# Applications and examples of mixed Hodge structures; Hodge theory for all varieties

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November 18, 2021

The first part of this talk is based on Deligne's *Théorie de Hodge, II* (Section 4) and Cirici's notes "Weight filtration: some examples." The second part is based on Deligne's *Théorie de Hodge, III* (Sections 5, 6.2, 8).

## 1 Applications of *Hodge II*

**Theorem 1.** The weight spectral sequence $E_1^{-p,q} = H^{-2p+q}(D^{(p)};\mathbb{Q})(-p) \implies H^{-p+q}(U;\mathbb{Q})$ degenerates at $E_2$. The pure Hodge structures on the $E_2$ page induce a mixed Hodge structure on $H^{-p+q}(U;\mathbb{Q})$.

**Corollary 1.** If $H^k(U;\mathbb{C})$ has a nonzero weight space $H^{p,q}$ in some associated graded, then $p, q \leqslant k$, and $p + q \geqslant k$.

*Proof.* Only the terms $E_1^{-p,q} = H^{-2p+q}(D^{(p)};\mathbb{Q})(-p)$ with $-2p + q \geqslant 0$ can be nonzero. This is a pure Hodge structure of weight $q$. The bounds then follow from some bookkeeping involving the spectral sequence. □

**Corollary 2.** If $X$ is any smooth compactification of $U$, then the image of $H^k(X;\mathbb{Q}) \to H^k(U;\mathbb{Q})$ is the bottom weight part $W_k(H^k(U;\mathbb{Q}))$. If we have maps $Y \to U \hookrightarrow X$ with $Y$ smooth and proper, then the image of $H^k(X;\mathbb{Q}) \to H^k(Y;\mathbb{Q})$ equals the image of $H^k(U;\mathbb{Q}) \to H^k(Y;\mathbb{Q})$.

*Proof.* By a theorem of Grothendieck, the image $H^k(X;\mathbb{Q}) \to H^k(U;\mathbb{Q})$ is independent of smooth compactification $X$. By resolution of singularities, we may assume that $X - U$ is a simple normal crossings divisor, so that we can use $X$ to compute the weight spectral sequence of $U$. The map of weight spectral sequences associated to $U$ and $X$ identifies $H^k(X;\mathbb{Q})$ with $E_1^{0,k}$, so $\mathrm{im}(H^k(X;\mathbb{Q}) \to H^k(U;\mathbb{Q}))$ is identified with $E_2^{0,k}$, which is precisely the lowest weight part $W_k(H^k(U;\mathbb{Q}))$.

If $Y$ is smooth and proper, then $H^k(Y;\mathbb{Q})$ is pure of weight $k$. Thus, if we have a map $Y \to U$ with $Y$ smooth and proper, then the image $\mathrm{im}(H^k(U;\mathbb{Q}) \to H^k(Y;\mathbb{Q}))$ is the image $\mathrm{im}(W_k(H^k(U;\mathbb{Q})) \to H^k(Y;\mathbb{Q}))$ (since a map of mixed Hodge structures is strict), which is $\mathrm{im}(H^k(X;\mathbb{Q}) \to H^k(Y;\mathbb{Q}))$ by the previous paragraph. □

**Theorem 2** (Global invariant cycles theorem)**.** If $f : U \to S$ is a smooth proper map with $S$ smooth and separated and $U \hookrightarrow X$ is a smooth compactification, then the image of $H^k(X;\mathbb{Q}) \to H^k(X_s;\mathbb{Q})$ ($X_s$ is a smooth fiber) is the monodromy invariants $H^k(X_s;\mathbb{Q})^{\pi_1(S,s)}$.

*Proof of projective case.* The surjectivity of $H^k(U;\mathbb{Q}) \to H^k(X_s;\mathbb{Q})^{\pi_1(S,s)}$ follows from the degeneration of the Leray spectral sequence for $f$, since the composite $H^k(U;\mathbb{Q}) \to H^k(X_s;\mathbb{Q})^{\pi_1(S,s)} \cong H^0(U, R^k f_*\mathbb{Q})$ is an edge homomorphism of the Leray spectral sequence. We then apply the previous corollary to prove the projective case. $\square$

# 2 Some computations

I'm too lazy to draw the spectral sequences in LaTeX, so I'll just refer to Joana Cirici's notes.

# 3 Review of simplicial stuff

We move on to *Hodge III*, where Deligne constructs mixed Hodge structures for all varieties.

To deal with singular varieties, we have to go simplicial. I'm going to recall a bunch of definitions from Caleb's talk. A **simplicial variety** is a contravariant functor from the simplex category to the category of varieties, i.e. a collection $Y_\bullet = (Y_n)_{n\geqslant 0}$ with face and degeneracy maps. An **augmented simplicial variety**, written $Y_\bullet \to X$, is a simplicial variety equipped with a map $Y_0 \to X$. For each $n \geqslant 0$, there is an $n$-truncated variant of these categories, equipped with an adjunction $(\mathrm{sk}_n, \mathrm{cosk}_n)$.

A **sheaf** on a simplicial variety $Y_\bullet$ is a collection $\mathcal{F}^\bullet$ ($\mathcal{F}^n$ is a sheaf on $Y_n$) equipped with maps $\mathcal{F}^n(p) : \mathcal{F}^n \to p_*\mathcal{F}^m$ ($p : [n] \to [m]$) that are functorial (i.e. $p_*\mathcal{F}^n(q) \circ \mathcal{F}^n(p) = \mathcal{F}^n(q \circ p)$). Given a map $f : Y^1_\bullet \to Y^2_\bullet$, we can define the usual sheaf operations $f_*, f^*$ pointwise. Given an augmentation $a : Y_\bullet \to X$, we also define operations $a_*, a^*$, but they're defined kinda weirdly:

$$\begin{aligned}(a^*\mathcal{F})_n &= a_n^*\mathcal{F}\\ a_*\mathcal{F}^\bullet &= \ker(a_{0*}\mathcal{F}_0 \rightrightarrows a_{1*}\mathcal{F}_1).\end{aligned}$$

Taking $X = \mathrm{pt}$, we get the **global sections functor** $\Gamma$:

$$\Gamma(\mathcal{F}^\bullet) = \ker(\Gamma(\mathcal{F}_0) \rightrightarrows \Gamma(\mathcal{F}_1))$$

The **cohomology of a sheaf on a simplicial variety** is defined to be the right derived functors of $\Gamma$. Given a variety $S$, we can consider the **constant simplicial variety** $S_\bullet$ where $S_n = S$ for all $n$ and all the simplicial maps are the identity. Then a sheaf on $S_\bullet$ is equivalent to a cosimplicial sheaf on $S$.

An augmented simplicial variety $a : Y_\bullet \to X$ is a **proper hypercovering** if the map $Y_{n+1} \to (\mathrm{cosk}_n \mathrm{sk}_n Y)_{n+1}$ is proper and surjective for all $n \geqslant -1$. Proper hypercoverings are important because they satisfy **cohomological descent**, i.e. $\mathcal{F} \xrightarrow{\sim} Ra_*a^*\mathcal{F}$ for all sheaves $\mathcal{F}$ on $X$. Since $\Gamma \circ a = \Gamma$, we get

$$H^k(X, \mathcal{F}) \cong H^k(X, Ra_*a^*\mathcal{F}) \cong H^k(Y_\bullet, a^*\mathcal{F}).$$

In other words, we can compute the cohomology of $X$ by computing the cohomology of $Y_\bullet$.

# 4 Overview of *Hodge III*

The main idea of *Hodge III* is to replace a variety $Y$ by a proper hypercovering $Y_\bullet$ where each $Y_n$ is smooth. *Hodge II* tells us that the cohomology of each $Y_n$ is a mixed Hodge structure. As we will explain below, these mixed Hodge structures come from mixed Hodge complexes of sheaves. With some homological algebra, we can glue these mixed Hodge structures to get a mixed Hodge structure on the cohomology of $Y_\bullet$. By the theory of cohomological descent, the pullback $H^k(X;\mathbb{Z}) \to H^k(Y_\bullet;\mathbb{Z})$ is an isomorphism, so we get a mixed Hodge structure on $H^k(X;\mathbb{Z})$.

## 5 Mixed Hodge complexes

Deligne's proof of the main theorem in *Hodge II* is actually very general. In *Hodge III*, he introduces the formalism of mixed Hodge complexes, which provide a systematic way to produce mixed Hodge structures.

We talk about objects in the **filtered derived category** of an abelian category $C$, which we denote $DF(C)$. $DF(C)$ is formed from the category of filtered complexes in $C$, identifying homotopic maps (where the homotopy respects the filtration), and inverting filtered quasi-isomorphisms. There are also the standard variants $D^+F(C), D^-F(C), D^bF(C)$. We also consider the bi-filtered derived category $DF_2(C)$.

**Definition 1.** For $A \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}\}$, an ***A*-Hodge complex of weight** $n$ consists of the following data:

(1) A complex $K_A \in D^+(A)$ such that each $H^k(K_A)$ is finitely generated.

(2) A filtered complex $(K_\mathbb{C}, F) \in D^+F(\mathbb{C})$ ($F$ is decreasing) and an isomorphism $\alpha : K_A \otimes_A \mathbb{C} \simeq K_\mathbb{C}$ in the derived category (i.e. a zig-zag of quasi-isomorphisms).

This data must satisfy the following conditions:

(1) The differential $d$ on $K_\mathbb{C}$ is strictly compatible with $F$.

(2) For all $k$, $(H^k(K_A), F)$ is a Hodge structure of weight $n + k$.

We can upgrade this definition to the sheaf level.

**Definition 2.** Let $X$ be a topological space. An ***A*-Hodge complex of sheaves of weight** $n$ consists of the following data:

(1) A complex $K_A \in D^+(X, A)$.

(2) A filtered complex $(K_\mathbb{C}, F) \in D^+F(X, \mathbb{C})$ and an isomorphism $\alpha : K_A \otimes_A \mathbb{C} \simeq K_\mathbb{C}$ in the derived category.

This data must satisfy the following condition: $(R\Gamma(K_A), R\Gamma((K_\mathbb{C}, F)), R\Gamma(\alpha))$ is an $A$-Hodge complex of weight $n$.

When we omit $A$, we assume that $A = \mathbb{Z}$.

**Example 1.** If $X$ is a smooth proper variety, then $(\mathbb{Z}_X, (\Omega_X^\bullet, F), \alpha)$ is a Hodge complex of sheaves of weight 0. Here, $F$ is the bête filtration, and $\alpha$ is the map $\mathbb{C}_X \to \Omega_X^\bullet$, which is a quasi-isomorphism by the holomorphic Poincaré lemma. This is the content of classical Hodge theory.

We get analogous definitions for mixed Hodge theory.

**Definition 3.** A **mixed Hodge complex** consists of the following data:

(1) A complex $K \in D^+(\mathbb{Z})$ such that each $H^k(K)$ is finitely generated.

(2) A filtered complex $(K_\mathbb{Q}, W) \in D^+F(\mathbb{Q})$ ($W$ is increasing) and an isomorphism $\alpha : K \otimes_\mathbb{Z} \mathbb{Q} \simeq K_\mathbb{Q}$ in the derived category.

(3) A bi-filtered complex $(K_\mathbb{C}, W, F) \in D^+F_2(\mathbb{C})$ and an isomorphism $\beta : (K_\mathbb{Q}, W) \otimes_\mathbb{Q} \mathbb{C} \simeq (K_\mathbb{C}, W)$ in the filtered derived category.

This data must satisfy the following condition: for each $n$, $(\mathrm{Gr}_n^W K_{\mathbb{Q}}, (\mathrm{Gr}_n^W K_{\mathbb{C}}, F), \beta)$ is a $\mathbb{Q}$-Hodge complex of weight $n$.

**Definition 4.** Let $X$ be a topological space. A **mixed Hodge complex of sheaves** consists of the following data:

(1) A complex $K \in D^+(X, \mathbb{Z})$ such that each $\mathbb{H}^k(X, K)$ is finitely generated.

(2) A filtered complex $(K_{\mathbb{Q}}, W) \in D^+F(X, \mathbb{Q})$ and an isomorphism $\alpha : K \otimes_{\mathbb{Z}} \mathbb{Q} \simeq K_{\mathbb{Q}}$ in the derived category.

(3) A bi-filtered complex $(K_{\mathbb{C}}, W, F) \in D^+F_2(X, \mathbb{C})$ and an isomorphism $\beta : (K_{\mathbb{Q}}, W) \otimes_{\mathbb{Q}} \mathbb{C} \simeq (K_{\mathbb{C}}, W)$ in the filtered derived category.

This data must satisfy the following condition: for each $n$, $(\mathrm{Gr}_n^W K_{\mathbb{Q}}, (\mathrm{Gr}_n^W K_{\mathbb{C}}, F), \beta)$ is a $\mathbb{Q}$-Hodge complex of sheaves of weight $n$.

The following proposition follows from the definitions. It provides a great source of mixed Hodge complexes.

**Proposition 1.** If

$$(K, (K_{\mathbb{Q}}, W), \alpha, (K_{\mathbb{C}}, W, F), \beta)$$

is a mixed Hodge complex of sheaves, then

$$(R\Gamma(K), (R\Gamma(K_{\mathbb{Q}}), W), R\Gamma(\alpha), (R\Gamma(K_{\mathbb{C}}), W, F), \beta)$$

is a mixed Hodge complex.

**Example 2.** Let $U$ be the complement of a simple normal crossings divisor $D$ in a smooth proper variety $X$, and let $j$ denote the inclusion $U \hookrightarrow X$. Then

$$(Rj_*\mathbb{Z}_U, (Rj_*\mathbb{Q}_U, \tau), \alpha, (\Omega_X^\bullet(\log D), W, F), \beta)$$

is a mixed Hodge complex of sheaves. Here, $\alpha$ is the natural map $Rj_*\mathbb{Z}_U \otimes_{\mathbb{Z}} \mathbb{Q} \simeq Rj_*\mathbb{Q}_U$, and $\beta$ is the zig-zag

$$(Rj_*\mathbb{Q}_U, \tau) \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} (Rj_*\mathbb{C}_U, \tau) \xrightarrow{\sim} (Rj_*\Omega_U^\bullet, \tau) \xleftarrow{\sim} (j_*\Omega_U^\bullet, \tau) \xleftarrow{\sim} (\Omega_X^\bullet(\log D), \tau) \xrightarrow{\sim} (\Omega_X^\bullet(\log D), W)$$

from *Hodge II*. The associated gradeds are Hodge complexes of sheaves because

$$(\mathrm{Gr}_n^W Rj_*\mathbb{Q}_U, (\mathrm{Gr}_n^W \Omega_X^\bullet(\log D), F), \alpha) \cong (i_{n*}\mathbb{Q}_{D^{(n)}}[-n](-n), (i_{n*}\Omega_{D^{(n)}}^\bullet[-n], F), \alpha).$$

The main theorem of *Hodge II* (the big mess of homological algebra) can be adapted to any mixed Hodge complex.

**Theorem 3** (*Hodge II*)**.** If $(K, (K_{\mathbb{Q}}, W), \alpha, (K_{\mathbb{C}}, W, F), \beta)$ is a mixed Hodge complex, then the filtrations $W[n]$ ($W$ shifted by $n$) and $F$ make $H^n(K)$ a mixed Hodge structure.

Our goal is to get a mixed Hodge complex from a proper hypercovering $Y_\bullet \to X$. We will see how to do this next time.