

Categorification Homework

MSRI Introductory Workshop: Homology Theories of Knots and Links

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- 1) (**Adjoint functors**) Look at the Wikipedia page on adjoint functors.
- 2) In \mathcal{U} we have

$$\deg \left(\begin{array}{c} \text{[Diagram: A complex link with crossings and bubbles, labeled } n \text{]} \\ n \end{array} \right) = 28 - 6n.$$

- 3) (**Fake Bubbles**) Assume that $n > 1$. Write the following fake bubbles in terms of real bubbles:

$$\text{a) } \begin{array}{c} n \\ \text{[Bubble]} \\ -n-1+0 \end{array} := 1$$

$$\text{b) } \begin{array}{c} n \\ \text{[Bubble]} \\ -n-1+1 \end{array} := - \begin{array}{c} n \\ \text{[Bubble]} \\ n-1+1 \end{array}$$

$$\text{c) } \begin{array}{c} n \\ \text{[Bubble]} \\ -n-1+2 \end{array} := - \begin{array}{c} n \\ \text{[Bubble]} \\ n-1+2 \end{array} + \begin{array}{c} n \\ \text{[Bubble]} \\ n-1+1 \end{array} \begin{array}{c} n \\ \text{[Bubble]} \\ n-1+1 \end{array}$$

Note that in general, the fake bubbles are inductively defined as

$$\begin{array}{c} n \geq 0 \\ \text{[Bubble]} \\ -n-1+j \end{array} = - \sum_{\ell_1+\ell_2=j} \begin{array}{c} n \\ \text{[Bubble]} \\ n-1+\ell_1 \end{array} \begin{array}{c} n \\ \text{[Bubble]} \\ -n-1+\ell_2 \end{array}$$

- 4) (**Curls**) Note that in general

$$\begin{array}{c} n \\ \text{[Curl]} \end{array} = - \sum_{\substack{f_1+f_2 \\ =-n}} \begin{array}{c} f_2 \\ \text{[Bubble]} \\ (n-1)+f_1 \end{array}$$

$$\text{a) If } n > 0 \Rightarrow \begin{array}{c} n \\ \text{[Curl]} \end{array} = 0$$

- b) If $n = 0$ then

$$\begin{array}{c} 0 \\ \text{[Curl]} \end{array} = - \begin{array}{c} 0 \\ \text{[Bubble]} \\ -1 \end{array} = - \begin{array}{c} 0 \\ \text{[Bubble]} \\ -1 \end{array}$$

since $\deg \left(\begin{array}{c} 0 \\ \text{[Bubble]} \\ -1 \end{array} \right) = 2(1-0) - 2 = 0$ so that $\begin{array}{c} 0 \\ \text{[Bubble]} \\ -1 \end{array} := 1.$

c) If $n = -1$ then

$$\begin{aligned}
 & \text{Diagram with two crossings}^{-1} = - \text{Diagram with a dot and a loop}^{-1} - \text{Diagram with a dot and a loop}^{-1} \\
 & \deg \left(\text{Diagram with a loop}^{-1}_{-2} \right) = 0 \quad \text{and} \quad \deg \left(\text{Diagram with a loop}^{-1}_{-1} \right) = 1 \\
 & \Rightarrow \text{Diagram with a loop}^{-1}_{-2} := 1 \quad \text{and} \quad \text{Diagram with a loop}^{-1}_{-1} = - \text{Diagram with a loop}^{-1}_1 \\
 & \Rightarrow \text{Diagram with two crossings}^{-1} = - \text{Diagram with a dot and a line}^{-1} + \text{Diagram with a dot and a loop}^{-1}_1
 \end{aligned}$$

5) Using the nilHecke relation

$$\text{Diagram with two crossings}^n = - \sum_{\substack{f_1+f_2 \\ =1-n}} \text{Diagram with a dot and a loop}^n$$

6) (\mathfrak{sl}_2 relations) Use the relations in \mathcal{U} to show that

$$\mathcal{EF}1_1 \cong \mathcal{FE}1_1 \oplus 1_1.$$

Solution: If $n = 1$ then the isomorphism given in lecture takes the form

$$\begin{array}{ccccc}
 \alpha_2 = & \text{Diagram with two crossings}^1 & \mathcal{EF}1_1 & \beta_2 = & \text{Diagram with a loop}^1 \\
 & \nearrow & & \nwarrow & \\
 & \mathcal{FE}1_1 & & & 1_1\{0\} \\
 & \nwarrow & & \nearrow & \\
 \alpha_1 = - & \text{Diagram with two crossings}^1 & \mathcal{EF}1_1 & \beta_1 = & \text{Diagram with a loop}^1
 \end{array}$$

Checking that these 2-morphisms define an isomorphism $\mathcal{EF}1_1 \cong \mathcal{FE}1_1 \oplus 1_1$ amounts to checking that

$$\begin{aligned}
 \alpha_2 \circ \alpha_1 + \beta_2 \circ \beta_1 &= Id_{\mathcal{EF}1_1}, \quad \alpha_1 \circ \alpha_2 = Id_{\mathcal{FE}1_1}, \quad \beta_1 \circ \beta_2 = Id_{1_1}, \\
 \beta_1 \circ \alpha_2 &= 0, \quad \beta_2 \circ \alpha_1 = 0.
 \end{aligned}$$

All of these equations follow from the relations in \mathcal{U} :

$$\begin{aligned}
 \alpha_1 \circ \alpha_2 &= - \text{Diagram with two crossings}^1 = 1 \text{Diagram with two vertical lines}^1 = Id_{\mathcal{FE}1_1} \\
 \beta_1 \circ \beta_2 &= \text{Diagram with a loop}^1 = 1 = Id_{1_1} \\
 \beta_1 \circ \alpha_2 &= \text{Diagram with two crossings}^1 = - \sum_{\ell_1+\ell_2=-2} \text{Diagram with a loop}^1 = 0 \\
 \alpha_2 \circ \alpha_1 + \beta_2 \circ \beta_1 &= - \text{Diagram with two crossings}^1 + 1 \text{Diagram with two vertical lines}^1 = Id_{\mathcal{EF}1_1}.
 \end{aligned}$$