Categorification Homework

MSRI Introductory Workshop: Homology Theories of Knots and Links

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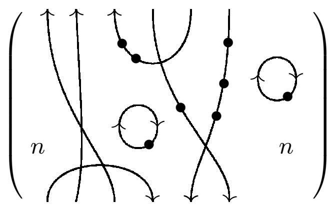
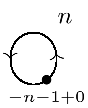
1) **(Adjoint functors)** Look at the Wikipedia page on adjoint functors.

2) In $\mathcal{U}$ we have

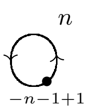
deg 

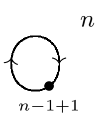
$= 28 - 6n.$

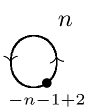
3) **(Fake Bubbles)** Assume that $n > 1$. Write the following fake bubbles in terms of real bubbles:

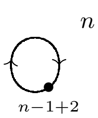
a) 

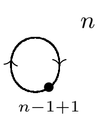
$:= 1$

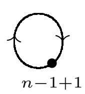
b) 

$:= -$ 

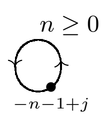
c) 

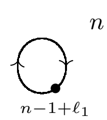
$:= -$ 

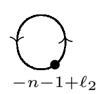
$+$ 



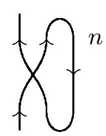
Note that in general, the fake bubbles are inductively defined as

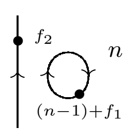


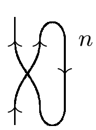
$= - \sum_{\ell_1+\ell_2=j}$ 



4) **(Curls)** Note that in general

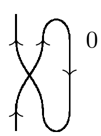


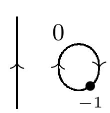
$= - \sum_{\substack{f_1+f_2 \\ =-n}}$ 

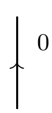
a) If $n > 0 \Rightarrow$ 

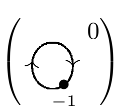
$= \quad 0$

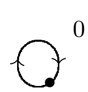
b) If $n = 0$ then



$= \; -$ 

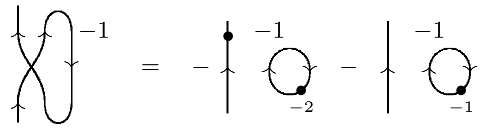
$= \; -$ 

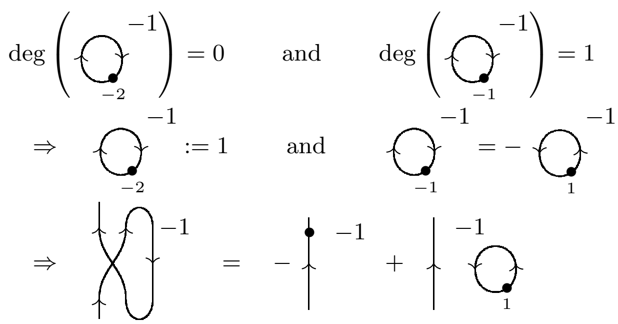
since deg 

$= 2(1-0) - 2 = 0$ so that 

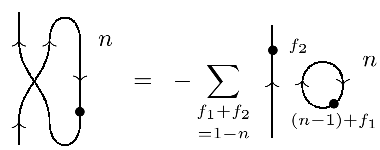
$:= 1.$

c) If $n = -1$ then





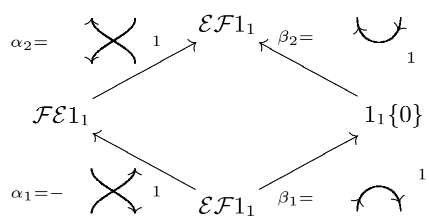
5) Using the nilHecke relation



6) **($\mathfrak{sl}_2$ relations)** Use the relations in $\mathcal{U}$ to show that

$$\mathcal{E}\mathcal{F}1_1 \cong \mathcal{F}\mathcal{E}1_1 \oplus 1_1.$$

Solution: If $n = 1$ then the isomorphism given in lecture takes the form



Checking that these 2-morphisms define an isomorphism $\mathcal{E}\mathcal{F}1_1 \cong \mathcal{F}\mathcal{E}1_1 \oplus 1_1$ amounts to checking that

$$\alpha_2 \circ \alpha_1 + \beta_2 \circ \beta_1 = Id_{\mathcal{E}\mathcal{F}1_1}, \quad \alpha_1 \circ \alpha_2 = Id_{\mathcal{F}\mathcal{E}1_1}, \quad \beta_1 \circ \beta_2 = Id_{1_1},$$

$$\beta_1 \circ \alpha_2 = 0, \qquad \beta_2 \circ \alpha_1 = 0.$$

All of these equations follow from the relations in $\mathcal{U}$: