The image of the cohomology of a compactification

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See Section 9.2 of Grothendieck's paper. This starts on page 153 of the paper/page 66 of the PDF. Grothendieck does things in the étale topology, but we'll work over $\mathbb{C}$ and use the analytic topology.

Proposition 1. *Let U be a smooth variety, and let $f : X' \to X$ be a map of smooth compactifications of U that is an isomorphism above $U \subset X$. Let $\mathcal{F}$ be a local system on X. Then*

$$\operatorname{im}(H^k(X, \mathcal{F}) \to H^k(U, \mathcal{F}|_U)) = \operatorname{im}(H^k(X', f^*\mathcal{F}) \to H^k(U, \mathcal{F}|_U)).$$

Proof. Let $Y = X - U$ and $Y' = X' - U$. We have a map of exact sequences in cohomology:

$$\begin{tikzcd} H^k(X, \mathcal{F}) \arrow[r] \arrow[d] & H^k(U, \mathcal{F}|_U) \arrow[r] \arrow[d, "\cong"] & H^{k+1}_Y(X, \mathcal{F}) \arrow[d] \\ H^k(X', f^*\mathcal{F}) \arrow[r] & H^k(U, \mathcal{F}|_U) \arrow[r] & H^{k+1}_{Y'}(X', f^*\mathcal{F}) \end{tikzcd}$$

It suffices to prove that $H^{k+1}_Y(X, \mathcal{F}) \to H^{k+1}_{Y'}(X', f^*\mathcal{F})$ is an injection. Consider the composite

$$r : \mathcal{F} \to Rf_*f^*\mathcal{F} \xrightarrow{\sim} Rf_*f^!\mathcal{F} \to \mathcal{F}$$

in $D^+(X, \mathbb{Z})$. The first and last maps are induced by the adjunctions ($f_* = f_!$ because f is proper), and the middle map is induced by the isomorphism $f^*\mathcal{F} \xrightarrow{\sim} f^!\mathcal{F}$ (here, we use that $\mathcal{F}$ is a local system, that $f^! = \mathbb{D}f^*\mathbb{D}$, and that X and X' are the same dimension). Since $f|_U = \operatorname{id}|_U$, $r|_U = \operatorname{id}_{\mathcal{F}|_U}$. Since $\mathcal{F}$ is a local system, r is determined by its value on any stalk. In particular, since r is the identity on any stalk in U, $r = \operatorname{id}_{\mathcal{F}}$.

We apply $R^{k+1}\Gamma_Y$ to the maps $\mathcal{F} \to Rf_*f^*\mathcal{F} \to \mathcal{F}$ to get

$$H^{k+1}_Y(X, \mathcal{F}) \to H^{k+1}_{Y'}(X', f^*\mathcal{F}) \to H^{k+1}_Y(X, \mathcal{F}),$$

where the composite is the identity. Here, we use that $\Gamma_Y f_* = \Gamma_{f^{-1}(Y)} = \Gamma_{Y'}$. We conclude that $H^{k+1}_Y(X, \mathcal{F}) \to H^{k+1}_{Y'}(X', f^*\mathcal{F})$ is an injection, which implies the proposition. □

Corollary 1. *Let X be a smooth compactification of a smooth variety U. The image of $H^k(X; \mathbb{Z}) \to H^k(U; \mathbb{Z})$ is independent of choice of X.*

Proof. Given any two compactifications X_1 and X_2, we can get a smooth compactification X' dominating both X_1 and X_2 by taking a resolution of singularities of the closure of the image of $U \overset{\Delta}{\hookrightarrow} X_1 \times X_2$. Hence, we just need to apply the proposition to $X' \to X_1$ and $X' \to X_2$ where our local system is the constant sheaf $\mathbb{Z}$. □