Hodge theory for all varieties II

Kevin Chang

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This talk is based on Deligne's *Théorie de Hodge, III* (Section 8).

1 Mixed Hodge complexes from simplicial resolutions

Recall the following definition from last time.

Definition 1. A **mixed Hodge complex** consists of the following data:

(1) A complex $K \in D^+(\mathbb{Z})$ such that each $H^k(K)$ is finitely generated.

(2) A filtered complex $(K_\mathbb{Q}, W) \in D^+F(\mathbb{Q})$ (W is increasing) and an isomorphism $\alpha : K \otimes_\mathbb{Z} \mathbb{Q} \simeq K_\mathbb{Q}$ in the derived category.

(3) A bi-filtered complex $(K_\mathbb{C}, W, F) \in D^+F_2(\mathbb{C})$ and an isomorphism $\beta : (K_\mathbb{Q}, W) \otimes_\mathbb{Q} \mathbb{C} \simeq (K_\mathbb{C}, W)$ in the filtered derived category.

This data must satisfy the following condition: for each n, $(\operatorname{Gr}_n^W K_\mathbb{Q}, (\operatorname{Gr}_n^W K_\mathbb{C}, F), \beta)$ is a $\mathbb{Q}$-Hodge complex of weight n.

Recall the geometric setup. We have X, an arbitrary variety, equipped with a proper hypercovering $Y_\bullet \to X$ (a proper hypercovering is a simplicial variety such that each $Y_{n+1} \to (\operatorname{cosk}_n \operatorname{sk}_n Y_\bullet)_{n+1}$ is proper and surjective). We can pick the $Y_\bullet$ such that there is a simplicial compactification $(Y_\bullet \to X) \hookrightarrow (\overline{Y}_\bullet \to \overline{X})$ such that each $\overline{Y}_n$ is smooth and each $D_n := \overline{Y}_n - Y_n$ is a simple normal crossings divisor.

The main theorem of *Hodge II* is that the induced filtrations (with the weight filtration shifted) make the cohomology groups of a mixed Hodge complex into mixed Hodge structures. Thus, we want to get a mixed Hodge complex from our proper hypercovering.

First, we need to introduce some more definitions that are suited to the simplicial setting. A **mixed Hodge complex of sheaves on a simplicial space** $Y_\bullet$ is defined to be a tuple of data $(K, (K_\mathbb{Q}, W), \alpha, (K_\mathbb{C}, W, F), \beta)$ on $X_\bullet$ as in the definition of a mixed Hodge complex of sheaves on a space, where we require the restriction to each X_n to be a mixed Hodge complex of sheaves on the space. The key example in our geometric setup is $(Rj_*\mathbb{Z}_{Y_\bullet}, (Rj_*\mathbb{Q}_{Y_\bullet}, \tau_\leqslant), \alpha, (\Omega^\bullet_{\overline{Y}^\bullet}(\log D_\bullet), W, F), \beta)$ on $\overline{Y}_\bullet$, where j is the inclusion $Y_\bullet \hookrightarrow \overline{Y}_\bullet$ and W, F, α, β are the usual filtrations and comparison maps on each $\overline{Y}_n$.

From a mixed Hodge complex of sheaves on an augmented simplicial space $Y_\bullet \to X$, we can get a **cosimplicial mixed Hodge complex** taking the right derived functor of global sections on each piece, i.e.

$$(R\Gamma^\bullet Rj_*\mathbb{Z}_{Y_\bullet}, (R\Gamma^\bullet Rj_*\mathbb{Q}_{Y_\bullet}, \tau_\leqslant), R\Gamma^\bullet(\alpha), (R\Gamma^\bullet(\Omega^\bullet_{\overline{Y}^\bullet}(\log D_\bullet)), W, F), R\Gamma^\bullet(\beta)).$$

Let's recall how this cosimplicial complex relates to the cohomology of X. For any sheaf $\mathcal{F}$ on X, $H^*(X,\mathcal{F}) \cong H^*(Y_\bullet,\mathcal{F}^\bullet)$ ($\mathcal{F}^\bullet$ is the pullback of $\mathcal{F}$ to $Y_\bullet$). Let's suppose we have a bounded-below complex of abelian sheaves $\mathcal{F}^\bullet$ on $Y_\bullet$. The global sections of $\mathcal{F}^\bullet$ are $\ker(\Gamma(X_0,\mathcal{F}^0) \rightrightarrows \Gamma(X_1,\mathcal{F}^1))$. Thus, to get the derived global sections, we replace $\mathcal{F}^\bullet$ by an injective resolution $\mathcal{K}^{\bullet\bullet}$ (all the $\mathcal{K}^{p\bullet}$ are injective; the first coordinate tracks the degree in the resolution; the second coordinate tracks the simplicial degree) and then compute the cohomology of the complex whose qth term is $\ker(\Gamma(X_0,\mathcal{K}^{p0}) \rightrightarrows \Gamma(X_1,\mathcal{K}^{p1}))$. We can also consider the cosimplicial complex $K^{pq} := \Gamma(X_q,\mathcal{K}^{pq})$, which becomes a double complex when we use the alternating sums of the faces as the differentials in the q-direction. We can get the total complex $\mathbf{s}K^\bullet$:

$$\mathbf{s}K^n = \bigoplus_{p+q} \Gamma(X_q,\mathcal{K}^{pq}),$$

with differential

$$d(x^{pq}) = d_{\mathcal{K}}(x^{pq}) + (-1)^p \sum_i (-1)^i \delta_i x^{pq}.$$

It follows from the usual homological algebra that $\Gamma(X_\bullet,\mathcal{K}^{\bullet\bullet}) \simeq \mathbf{s}(K^{\bullet\bullet}) = \mathbf{s}\Gamma^\bullet(X_\bullet,\mathcal{K}^{\bullet\bullet}$. Thus, $R\Gamma(X_\bullet,\mathcal{F}^\bullet) \simeq sR\Gamma^\bullet(X_\bullet,\mathcal{F}^\bullet)$.

There is a filtration L on $\mathbf{s}K^\bullet$ given by $L^r(\mathbf{s}K)^\bullet = \bigoplus_{q\geqslant r} K^{pq}$. The rth associated graded of L is just the complex $0 \to K^{0r} \to K^{1r} \to \cdots$ of global sections on X_r, which has cohomology $H^*(X_r,\mathcal{F}^r)$. Thus, we have a spectral sequence

$$E_1^{st} = H^t(X_s,\mathcal{F}^s) \implies H^{s+t}(X_\bullet,\mathcal{F}^\bullet).$$

Everything I just said about sheaves $\mathcal{F}^\bullet$ applies to bounded-below complexes of sheaves, such as $Rj_*\mathbb{Z}$ in the geometric situation at hand.

Let's look at the cosimplicial mixed Hodge complex $R\Gamma^\bullet Rj_*\mathbb{Z}_{Y_\bullet}$. According to what I just said, the cohomology of $\mathbf{s}R\Gamma^\bullet Rj_*\mathbb{Z}_{Y_\bullet}$ computes the cohomology of X. Thus, to construct a mixed Hodge structure on the cohomology of X, it suffices to place the structure of a mixed Hodge complex on $\mathbf{s}R\Gamma^\bullet Rj_*\mathbb{Z}_{Y_\bullet}$. This is the main theorem of *Hodge III.*

Theorem 1. Let $K^{\bullet\bullet}$ be a cosimplicial mixed Hodge complex (the other data is omitted because I'm lazy). There exists a natural filtration $\delta(W,L)$ such that $(\mathbf{s}K^\bullet,\delta(W,L),F)$ is a mixed Hodge complex.

Proof. Define

$$\delta(W,L)_n(\mathbf{s}K^\bullet_{\mathbb{Q}}) := \bigoplus_{p,q} W_{n+p}(K^{pq}_{\mathbb{Q}}).$$

Then

$$\operatorname{Gr}_n^{\delta(W,L)}(\mathbf{s}K^\bullet_{\mathbb{Q}}) \simeq \bigoplus_p \operatorname{Gr}^W_{n+p} K^{\bullet p}_{\mathbb{Q}}[-p].$$

Because each $K^{\bullet p}$ is a mixed Hodge complex, each $\operatorname{Gr}^W_{n+p} K^{\bullet p}_{\mathbb{Q}}[-p]$ is a $\mathbb{Q}$-Hodge complex of weight n. Thus, $\operatorname{Gr}_n^{\delta(W,L)}(\mathbf{s}K^\bullet_{\mathbb{Q}})$ is a $\mathbb{Q}$-Hodge complex of weight n, and $(\mathbf{s}K^\bullet,\delta(W,L),F)$ is a mixed Hodge complex. □

Applying this to our cosimplicial mixed Hodge complex $R\Gamma^\bullet Rj_*\mathbb{Z}_{Y_\bullet}$ gets us our mixed Hodge structure on $H^*(Y_\bullet,\mathbb{Z}^\bullet) \cong H^*(X;\mathbb{Z})$. This mixed Hodge structure is functorial because we can pick simplicial resolutions compatible with maps (this is pretty hard apparently, and Deligne blackboxes it), and we can prove uniqueness in the same way as in *Hodge II.*

We can describe the E_1 page of a spectral sequence converging to the cohomology $H^*(X;\mathbb{Q})$, using the theory of mixed Hodge complexes. If you do the bookkeeping, letting $D_n^{(r)}$ denote the disjoint union of the r-wise intersections of D_n, we have

$$E_1^{-a,b} = \bigoplus_{\substack{p+2r=b \\ q-r=-a}} H^p(D_q^{(r)};\mathbb{Q})(-r) \implies H^{-a+b}(X;\mathbb{Q}).$$

Note that it is possible for a to be negative. By the properties of mixed Hodge complexes established in *Hodge II*, this spectral sequence degenerates at the E_2 page. Moreover, it is possible to characterize the d_1 differentials as Gysin maps, as they are in *Hodge II*.

$E_1^{-a,b}$ is a $\mathbb{Q}$-Hodge structure of weight b. However, unlike in *Hodge II*, we have different restrictions on which terms can be nonzero based on the properties of X.

Proposition 1. All the Hodge numbers h^{st} that appear in $H^n(X;\mathbb{Q})$ satisfy $0 \leqslant s,t \leqslant n$.

Proof. All the terms $H^p(D_q^{(r)};\mathbb{Q})(-r)$ have nonnegative h^{st} only (recall that twisting by $-r$ raises both s and t by $r \geqslant 0$), so $s,t \geqslant 0$.

If $H^p(D_q^{(r)};\mathbb{Q})(-r)$ contributes to $H^n(X;\mathbb{Q})$, then $p+q+r=n$, so $p+r \leqslant n$. For any $h^{s't'}$ appearing in $H^p(D_q^{(r)};\mathbb{Q})$, $s'+t'=p$, so $s',t' \leqslant p$. Thus, $s,t \leqslant p+r \leqslant n$. $\square$

Proposition 2. If X is proper, all the weights that appear in $H^n(X;\mathbb{Q})$ are between 0 and n, inclusive. Moreover, $W_{n-1}(H^n(X;\mathbb{Q})) = \ker(H^n(X;\mathbb{Q}) \to H^n(Y_0;\mathbb{Q}))$.

Proof. There are no D_q, so $r=0$ for all terms appearing in the spectral sequence. The terms that contribute to $H^n(X;\mathbb{Q})$ are $H^p(Y_q;\mathbb{Q})$ with $p+q=n$. The weights of $H^p(Y_q;\mathbb{Q})$ are all $p \leqslant n$, so $H^n(X;\mathbb{Q})$ has weights $\leqslant n$. The second claim follows from the spectral sequence. $\square$

2 The Hodge characteristic

Given a variety X and a decomposition $X = U \sqcup Z$, where U is open and Z is closed, it's true that $\chi^c(X) = \chi^c(U) + \chi^c(Z)$, where χ^c is the Euler characteristic with compact support. This follows from the long exact sequence in relative cohomology. Combined with the Künneth formula, this means that χ^c defines a ring homomorphism $K_0(\mathrm{Var}) \to \mathbb{Z}$, where the Grothendieck ring of varieties $K_0(\mathrm{Var})$ is generated by complex varieties with additive relations $[X] = [U] + [Z]$ (the **scissor relations**) and multiplicative relations $[X \times Y] = [X] \cdot [Y]$.

We would like to upgrade this to something involving Hodge numbers. For this, we first need that there are mixed Hodge structures on cohomology groups with compact support. Thankfully, Deligne has our back.

Theorem 2. Let $Y \subset X$ be a locally closed subvariety. There is a natural mixed Hodge structure on $H^*(X,Y;\mathbb{Z})$. Moreover, the maps in the long exact sequence

$$\cdots \to H^{n-1}(Y;\mathbb{Z}) \to H^n(X,Y;\mathbb{Z}) \to H^n(X;\mathbb{Z}) \to \cdots$$

are maps of mixed Hodge structures.

Theorem 3. The Künneth isomorphism

$$\bigoplus_{p+q=n} H^p(X;\mathbb{Q}) \otimes H^q(Y;\mathbb{Q}) \xrightarrow{\cong} H^n(X \times Y;\mathbb{Q})$$

is an isomorphism of mixed Hodge structures. The same holds for the relative Künneth isomorphism.

Example 1. We can use the compactification $\mathbb{P}^1$ of $\mathbb{A}^1$ to compute the mixed Hodge structure on $H_c^*(\mathbb{A}^1;\mathbb{Z}) \cong H^*(\mathbb{P}^1, \mathrm{pt}; \mathbb{Z})$. The only nonzero group is $H_c^2(\mathbb{A}^1)$, and the long exact sequence in cohomology yields an isomorphism $H_c^2(\mathbb{A}^1;\mathbb{Z}) \xrightarrow{\cong} H^2(\mathbb{P}^1;\mathbb{Z})$. Thus, $\mathrm{Gr}_2^W H_c^2(\mathbb{A}^1;\mathbb{Q}) \cong \mathbb{Q}$. By the Künneth theorem, we get $\mathrm{Gr}_{2n}^W H_c^{2n}(\mathbb{A}^{2n};\mathbb{Q}) \cong \mathbb{Q}$.

Any mixed Hodge structure V defines an element $[V_\mathbb{Q}] \in K_0(\mathfrak{hs})$, the Grothendieck group of $\mathbb{Q}$-Hodge structures, just by taking the graded pieces. V yields numerical invariants by taking the Hodge numbers: define $P_{\mathrm{hn}}([M]) = \sum_{p,q} h^{p,q} u^p v^q$. Now define the **Hodge characteristic with compact support** $\chi^c_{\mathrm{Hdg}} : K_0(\mathrm{Var}) \to K_0(\mathfrak{hs})$ by $\chi^c_{\mathrm{Hdg}}(X) := \sum_i (-1)^i [H_c^i(X;\mathbb{Q})]$. This is a ring homomorphism by the above theorems. We can also define the **Hodge-Euler polynomial with compact support** $e^c_{\mathrm{Hdg}} : K_0(\mathrm{Var}) \to \mathbb{Z}[u,v]$ by $e^c_{\mathrm{Hdg}}(X) := P_{hn}(\chi^c_{\mathrm{Hdg}}(X))$. This is also a ring homomorphism because P_{hn} is. If X is smooth and proper, X has pure Hodge structures on cohomology, so $e^c_{\mathrm{Hdg}}(X)$ actually determines the cohomology.

Example 2. Let X be a toric variety with s_k orbits of dimension k. Each k-dimensional orbit is isomorphic to $(\mathbb{C}^*)^k$, which has Hodge-Euler polynomial $(uv-1)^k$ (we can check that $e^c_{\mathrm{Hdg}}(\mathbb{C}^*) = uv - 1$ by applying the scissor relations to $\mathbb{A}^1 = \mathbb{C}^* \sqcup \mathrm{pt}$). Thus, $e^c_{\mathrm{Hdg}}(X) = \sum_k s_k(uv-1)^k$. If X is a smooth proper toric variety (or even proper with quotient singularities), this tells us that the Poincaré polynomial of X is $\sum_k s_k(t^2-1)^k$.

Example 3. Let X be an irreducible nodal curve whose normalization has genus g. This isn't the best way to do this, but we can use scissor relations to compute the mixed Hodge structures on $H^*(X;\mathbb{Q})$. First off, $H^0(X;\mathbb{Q}) \cong H^2(X;\mathbb{Q}) \cong \mathbb{Q}$, with both groups of the "correct" weight. The scissor relations give us

$$\begin{aligned}
e^c_{\mathrm{Hdg}}(X) &= e^c_{\mathrm{Hdg}}(X - \{2p \text{ points}\}) + e^c_{\mathrm{Hdg}}(p \text{ points}) \\
&= e^c_{\mathrm{Hdg}}(X) - 2p + p \\
&= (1 - gu - gv + uv) - p \\
&= (1 + uv) - (p + gu + gv).
\end{aligned}$$

Thus, the Hodge numbers appearing in $H^1(X;\mathbb{Q})$ are $h^{00} = p$, $h^{10} = g$, and $h^{01} = g$.