# Hodge theory for smooth varieties II

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This talk is based on Deligne's *Théorie de Hodge, II* (Sections 3 and 4) and Peters-Steenbrink's *Mixed Hodge Structures* (Chapter 4).

## 1 Recap

Recall the setup. We have a smooth variety $U$ with a smooth compactification $X$ such that $D = X - U$ is a simple normal crossings divisor. The inclusion $U \hookrightarrow X$ is denoted $j$. The logarithmic de Rham complex $\Omega_X^\bullet(\log D)$ is generated by differential forms locally of the form $\frac{dz_i}{z_i}$ (near $z_i = 0$) and $dz_i$ (otherwise). We denote the indivudal $m$-fold intersections of the components $D_1, \dots, D_s$ by $D_I$ ($I \subset \{1, \dots, s\}$) and denote the closed embeddings $D_I \hookrightarrow X$ by $a_I$. We denote the disjoint union of the $m$-fold intersections $D_I$ ($|I| = m$) by $D^{(m)}$, and we denote the natural map $D^{(m)} \to X$ by $a_m$.

Our goal is the following theorem.

**Theorem 1.** *$H^*(U; \mathbb{Z})$ carries a natural mixed Hodge structure. These mixed Hodge structures are functorial with respect to maps of smooth varieties $U \to V$.*

## 2 The logarithmic de Rham complex

We can use $\Omega_X^\bullet(\log D)$ to compute the cohomology of $U$ by the following proposition, which can be proven using the residue map (we will introduce this in a bit).

**Lemma 1.** *The inclusion $\Omega_X^\bullet(\log D) \hookrightarrow j_*\Omega_U^\bullet$ is a quasi-isomorphism.*

To summarize, we have a zig-zag of quasi-isomorphisms

$$Rj_*\mathbb{C} \xrightarrow{\sim} j_*\Omega_U \xleftarrow{\sim} \Omega_X^\bullet(\log D),$$

so we have a natural isomorphism $H^*(U; \mathbb{C}) \cong \mathbb{H}^*(X, \Omega_X^\bullet(\log D))$, along with a natural isomorphism of hypercohomology spectral sequences with respect to the canonical filtration.

Why $\Omega_X^\bullet(\log D)$? There is a natural **weight filtration** $W$ on $\Omega_X^\bullet(\log D)$, which is the increasing filtration where $W_m(\Omega_X^\bullet(\log D))$ is locally spanned by forms of the form $\alpha \wedge \frac{dz_{i_1}}{z_{i_1}} \wedge \cdots \wedge \frac{dz_{i_m}}{z_{i_{m'}}}$, where $\alpha$ is holomorphic on $X$ and $m' \leqslant m$. The key feature of the weight filtration is that it captures the geometry of the $D_I$ via the **residue map**. For each $I \subset \{1, \dots, s\}$, define a map

$$\begin{aligned} \mathrm{res}_I : W_m\Omega_X^\bullet(\log D) &\to a_{I*}\Omega_{D_I}^\bullet[-m] \\ \alpha \wedge \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_m}{z_m} + \alpha' &\mapsto \alpha|_{D_I}, \end{aligned}$$

where $z_1, \dots, z_n$ are local coordinates on $X$ such that $D_I$ is cut out by $z_1 = \cdots = z_m = 0$ (these must be ordered compatibly with $I$). Note that $\mathrm{res}_I(W_{m-1}\Omega_X^\bullet(\log D)) = 0$, so $\mathrm{res}_I$ descends to a map $\mathrm{res}_I : \mathrm{Gr}_m^W \Omega_X^\bullet(\log D) \to a_{I*}\Omega_{D_I}^\bullet$. We can patch together all the $\mathrm{res}_I$ to get $\mathrm{res}_m = \bigoplus_{|I|=m} \mathrm{res}_I : \mathrm{Gr}_m^W \Omega_X^\bullet(\log D) \to a_{m*}\Omega_{D^{(m)}}^\bullet$.

**Lemma 2.** $\mathrm{res}_m$ defines an isomorphism $\mathrm{Gr}_m^W \Omega_X^\bullet(\log D) \xrightarrow{\cong} a_{m*}\Omega_{D^{(m)}}^\bullet$.

**Corollary 1.** (a) The cohomology sheaves of $\mathrm{Gr}_m^W \Omega_X^\bullet(\log D)$ are all 0 except the $m$th cohomology, which is $a_{m*}\mathbb{C}$.

(b) The identity map is a filtered quasi-isomorphism $(\Omega_X^\bullet(\log D), \tau) \xrightarrow{\sim} (\Omega_X^\bullet(\log D), W)$.

To sum up, we have a zig-zag of filtered quasi-isomorphisms

$$(Rj_*\mathbb{C}, \tau) \xrightarrow{\sim} (j_*\Omega_U, \tau) \xleftarrow{\sim} (\Omega_X^\bullet(\log D), \tau) \xrightarrow{\sim} (\Omega_X^\bullet(\log D), W).$$

From this, we get an identification of the hypercohomology spectral sequences of $(Rj_*\mathbb{C}, \tau)$ and $(\Omega_X^\bullet(\log D), W)$.

The last piece of geometry we need is the following. This can be proven by looking at explicit generators of cohomology near points of $X$.

**Lemma 3.** Consider the identification of cohomology sheaves $R^m j_*\mathbb{C}_U \cong a_{m*}\mathbb{C}_{D^{(m)}}$. The image of $R^m j_*\mathbb{Z}_U \to R^m j_*\mathbb{C}_U$ is identified with $a_{m*}\mathbb{Z}_{D^{(m)}}(-m) \subset a_{m*}\mathbb{C}_{D^{(m)}}$.

# 3 The weight spectral sequence

The rest of the proof is homological algebra.

The main object here is the **weight spectral sequence**, which is the hypercohomology spectral sequence for $(\Omega_X^\bullet(\log D), W)$:

$$E_1^{-p,q} = H^{-p+q}(X, \mathrm{Gr}_p^W \Omega_X^\bullet(\log D)) \cong H^{-2p+q}(D^{(p)}; \mathbb{C}) \implies H^{-p+q}(U; \mathbb{C}).$$

Since $(Rj_*\mathbb{C}, \tau) \simeq (\Omega_X^\bullet(\log D), W)$ and $(Rj_*\mathbb{C}, \tau) \simeq (Rj_*\mathbb{Q}, \tau) \otimes_{\mathbb{Q}} \mathbb{C}$, the weight spectral sequence is defined over $\mathbb{Q}$. By Lemma 3, the weight spectral sequence over $\mathbb{Q}$ is given by

$$E_1^{-p,q} = H^{-2p+q}(D^{(p)}; \mathbb{Q})(-p) \implies H^{-p+q}(U; \mathbb{Q}).$$

Since it is defined over $\mathbb{Q}$, the weight spectral sequence (over $\mathbb{C}$) has an action by complex conjugation. Recall that $\Omega_X^\bullet(\log D)$ has a second filtration $F$, the bête filtration. $F$ induces three filtrations on the terms of the spectral sequence, called the **first direct filtration** $F_d$, the second direct filtration $F_{d*}$, and the **inductive filtration** $F_r$. These filtrations do not agree in general. Rather than define them and waste your time, I will state the key properties of the filtrations.

**Lemma 4.** Suppose $(K, W, F)$ is a doubly filtered complex, where $F$ has finitely many associated gradeds in each component (i.e. it is **biregular**). Then there are filtrations $F_d, F_r, F_{d*}$ on the terms of the spectral sequence of the filtered complex $(K, W)$.

(a) $F_d \subset F_r \subset F_{d*}$.

(b) The differentials $d_r$ are compatible with $F_d$ and $F_{d*}$.

(c) Suppose the $d_r$ are strictly compatible with $F_r$ for $r \in \{0, \ldots, r_0 - 2\}$. Then for all $0 \leqslant r \leqslant r_0$, $F_d = F_r = F_{d*}$. In particular, $F_d = F_r = F_{d*}$ automatically on $E_0$ and $E_1$.

(d) Suppose the $d_r$ are strictly compatible with $F_r$ for all $r$. Then $F_d = F_r = F_{d*}$ on $E_\infty$, and this filtration is the filtration induced by $F$ on the associated gradeds $\mathrm{Gr}_m^W H^*(K)$.

Now I'll state the actual result.

**Theorem 2.** (a) $F_d = F_r = F_{d*}$ for all terms of the weight spectral sequence.

(b) The filtration on $H^*(U; \mathbb{C})$ obtained from the weight spectral sequence is induced by a filtration $W$ on $H^*(U; \mathbb{Q})$. Neither $W$ nor $F$ (the Hodge filtration, which is obtained from the bête filtration) depends on the choice of compactification $X$.

(c) The filtrations $W[k]$ ($W[k]_m = W_{m-k}$) and $F$ make $H^k(U; \mathbb{Z})$ into a mixed Hodge structure.

*Proof.* We have already proven the first claim of (b), since the weight spectral sequence is defined over $\mathbb{Q}$.

The proof proceeds in a series of lemmas.

**Lemma 5.** The hypercohomology spectral sequence of $(\mathrm{Gr}_m^W \Omega_X^\bullet(\log D), F)$ degenerates at $E_1$. The filtration on $E_1^{-p,q} \cong H^{-2p+q}(D^{(p)}; \mathbb{C})$ induced by this spectral sequence (i.e. induced by $F$) is $q$-opposite with respect to complex conjugation (note that the complex conjugation comes from $H^{-2p+q}(D^{(p)}; \mathbb{Q})(-p)$, not $H^{-2p+q}(D^{(p)}; \mathbb{Q})$).

*Proof.* Recall that $\mathrm{Gr}_m^W \Omega_X^\bullet(\log D) \cong a_{m*}\Omega^\bullet_{D(m)}[-m]$, so the hypercohomology spectral sequence here is just the Hodge-de Rham spectral sequence, which degenerates at $E_1$. The second claim follows because the Hodge filtration on $H^{-2p+q}(D^{(p)}; \mathbb{C}) = H^{-2p+q}(D^{(p)}; \mathbb{Q})(-p) \otimes_{\mathbb{Q}} \mathbb{C}$ is $q$-opposite. □

From this lemma, we see that the terms $E_1^{-p,q}$ over $\mathbb{Q}$ are actually $H^{-2p+q}(D^{(p)}; \mathbb{Q})(-p)$ as $\mathbb{Q}$-Hodge structures, with the filtration induced by $F$.

**Lemma 6.** $d_1$ (on the weight spectral sequence) is strictly compatible with the filtration induced by $F$ (we will call this filtration $F$).

*Proof.* $d_1 : H^{-2p+q}(D^{(p)}; \mathbb{C}) \to H^{-2p+q+2}(D^{(p-1)}; \mathbb{C})$ is automatically compatible with $F$, if we view $F$ as the first direct filtration. Thus, $d_1 : H^{-2p+q}(D^{(p)}; \mathbb{Q})(-p) \to H^{-2p+q+2}(D^{(p-1)}; \mathbb{Q})(-p+1)$ is a morphism of $\mathbb{Q}$-Hodge structures, which must automatically be strict. □

**Lemma 7.** The inductive filtration on $E_2^{-p,q}$ is $q$-opposite.

*Proof.* This follows from the strict compatibility of $d_1$ with $F$. More precisely, the inductive filtration is the filtration induced on $E_{r+1}^{-p,q}$ by $E_r^{-p,q}$, where we treat $E_{r+1}$ as the cohomology of $E_r$. The lemma then follows because the $d_1$ differentials are morphisms of $\mathbb{Q}$-Hodge structures. □

**Lemma 8.** For $r \geqslant 0$, $d_r$ is strictly compatible with the inductive filtration. For $r \geqslant 2$, $d_r = 0$.

*Proof.* For $r = 0$ (I haven't exactly explained what the $E_0$ page is, but trust me), the claim follows from Lemma 5. For $r = 1$, we have already proven the claim in Lemma 6. Thus, it suffices to show that $d_r = 0$ for all $r \geqslant 2$.

Induct on $r$, starting from $r = 2$. By induction, $F_d = F_r = F_{d*}$ on $E_r^{-p,q}$ (use Lemma 4(c)), so $d_r$ is compatible with $F_r$. Call this common filtration $F$. $F$ is the same filtrations as on $E_2^{-p,q} = E_r^{-p,q}$. Thus, Lemma 7 implies that $F$ is $q$-opposite with respect to complex conjugation. Now

$$\begin{aligned} d_r(E_r^{-p,q}) &= d_r\left(\sum_{a+b=q} F^a(E_r^{-p,q}) \cap \overline{F}^b(E_r^{-p,q})\right) \\ &\subset \sum_{a+b=q} F^a(E_r^{-p+r,q-r+1}) \cap \overline{F}^b(E_r^{-p+r,q-r+1}) \\ &= 0, \end{aligned}$$

where in the last step, we use that $F$ is $(q-r+1)$-opposite on $E_r^{p+r,q-r+1}$ and $q-r+1 < q$. This proves that $d_r = 0$.

□

We are basically done. Lemma 4(d) implies that the filtration $F$ on $E_2^{-p,q} = E_\infty^{-p,q}$ agrees with the filtration induced by $F$ on $H^{-p+q}(U;\mathbb{C})$ (it also implies (a)). Hence, the associated graded $\operatorname{Gr}_q^{W[-p+q]} H^{-p+q}(U;\mathbb{Q}) = \operatorname{Gr}_p^W H^{-p+q}(U;\mathbb{Q}) = \operatorname{Gr}_W^{-p} H^{-p+q}(U;\mathbb{Q})$ is $E_2^{-p,q}$, which is a $\mathbb{Q}$-Hodge structure of weight $q$ (more precisely, recall that $E_2^{-p,q}$ is a subquotient of $H^{-2p+q}(D^{(p)};\mathbb{Q})(-p)$). We have thus constructed a mixed Hodge structure on $H^k(U;\mathbb{Q})$.

What remains is to prove the independence of choice of compactification in (b) and the functoriality in (c). We prove the functoriality first. Suppose $f : U \to V$ is a map of smooth varieties. We can find some good compactifications $X$ and $Y$ (of $U$ and $V$, respectively) with a compatible map $\overline{f} : X \to Y$ (first get a rational map $X' \dashrightarrow Y$ and then take a resolution of the closure of the graph of this rational map). Then we get a map $\overline{f}^* \Omega_Y^\bullet(\log D_Y) \to \Omega_X^\bullet(\log D_X)$ compatible with the weight and bête filtrations, so we get a map on hypercohomology $\mathbb{H}^*(Y, \Omega_Y^\bullet(\log D_Y)) \to \mathbb{H}^*(X, \Omega_X^\bullet(\log D_X)$ compatible with the filtrations. This map is a map of mixed Hodge structures, so we have our functoriality, and we have proven (c).

Finally, we prove independence of choice of compactification, using the functoriality. Given two compactifications $X_1, X_2$ of $U$, we can pick some compactification $X$ with maps $X \to X_1$ and $X \to X_2$ (take a resolution of the closure of $U \overset{\Delta}{\hookrightarrow} X_1 \times X_2$). Then the maps $\mathbb{H}^*(X, \Omega_X^\bullet(\log D_X)) \to \mathbb{H}^*(X_i, \Omega_{X_i}^\bullet(\log D_{X_i}))$ are bijective maps compatible with the filtrations. By strictness of maps of mixed Hodge structures, bijective maps compatible with the filtrations must be isomorphisms of mixed Hodge structures, so we have proven (b).

□

**Corollary 2.** (a) The hypercohomology spectral sequence of $(\Omega_X^\bullet(\log D), W)$ degenerates at $E_2$. The differential $d_1 : H^{-2p+q}(D^{(p)};\mathbb{Q})(-p) \to H^{-2p+q+2}(D^{(p-1)};\mathbb{Q})(-p+1)$ is identified with the alternating sum of the Gysin maps with respect to $D_I \hookrightarrow D_I - \{i\}$.

(b) The hypercohomology spectral sequence of $(\Omega_X^\bullet(\log D), F)$ degenerates at $E_1$.

**Corollary 3.** If $H^k(U;\mathbb{C})$ has a nonzero weight space $H^{p,q}$ in some associated graded, then $p, q \leqslant k$, and $p + q \geqslant k$.

**Corollary 4.** If $X$ is any smooth compactification of $U$, then the image of $H^k(X;\mathbb{Q}) \to H^k(U;\mathbb{Q})$ is the bottom weight part $W_k(H^k(U;\mathbb{Q}))$. If we have maps $Y \to U \hookrightarrow X$ with $Y$ smooth and proper, then the image of $H^k(X) \to H^k(Y)$ equals the image of $H^k(U) \to H^k(Y)$.

**Theorem 3** (Global invariant cycles theorem)**.** If $U \to S$ is a smooth proper map with $S$ smooth and separated and $U \hookrightarrow X$ is a smooth compactification, then the image of $H^k(X;\mathbb{Q}) \to H^k(X_s;\mathbb{Q})$ ($X_s$ is a smooth fiber) is the monodromy invariants $H^k(X_s;\mathbb{Q})^{\pi_1(S,s)}$.

*Sketch.* The surjectivity of $H^k(U;\mathbb{Q}) \to H^k(X_s;\mathbb{Q})^{\pi_1(S,s)}$ $U \to S$ is projective was proven by Deligne in an earlier paper (this is a corollary of the degeneration of the Leray spectral sequence for smooth proper maps). We then apply the previous corollary to prove the projective case. To prove the general proper case, we can use Chow's lemma.

□