Categorification Homework

MSRI Introductory Workshop: Homology Theories of Knots and Links

Aaron Lauda   email: lauda@math.columbia.edu

1) **(Adjoint functors)** Given categories $\mathcal{C}$ and $\mathcal{D}$ and functors $\mathcal{C} \underset{F}{\overset{U}{\rightleftarrows}} \mathcal{D}$, we say that F is the left adjoint of U when there exists a natural bijection $\Phi_{Y,X} \colon \mathrm{Hom}_{\mathcal{C}}(FY, X) \cong \mathrm{Hom}_{\mathcal{D}}(Y, UX)$. Show that this is equivalent to the following:

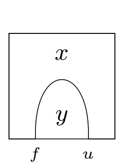
   (a) There exist natural transformations

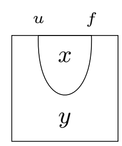
   - $1_{\mathcal{C}} \Leftarrow F \circ U \colon \eta$
   - $U \circ F \Leftarrow 1_{\mathcal{D}} \colon \varepsilon$

   such that for each X in $\mathcal{C}$ and Y in $\mathcal{D}$

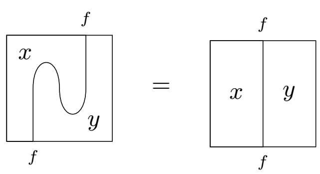
   - $1_{UX} = U(\varepsilon_X) \circ \eta_{UX}$ or $(\varepsilon 1_U) \circ (1_U \eta) = 1_U$
   - $1_{FY} = \varepsilon_{FY} \circ F(\eta_Y)$ or $(1_F \varepsilon) \circ (\eta 1_F) = 1_F$

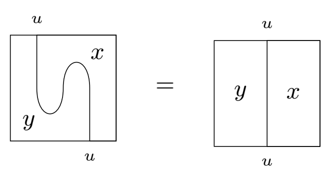
   (b) There exist natural transformations

   $\varepsilon \; :=$ 

   $\eta \; :=$ 

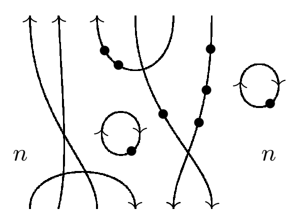
   such that the equalities





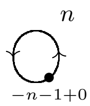
   hold.

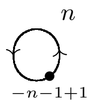
2) Compute the degree of the following diagram in $\mathcal{U}$

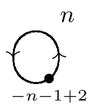


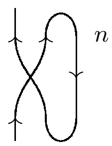
Hint: The relations in $\mathcal{U}$ imply that diagrams related by planar isotopy represent the same 2-morphism and therefore have the same degree.

3) **(Fake Bubbles)** Assume that $n > 1$. Write the following fake bubbles in terms of real bubbles:

a) 

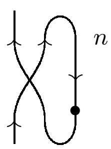
b) 

c) 

4) **(Curls)** Simplify the diagram 

and write the result in terms of real bubbles when

a) $n > 0$,
b) $n = 0$,
c) $n = -1$.

5) Simplify the diagram 

for all n. You do not need to express real bubbles in terms of fake bubbles.

6) **($\mathfrak{sl}_2$ relations)** Use the relations in $\mathcal{U}$ to show that

$$\mathcal{E}\mathcal{F}1_1 \cong \mathcal{F}\mathcal{E}1_1 \oplus 1_1.$$