

Diagrammatic categorification of quantum groups I

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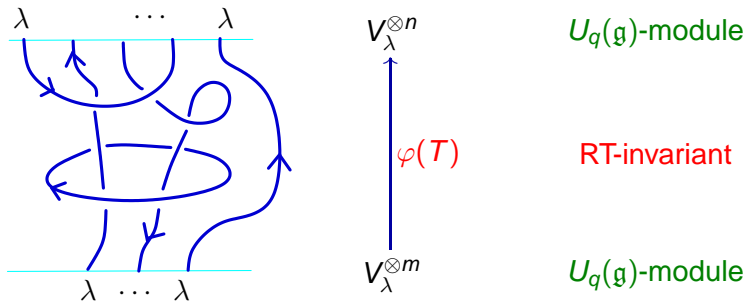
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Available at <http://www.math.columbia.edu/~lauda/talks/MSRI>

Reshetikhin-Turaev invariant

For \mathfrak{g} a simple Lie algebra the quantum deformation $U_q(\mathfrak{g})$ of the enveloping algebra of \mathfrak{g} gives link/tangle invariants.

Colour the strands of a tangle by a representation V_λ of $U_q(\mathfrak{g})$



The invariant $\varphi(T)$ is a map of $U_q(\mathfrak{g})$ -representations.

Example

- $\mathfrak{g} = \mathfrak{sl}_2$ Jones polynomial, coloured Jones polynomial
- $\mathfrak{g} = \mathfrak{sl}_n$ specializations of the HOMFLYPT polynomial

Quantum $sl(2)$

Definition

The quantum group $U_q(sl_2)$ is the associative algebra (with unit) over $\mathbb{Q}(q)$ with generators E, F, K, K^{-1} and relations

- $KK^{-1} = 1 = K^{-1}K,$
- $KE = q^2EK, \quad KF = q^{-2}FK,$
- $EF - FE = \frac{K - K^{-1}}{q - q^{-1}}$

Consider a finite-dimensional representation V with a weight decomposition

$$V(n+2)$$

$$E \left(\begin{array}{c} \uparrow \\ \\ \downarrow \end{array} \right) F$$

$$V(n)$$

$$E \left(\begin{array}{c} \uparrow \\ \\ \downarrow \end{array} \right) F$$

$$V(n-2)$$

$$V = \bigoplus_{n \in \mathbb{Z}} V(n)$$

$$Kv = q^n v, \quad v \in V(n)$$

Why categorify quantum groups?

Conjectured applications to low-dimensional topology

- Representation theoretic explanation of Khovanov homology
- Categorification of the Reshetikhin-Turaev quantum knot invariants.
- Crane-Frenkel conjectured categorified quantum groups would give 4-dimensional TQFTs

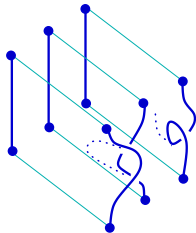
Categorified representation theory should provide new insights for ordinary representation theory.

- Geometric representation theory (algebraic/combinatorial analog of perverse sheaves)
- Positivity and integrality properties for quantum groups
- Representation theory of the symmetric group!

Quantum Group $\xrightarrow{\text{representation category}}$ Braided monoidal category with duals



Categorified Quantum Group $\xrightarrow{\text{representation 2-category}}$ Braided monoidal 2-category with duals



Road map to categorification

There are several hints that suggest that quantum groups are really just shadows of a richer algebraic structure

- Lusztig's discovery of canonical basis for idempotent version $\dot{\mathbf{U}}$ of quantum groups.
- Geometric constructions of categorical quantum group actions
- The existence of semilinear forms $\langle, \rangle: \mathbf{U} \times \mathbf{U} \rightarrow \mathbb{Q}(q)$ with nice properties.

Beilinson, Lusztig, and MacPherson added orthogonal idempotents 1_n for projection onto $V(n)$ to produce $\dot{\mathbf{U}} := \dot{\mathbf{U}}_q(\mathfrak{sl}_2)$, a $\mathbb{Q}(q)$ -algebra without unit

$$\mathbf{U}_q(\mathfrak{sl}_2) \longrightarrow \dot{\mathbf{U}}$$

collection of
orthogonal idempotents
 1_n for $n \in \mathbb{Z}$

$$K1_n = q^n 1_n \Rightarrow \text{no more } K$$

$$E1_n = 1_{n+2}E = 1_{n+2}E1_n$$

$$F1_n = 1_{n-2}F = 1_{n-2}F1_n$$

$$EF1_n - FE1_n = [n]1_n \quad [n] = \frac{q^n - q^{-n}}{q - q^{-1}}$$

$\dot{\mathbf{U}}$ has a basis $\{E^a F^b 1_n\}$ for $n \in \mathbb{Z}$, $a, b \geq 0$

Integral forms

The integral form $\dot{\mathbf{U}}_{\mathbb{Z}}$ of $\dot{\mathbf{U}}_q(\mathfrak{sl}_2)$ is spanned by products of divided differences

$$E^{(a)}1_n := \frac{E^a}{[a]!}1_n \quad F^{(b)}1_n := \frac{F^b}{[b]!}1_n$$

Crane and Frenkel conjectured (1994) that $\dot{\mathbf{U}}_{\mathbb{Z}}$ could be categorified using Lusztig's canonical basis

$$\begin{aligned} E^{(a)}F^{(b)}1_n, & \quad n \leq b - a \\ F^{(b)}E^{(a)}1_n, & \quad n \geq b - a \end{aligned}$$

Structure constants are in $\mathbb{N}[q, q^{-1}]$

$\dot{\mathbf{U}}_{\mathbb{Z}}$ is the Grothendieck ring of some higher structure \mathcal{U} .

Why 2-categories?

$\dot{\mathbf{U}}$ is a nonunital algebra with a collection of mutually-orthogonal idempotents

(small) pre-additive categories \longleftrightarrow **idempotent rings**

$\Rightarrow \dot{\mathbf{U}}$ is a pre-additive category

- objects: $n \in \mathbb{Z}$
- morphisms $n \rightarrow m$: abelian group $1_m \dot{\mathbf{U}} 1_n$
 - ▶ identities: 1_n
 - ▶ composition: $1_{m'} \dot{\mathbf{U}} 1_m \otimes 1_{n'} \dot{\mathbf{U}} 1_n \rightarrow \delta_{n',m} 1_{m'} \dot{\mathbf{U}} 1_n$ given by multiplication

\Rightarrow the categorification $\dot{\mathcal{U}}$ of $\dot{\mathbf{U}}$ should be a 2-category

(small) pre-additive 2-categories \longleftrightarrow **idempotent additive monoidal categories**

Grothendieck
category/ring

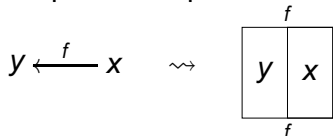


(small) pre-additive categories \longleftrightarrow **idempotent rings**

2-categories and string diagrams

A 2-category is given by

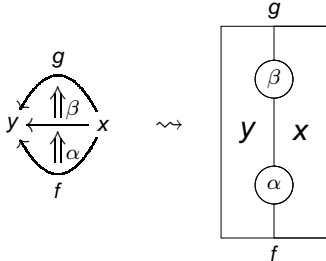
- objects: represented by regions in the plane \boxed{x} or \boxed{y}
- morphisms: represented by lines separating regions of the plane



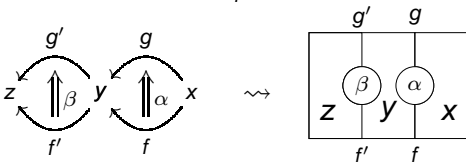
- 2-morphisms: $y \begin{array}{c} \curvearrowright \\ \uparrow \alpha \\ \curvearrowleft \end{array} x \quad \rightsquigarrow \quad \begin{array}{|c|c|} \hline & g & \\ \hline y & \alpha & x \\ \hline & & \\ \hline \end{array}$

together with composition operations and identity 1 and 2-morphisms.

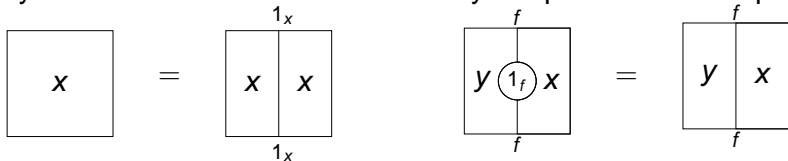
vertical composition



horizontal composition



By convention we do not draw identity morphisms or 2-morphisms:



Examples

1 Cat:

- ▶ objects: categories
- ▶ morphisms: functors
- ▶ 2-morphisms: natural transformations

1 Bim:

- ▶ objects: commutative rings R, S, T, \dots
- ▶ morphisms: (S, R) -bimodules

$$\text{composition: } T \xleftarrow{\tau N_S} S \xleftarrow{s M_R} R := T \xleftarrow{\tau N_S \otimes_S s M_R} R$$

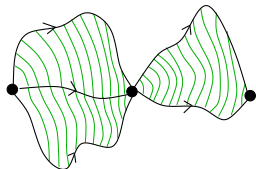
- ▶ 2-morphisms: bimodule homomorphisms

1 $\Pi(X)$:

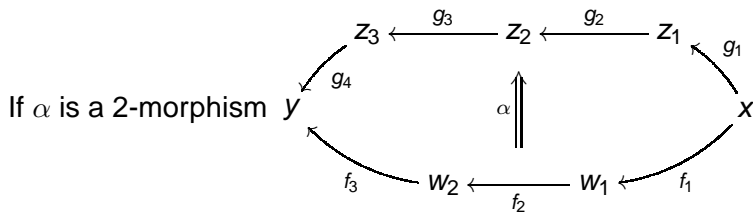
objects: points of a topological space X

morphisms: paths in X

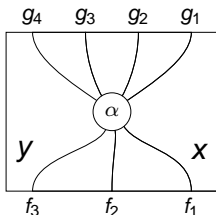
2-morphisms: homotopies between paths



The last two examples are really *weak* 2-categories or bicategories.



then α becomes the string diagram:

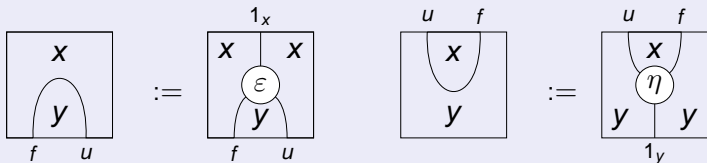


Now let's apply string diagrams to adjoint functors!

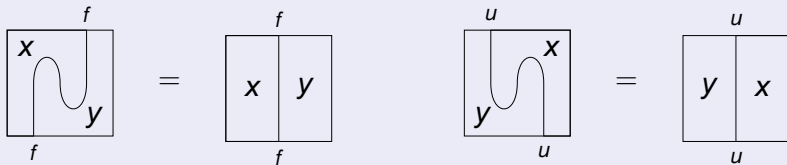
Definition

An adjunction in a 2-category consists of

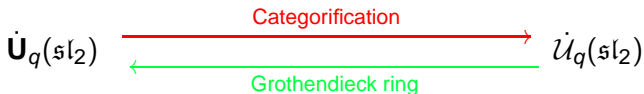
- objects x and y
- morphisms $y \xleftarrow{f} x$ and $x \xleftarrow{u} y$
- 2-morphisms $1_x \Leftarrow f \circ u: \varepsilon$ and $u \circ f \Leftarrow 1_y: \eta$



such that the equalities



hold.



n weight

object n of $\mathcal{U}_q(\mathfrak{sl}_2)$

b basis element

1-morphism of $\mathcal{U}_q(\mathfrak{sl}_2)$

i.e., $1_n, E1_n, F1_n$

$1_n, \mathcal{E}1_n, \mathcal{F}1_n$

$q^a b$

$b\{a\} \Rightarrow$ (1-morphisms should be graded)

$$x \cdot y = \sum_z m_{xy}^z z$$

m_{xy}^z structure constants
in $\mathbb{N}[q, q^{-1}]$

$$x \circ y = \bigoplus_z m_{xy}^z z$$

2-morphisms ???

Since the morphisms are graded, there is an internal 2-HOM given by taking homomorphisms of all degrees

$$\mathrm{HOM}_{\dot{\mathcal{U}}}(,): 1\mathrm{morph}(\dot{\mathcal{U}}) \times 1\mathrm{morph}(\dot{\mathcal{U}}) \longrightarrow \mathbf{GrVect}_k$$

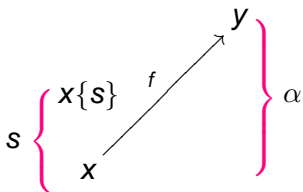
Decategorification

$$\langle , \rangle: \dot{\mathbf{U}} \times \dot{\mathbf{U}} \longrightarrow \mathbb{Z}[q, q^{-1}]$$

$\left. \begin{array}{c} \{ \\ \downarrow \\ K_0 \end{array} \right\}$
 $\left. \begin{array}{c} \{ \\ \downarrow \\ K_0 \end{array} \right\}$
 $\left. \begin{array}{c} \{ \\ \downarrow \\ \mathrm{gdim} \end{array} \right\}$

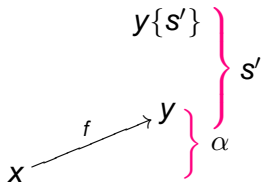
Choice of 2-morphisms is controlled by such a form on $\dot{\mathbf{U}}$

This form will be semilinear (or sesquilinear) since $x \xrightarrow{f} y$
 $\text{deg} = \alpha$



$$x\{s\} \longrightarrow y$$

$$\text{deg} = \alpha - s$$



$$x \longrightarrow y\{s'\}$$

$$\text{deg} = \alpha + s'$$

$$\langle q^s x, y \rangle = \text{gdim} (\text{HOM}(x\{s\}, y)) = q^{-s} \text{gdim} (\text{HOM}(x, y)) = q^{-s} \langle x, y \rangle$$

$$\langle x, q^{s'} y \rangle = \text{gdim} (\text{HOM}(x, y\{s'\})) = q^{s'} \text{gdim} (\text{HOM}(x, y)) = q^{s'} \langle x, y \rangle$$

From geometric considerations we expect that

$$\langle E^{(a)}1_n, E^{(a)}1_n \rangle = \text{grdim} H^*(Gr(a, \infty)) = \prod_{j=1}^a \frac{1}{1 - q^{2j}}$$

Example

- $\langle E1_n, E1_n \rangle = \frac{1}{1 - q^2} = 1 + q^2 + q^4 + q^6 + \dots$
- $\langle E^21_n, E^21_n \rangle = [2][2] \frac{1}{1 - q^2} \frac{1}{1 - q^4} = (1 - q^{-2}) \left(\frac{1}{1 - q^2} \right)$

Idea: use the semilinear form to guess generating 2-morphisms

Example

$$\text{gdim}(\text{HOM}_{\mathcal{U}}(\mathcal{E}1_n, \mathcal{E}1_n)) = \langle E1_n, E1_n \rangle = \frac{1}{1 - q^2} = 1 + q^2 + q^4 + q^6 + \dots$$

The identity 2-morphism $E1_n \Rightarrow E1_n$ must be degree zero

$$\text{deg} \left(\begin{array}{c} | \\ \vdots \\ | \end{array} \begin{array}{c} n \\ | \\ \vdots \\ | \end{array} \right) = 0$$

contributes $q^0 = 1$ to graded dimension

The $q^2 \Rightarrow$ new 2-morphism $E1_n \Rightarrow E1_n$

$$\begin{array}{c} | \\ \vdots \\ \bullet \\ | \end{array} \begin{array}{c} n \\ | \\ \vdots \\ | \end{array} \quad \text{with} \quad \text{deg} \left(\begin{array}{c} | \\ \vdots \\ \bullet \\ | \end{array} \begin{array}{c} n \\ | \\ \vdots \\ | \end{array} \right) = 2$$

contributes q^2 to graded dimension

Example (cont.)

Vertically composing the dot with itself we get

$$\deg \left(\begin{array}{c} \uparrow \\ \alpha \bullet \\ | \\ n \end{array} \right) := \deg \left(\left(\begin{array}{c} \uparrow \\ (n+2) \bullet \\ | \\ n \end{array} \right)^\alpha \right) = 2\alpha,$$

$$\text{gdim} (\text{HOM}_{\mathcal{U}}(\mathcal{E}1_n, \mathcal{E}1_n))$$

$$= \deg \left(\begin{array}{c} \uparrow \\ | \\ n \end{array} \right) + \deg \left(\begin{array}{c} \uparrow \\ \bullet \\ | \\ n \end{array} \right) + \deg \left(\begin{array}{c} \uparrow \\ 2 \bullet \\ | \\ n \end{array} \right) + \dots$$

$$= 1 + q^2 + q^4 + \dots$$

$$= \frac{1}{1 - q^2} = \langle \mathcal{E}1_n, \mathcal{E}1_n \rangle$$

Example ($\text{HOM}_{\dot{U}}(\mathcal{E}^2 1_n, \mathcal{E}^2 1_n)$)

$$\sum_{0 \leq \alpha_1, 0 \leq \alpha_2} \text{deg} \left(\begin{array}{c} n+4 \\ \alpha_2 \uparrow \quad \uparrow n \\ \bullet \quad \bullet \\ \downarrow \alpha_1 \end{array} \right) = (1 + q^2 + q^4 + \dots)(1 + q^2 + \dots)$$

$$= \left(\frac{1}{1 - q^2} \right)^2$$

The semilinear form gives $\text{gdim } \dot{U}(\mathcal{E}\mathcal{E}1_n, \mathcal{E}\mathcal{E}1_n)$

$$= \langle EE1_n, EE1_n \rangle = [2][2] \langle \mathcal{E}^{(2)} 1_n, \mathcal{E}^{(2)} 1_n \rangle = (1 + q^{-2}) \left(\frac{1}{1 - q^2} \right)^2$$

\Rightarrow an additional generating 2-morphism of degree -2:

$$\begin{array}{c} \nearrow \\ \times \\ \searrow \end{array} \quad n \quad := \quad n+4 \quad \begin{array}{c} \nearrow \\ \bullet \\ \searrow \end{array} \quad n$$

Example (cont.)

$$\text{gdim } \dot{U}(\mathcal{E}\mathcal{E}1_n, \mathcal{E}\mathcal{E}1_n) = (1 - q^{-2})(1 + q^2 + q^4 + \dots)(1 + q^2 + q^4 + \dots)$$

$$\text{deg} \left(\begin{array}{c} \text{Diagram of two strands crossing twice} \\ n \end{array} \right) = -4 \quad \Rightarrow \quad \begin{array}{c} \text{Diagram of two strands crossing twice} \\ n \end{array} = 0$$

The space of degree zero 2-morphisms is 3-dimensional

$$\Rightarrow \begin{array}{cccccc} \uparrow & \uparrow & \text{Diagram of two strands crossing once with a dot on the bottom-left strand} & \text{Diagram of two strands crossing once with a dot on the top-right strand} & \text{Diagram of two strands crossing once with a dot on the top-left strand} & \text{Diagram of two strands crossing once with a dot on the bottom-right strand} \\ & n & n & n & n & n \end{array}$$

are not linearly independent.

\Rightarrow Add relations on these 2-morphisms

Considerations of adjoints in the geometric setting suggest that

$$\langle ux, y \rangle = \langle x, \tau(u)y \rangle \quad \text{for } u \in \mathbf{U} \text{ and } x, y \in \dot{\mathbf{U}},$$

where $\tau: U \rightarrow U$ is such that



- $\tau(xy) = \tau(y)\tau(x)$ for all $x \in \dot{\mathbf{U}}$ (antihomomorphism)
- $\tau(1_n) = 1_n$
- $\tau(1_n E 1_{n-2}) = q^{1-n} 1_n F 1_{n+2}$
- $\tau(1_n F 1_{n+2}) = q^{1+n} 1_{n+2} E 1_n$

for all $n \in \mathbb{Z}$.

Example

$$\text{gdim } \text{HOM}_{\dot{\mathcal{U}}}(\mathcal{F}\mathcal{E}1_n, 1_n) = \langle FE1_n, 1_n \rangle = \langle E1_n, \tau(F)1_n \rangle = \frac{q^{1+n}}{1-q^2}$$

$$\text{gdim } \text{HOM}_{\dot{\mathcal{U}}}(\mathcal{E}\mathcal{F}1_n, 1_n) = \langle EF1_n, 1_n \rangle = \langle F1_n, \tau(E)1_n \rangle = \frac{q^{1-n}}{1-q^2}$$

generator		
degree	1+n	1-n

Add generators



and dots so that

$$\begin{aligned} \text{gdim HOM}_{\mathcal{U}}(\mathcal{F}\mathcal{E}1_n, 1_n) &= \text{gdim} \sum_{\alpha=0}^{\infty} \left(\begin{array}{c} \alpha \\ \text{cup with } \alpha \text{ dots} \end{array} \begin{array}{c} n \end{array} \right) \\ &= q^{1-n}(1 + q^2 + q^4 + \dots) = \frac{q^{1-n}}{1 - q^2} = \langle FE1_n, 1_n \rangle \end{aligned}$$

and

$$\begin{aligned} \text{gdim HOM}_{\mathcal{U}}(\mathcal{E}\mathcal{F}1_n, 1_n) &= \text{gdim} \sum_{\alpha=0}^{\infty} \left(\begin{array}{c} \alpha \\ \text{cap with } \alpha \text{ dots} \end{array} \begin{array}{c} n \end{array} \right) \\ &= q^{1-n}(1 + q^2 + q^4 + \dots) = \frac{q^{1-n}}{1 - q^2} = \langle EF1_n, 1_n \rangle \end{aligned}$$

Similar calculations for $\text{HOM}_{\dot{\mathcal{U}}}(1_n, \mathcal{F}\mathcal{E}1_n)$ and $\text{HOM}_{\dot{\mathcal{U}}}(1_n, \mathcal{E}\mathcal{F}1_n)$ suggest generators

generator		
degree	1+n	1-n

Using these 2-morphisms we can construct new 2-morphisms $\mathcal{E}1_n \Rightarrow \mathcal{E}1_n$ of the form



Recall that

$$\text{gdim}(\text{HOM}_{\mathcal{U}}(\mathcal{E}1_n, \mathcal{E}1_n)) = \langle E1_n, E1_n \rangle = \frac{1}{1-q^2} = 1 + q^2 + q^4 + q^6 + \dots$$

But

$$\begin{aligned} \text{deg} \left(\begin{array}{c} n+2 \\ \uparrow \quad \downarrow \\ \text{loop} \\ \uparrow \quad \downarrow \\ n \end{array} \right) &= \text{deg} \left(\begin{array}{c} | \\ \cup \\ n \end{array} \right) + \text{deg} \left(\begin{array}{c} n+2 \\ \text{loop} \\ | \end{array} \right) \\ &= 1 + n + 1 - (n+2) = 0 \end{aligned}$$

similarly

$$\text{deg} \left(\begin{array}{c} \downarrow \quad \uparrow \\ \text{loop} \\ \downarrow \quad \uparrow \\ n+2 \end{array} \right) = 0$$

Hence these two 2-morphisms must be a multiple of the identity 2-morphism $\mathcal{E}1_n \Rightarrow \mathcal{E}1_n$

$$\begin{array}{c} n+2 \\ | \\ n \end{array}$$

Definition

\mathcal{U} is an additive \mathbb{k} -linear 2-category. The 2-category \mathcal{U} consists of

- objects: n for $n \in \mathbb{Z}$.

The homs $\mathcal{U}(n, n')$ between two objects n, n' are additive \mathbb{k} -linear categories consisting of:

- objects of $\mathcal{U}(n, n')$: a 1-morphism in \mathcal{U} from n to n' is a formal finite direct sum of 1-morphisms

$$\mathcal{E}^{\alpha_1} \mathcal{F}^{\beta_1} \dots \mathcal{E}^{\alpha_m} \mathcal{F}^{\beta_m} \mathbf{1}_n\{\mathbf{s}\} = \mathbf{1}_{n'} \mathcal{E}^{\alpha_1} \mathcal{F}^{\beta_1} \dots \mathcal{E}^{\alpha_m} \mathcal{F}^{\beta_m} \mathbf{1}_n\{\mathbf{s}\}$$

for any $\mathbf{s} \in \mathbb{Z}$ and $n' = n + 2 \sum \alpha_i - 2 \sum \beta_i$.

- morphisms of $\mathcal{U}(n, n')$: the \mathbb{k} -vector space of 2-morphisms

$$\mathcal{E}^{\alpha_1} \mathcal{F}^{\beta_1} \dots \mathcal{E}^{\alpha_m} \mathcal{F}^{\beta_m} \mathbf{1}_n\{\mathbf{s}\} \implies \mathcal{E}^{\alpha'_1} \mathcal{F}^{\beta'_1} \dots \mathcal{E}^{\alpha'_m} \mathcal{F}^{\beta'_m} \mathbf{1}_n\{\mathbf{s}'\}$$

given by linear combinations of degree $\mathbf{s} - \mathbf{s}'$ diagrams, modulo certain relations, built from composites of:

Generating 2-morphisms: identities

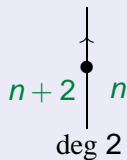
Degree zero identity 2-morphisms 1_x for each 1-morphism x in \mathcal{U} ; we write

$$\begin{array}{ccc}
 1_{\mathcal{E}1_n\{s\}} & & 1_{\mathcal{F}1_\lambda\{s\}} \\
 n+2 \quad \left| \quad n & & n-2 \quad \left| \quad n \\
 \uparrow & & \downarrow \\
 & &
 \end{array}$$

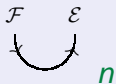
and more generally, the identity 2-morphism $1_{\mathcal{E}^{\alpha_1}\mathcal{F}^{\beta_1}\dots\mathcal{E}^{\alpha_m}\mathcal{F}^{\beta_m}1_n\{s\}1_\lambda\{s\}}$ is represented as

$$\begin{array}{ccccccc}
 n' & & & & & & n \\
 \uparrow & \uparrow & \uparrow & \downarrow & \downarrow & \dots & \uparrow & \uparrow & \uparrow & \downarrow & \downarrow & \downarrow & \downarrow \\
 \underbrace{\hspace{2cm}} & \underbrace{\hspace{2cm}} & & & \underbrace{\hspace{2cm}} & & \underbrace{\hspace{2cm}} & \underbrace{\hspace{2cm}} & & & & & \\
 \alpha_1 & \beta_1 & & & \alpha_m & & \beta_m & & & & & &
 \end{array}$$

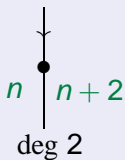
2-morphisms II



deg 2



deg $n+1$



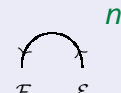
deg 2



deg $1-n$



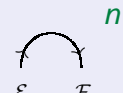
deg -2



deg $n+1$

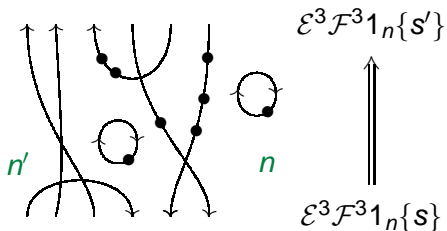


deg -2



deg $1-n$

For example



take degree $s - s'$ diagrams
this makes the total degree = 0

Local relations

$\mathcal{E}1_n$ and $\mathcal{F}1_n$ are biadjoint up to grading shift

$$\begin{array}{c} n+2 \\ \uparrow \end{array} \text{ (loop) } \begin{array}{c} \downarrow \\ n \end{array} = \begin{array}{c} n+2 \\ \uparrow \end{array} \begin{array}{c} \downarrow \\ n \end{array}$$

$$\begin{array}{c} n \\ \uparrow \end{array} \text{ (loop) } \begin{array}{c} \downarrow \\ n+2 \end{array} = \begin{array}{c} n \\ \uparrow \end{array} \begin{array}{c} \downarrow \\ n+2 \end{array}$$

$$\begin{array}{c} \uparrow \\ n+2 \end{array} \text{ (loop) } \begin{array}{c} \downarrow \\ n \end{array} = \begin{array}{c} \uparrow \\ n+2 \end{array} \begin{array}{c} \downarrow \\ n \end{array}$$

$$\begin{array}{c} \uparrow \\ n \end{array} \text{ (loop) } \begin{array}{c} \downarrow \\ n+2 \end{array} = \begin{array}{c} \uparrow \\ n \end{array} \begin{array}{c} \downarrow \\ n+2 \end{array}$$

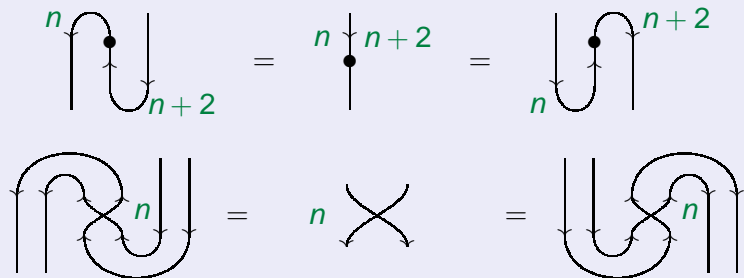
NilHecke relations

$$\text{crossing} \begin{array}{c} \downarrow \\ n \end{array} = 0,$$

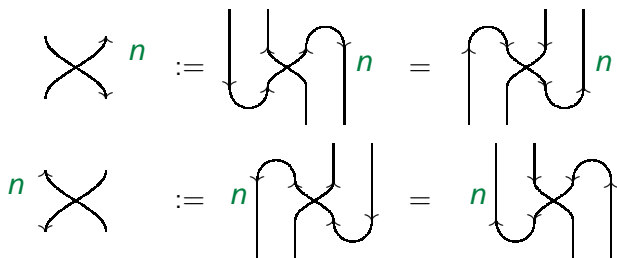
$$\text{crossing} \begin{array}{c} \downarrow \\ n \end{array} \text{ (loop) } = \text{crossing} \begin{array}{c} \downarrow \\ n \end{array} \text{ (loop) }$$

$$\begin{array}{c} \uparrow \\ n \end{array} \begin{array}{c} \uparrow \\ n \end{array} = \text{crossing} \begin{array}{c} \downarrow \\ n \end{array} \text{ (dot) } - \text{crossing} \begin{array}{c} \downarrow \\ n \end{array} \text{ (dot) } = \text{crossing} \begin{array}{c} \downarrow \\ n \end{array} \text{ (dot) } - \text{crossing} \begin{array}{c} \downarrow \\ n \end{array} \text{ (dot) }$$

Topological invariance



We can define



Positivity of bubbles

All dotted bubbles of negative degree are zero. That is,

$$\deg \left(\begin{array}{c} n \\ \circlearrowleft \\ \beta \end{array} \right) = 2(1 - n) + 2\beta$$

$$\deg \left(\begin{array}{c} n \\ \circlearrowright \\ \beta \end{array} \right) = 2(1 + n) + 2\beta$$

$$\Rightarrow \begin{array}{c} n \\ \circlearrowleft \\ \beta \end{array} = 0 \quad \text{if } \beta < n - 1$$

$$\begin{array}{c} n \\ \circlearrowright \\ \beta \end{array} = 0 \quad \text{if } \beta < -n - 1$$

It is convenient to emphasize the degree of a bubble on its label $\alpha \geq 0$

$$\deg \left(\begin{array}{c} n \\ \circlearrowleft \\ (n-1)+\alpha \end{array} \right) = 2\alpha$$

$$\deg \left(\begin{array}{c} n \\ \circlearrowright \\ (-n-1)+\alpha \end{array} \right) = 2\alpha$$

sometimes this is a negative number!

Fake bubbles



$$\deg \left(\begin{array}{c} n \\ \text{bubble} \\ \alpha \end{array} \right) \geq 0 \quad \alpha < 0$$



$$\deg \left(\begin{array}{c} n \\ \text{bubble} \\ \alpha \end{array} \right) \geq 0 \quad \alpha < 0$$

These formal symbols are inductively defined

Infinite Grassmannian equation:

$$\left(\begin{array}{c} n \\ \text{bubble} \\ -n-1 \end{array} + \begin{array}{c} n \\ \text{bubble} \\ -n-1+1 \end{array} t + \dots + \begin{array}{c} n \\ \text{bubble} \\ -n-1+\alpha \end{array} t^\alpha + \dots \right) \times$$

$$\left(\begin{array}{c} n \\ \text{bubble} \\ n-1 \end{array} + \dots + \begin{array}{c} n \\ \text{bubble} \\ n-1+\alpha \end{array} t^\alpha + \dots \right) = 1.$$

We set the degree zero bubbles equal to 1 for convenience.

Reduction to bubbles

$$\begin{aligned}
 \text{Diagram 1} \quad n &= - \sum_{\substack{f_1+f_2 \\ =-n}} \text{Diagram 2} \quad n \\
 \text{Diagram 3} \quad n &= \sum_{\substack{g_1+g_2 \\ =n}} \text{Diagram 4} \quad n
 \end{aligned}$$

The first equation shows a diagram with two vertical lines crossing twice (forming a figure-eight shape) with arrows pointing upwards, labeled n . This is equal to a negative sum over all pairs of integers f_1, f_2 such that $f_1 + f_2 = -n$. The summand is a diagram consisting of a vertical line with a dot at the top, a bubble (circle) with a dot at the bottom, and another vertical line with a dot at the top. The bubble has two arrows pointing clockwise. The top line has f_2 arrows pointing up, and the bottom line has $(n-1) + f_1$ arrows pointing up. The entire summand is labeled n .

The second equation shows a diagram with two vertical lines crossing twice (forming a figure-eight shape) with arrows pointing downwards, labeled n . This is equal to a sum over all pairs of integers g_1, g_2 such that $g_1 + g_2 = n$. The summand is a diagram consisting of a vertical line with a dot at the bottom, a bubble (circle) with a dot at the top, and another vertical line with a dot at the bottom. The bubble has two arrows pointing clockwise. The bottom line has g_2 arrows pointing down, and the top line has $(-n-1) + g_1$ arrows pointing down. The entire summand is labeled n .

EF decomposition

$$\begin{aligned}
 \text{Diagram 1} \quad n &= - \text{Diagram 2} \quad n + \sum_{\substack{f_1+f_2+f_3 \\ =n-1}} \text{Diagram 3} \quad n \\
 \text{Diagram 4} \quad n &= - \text{Diagram 5} \quad n + \sum_{\substack{g_1+g_2+g_3 \\ =-n-1}} \text{Diagram 6} \quad n
 \end{aligned}$$

The first equation shows two vertical lines with arrows pointing upwards, labeled n . This is equal to a negative diagram with two vertical lines crossing twice (figure-eight) with arrows pointing upwards, labeled n , plus a sum over all triples of integers f_1, f_2, f_3 such that $f_1 + f_2 + f_3 = n - 1$. The summand is a diagram with two vertical lines crossing twice (figure-eight) with arrows pointing upwards, labeled n . The top line has f_1 arrows pointing up, the bottom line has $(-n-1) + f_2$ arrows pointing up, and there are two bubbles (circles) with dots at the top and bottom respectively, each with two arrows pointing clockwise. The top bubble has f_3 arrows pointing up, and the bottom bubble has f_1 arrows pointing up.

The second equation shows two vertical lines with arrows pointing downwards, labeled n . This is equal to a negative diagram with two vertical lines crossing twice (figure-eight) with arrows pointing downwards, labeled n , plus a sum over all triples of integers g_1, g_2, g_3 such that $g_1 + g_2 + g_3 = -n - 1$. The summand is a diagram with two vertical lines crossing twice (figure-eight) with arrows pointing downwards, labeled n . The bottom line has g_2 arrows pointing down, the top line has $(n-1) + g_3$ arrows pointing down, and there are two bubbles (circles) with dots at the top and bottom respectively, each with two arrows pointing clockwise. The top bubble has g_1 arrows pointing down, and the bottom bubble has g_3 arrows pointing down.

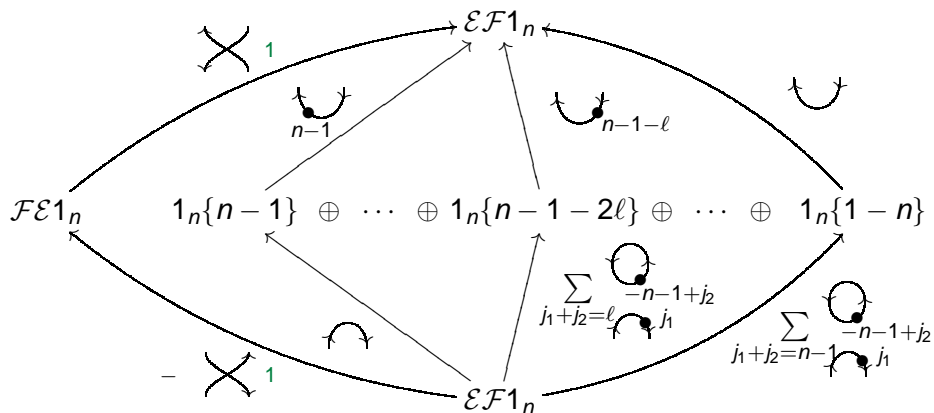
For $[n] = q^{n-1} + q^{n-3} + \dots + q^{1-n}$ write

$$\oplus_{[n]} 1_n := 1_n\{n-1\} \oplus 1_n\{n-3\} \oplus \dots \oplus 1_n\{1-n\}$$

Lifting \mathfrak{sl}_2 relations

$$\mathcal{EF}1_n \cong \mathcal{F}E1_n \oplus_{[n]} 1_n \quad \text{for } n \geq 0$$

$$\mathcal{F}E1_n \cong \mathcal{EF}1_n \oplus_{[-n]} 1_n \quad \text{for } n \leq 0$$



Example ($n = 0$, $\mathcal{E}\mathcal{F}1_0 \cong \mathcal{F}\mathcal{E}1_0$)

$$\mathcal{F}\mathcal{E}1_0 \begin{array}{c} \xrightarrow{\quad \text{X}_0 \quad} \\ \xleftarrow{\quad -\text{X}_0 \quad} \end{array} \mathcal{E}\mathcal{F}1_0$$

These maps are isomorphisms since for $n = 0$

$$-\text{X}_0 = \begin{array}{c} \downarrow \quad \uparrow \\ \uparrow \quad \downarrow \end{array} 0$$

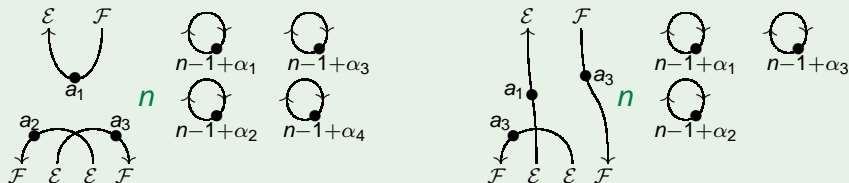
$$-\text{X}_0 = \begin{array}{c} \uparrow \quad \downarrow \\ \downarrow \quad \uparrow \end{array} 0$$

Spanning sets

Using relations one can find spanning sets for the space of 2-morphisms:

Example

$$\mathcal{U}(\mathcal{F}\mathcal{E}^2\mathcal{F}1_n, \mathcal{E}\mathcal{F}1_n)$$



- no strand intersects itself
- all closed diagrams are reduced to non-nested bubbles with the same orientation and are moved to the far right of each diagram
- all dots are confined to a small interval on each strand

Iterated flag varieties

$$\begin{array}{ccc} & FI(k, k+1, N) & \\ \swarrow & \{0 \subset \mathbb{C}^k \subset \mathbb{C}^{k+1} \subset \mathbb{C}^N\} & \searrow \\ Gr(k, N) & & Gr(k+1, N) \\ \{0 \subset \mathbb{C}^k \subset \mathbb{C}^N\} & & \{0 \subset \mathbb{C}^{k+1} \subset \mathbb{C}^N\} \end{array}$$

$$\begin{array}{ccc} & H_{k,k+1} := H^*(FI(k, k+1, N)) & \\ \nearrow & & \nwarrow \\ H_k := H^*(Gr(k, N)) & & H_{k+1} := H^*(Gr(k+1, N)) \end{array}$$

$$\begin{array}{c}
 H_N \cong \mathbb{C} \\
 \vdots \\
 H_{k+1} \\
 \begin{array}{c} \left(\begin{array}{c} \nearrow \\ \searrow \end{array} \right) \\ H_{k+1,k} \quad H_{k+1,k} \end{array} \\
 H_k \\
 \begin{array}{c} \left(\begin{array}{c} \nearrow \\ \searrow \end{array} \right) \\ H_{k,k-1} \quad H_{k,k-1} \end{array} \\
 H_{k-1} \\
 \vdots \\
 H_0 \cong \mathbb{C}
 \end{array}$$

$$\begin{array}{l}
 \Gamma_N: \mathcal{U} \rightarrow \text{Flag}_N \\
 n \mapsto \begin{cases} H_k & n = 2k - N \\ \mathbf{0} & n \neq 2k - N \end{cases} \\
 \mathcal{E}1_n \mapsto H_{k+1,k} \\
 \mathcal{F}1_n \mapsto H_{k-1,k} \\
 \mathcal{E}1_{n+2}\mathcal{E}1_n \mapsto H_{k+2,k+1} \otimes_{H_{k+1}} H_{k+1,k} \\
 \begin{array}{c} \nearrow \\ \searrow \end{array} n \mapsto \begin{array}{c} H_{k+2,k+1} \otimes_{H_{k+1}} H_{k+1,k} \\ \uparrow \text{bimodule map} \\ H_{k+2,k+1} \otimes_{H_{k+1}} H_{k+1,k} \end{array}
 \end{array}$$

Theorem

Γ_N is a 2-functor, all relations in \mathcal{U} hold in the iterated flag category \mathbf{Flag}_N . Γ_N categorifies the irreducible $N + 1$ -dimensional representation.

Theorem (arXiv:0803.3652)

This graphical calculus is consistent and categorifies $\dot{\mathbf{U}}_{\mathbb{Z}}$

- $\dot{\mathbf{U}}_{\mathbb{Z}} \cong K_0(\dot{\mathcal{U}})$ the Grothendieck ring/category of this 2-category

$$x \oplus y \in \dot{\mathcal{U}} \rightsquigarrow [x] + [y] \in K_0(\dot{\mathcal{U}}) \quad x\{s\} \rightsquigarrow q^s[x] \in K_0(\dot{\mathcal{U}})$$

- Indecomposable 1-morphisms \Leftrightarrow Lusztig canonical basis element
- Graded 2Hom $\text{HOM}_{\dot{\mathcal{U}}}(x, y)$ categorifies the semilinear form $\langle x, y \rangle$
- The 2-category $\dot{\mathcal{U}}$ acts on cohomology of iterated flag varieties, categorifying the irreducible N -dimensional rep of $\mathbf{U}_q(\mathfrak{sl}_2)$
- Various known (anti) linear (anti)automorphism of $\mathbf{U}_q(\mathfrak{sl}_2)$ have categorifications that are given on 2-morphisms by symmetries of the graphical calculus.

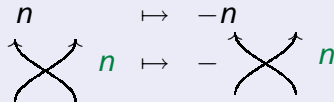
$$\omega: \dot{\mathbf{U}} \rightarrow \mathbf{U}$$

$$E1_n \mapsto F1_{-n}$$

$$F1_n \mapsto E1_{-n}$$

\rightsquigarrow

Invert orientation



Joint with Mikhail Khovanov

- 2-category $\dot{\mathcal{U}}$ has an extension to a categorification of $\dot{\mathbf{U}}(\mathfrak{sl}_n)$.
- There is a categorification of $\dot{\mathbf{U}}^+(\mathfrak{g})$ for any Kac-Moody algebra \mathfrak{g} using a similar diagrammatic calculus (arXiv:0803.4121, arXiv:0804.2080).
- Conjectural categorification of the integral form of $\dot{\mathbf{U}}(\mathfrak{g})$ for any Kac-Moody algebra (arXiv:0807.3250).

arxiv.0407205, arXiv:0812.5023

Closely related 2-categories were studied by Chuang and Rouquier.

arXiv:0902.1796

Geometric notions of categorical \mathfrak{sl}_2 -actions and their connection with the 2-categories above have been studied by Cautis-Kamnitzer-Licata.