Intro to mixed Hodge structures; Hodge theory for smooth varieties

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This talk is based on Deligne's *Théorie de Hodge, II* (up to 3.I) and Peters-Steenbrink's *Mixed Hodge Structures* (Chapter 4).

1 Introduction

Hodge theory is a powerful tool for studying the topology of complex varieties. For smooth proper X, we have a decomposition of each cohomology group, a.k.a. a pure Hodge structure:

$$H^n(X;\mathbb{C}) = \bigoplus_{p+q=n} H^{p,q}(X).$$

In *Hodge II*, Deligne proves that all smooth varieties X carry mixed Hodge structures. The main trick is to pick a simple normal crossings compactification of X and to study its boundary; the Hodge structures on the compactification and the boundary components will give us a mixed Hodge structure on X. I will explain the first part of the story today.

2 Review of Hodge theory

Definition 1. There are a couple ways to define **Hodge structures**:

(1) For $A \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}\}$, an A-Hodge structure is a finitely generated A-module H with a decomposition of $H_{\mathbb{C}} := H \otimes_A \mathbb{C}$ of the form

$$H_{\mathbb{C}} = \bigoplus_{p,q\in\mathbb{Z}} H^{p,q}.$$

In this decomposition, we must have $H^{q,p} = \overline{H^{p,q}}$. We say that H is of **weight** n when $H^{p,q} = 0$ for all $p + q \neq n$.

(2) An A-Hodge structure of weight n is a finitely generated A-module H with a finite decreasing filtration F on $H_{\mathbb{C}}$. This filtration F is required to be n-**opposite**, meaning that whenever $p + q = n + 1$, $F^p \cap \overline{F}^q = 0$ and $F^p \oplus \overline{F}^q = H_{\mathbb{C}}$. In the language of definition (1), $F^p(H_{\mathbb{C}}) = \bigoplus_{i\geqslant p} H^{i,n-i}$.

(3) An A-Hodge structure is a finitely generated A-module with an action of the **Deligne torus** $S := \mathrm{Res}^{\mathbb{C}}_{\mathbb{R}}\, \mathbb{G}_{m,\mathbb{C}}$ on $H_{\mathbb{R}}$. H is of weight n when the representation is of weight n (I am too lazy to write down what this means).

When we say "Hodge structure," we really mean "$\mathbb{Z}$-Hodge structure." When we say "pure A-Hodge structure," we mean "an A-Hodge structure of some weight n."

A morphism of A-Hodge structures is a map of the underlying A-modules respecting the filtration after tensoring with $\mathbb{C}$ (note that this is enough to guarantee that the map respects the bi-grading).

Tensor products and duals of Hodge structures (maybe assume torsion-free) are naturally Hodge structures. The degrees behave as you'd expect.

Example 1. (1) Let X be a smooth projective variety. The main theorem of Hodge theory is that $H^n(X;\mathbb{Z})$ is naturally a Hodge structure of weight n:

$$H^n(X;\mathbb{C}) = \bigoplus_{p+q=n} H^{p,q}(X).$$

This is proven by placing a Kähler metric on X using a projective embedding. When we do this, we get an isomorphism between $H^n(X;\mathbb{C})$ and the space of harmonic n-forms on X, i.e. forms killed by the Hodge Laplacian. Moreover, because the metric is Kähler, the Hodge Laplacian respects bi-degrees on differential forms, so we get the Hodge decomposition above.

(2) In *Théorème de Lefschetz et critères de dégénérescence de suites spectrales*, Deligne uses Chow's lemma to prove that $H^n(X;\mathbb{Z})$ is naturally a Hodge structure of weight n when X is smooth and proper.

(3) The **Tate-Hodge structure** $\mathbb{Z}(1)$ is the Hodge structure of weight -2 with underlying $\mathbb{Z}$-module $2\pi i\mathbb{Z}$ and $\mathbb{Z}(1)\otimes_{\mathbb{Z}}\mathbb{C} = H^{-1,-1}$. Tensoring Hodge structures with powers of $\mathbb{Z}(1)$ (denoted $\mathbb{Z}(n)$) allows us to raise/lower their weights. We write $H(n) := H \otimes \mathbb{Z}(n)$. For instance, if $Z \hookrightarrow X$ is a codimension r inclusion of smooth projective varieties, the Gysin map is a map of Hodge structures $H^{n-2r}(Z;\mathbb{Z})(-r) \to H^n(X;\mathbb{Z})$.

Speaking of the Gysin map, let's look at the Gysin sequence (let $U := X - Z$):

$$\cdots \to H^{n-2r}(Z;\mathbb{Z})(-r) \to H^n(X;\mathbb{Z}) \to H^n(U;\mathbb{Z}) \to H^{n-2r+1}(Z;\mathbb{Z})(-r) \to \cdots$$

The fact that $H^n(U;\mathbb{Z})$ is sandwiched between two Hodge structures of weights n and $n+1$ strongly suggests that there should be some kind of "mixed" Hodge structure on $H^n(U;\mathbb{Z})$. The "weight n part" of $H^n(U;\mathbb{C})$ should be the image of $H^n(X;\mathbb{C})$, and the quotient, coming from $H^{n-2r+1}(Z;\mathbb{Z})(-r)$, should have "weight $n+1$." If all smooth varieties arose as complements of smooth closed subvarieties, we would have our mixed Hodge structures, but sadly, the situation isn't that simple.

This last example motivates the following definition.

Definition 2. A **mixed Hodge structure** is a finitely generated $\mathbb{Z}$-module H with an increasing filtration W (the **weight filtration**) on $H_{\mathbb{Q}}$ and a decreasing filtration F (the **Hodge filtration**) on $H_{\mathbb{C}}$ such that for all n, $(\mathrm{Gr}_n^W H_{\mathbb{Q}}, F)$ is a pure Hodge structure of weight n.

Example 2. A pure Hodge structure H of weight n is naturally a mixed Hodge structure: set $W_{n-1}H_{\mathbb{Q}} = 0$ and $W_nH_{\mathbb{Q}} = H_{\mathbb{Q}}$.

Definition 3. A **morphism of mixed Hodge structures** $(H, W, F) \to (H', W', F')$ is a map of $\mathbb{Z}$-modules $H \to H'$ compatible with both filtrations.

An important feature of mixed Hodge structures is that morphisms of mixed Hodge structures are automatically **strictly compatible** with both filtrations: if $b \in W'_n(H')$ is in the image of some morphism $f : (H, W, F) \to (H', W', F')$, then $b = f(a)$ for some $a \in W_n(H)$ (the analogous thing holds for F as well).

Here is the main theorem of *Hodge II*.

Theorem 1. If U is a smooth variety, then $H^n(U; \mathbb{Z})$ carries a natural mixed Hodge structure. This mixed Hodge structure is functorial.

3 Hypercohomology spectral sequences

Let's think about the Hodge decomposition $H^n(X; \mathbb{C}) = \bigoplus_{p+q=n} H^{p,q}(X)$ in terms of homological algebra. The Hodge decomposition implies the degeneration of the **Hodge-de Rham spectral sequence** at E_1:

$$E_1^{pq} = H^q(X, \Omega^p_X) \implies H^{p+q}(X; \mathbb{C}),$$

identifying $H^{p,q}(X) \cong H^q(X, \Omega^p_X)$. The Hodge filtration F is then the filtration given by this spectral sequence, i.e. $F^p(H^*(X; \mathbb{C})) = \operatorname{im}(\mathbb{H}^*(X, \Omega_X^{\geqslant p}) \to \mathbb{H}^*(X, \Omega^\bullet_X))$, where we identify $\mathbb{H}^*(X, \Omega^\bullet_X) \cong H^*(X; \mathbb{C})$ ($\mathbb{C} \xrightarrow{\sim} \Omega^\bullet_X$ by the holomorphic Poincaré lemma).

The Hodge-de Rham spectral sequence is an example of a **hypercohomology spectral sequence**: for any bounded filtered complex of abelian sheaves (K, F) on a space X such that F is finite on each component of K and decreasing, we get a spectral sequence

$$E_1^{pq} = \mathbb{H}^{p+q}(X, \operatorname{Gr}^p_F K) \implies \mathbb{H}^{p+q}(X, K).$$

To get the Hodge-de Rham spectral sequence, we take K to be the de Rham complex $\Omega^\bullet_X$ and F to be the **bête filtration** ("stupid" filtration), which is denoted σ:

$$\sigma^p(K)^n = \begin{cases} K^n & \text{if } n \geqslant p \\ 0 & \text{if } n < p. \end{cases}$$

Then

$$E_1^{pq} = \mathbb{H}^{p+q}(X, \operatorname{Gr}^p_F K) = \mathbb{H}^{p+q}(X, K^p[-p]) = H^q(X, K^p) = H^q(X, \Omega^p_X),$$

and $H^{p+q}(X; \mathbb{C}) \cong \mathbb{H}^{p+q}(X; \Omega^\bullet_X)$ because $\mathbb{C} \xrightarrow{\sim} \Omega^\bullet_X$.

Hypercohomology spectral sequences are everywhere in this subject, so we say a bit about them here. As stated above, the input for a hypercohomology spectral sequence is a filtered complex of abelian sheaves (K, F). A map of filtered complexes $(K, F) \to (K', F')$ induces a map on the corresponding hypercohomology spectral sequences. If $(K, F) \xrightarrow{\sim} (K', F')$ is a **filtered quasi-isomorphism** (i.e. the induced map on every Gr is a quasi-isomorphism), then the induced map on spectral sequences is an isomorphism from E_1 onward. F induces a filtration on $\mathbb{H}^*(X, K)$ given by $F^p(\mathbb{H}^*(X, K)) = \operatorname{im}(\mathbb{H}^*(X, F^p(K)) \to \mathbb{H}^*(X, K))$; this is reflected by the hypercohomology spectral sequence.

In addition to the bête filtration, any complex K carries a **canonical filtration** τ, which is an increasing filtration:

$$\tau_p(K)^n = \begin{cases} 0 & \text{if } n > p \\ \ker d^p & \text{if } n = p \\ K^n & \text{if } n < p. \end{cases}$$

The associated graded pieces are $\mathrm{Gr}_p^\tau K \simeq H^p(K)[-p]$. This implies that any quasi-isomorphism $K \xrightarrow{\sim} K'$ induces a filtered quasi-isomorphism $(K, \tau_K) \xrightarrow{\sim} (K', \tau_{K'})$, so it makes sense to talk about the canonical filtration on a bounded below derived category object.

4 Hodge theory on arbitrary smooth varieties

Let U be an arbitrary (connected) smooth variety. We bust out some heavy geometric machinery to get a nice compactification of U. First, Nagata's compactification theorem implies that we can embed U as an open subvariety of some proper X'. Then by resolution of singularities, we can find some resolution $f : X \to X'$ for which $f^{-1}(U) \cong U$ and $D := X - U$ is a **simple normal crossings divisor**, i.e. D analytic locally looks like an intersection of hyperplanes $z_1 \cdots z_r = 0$ in X. We order the components of D: $D_1, \ldots, D_s$. For each subset $I \subset \{1, \ldots, s\}$, let D_I denote the intersection $\bigcap_{i \in I} D_i$. Here, $D_\varnothing = X$. Note that each D_I is smooth and proper, so each of them has a Hodge decomposition. We expect the cohomology of U to be related to the cohomology of the D_I. For each $0 \leqslant i \leqslant n$, let $D^{(i)}$ denote the disjoint union of the i-wise intersections:

$$D^{(i)} := \bigsqcup_{|I|=i} D_I.$$

By the holomorphic Poincaré lemma, there is a quasi-isomorphism $\mathbb{C}_U \xrightarrow{\sim} \Omega_U^\bullet$. We can push this forward to X along the inclusion $j : U \hookrightarrow X$: $Rj_*\mathbb{C}_U \xrightarrow{\sim} Rj_*\Omega_U^\bullet$. Since the Ω_U^k are coherent sheaves on U and j locally looks like a map of Stein manifolds (this is the complex analytic version of being an affine morphism; more precisely, X locally looks like a polydisc (product of open discs)), all the higher pushforwards $R^i j_* \Omega_U^k$ vanish. Thus, $Rj_*\Omega_U^\bullet \simeq j_*\Omega_U^\bullet$. Moreover, these complexes carry canonical filtrations: $(Rj_*\mathcal{C}_U, \tau) \simeq (j_*\Omega_U^\bullet, \tau)$.

The nice thing about the boundary being a simple normal crossings divisor is that we can consider the **logarithmic de Rham complex** $\Omega_X^\bullet(\log D) \subset j_*\Omega_U^\bullet$. The sections of the logarithmic de Rham complex are called logarithmic forms, and they are defined as follows. Near each point of X, we can pick coordinates $z_1, \ldots, z_n$ on X such that D is locally cut out by $z_1 \cdots z_t = 0$ for some t. The sheaf of logarithmic 1-forms $\Omega_X^1(\log D)$ is the $\mathcal{O}_X$-module locally spanned by $\frac{dz_i}{z_i}$ $(1 \leqslant i \leqslant t)$ and dz_i $(t+1 \leqslant i \leqslant n)$. We then define $\Omega_X^k(\log D)$ to be the image of $\bigwedge^k \Omega_X^1(\log D)$ in $j_*\Omega_X^k$. We see that the $\Omega_X^k(\log D)$ are locally free and that they form a complex.

Proposition 1. The inclusion $\Omega_X^\bullet(\log D) \hookrightarrow j_*\Omega_U^\bullet$ is a quasi-isomorphism.

Proof. This is a local statement, so since every point of X has a fundamental system of neighborhoods that look like polydiscs, we just need to check this for $j : \Delta_k^n \hookrightarrow \Delta^n$, where Δ^n is a polydisc with coordinates $z_1, \ldots, z_n$ and Δ_k^n is the complement of $z_1 \cdots z_k = 0$. Let D_k^n denote the divisor $z_1 \cdots z_k = 0$, i.e. $D_k^n = \Delta^n - \Delta_k^n$. Let $K_{n,k}^\bullet := \Gamma(\Delta^n, \Omega_{\Delta_n}^\bullet(\log D_n^k))$. It suffices to show that the natural map $K_{n,k}^\bullet \hookrightarrow \Gamma(\Delta_k^n, \Omega_{\Delta_n^k})$ is a quasi-isomorphism.

Since Δ_k^n is Stein, $H^*(\Gamma(\Delta_k^n, \Omega_{\Delta_n^k})) \cong H^*(\Delta_k^n; \mathbb{C})$ (by the natural map). We find representatives for the cohomology classes of Δ_k^n in $K_{n,k}^\bullet$ as follows. Let $R_{n,k}^\bullet$ be the subcomplex of $K_{n,k}^\bullet$ with $R_{n,k}^1 = \mathbb{C}\frac{dz_1}{z_1} \oplus \cdots \oplus \mathbb{C}\frac{dz_k}{z_k}$ and $R_{n,k}^* = \bigwedge^* R_{n,k}^1$. Then $R_{n,k}^\bullet$ has trivial differential, and moreover, $R_{n,k}^\bullet \to \Gamma(\Delta_k^n, \Omega_{\Delta_n^k}^\bullet)$ is a quasi-isomorphism (Δ_k^n is a product of punctured discs and discs, so we know its cohomology).

It remains to show that $R_{n,k}^\bullet \to K_{n,k}^\bullet$ is a quasi-isomorphism. We use induction. For $k = 0$, we are done. To induct, we use the **residue map** $\mathrm{res} : K_{n,k}^\bullet \to K_{n-1,k-1}^\bullet[-1]$: for $\omega = \eta \wedge \frac{dz_k}{z_k} + \eta'$

with η, η' not containing dz_k, we define $\mathrm{res}(\omega) = \eta|_{\Delta^{n-1}_{k-1}}$, where Δ^{n-1}_{k-1} is defined by $z_k = 0$. We then induct by spamming the five lemma on this commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & R^\bullet_{n,k-1} \arrow[r] \arrow[d] & R^\bullet_{n,k} \arrow[r, "\mathrm{res}"] \arrow[d] & R^\bullet_{n-1,k-1}[-1] \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K^\bullet_{n,k-1} \arrow[r] & K^\bullet_{n,k} \arrow[r, "\mathrm{res}"] & K^\bullet_{n-1,k-1}[-1] \arrow[r] & 0
\end{tikzcd}$$

□

To summarize, we have a zig-zag of quasi-isomorphisms

$$Rj_*\mathbb{C} \xrightarrow{\sim} j_*\Omega_U \xleftarrow{\sim} \Omega^\bullet_X(\log D),$$

so we have a natural isomorphism $H^*(U;\mathbb{C}) \cong \mathbb{H}^*(X, \Omega^\bullet_X(\log D))$, along with a natural isomorphism of hypercohomology spectral sequences with respect to the canonical filtration.

What was the point of considering $\Omega^\bullet_X(\log D)$? There is a natural **weight filtration** W on $\Omega^\bullet_X(\log D)$, which is the increasing filtration where $W_n(\Omega^\bullet_X(\log D))$ is locally spanned by forms of the form $\alpha \wedge \frac{dz_{i_1}}{z_{i_1}} \wedge \cdots \wedge \frac{dz_{i_m}}{z_{i_m}}$, where α is holomorphic on X and $m \leqslant n$. I will prove next time that the identity $(\Omega^\bullet_X(\log D), \tau) \to (\Omega^\bullet_X(\log D), W)$ is a filtered quasi-isomorphism. Thus, we have a zig-zag of filtered quasi-isomorphisms

$$(Rj_*\mathbb{C}, \tau) \xrightarrow{\sim} (j_*\Omega_U, \tau) \xleftarrow{\sim} (\Omega^\bullet_X(\log D), \tau) \xrightarrow{\sim} (\Omega^\bullet_X(\log D), W).$$

We get an identification of hypercohomology spectral sequences. Next time, we will see what that gets us.