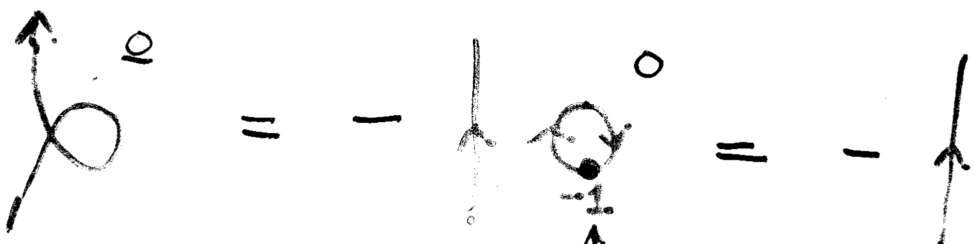


zero unless $-n \geq 0$

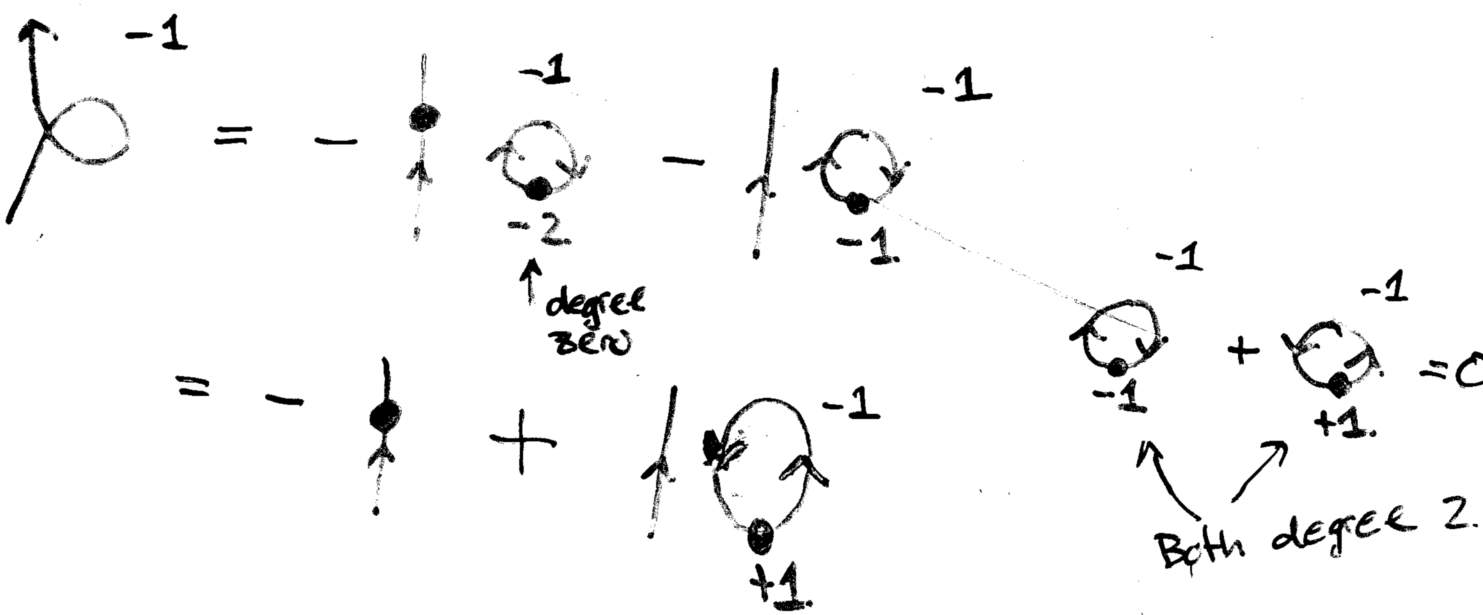
Fake bubble!

For example,





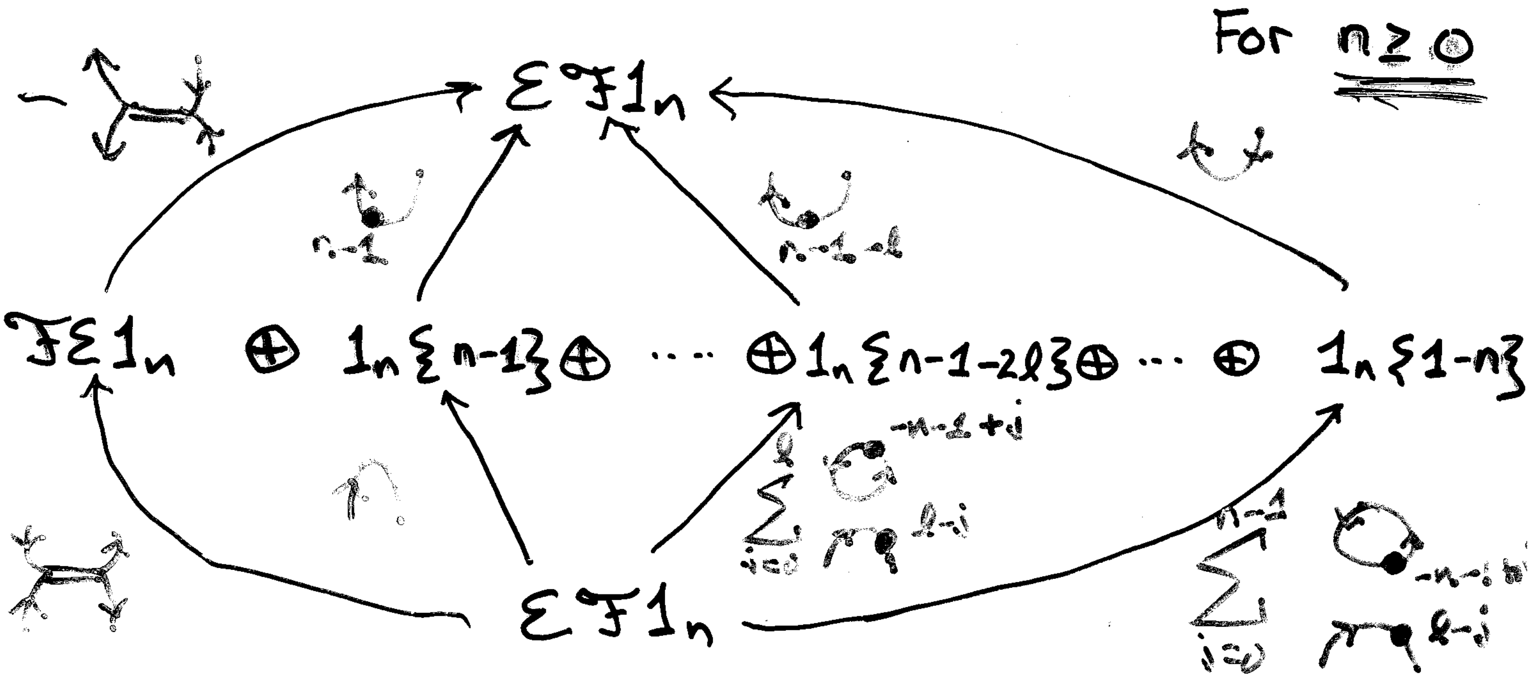
degree zero



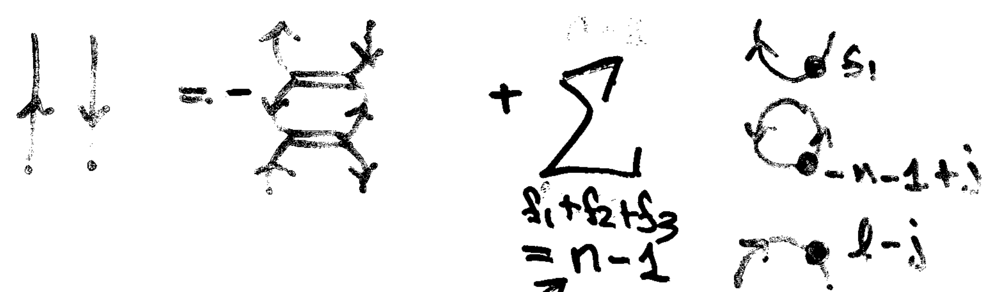
degree zero

Both degree 2.

For $\underline{\underline{n \geq 0}}$



Decomposition of $\mathcal{E}\mathcal{F}1_n$ into orthogonal idempotents



If Idempotents split $\Rightarrow$ $\mathcal{E}\mathcal{F}1_n = \mathcal{F}\mathcal{E}1_n \oplus 1_n\{n-1\} \oplus \cdots \oplus 1_n\{1-n\}$

$= \mathcal{F}\mathcal{E}1_n \oplus_{[n]} 1_n \quad n \geq 0$

LIFTING OF sl_2-RELATIONS

$$\mathcal{E}\mathcal{F}1_n = \mathcal{F}\mathcal{E}1_n \oplus_{[n]} 1_n \qquad \mathcal{F}\mathcal{E}1_n = \mathcal{E}\mathcal{F}1_n \oplus_{[-n]} 1_n$$

$n \geq 0 \qquad\qquad n \leq 0$

THEOREM

- Indecomposable 1-morphism $\longleftrightarrow$ Lusztig canonical basis element

- Split Grothendieck ring

$$K_0(\dot{\mathcal{U}}) \cong {}_{\mathcal{A}}\dot{U}$$

$$\underset{\substack{\uparrow \\ \text{1-morphism}}}{A} \oplus B = C \longleftrightarrow [A]+[B]=C$$

- Graded Hom $\dot{\mathcal{U}}(x,y)$ $\longleftrightarrow$ semilinear form

- Symmetries of graphical calculus lift various (anti) automorphisms of ${}_{\mathcal{A}}\dot{U}$

- $\dot{\mathcal{U}}$ acts on a 2-category constructed using cohomology of Grassmanians and iterated flag varieties $\Rightarrow$ Categorification of irreps of quantum sl_2

Action on flag varieties

$n = 2k - N$ | $V(N+1)$

N $\bullet$ $k=N$ $\qquad\qquad$ $\bullet\ H^*(Gr(N,N)) =: H_N$

$\vdots$ $\qquad\qquad$ $\vdots$

$$\xrightarrow{\Gamma_N}$$

$$\begin{tikzcd}
n+2 \quad \bullet\, k+1 \arrow[d, bend right, "F"'] \\
n \quad \bullet\, k \arrow[u, bend right, "E"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\bullet\ H^*(Gr(k+1,N)) =: H_{k+1} \arrow[d, bend right, "F"'] \\
\bullet\ H^*(Gr(k,N)) =: H_k \arrow[u, bend right, "E = H_{k+1,k}"']
\end{tikzcd}$$

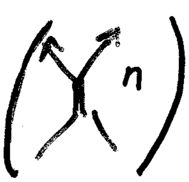
$\vdots$ $\qquad\qquad$ $\vdots$

$-N$ $\bullet$ $k=0$ $\qquad\qquad$ $\bullet\ H^*(Gr(0,N)) =: H_0$

$H_{k+1,k} := H^*(Fl(k,k+1,N))$ $\qquad$ (H_{k+1}, H_k)-bimodule

$$\begin{tikzcd}
& Fl(k,k+1,N) \arrow[dl] \arrow[dr] & \\
Fl(k,N) & & Fl(k+1,N)
\end{tikzcd}
\Rightarrow
\begin{tikzcd}
& H_{k+1,k} & \\
H_k \arrow[ur] & & H_{k+1} \arrow[ul]
\end{tikzcd}$$

Get an action of 2-category $\mathcal{U}$

$\Gamma_N\Big($  $\Big)$: $H_{k+2,k+1} \otimes_{H_{k+1}} H_{k+1,k} \longrightarrow H_{k+2,k+1} \otimes_{H_{k+1}} H_{k+1,k}$

bimodule map given by divided differences on Chern classes of line bundles

$\Gamma_N\Big($  $\Big)$: $H_k \longrightarrow H_{k,k+1} \otimes_{H_{k+1}} H_{k+1,k}$

Generalizations (Joint w/ M. Khovanov)

- Can define 2-category $\dot{\mathcal{U}}$ for any simple Lie algebra (Kac-Moody Lie Algebra)

$$ {}_A\dot{U} \xrightarrow[\text{SURJECTIVE}]{\gamma} K_0(\dot{\mathcal{U}}) $$

- γ is an isomorphism in sl_n-case.
(All structure of previous slide lifts as well)
$\Rightarrow$ Basis of indecomposables gives a basis of $\dot{U}(sl_n)$ where structure constants are in $\mathbb{N}[q,q^{-1}]$. (Lusztig basis?)

- U_q^+ (positive nilpotent cone) can be categorified for any Kac-Moody Lie algebra
Leads to a conjectural categorification of irreducible heighest weight reps of $U_q(\mathfrak{g})$ by cyclotomic quotients.

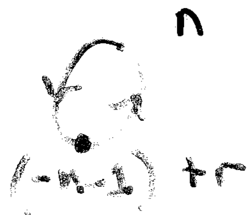
Partial results: • sl_n, $\widehat{sl_n}$ Brundan–Kleshchev

Symmetric Functions

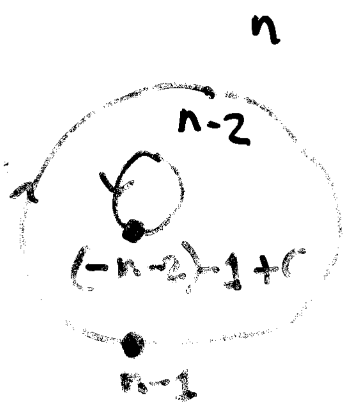
$$\mathcal{U}(1_n, 1_n) \xrightarrow{\sim} Sym(x_1, x_2, x_3, \dots)$$

(closed diagram)

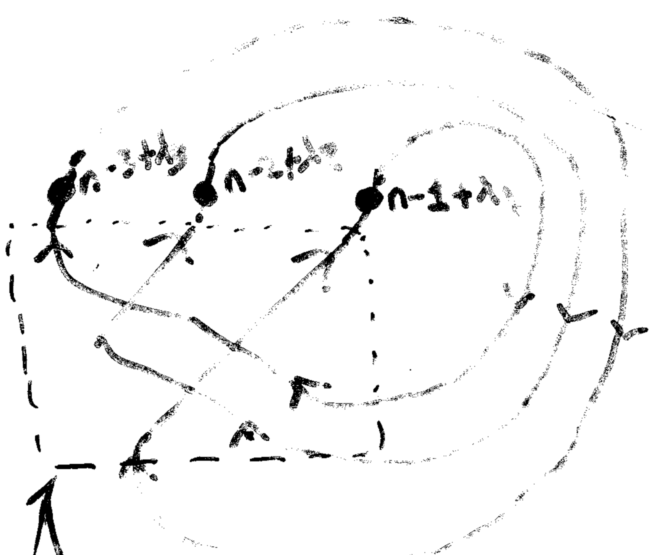
$\longmapsto e_r$ (elementary symmetric function)



$\longmapsto (-1)^r h_r$ (complete symmetric function)



$\longmapsto (-1)^{n-1} p_n$ (Power sum symmetric functions)



$\longmapsto s_{\lambda_1, \lambda_2, \lambda_3}$ (schur function)

longest braid on l-strands for partition $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \dots \geq \lambda_l$