

Diagrammatic categorification of quantum groups II

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The goal: categorify $U_q^+(\mathfrak{g})$

The quantum enveloping algebra $U_q(\mathfrak{g})$ of a symmetrizable Kac-Moody Lie algebra \mathfrak{g} has a decomposition

$$U_q(\mathfrak{g}) = U_q^- \otimes U_q(\mathfrak{h}) \otimes U_q^+$$

U_q^+ has the structure of a bialgebra: **try to categorify the bialgebra U_q^+**

The plan: define a new algebra R

$$\begin{array}{c} R\text{-mod} - \left(\begin{array}{l} \text{category of finitely generated} \\ \text{graded projective modules} \end{array} \right) \\ \downarrow \text{Decategorification} \\ \text{(Grothendieck group)} \\ K_0(R\text{-mod}) \cong U_q^+(\mathfrak{g}) \end{array}$$

Grothendieck groups

If R is a graded ring then $K_0(R)$ is the Grothendieck group of the category $R\text{-pmod}$ of graded projective finitely-generated R -modules.

$K_0(R)$ has generators $[M]$ over all objects of $R\text{-pmod}$ and defining relations

$$\begin{aligned} [M] &= [M_1] + [M_2] && \text{if } M \cong M_1 \oplus M_2 \\ [M\{s\}] &= q^s[M] && s \in \mathbb{Z} \end{aligned}$$

$K_0(R)$ is a $\mathbb{Z}[q, q^{-1}]$ -module.

A useful trick

If e is an idempotent in R , then

$$R \cong Re \oplus R(1 - e).$$

$$U_q^+ \subset U_q(\mathfrak{g})$$

$$\mathfrak{g} = \mathfrak{sl}_n$$

$$E_i = e_{i,i+1} = \begin{matrix} & & & i+1 \\ i & \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 \\ \vdots & \ddots & & \\ 0 & \dots & 0 & \end{pmatrix} \end{matrix}$$

Lie algebra relations:

$$[E_i, E_j] = 0 \quad |i - j| > 1 \qquad [E_i, [E_i, E_j]] = 0 \quad |i - j| = 1$$

Enveloping algebra relations for $U^+(\mathfrak{sl}_n)$

$$E_i E_j = E_j E_i \quad |i - j| > 1$$

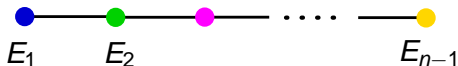
$$2E_i E_j E_i = E_i^2 E_j + E_j E_i^2 \quad j = i \pm 1$$

Quantum enveloping algebra $U_q^+(\mathfrak{sl}_n)$

$$E_i E_j = E_j E_i \quad |i - j| > 1$$

$$\text{quantum 2} \rightarrow (q + q^{-1})E_i E_j E_i = E_i^2 E_j + E_j E_i^2 \quad j = i \pm 1$$

$U_q^+(\mathfrak{sl}_n)$ has a generator E_i for each vertex of the Dynkin graph



U_q^+ for any Γ

Let Γ be an unoriented graph with set of vertices I .

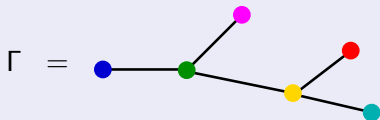
U_q^+ is the $\mathbb{Q}(q)$ -algebra with:

- generators: $E_i \quad i \in I$

- relations: $E_i E_j = E_j E_i$ if $\begin{matrix} i & j \\ \bullet & \bullet \end{matrix}$

- $(q + q^{-1})E_i E_j E_i = E_i^2 E_j + E_j E_i^2$ if $\begin{matrix} i & j \\ \bullet & \bullet \end{matrix}$

U_q^+ is $\mathbb{N}[I]$ graded with $\deg(E_i) = i$.



Integral form of U_q^+

Define quantum integers and quantum factorials:

$$[a] := \frac{q^a - q^{-a}}{q - q^{-1}} \qquad [a]! := [a][a-1] \dots [1]$$

Example

- $[1] = 1$
- $[2] = \frac{q^2 - q^{-2}}{q - q^{-1}} = q + q^{-1}$
- $[3] = \frac{q^3 - q^{-3}}{q - q^{-1}} = q^2 + 1 + q^{-2}$

The algebra $U_{\mathbb{Z}}^+$ is the $\mathbb{Z}[q, q^{-1}]$ -subalgebra of U_q^+ generated by all products of quantum divided powers:

$$E_i^{(a)} := \frac{E_i^a}{[a]!}$$

Since

$$E_i^{(2)} = \frac{E_i^2}{q + q^{-1}}$$

we can write the U_q^+ relation

$$(q + q^{-1})E_i E_j E_i = E_i^2 E_j + E_j E_i^2 \quad \text{if } \begin{array}{c} i \quad j \\ \bullet \text{---} \bullet \end{array}$$

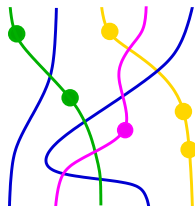
as

$$E_i E_j E_i = E_i^{(2)} E_j + E_j E_i^{(2)} \quad \text{if } \begin{array}{c} i \quad j \\ \bullet \text{---} \bullet \end{array}$$

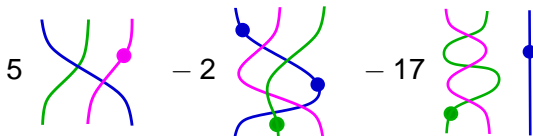
Categorification of U_q^+

Associated to graph Γ consider braid-like diagrams with dots whose strands are labelled by the vertices $i \in I$ of the graph Γ .

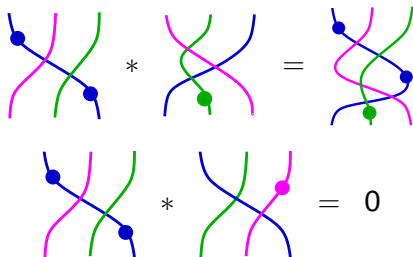
Let $\nu = \sum_{i \in I} \nu_i \cdot i$, for $\nu_i = 0, 1, 2, \dots$
 ν keeps track of how many strands of each color occur in a diagram



Form an abelian group by taking \mathbb{Z} -linear (or \mathbb{k} -linear) combinations of diagrams:



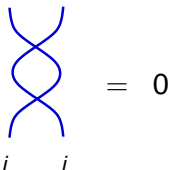
Multiplication is given by stacking diagrams on top of each other when the colors match:

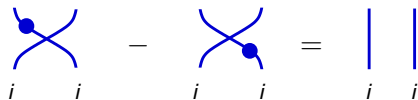


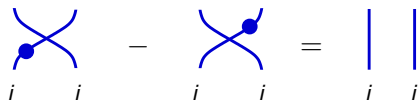
Definition

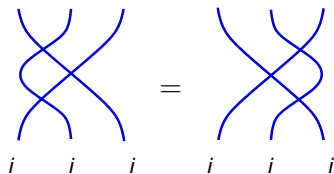
Given $\nu \in \mathbb{N}[I]$ define the ring $R(\nu)$ as the set of planar diagrams colored by ν , modulo planar braid-like isotopies and the following local relations:

Local relations I


$$\begin{array}{c} \text{crossing} \\ i \quad i \end{array} = 0$$


$$\begin{array}{c} \text{crossing with dot on top-left} \\ i \quad i \end{array} - \begin{array}{c} \text{crossing with dot on bottom-right} \\ i \quad i \end{array} = \begin{array}{c} | \\ i \end{array} \begin{array}{c} | \\ i \end{array}$$

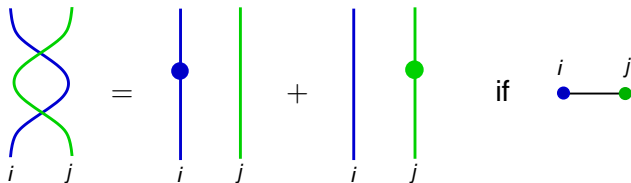

$$\begin{array}{c} \text{crossing with dot on bottom-left} \\ i \quad i \end{array} - \begin{array}{c} \text{crossing with dot on top-right} \\ i \quad i \end{array} = \begin{array}{c} | \\ i \end{array} \begin{array}{c} | \\ i \end{array}$$


$$\begin{array}{c} \text{crossing} \\ i \quad i \end{array} = \begin{array}{c} \text{crossing} \\ i \quad i \end{array}$$

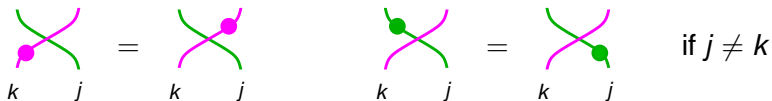
Local relations II



A diagrammatic equation showing a crossing of two strands, labeled i (blue) and k (magenta). The left side shows the crossing. The right side shows two parallel vertical strands, one blue and one magenta. To the right of the equals sign is the text "if" followed by two colored dots: a blue dot labeled i and a magenta dot labeled k .

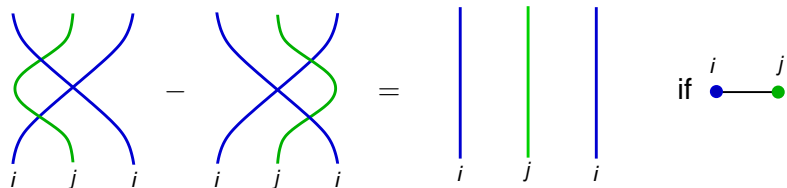


A diagrammatic equation showing a crossing of two strands, labeled i (blue) and j (green). The left side shows the crossing. The right side shows the sum of two diagrams: the first has a blue dot on the blue strand and a green strand; the second has a blue strand and a green dot on the green strand. To the right of the plus sign is the text "if" followed by a blue dot labeled i and a green dot labeled j connected by a horizontal line.

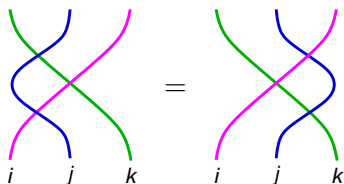


Two diagrammatic equations for crossings of strands k (magenta) and j (green). The first equation shows a crossing with a magenta dot on the magenta strand equal to a crossing with a magenta dot on the green strand. The second equation shows a crossing with a green dot on the magenta strand equal to a crossing with a green dot on the green strand. To the right of the second equation is the text "if $j \neq k$ ".

Local relations III



A diagrammatic equation showing the cancellation of crossings. On the left, two diagrams are subtracted: the first has a blue strand crossing over a green strand, and the second has a green strand crossing over a blue strand. The strands are labeled i , j , and i at the bottom. This is equal to three parallel vertical strands: blue, green, and blue, labeled i , j , and i . To the right, the text "if" is followed by a diagram of a blue dot and a green dot connected by a horizontal line, with i above the blue dot and j above the green dot.



A diagrammatic equation showing a crossing between two strands. On the left, a blue strand crosses over a green strand. On the right, a green strand crosses over a blue strand. The strands are labeled i , j , and k at the bottom.

otherwise,

some of i, j, k may be equal

Grading

$q \longrightarrow$ grading shift

$$\deg \left(\begin{array}{c} | \\ \bullet \\ | \end{array} \right) = 2$$
$$\deg \left(\begin{array}{cc} & \\ i & j \\ & \end{array} \right) = \begin{cases} -2 & \text{if } i = j \\ 0 & \text{if } \begin{array}{cc} & \\ \bullet & \bullet \end{array} \\ 1 & \text{if } \begin{array}{cc} & \\ \bullet & \text{---} \bullet \end{array} \end{cases}$$

The $R(\nu)$ relations are homogeneous with respect to this grading.

Example

- If $\nu = 0$ then $R(0) = \mathbb{Z}$ with unit element given by the empty diagram.
- If $\nu = i$ for some vertex i , then a diagram is a line with some number $a \geq 0$ of dots on it.

$$a \begin{array}{c} | \\ \bullet \\ | \\ i \end{array} := \left(\begin{array}{c} | \\ \bullet \\ | \\ i \end{array} \right)^a$$

Hence, $R(i) \cong \mathbb{Z}[x]$ where the isomorphism maps

$$a \begin{array}{c} | \\ \bullet \\ | \\ i \end{array} \mapsto x^a$$

R_ν is the associative, F -algebra on generators $1_{\underline{i}}$, $x_{a,\underline{i}}$; $\psi_{b,\underline{i}}$ for $1 \leq a \leq m$, $1 \leq b \leq m-1$ and $\underline{i} \in \text{Seq}(\nu)$ subject to the following relations for $\underline{i}, \underline{j} \in \text{Seq}(\nu)$:

$$1_{\underline{i}}1_{\underline{j}} = \delta_{\underline{i},\underline{j}}1_{\underline{i}},$$

$$x_{a,\underline{i}} = 1_{\underline{i}}x_{a,\underline{i}}1_{\underline{i}},$$

$$\psi_{a,\underline{i}} = 1_{s_a(\underline{i})}\psi_{a,\underline{i}}1_{\underline{i}},$$

$$x_{a,\underline{i}}x_{b,\underline{i}} = x_{b,\underline{i}}x_{a,\underline{i}},$$

$$\psi_{a,s_a(\underline{i})}\psi_{a,\underline{i}} = \begin{cases} 0 & \text{if } i_r = i_{r+1} \\ 1_{\underline{i}} & \text{if } (\alpha_{i_a}, \alpha_{i_{a+1}}) = 0 \\ \left(x_{a,\underline{i}}^{-\langle i_a, i_{a+1} \rangle} + x_{a+1,\underline{i}}^{-\langle i_{a+1}, i_a \rangle} \right) 1_{\underline{i}} & \text{if } (\alpha_{i_a}, \alpha_{i_{a+1}}) \neq 0 \text{ and } i_a \neq i_{a+1} \end{cases},$$

$$\psi_{b,s_a(\underline{i})}\psi_{a,\underline{i}} = \psi_{a,s_b(\underline{i})}\psi_{b,\underline{i}} \quad \text{if } |a-b| > 1,$$

$$\psi_{a,s_{a+1}s_a(\underline{i})}\psi_{a+1,s_a(\underline{i})}\psi_{a,\underline{i}} - \psi_{a+1,s_a s_{a+1}(\underline{i})}\psi_{a,s_{a+1}(\underline{i})}\psi_{a+1,\underline{i}} =$$

$$= \begin{cases} \sum_{r=0}^{-\langle i_a, i_{a+1} \rangle - 1} x_{a,\underline{i}}^r x_{a+2,\underline{i}}^{-\langle i_a, i_{a+1} \rangle - 1 - r} & \text{if } i_a = i_{a+2} \text{ and } (\alpha_{i_a}, \alpha_{i_{a+1}}) \neq 0 \\ 0 & \text{otherwise,} \end{cases}$$

$$\psi_{a,\underline{i}}x_{b,\underline{i}} - x_{s_a(b),s_a(\underline{i})}\psi_{a,\underline{i}} = \begin{cases} 1_{\underline{i}} & \text{if } a = b \text{ and } i_a = i_{a+1} \\ -1_{\underline{i}} & \text{if } a = b + 1 \text{ and } i_a = i_{a+1} \\ 0 & \text{otherwise.} \end{cases}$$

Let $R = \bigoplus_{\nu} R(\nu)$. For each product of E_i 's in U_q^+ we have an idempotent in R :

$$E_i E_j E_k E_i E_j E_\ell \quad \mapsto \quad 1_{ijkij\ell} := \begin{array}{cccccc} | & | & | & | & | & | \\ i & j & k & i & j & \ell \end{array}$$

This gives rise to a projective module

$$\mathcal{E}_i \mathcal{E}_j \mathcal{E}_k \mathcal{E}_i \mathcal{E}_j \mathcal{E}_\ell := R 1_{ijkij\ell} = R(2i + 2j + k + \ell) 1_{ijkij\ell}$$

corresponding to the idempotent $1_{ijkij\ell}$ above.

Example

For a given $i \in I$ we write \mathcal{E}_i^m for the projective module $R(mi) \cong \text{NH}_m$

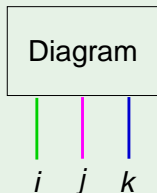
corresponding to the idempotent $1_{i^m} = \begin{array}{ccc} | & | & | \\ i & i & i \end{array} \cdots \begin{array}{c} | \\ i \end{array}$, where $i^m := i \dots i$.

Example

Consider

$$R1_{ijk} = R(i+j+k)1_{ijk}$$

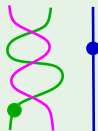
The projective module $\mathcal{E}_i\mathcal{E}_j\mathcal{E}_k := R(i+j+k)1_{ijk}$ consists of linear combinations of diagrams that have the sequence ijk at the bottom



i.e.



and



$\in R(i+j+k)1_{ijk}$

But

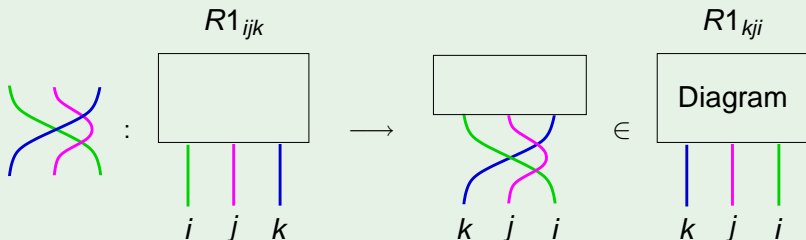


$\notin R(i+j+k)1_{ijk}$

We can construct maps between projective modules by adding diagrams at the *bottom*

Example

We get a module map from $\mathcal{E}_i \mathcal{E}_j \mathcal{E}_k := R(i+j+k)1_{ijk}$ to $\mathcal{E}_k \mathcal{E}_j \mathcal{E}_i := R(i+j+k)1_{kji}$ as follows:



Given a graded module M and a Laurent polynomial $f = \sum f_a q^a \in \mathbb{Z}[q, q^{-1}]$ write

$$M^{\oplus f} \quad \text{or} \quad \bigoplus_f M$$

to denote the direct sum over $a \in \mathbb{Z}$ of f_a copies of $M\{a\}$

Example

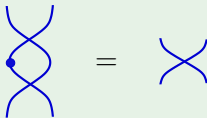
Since $[3] = q^2 + 1 + q^{-2} \in \mathbb{Z}[q, q^{-1}]$, for a graded module M


$$\bigoplus_{[3]} M = M\{2\} \oplus M\{0\} \oplus M\{-2\}$$

Example ($n = 2$)

$$E_i^{(2)} = \frac{E_i^2}{q+q^{-1}} \quad \text{or} \quad E_i^2 = (q + q^{-1})E_i^{(2)}$$

Recall that



so that $e_2 =$  is an idempotent.

$\mathcal{E}_i^{(2)}$ is the projective module for this idempotent

$$\mathcal{E}_i^{(2)} := R(2i)e_2\{1\}$$

$$\mathcal{E}_i^2 \cong \mathcal{E}_i^{(2)}\{1\} \oplus \mathcal{E}_i^{(2)}\{-1\}$$

Categorification of $E_i E_j E_i = E_i^{(2)} E_j + E_j E_i^{(2)}$

if $\bullet_i \text{---} \bullet_j$

Let $e' =$

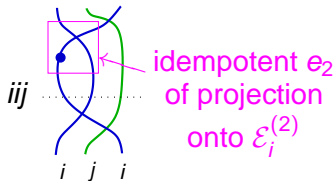
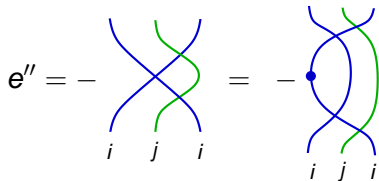
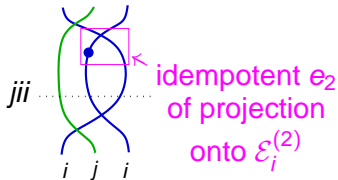
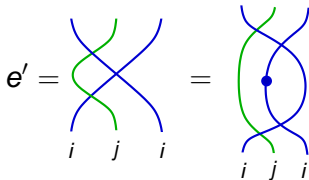
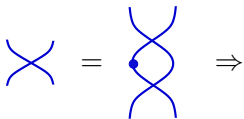
$(e')^2 =$ $=$ $+$ $=$ $=$ $= e'$

$e'' = 1_{jji} - e' = -$ is idempotent too $(e'')^2 = e''$

Orthogonality $e'e'' = e''e' = 0$ and $1_{jji} = e' + e''$ imply

$$\mathcal{E}_i \mathcal{E}_j \mathcal{E}_i \cong \mathcal{E}_i \mathcal{E}_j \mathcal{E}_i e' \oplus \mathcal{E}_i \mathcal{E}_j \mathcal{E}_i e''$$

But



Therefore,

$$\varepsilon_i \varepsilon_j \varepsilon_i e' \cong \varepsilon_j \varepsilon_i^{(2)}$$

$$\varepsilon_i \varepsilon_j \varepsilon_i e'' \cong \varepsilon_i^{(2)} \varepsilon_j$$

so that the relation

if i j

together with the other relations imply

$$\varepsilon_i \varepsilon_j \varepsilon_i \cong \varepsilon_j \varepsilon_i^{(2)} \oplus \varepsilon_i^{(2)} \varepsilon_j$$

There are induction and restriction functors corresponding to inclusions $R(\nu) \otimes R(\nu') \subset R(\nu + \nu')$

$$\text{Ind}_{\nu, \nu'}^{\nu + \nu'} : R(\nu) \otimes R(\nu')\text{-pmod} \rightarrow R(\nu + \nu')\text{-pmod}$$

$$\text{Res}_{\nu, \nu'}^{\nu + \nu'} : R(\nu + \nu')\text{-pmod} \rightarrow R(\nu) \otimes R(\nu')\text{-pmod}$$

Summing over all ν, ν' gives functors

$$\text{Ind} : (R \otimes R)\text{-pmod} \rightarrow R\text{-pmod} \qquad \text{Res} : R\text{-pmod} \rightarrow (R \otimes R)\text{-pmod}$$

These map projectives to projectives \Rightarrow

$$[\text{Ind}] : K_0(R) \otimes K_0(R) \rightarrow K_0(R) \qquad [\text{Res}] : K_0(R) \rightarrow K_0(R) \otimes K_0(R)$$

Write $[\text{Ind}](x_1, x_2)$ for $x_1, x_2 \in K_0(R)$ as $x_1 x_2$

Work over a field \mathbb{k} .

Theorem (M.Khovanov, A. L. arXiv:0803.4121)

There is an isomorphism of twisted bialgebras:

$$\begin{aligned} \gamma: U_{\mathbb{Z}}^+ &\longrightarrow K_0(R) \\ E_{i_1}^{(a_1)} E_{i_2}^{(a_2)} \cdots E_{i_k}^{(a_k)} &\mapsto \left[\mathcal{E}_{i_1}^{(a_1)} \mathcal{E}_{i_2}^{(a_2)} \cdots \mathcal{E}_{i_k}^{(a_k)} \right] \end{aligned}$$

multiplication \mapsto multiplication given by [Ind]

comultiplication \mapsto comultiplication given by [Res]

The semilinear form on $U_{\mathbb{Z}}^+$ maps to the HOM form on $K_0(R)$

$$(x, y) = (\gamma(x), \gamma(y))$$

Injectivity of γ

Injectivity of the map $\gamma: U_{\mathbb{Z}}^+ \rightarrow K_0(R)$ uses that U_q^+ is the quotient of a free associative algebra by the radical of the semilinear form. This follows from the quantum version of the Gabber-Kac theorem (proof, due to Lusztig for an arbitrary graph, uses perverse sheaves).

Surjectivity of γ

Surjectivity follows by mirroring the work of Grojnowski and Vazirani.

M.Khovanov, A. L. (arXiv:0804.2080)

This theorem has an extension to the non-simply laced case. The basis of indecomposable gives a new basis for $U_{\mathbb{Z}}^+$ where structure constants are necessarily positive.

Theorem (simply-laced)

$$U_{\mathbb{Z}}^+ \xrightarrow{\sim} K_0(R)$$

Lusztig-Kashiwara canonical basis \longleftrightarrow indecomposable projective $[P]$

arXiv:0901.4450

Brundan and Kleshchev gave an algebraic proof when Γ is a chain or a cycle.

arXiv:0901.3992

The general case (over \mathbb{C}) was proven by Varagnolo and Vasserot who showed that rings $R(\nu)$ in the simply-laced case were isomorphic to certain Ext-algebras of Perverse sheaves on Lusztig quiver varieties.

Cyclotomic quotients

For a given weight $\lambda = \sum_{i \in I} \lambda_i \cdot \Lambda_i$ define the cyclotomic quotient R_ν^λ of $R(\nu)$ by imposing the additional relations: for any sequence $i_1 i_2 \cdots i_m$ of vertices of Γ

λ_{i_1} dots on the first strand of any sequence is zero

$$\longrightarrow \begin{array}{ccccccc} \lambda_{i_1} & & & & & & \\ | & | & | & \cdots & | & & \\ \bullet & & & & & & \\ | & | & | & & | & & \\ i_1 & i_2 & i_3 & & i_m & & \end{array} = 0$$

This is analogous to taking the Ariki-Koike cyclotomic quotient of the affine Hecke algebra:

$$H_d^\lambda := H_d / \left\langle \prod_{i \in I} (X_1 - q^i)^{\lambda_i} \right\rangle$$

Cyclotomic quotient conjecture

The category of finitely-generated graded modules over the ring

$$R^\lambda = \bigoplus_{\nu \in \mathbb{N}[I]} R_\nu^\lambda$$

categorifies the integrable version of the representation V_λ of $U_q(\mathfrak{g})$ of highest weight λ .

$$\begin{array}{ccc} V(\lambda) & \xrightarrow{\sim} & K_0(R^\lambda) \\ \text{Lusztig-Kashiwara} & \xrightarrow{\quad} & \text{indecomposable} \\ \text{canonical basis} & & \text{projective } [P] \end{array}$$

Theorem (Brundan-Kleshchev, arXiv:0808.2032)

There is an isomorphism $R_\nu^\lambda \longrightarrow H_\nu^\lambda$ where H_ν^λ is a single block of the cyclotomic Hecke algebra H_d^λ . Using this isomorphism they proved the cyclotomic quotient conjecture in type A and affine type A .

For level 2 quotients the result follows from earlier work of Brundan and Stroppel.

Corollary

There is a \mathbb{Z} -grading on blocks H_ν^λ of affine Hecke algebras.

Implies there is a new \mathbb{Z} -grading on blocks of the symmetric group.

Leads to graded Specht module theory, see Brundan-Kleshchev-Wang, arXiv:0901.0218.

Leads to a graded version of the generalized LLT-conjecture.

Generalizations

A.L. (arXiv:0803.3652)

There is a graphical 2-category categorifying the integral form of the idempotent completion of the entire quantum group $U_q(\mathfrak{sl}_2)$

- $\dot{\mathbf{U}}_{\mathbb{Z}} \cong K_0(\dot{\mathcal{U}})$ the Grothendieck ring/category of this 2-category
- Indecomposable 1-morphisms \Leftrightarrow Lusztig canonical basis element
- The 2-category $\dot{\mathcal{U}}$ acts on cohomology of iterated flag varieties, categorifying the irreducible N -dimensional rep of $\mathbf{U}_q(\mathfrak{sl}_2)$

M. Khovanov, A.L. (arXiv:0807.3250)

- 2-category $\dot{\mathcal{U}}$ has an extension to a categorification of $\dot{\mathbf{U}}(\mathfrak{sl}_n)$.
- Conjectural categorification of the integral form of $\dot{\mathbf{U}}(\mathfrak{g})$ for any Kac-Moody algebra.

arXiv:0812.5023

Closely related 2-categories were recently studied by Rouquier.