




Constrained Toda hierarchy and turning points of the Ruijsenaars–Schneider model

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Abstract

We introduce a new integrable hierarchy of nonlinear differential-difference equations which we call constrained Toda hierarchy (C-Toda). It can be regarded as a certain subhierarchy of the 2D Toda lattice obtained by imposing the constraint $\tilde{\mathcal{L}} = \mathcal{L}^\dagger$ on the two Lax operators (in the symmetric gauge). We prove the existence of the tau function of the C-Toda hierarchy and show that it is the square root of the 2D Toda lattice tau function. In this and some other respects, the C-Toda is a Toda analogue of the CKP hierarchy. It is also shown that zeros of the tau function of elliptic solutions satisfy the dynamical equations of the Ruijsenaars–Schneider model restricted to turning points in the phase space. The spectral curve has holomorphic involution which interchanges the marked points in which the Baker–Akhiezer function has essential singularities.

Keywords Toda hierarchy · Integrable many-body systems

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1 Introduction

The 2D Toda lattice hierarchy [1] is perhaps the most fundamental in the theory of integrable systems. The commuting flows of the hierarchy are parametrized by infinite

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sets of time variables $\mathbf{t} = \{t_1, t_2, t_3, \dots\}$ (“positive times”) and $\bar{\mathbf{t}} = \{\bar{t}_1, \bar{t}_2, \bar{t}_3, \dots\}$ (“negative times”), together with the “zeroth time” $t_0 = x$. Equations of the hierarchy are differential in the times $\mathbf{t}, \bar{\mathbf{t}}$ and difference in x with a lattice spacing η . A common solution is provided by the tau function $\tau = \tau(x, \mathbf{t}, \bar{\mathbf{t}})$ which satisfies an infinite set of bilinear differential-difference equations of Hirota type [2,3]. All dependent variables are expressed through the tau function in one or another way.

Equally fundamental is the Kadomtsev–Petviashvili (KP) hierarchy with independent variables $\mathbf{t} = \{t_1, t_2, t_3, \dots\}$ which can be regarded as a subhierarchy of the 2D Toda lattice obtained by fixing the times $\bar{\mathbf{t}}$ and t_0 . Equations of the KP hierarchy are purely differential.

Many if not all known integrable nonlinear partial differential and difference equations are reductions or special cases of the 2D Toda lattice and KP hierarchies. Remarkably, they also contain most of the known finite-dimensional many-body integrable systems. For example, solutions of the KP hierarchy which are elliptic functions of t_1 with N poles in the fundamental domain (zeros of the tau function) give rise to the N -body elliptic Calogero–Moser system [4–6]: Zeros of the tau function as functions of t_2 move as Calogero–Moser particles (see [7–10] and [11] for a review). Later, it was shown that this correspondence can be extended to all commuting flows of the hierarchy: The t_j -dynamics of zeros of the tau function is the same as the Calogero–Moser dynamics with respect to the higher Hamiltonian H_j (see [12–14]). In their turn, poles of solutions of the 2D Toda lattice hierarchy which are elliptic functions of t_0 move as particles of the Ruijsenaars–Schneider model [15,16] which can be regarded as a relativistic extension of the Calogero–Moser model (see [17,18]).

Given an integrable hierarchy with a space of solutions \mathcal{M} , one can define a subhierarchy by imposing some constraints which restrict the space of solutions to $\mathcal{X} \subset \mathcal{M}$. In known examples, the constraints are preserved by only a part of the commuting flows of the hierarchy and are destroyed by the other part, so these time variables should be frozen.

Well-known examples of such situation are provided by the B- and C-versions of the KP hierarchy (BKP and CKP). In particular, the CKP hierarchy is introduced by imposing the constraint $\mathcal{L}^\dagger = -\mathcal{L}$ on the Lax operator of the KP hierarchy, where the operation \dagger is defined as $(f(x) \circ \partial_x^n)^\dagger = (-\partial_x)^n \circ f(x)$. The constraint is preserved by the “odd” flows and is destroyed by the “even” ones, so one should fix “even” times to zero values: $t_{2j} = 0$ for all j . The CKP hierarchy was introduced in the paper [19] and later different aspects of it were discussed in [20–23]. Recently, in [24], a characterization of the CKP hierarchy in terms of KP tau function was obtained: It was shown that the KP tau functions that provide solutions of the CKP hierarchy (with frozen “even” times) are characterized by the condition

$$\partial_{t_2} \log \tau \Big|_{t_{2j}=0} = 0. \tag{1.1}$$

This condition makes sense as defining “turning points” for zeros x_i of the tau function in the variable $x = t_1$: $\partial_{t_2} x_i = 0$ (the velocities vanish). For elliptic solutions, the zeros of the tau function move as particles of the elliptic Calogero–Moser system, so the condition (1.1) indeed defines the submanifold of turning points in the phase space,

where all momenta $p_i = 2\partial_{t_2}x_i$ are equal to zero. General algebraic geometrical solutions to the CKP hierarchy are obtained starting from algebraic curves which have a holomorphic involution, with the marked point on the curve (the point where the Baker–Akhiezer function has essential singularity) being a fixed point of the involution.

Moreover, one can prove that the CKP hierarchy possesses its own tau function τ^{CKP} which is a function of the “odd” times only, and this tau function is given by square root of the KP tau function restricted to the turning points.

In this paper, we suggest a Toda analogue of this story. To wit, we introduce a subhierarchy of the 2D Toda lattice which is related to it in the way much similar to the relation between the CKP and KP hierarchies. We call it C-Toda hierarchy.¹ (“C” is from “constrained” and simultaneously points to the similarity with CKP.) The constraint connects the two pseudo-difference Lax operators $\mathcal{L}, \tilde{\mathcal{L}}$ as follows:

$$\tilde{\mathcal{L}} = \mathcal{L}^\dagger \tag{1.2}$$

(in the symmetric gauge). This constraint is preserved by the flows $\partial_{t_j} - \partial_{\bar{t}_j}$ and is destroyed by the flows $\partial_{t_j} + \partial_{\bar{t}_j}$, so one should fix $t_j + \bar{t}_j = 0$ and vary only the times $T_j = \frac{1}{2}(t_j - \bar{t}_j)$. We show that solutions to the C-Toda hierarchy among all solutions to the 2D Toda lattice are characterized by the condition

$$(\partial_{t_1} + \partial_{\bar{t}_1}) \log \tau \Big|_{t_j + \bar{t}_j = 0} = 0. \tag{1.3}$$

Similarly to the CKP case, this condition makes sense as defining “turning points” for zeros x_i of the tau function in the variable x (the “zeroth time” of the 2D Toda lattice): $(\partial_{t_1} + \partial_{\bar{t}_1})x_i = 0$. For elliptic solutions, the zeros of the tau function move as particles of the elliptic Ruijsenaars–Schneider system, so the condition (1.3) indeed defines the submanifold of turning points in the phase space.

We also prove that the C-Toda hierarchy possesses its own tau function τ^{C} which is a function of the times T_j only, and this tau function is given by square root of the 2D Toda lattice tau function restricted to the turning points.

The analogies between the CKP and C-Toda hierarchies are summarized in the table:

	CKP	C-Toda
Evolution times	$t_1, t_3, t_5, \dots;$ $t_{2j} = 0$	$t_1 - \bar{t}_1, t_2 - \bar{t}_2, t_3 - \bar{t}_3, \dots;$ $t_j + \bar{t}_j = 0$
Constraints for $L -$ operators	$\mathcal{L}^\dagger = -\mathcal{L}$	$\tilde{\mathcal{L}} = \mathcal{L}^\dagger$
Tau functions	$\tau^{\text{CKP}} = \sqrt{\tau^{\text{KP}}}$	$\tau^{\text{C-Toda}} = \sqrt{\tau^{\text{Toda}}}$
Turning points conditions	$\partial_{t_2} \log \tau^{\text{CKP}} \Big _{t_{2j}=0} = 0$	$(\partial_{t_1} + \partial_{\bar{t}_1}) \log \tau^{\text{Toda}} \Big _{t_j + \bar{t}_j = 0} = 0$
Bilinear relations	$\oint_{C_\infty} \psi(\mathbf{t}, k) \psi(\mathbf{t}', -k) dk = 0$	$\left(\oint_{C_\infty} - \oint_{C_0} \right) \psi(\mathbf{t}, k) \psi(\mathbf{t}', k^{-1}) \frac{dk}{k} = 0$
Algebraic curves	involution $\iota,$ $\iota P_\infty = P_\infty$	involution $\iota,$ $\iota P_\infty = P_0, \iota P_0 = P_\infty$

¹ It is different from what is called Toda hierarchy of C-type in [1].

The paper is organized as follows: In Sect. 2.1, we briefly review the 2D Toda lattice hierarchy. In Sect. 2.2, the constrained Toda hierarchy (C-Toda) is introduced, and in Sect. 2.3 we prove the existence of the tau function for this hierarchy. Section 3 is devoted to the elliptic Ruijsenaars–Schneider model. We show that elliptic solutions of the C-Toda hierarchy generate the Ruijsenaars–Schneider dynamics of their poles (zeros of the tau function) restricted to the subspace in the phase space corresponding to turning points. We also prove that the spectral curve of the Lax matrix of the Ruijsenaars–Schneider model for turning points admits a holomorphic involution.

2 Constrained Toda hierarchy

2.1 2D Toda lattice

First of all, we briefly review the 2D Toda lattice hierarchy following [1]. Let us consider the pseudo-difference Lax operators:

$$\mathcal{L} = e^{\eta\partial_x} + \sum_{k \geq 0} U_k(x)e^{-k\eta\partial_x}, \quad \bar{\mathcal{L}} = c(x)e^{-\eta\partial_x} + \sum_{k \geq 0} \bar{U}_k(x)e^{k\eta\partial_x}, \quad (2.1)$$

where $e^{\eta\partial_x}$ is the shift operator acting as $e^{\pm\eta\partial_x} f(x) = f(x \pm \eta)$ and the coefficient functions U_k, \bar{U}_k are functions of $x, \mathbf{t}, \bar{\mathbf{t}}$. The Lax equations are

$$\partial_{t_m} \mathcal{L} = [\mathcal{B}_m, \mathcal{L}], \quad \partial_{t_m} \bar{\mathcal{L}} = [\mathcal{B}_m, \bar{\mathcal{L}}] \quad \mathcal{B}_m = (\mathcal{L}^m)_{\geq 0}, \quad (2.2)$$

$$\partial_{\bar{t}_m} \mathcal{L} = [\bar{\mathcal{B}}_m, \mathcal{L}], \quad \partial_{\bar{t}_m} \bar{\mathcal{L}} = [\bar{\mathcal{B}}_m, \bar{\mathcal{L}}] \quad \bar{\mathcal{B}}_m = (\bar{\mathcal{L}}^m)_{< 0}. \quad (2.3)$$

Here and below, given a subset $\mathbb{S} \subset \mathbb{Z}$, we denote $(\sum_{k \in \mathbb{Z}} U_k e^{k\eta\partial_x})_{\mathbb{S}} = \sum_{k \in \mathbb{S}} U_k e^{k\eta\partial_x}$. For example, $\mathcal{B}_1 = e^{\eta\partial_x} + U_0(x)$, $\bar{\mathcal{B}}_1 = c(x)e^{-\eta\partial_x}$. An equivalent formulation is through the zero curvature (Zakharov–Shabat) equations:

$$\partial_{t_n} \mathcal{B}_m - \partial_{t_m} \mathcal{B}_n + [\mathcal{B}_m, \mathcal{B}_n] = 0, \quad (2.4)$$

$$\partial_{\bar{t}_n} \mathcal{B}_m - \partial_{t_m} \bar{\mathcal{B}}_n + [\mathcal{B}_m, \bar{\mathcal{B}}_n] = 0, \quad (2.5)$$

$$\partial_{\bar{t}_n} \bar{\mathcal{B}}_m - \partial_{\bar{t}_m} \bar{\mathcal{B}}_n + [\bar{\mathcal{B}}_m, \bar{\mathcal{B}}_n] = 0. \quad (2.6)$$

For example, putting

$$c(x) = e^{\varphi(x) - \varphi(x - \eta)}, \quad (2.7)$$

we have from (2.5) at $m = n = 1$:

$$\partial_{t_1} \partial_{\bar{t}_1} \varphi(x) = e^{\varphi(x) - \varphi(x - \eta)} - e^{\varphi(x + \eta) - \varphi(x)}. \quad (2.8)$$

This is the famous 2D Toda lattice equation.

Note that from (2.2) and (2.3), it follows that

$$\partial_{t_m} \varphi = (\mathcal{L}^m)_0, \quad \partial_{\bar{t}_m} \varphi = -(\bar{\mathcal{L}}^m)_0. \tag{2.9}$$

The zero curvature equations are compatibility conditions for the auxiliary linear problems

$$\partial_{t_m} \psi = \mathcal{B}_m(x) \psi, \quad \partial_{\bar{t}_m} \psi = \bar{\mathcal{B}}_m(x) \psi, \tag{2.10}$$

where the wave function ψ depends on a spectral parameter k : $\psi = \psi(x, \mathbf{t}, \bar{\mathbf{t}}; k)$. The wave function has the following expansion in powers of k :

$$\psi(x, \mathbf{t}, \bar{\mathbf{t}}; k) = \begin{cases} k^{x/\eta} e^{\xi(\mathbf{t}, k)} \left(1 + \sum_{s \geq 1} \xi_s(x) k^{-s} \right), & k \rightarrow \infty, \\ k^{x/\eta} e^{\xi(\bar{\mathbf{t}}, k^{-1}) + \varphi(x)} \left(1 + \sum_{s \geq 1} \chi_s(x) k^s \right), & k \rightarrow 0, \end{cases} \tag{2.11}$$

where

$$\xi(\mathbf{t}, k) = \sum_{j \geq 1} t_j k^j. \tag{2.12}$$

The wave function satisfies the linear equation

$$\partial_{t_1} \psi(x, k) = \psi(x + \eta, k) + v(x) \psi(x, k), \tag{2.13}$$

where $v(x) = U_0(x)$.

The wave operators are pseudo-difference operators of the form

$$\begin{aligned} \mathcal{W}(x) &= 1 + \xi_1(x) e^{-\eta \partial_x} + \xi_2(x) e^{-2\eta \partial_x} + \dots \\ \bar{\mathcal{W}}(x) &= e^{\varphi(x)} (1 + \chi_1(x) e^{-\eta \partial_x} + \chi_2(x) e^{-2\eta \partial_x} + \dots) \end{aligned} \tag{2.14}$$

with the same coefficient functions ξ_j, χ_j as in (2.11), then the wave function can be written as

$$\begin{aligned} \psi &= \mathcal{W}(x) k^{x/\eta} e^{\xi(\mathbf{t}, k)}, \quad k \rightarrow \infty, \\ \psi &= \bar{\mathcal{W}}(x) k^{x/\eta} e^{\xi(\bar{\mathbf{t}}, k^{-1})}, \quad k \rightarrow 0. \end{aligned} \tag{2.15}$$

The dual wave function ψ^* is defined by

$$\psi^* = (\mathcal{W}^\dagger(x))^{-1} k^{-x/\eta} e^{-\xi(\mathbf{t}, k)}, \quad k \rightarrow \infty, \tag{2.16}$$

where the adjoint difference operator is defined according to the rule $(f(x) \circ e^{n\eta\partial_x})^\dagger = e^{-n\eta\partial_x} \circ f(x)$. The auxiliary linear problems for the dual wave function have the form

$$-\partial_{t_m} \psi^* = \mathcal{B}_m^\dagger(x) \psi^*. \tag{2.17}$$

The Lax operators (2.1) are obtained by ‘‘dressing’’ of the shift operators by $\mathcal{W}, \bar{\mathcal{W}}$:

$$\mathcal{L} = \mathcal{W} e^{\eta\partial_x} \mathcal{W}^{-1}, \quad \bar{\mathcal{L}} = \bar{\mathcal{W}} e^{-\eta\partial_x} \bar{\mathcal{W}}^{-1}. \tag{2.18}$$

So far we have used the standard gauge in which the coefficient of the first term of \mathcal{L} is fixed to be 1. In fact, there is a family of gauge transformations with $g = e^{\alpha\varphi(x)}$ [25,26]:

$$\begin{aligned} \mathcal{L} &\rightarrow g^{-1} \mathcal{L} g, & \bar{\mathcal{L}} &\rightarrow g^{-1} \bar{\mathcal{L}} g, \\ \mathcal{B}_n &\rightarrow g^{-1} \mathcal{B}_n g - g^{-1} \partial_{t_n} g, & \bar{\mathcal{B}}_n &\rightarrow g^{-1} \bar{\mathcal{B}}_n g - g^{-1} \partial_{\bar{t}_n} g \end{aligned}$$

of which $\alpha = 0$ corresponds to the standard gauge $\mathcal{L} = \mathcal{L}^{(0)}, \bar{\mathcal{L}} = \bar{\mathcal{L}}^{(0)}$. At $\alpha = \frac{1}{2}$, we have the so-called symmetric gauge:

$$\begin{aligned} \mathcal{L}^s &= c^s(x) e^{\eta\partial_x} + \sum_{k \geq 0} U_k^s(x) e^{-k\eta\partial_x}, \\ \bar{\mathcal{L}}^s &= c^s(x - \eta) e^{-\eta\partial_x} + \sum_{k \geq 0} \bar{U}_k^s(x) e^{k\eta\partial_x}, \end{aligned} \tag{2.19}$$

$$c^s(x) = e^{\frac{1}{2}(\varphi(x+\eta) - \varphi(x))}. \tag{2.20}$$

Hereafter, we write simply $\mathcal{L}^s, \bar{\mathcal{L}}^s$ instead of $\mathcal{L}^{(1/2)}, \bar{\mathcal{L}}^{(1/2)}$ for brevity. In the symmetric gauge, the generators of the t_m - and \bar{t}_m -flows $\mathcal{B}_m, \bar{\mathcal{B}}_m$ are

$$\mathcal{B}_m^s = ((\mathcal{L}^s)^m)_{>0} + \frac{1}{2} ((\mathcal{L}^s)^m)_0, \quad \bar{\mathcal{B}}_m^s = ((\bar{\mathcal{L}}^s)^m)_{<0} + \frac{1}{2} ((\bar{\mathcal{L}}^s)^m)_0. \tag{2.21}$$

Similar to (2.18), the Lax operators $\mathcal{L}^s, \bar{\mathcal{L}}^s$ are obtained by dressing of the shift operators:

$$\mathcal{L}^s = \mathcal{W}^s e^{\eta\partial_x} (\mathcal{W}^s)^{-1}, \quad \bar{\mathcal{L}}^s = \bar{\mathcal{W}}^s e^{-\eta\partial_x} (\bar{\mathcal{W}}^s)^{-1}, \tag{2.22}$$

where the wave operators are

$$\mathcal{W}^s(x) = e^{-\frac{1}{2}\varphi(x)} \mathcal{W}, \quad \bar{\mathcal{W}}^s(x) = e^{-\frac{1}{2}\varphi(x)} \bar{\mathcal{W}}. \tag{2.23}$$

We also note that the wave functions are given by

$$\psi(x, k) = e^{\frac{1}{2}\varphi(x)} \bar{\mathcal{W}}^s(x) k^{x/\eta} e^{\xi(\bar{\mathbf{t}}, k^{-1})}, \quad k \rightarrow 0, \tag{2.24}$$

$$\psi^*(x, k) = e^{-\frac{1}{2}\varphi(x)}(\mathcal{W}^{s^\dagger}(x))^{-1}k^{-x/\eta}e^{-\xi(\mathbf{t},k)}, \quad k \rightarrow \infty. \tag{2.25}$$

A common solution to the 2D Toda lattice hierarchy is provided by the tau function $\tau = \tau(x, \mathbf{t}, \bar{\mathbf{t}})$ [2,3]. The tau function satisfies the bilinear relation

$$\oint_{C_\infty} k^{\frac{x-x'}{\eta}-1} e^{\xi(\mathbf{t},k)-\xi(\mathbf{t}',k)} \tau(x, \mathbf{t} - [k^{-1}], \bar{\mathbf{t}}) \tau(x' + \eta, \mathbf{t}' + [k^{-1}], \bar{\mathbf{t}}) dk \\ = \oint_{C_0} k^{\frac{x-x'}{\eta}-1} e^{\xi(\bar{\mathbf{t}},k^{-1})-\xi(\bar{\mathbf{t}}',k^{-1})} \tau(x + \eta, \mathbf{t}, \bar{\mathbf{t}} - [k]) \tau(x', \mathbf{t}', \bar{\mathbf{t}}' + [k]) dk \tag{2.26}$$

valid for all $x, x', \mathbf{t}, \mathbf{t}', \bar{\mathbf{t}}, \bar{\mathbf{t}}'$. It is assumed that $x - x' \in \eta\mathbb{Z}$. The integration contour C_∞ in the left-hand side is a big circle around infinity separating the singularities coming from the exponential factor from those coming from the tau functions. The integration contour C_0 in the right-hand side is a small circle around zero separating the singularities coming from the exponential factor from those coming from the tau functions. The bilinear relation (2.26) encodes all differential-difference equations of the hierarchy.

Setting $x - x' = \eta, t_n - t'_n = \frac{1}{n}a^{-n}, \bar{t}_n - \bar{t}'_n = \frac{1}{n}b^{-n}$ in (2.26) and taking the residues, we get the three-term bilinear equation of the Hirota–Miwa type:

$$\tau(x, \mathbf{t} - [a^{-1}], \bar{\mathbf{t}}) \tau(x, \mathbf{t}, \bar{\mathbf{t}} - [b^{-1}]) - \tau(x, \mathbf{t}, \bar{\mathbf{t}}) \tau(x, \mathbf{t} - [a^{-1}], \bar{\mathbf{t}} - [b^{-1}]) \\ = (ab)^{-1} \tau(x - \eta, \mathbf{t} - [a^{-1}], \bar{\mathbf{t}}) \tau(x + \eta, \mathbf{t}, \bar{\mathbf{t}} - [b^{-1}]). \tag{2.27}$$

The functions $\varphi(x), U_0(x)$ are expressed through the tau function as follows:

$$\varphi(x) = \log \frac{\tau(x + \eta)}{\tau(x)}, \tag{2.28}$$

$$U_0(x) = \partial_{t_1} \log \frac{\tau(x + \eta)}{\tau(x)} = \partial_{t_1} \varphi(x). \tag{2.29}$$

The wave function $\psi(x, k)$ and its dual $\psi^*(x, k)$ are expressed through the tau function as follows [1–3]:

$$\psi(x, k) = k^{x/\eta} \exp\left(\sum_{j \geq 1} t_j k^j\right) \frac{\tau(x, \mathbf{t} - [k^{-1}], \bar{\mathbf{t}})}{\tau(x, \mathbf{t})}, \quad k \rightarrow \infty,$$

$$\psi(x, k) = k^{x/\eta} \exp\left(\sum_{j \geq 1} \bar{t}_j k^{-j}\right) \frac{\tau(x + \eta, \mathbf{t}, \bar{\mathbf{t}} - [k])}{\tau(x, \mathbf{t})}, \quad k \rightarrow 0,$$

$$\psi^*(x, k) = k^{-x/\eta} \exp\left(-\sum_{j \geq 1} t_j k^j\right) \frac{\tau(x + \eta, \mathbf{t} + [k^{-1}], \bar{\mathbf{t}})}{\tau(x + \eta, \mathbf{t})}, \quad k \rightarrow \infty,$$

$$\psi^*(x, k) = k^{-x/\eta} \exp\left(-\sum_{j \geq 1} \bar{t}_j k^{-j}\right) \frac{\tau(x, \mathbf{t}, \bar{\mathbf{t}} + [k])}{\tau(x + \eta, \mathbf{t})}, \quad k \rightarrow 0, \tag{2.30}$$

where

$$\mathbf{t} \pm [k] = \left\{t_1 \pm k, t_2 \pm \frac{1}{2}k^2, t_3 \pm \frac{1}{3}k^3, \dots\right\}. \tag{2.31}$$

Taking into account Formula (2.30), one can represent (2.26) as a bilinear relation for the wave functions:

$$\left(\oint_{C_\infty} - \oint_{C_0}\right) \psi(x, \mathbf{t}, \bar{\mathbf{t}}; k) \psi^*(x', \mathbf{t}', \bar{\mathbf{t}}'; k) \frac{dk}{2\pi i k} = 0, \quad x - x' \in \eta\mathbb{Z}. \tag{2.32}$$

2.2 The C-Toda hierarchy

The C-Toda hierarchy is defined by imposing the constraint

$$\bar{\mathcal{L}}^s = \mathcal{L}^{s\dagger} \tag{2.33}$$

(in the symmetric gauge). In the standard gauge, it looks as follows:

$$\bar{\mathcal{L}}e^\varphi = e^\varphi \mathcal{L}^\dagger. \tag{2.34}$$

This means that $\bar{U}_j^s(x) = U_j^s(x + j\eta)$ for $j \geq 0$. In terms of the wave operators, this is equivalent to the constraint

$$\bar{\mathcal{W}}^s \mathcal{W}^{s\dagger} = \mathcal{W}^s \bar{\mathcal{W}}^{s\dagger} = 1. \tag{2.35}$$

It is important to note that not all time flows of the full Toda hierarchy are consistent with the constraint. Let us introduce the following linear combinations of times:

$$T_j = \frac{1}{2}(t_j - \bar{t}_j), \quad y_j = \frac{1}{2}(t_j + \bar{t}_j), \tag{2.36}$$

then the corresponding vector fields are

$$\partial_{T_j} = \partial_{t_j} - \partial_{\bar{t}_j}, \quad \partial_{y_j} = \partial_{t_j} + \partial_{\bar{t}_j}. \tag{2.37}$$

One can see that the T_j -flows preserve the constraint. Indeed, we have:

$$\begin{aligned} \partial_{T_j}(\bar{\mathcal{L}}^s - \mathcal{L}^{s\dagger}) &= [\mathcal{B}_j^s, \bar{\mathcal{L}}^s] - [\mathcal{B}_j^s, \mathcal{L}^{s\dagger}] = [\mathcal{B}_j^s, \bar{\mathcal{L}}^s] + [\mathcal{B}_j^{s\dagger}, \mathcal{L}^{s\dagger}] \\ &= [\mathcal{B}_j^s + \bar{\mathcal{B}}_j^s, \bar{\mathcal{L}}^s] = (\partial_{t_j} + \partial_{\bar{t}_j})\bar{\mathcal{L}}^s. \end{aligned}$$

Similarly,

$$\begin{aligned} \partial_{\bar{t}_j}(\bar{\mathcal{L}}^s - \mathcal{L}^{s\dagger}) &= [\bar{\mathcal{B}}_j^s, \bar{\mathcal{L}}^s] - [\bar{\mathcal{B}}_j^s, \mathcal{L}]^{s\dagger} = [\bar{\mathcal{B}}_j^s, \bar{\mathcal{L}}^s] + [\bar{\mathcal{B}}_j^{s\dagger}, \mathcal{L}^{s\dagger}] \\ &= [\mathcal{B}_j^s + \bar{\mathcal{B}}_j^s, \bar{\mathcal{L}}^s] = (\partial_{t_j} + \partial_{\bar{t}_j})\bar{\mathcal{L}}^s, \end{aligned}$$

so

$$(\partial_{t_j} - \partial_{\bar{t}_j})(\bar{\mathcal{L}}^s - \mathcal{L}^{s\dagger}) = \partial_{T_j}(\bar{\mathcal{L}}^s - \mathcal{L}^{s\dagger}) = 0$$

for all T_j . At the same time, the y_j -flows destroy the constraint, so we should put $y_j = 0$ for all j . The situation is similar to the embedding of the CKP hierarchy into the KP one, where the constraint is preserved only by the ‘‘odd’’ times and all ‘‘even’’ times are fixed to be 0.

Set

$$\mathcal{A}_m = \mathcal{B}_m^s - \bar{\mathcal{B}}_m^s. \tag{2.38}$$

In particular,

$$\begin{aligned} \mathcal{A}_1 &= c^s(x)e^{\eta\partial_x} - c^s(x - \eta)e^{-\eta\partial_x}, \\ \mathcal{A}_2 &= c^s(x)c^s(x + \eta)e^{2\eta\partial_x} + c^s(x)(v(x) + v(x + \eta))e^{\eta\partial_x} \\ &\quad - c^s(x - \eta)(v(x) + v(x - \eta))e^{-\eta\partial_x} - c^s(x - \eta)c^s(x - 2\eta)e^{-2\eta\partial_x}, \end{aligned}$$

where $v(x) = U_0(x) = \frac{1}{2}\partial_{T_1}\varphi(x)$. The Zakharov–Shabat equations for the C-Toda hierarchy read

$$[\partial_{T_m} - \mathcal{A}_m, \partial_{T_n} - \mathcal{A}_n] = 0. \tag{2.39}$$

The simplest equation is obtained at $m = 1, n = 2$. It reads:

$$\begin{aligned} &(\partial_{T_2} - \partial_{T_1}^2)\varphi(x + \eta) - (\partial_{T_2} + \partial_{T_1}^2)\varphi(x) \\ &= 2e^{\varphi(x) - \varphi(x - \eta)} - 2e^{\varphi(x + 2\eta) - \varphi(x + \eta)} + \frac{1}{2}(\partial_{T_1}\varphi(x + \eta))^2 - \frac{1}{2}(\partial_{T_1}\varphi(x))^2. \end{aligned} \tag{2.40}$$

Eq. (2.15) together with the constraints (2.35) implies that the dual wave function ψ^* in the C-Toda hierarchy is expressed through the wave function ψ as follows:

$$\psi^*(x, k) = e^{-\varphi(x)}\psi(x, k^{-1})\Big|_{t_j + \bar{t}_j = 0}. \tag{2.41}$$

The bilinear relation (2.32) for the C-Toda hierarchy acquires the form

$$\left(\oint_{C_\infty} - \oint_{C_0}\right)\psi(x, \mathbf{t}, \bar{\mathbf{t}}; k)\psi(x', \mathbf{t}', \bar{\mathbf{t}}'; k^{-1})\frac{dk}{2\pi ik} = 0, \quad x - x' \in \eta\mathbb{Z}, \tag{2.42}$$

where it is assumed that $t_j + \bar{t}_j = t'_j + \bar{t}'_j = 0$.

Using relation (2.30), we see that Eq. (2.41) in terms of the tau function reads

$$\tau(x, \mathbf{t}, \bar{\mathbf{t}} - [k^{-1}]) = \tau(x, \mathbf{t} + [k^{-1}], \bar{\mathbf{t}}) \quad \text{at } t_k + \bar{t}_k = 0. \tag{2.43}$$

Expanding it in powers of k , we obtain, in the leading order:

$$(\partial_{t_1} + \partial_{\bar{t}_1}) \log \tau(x, \mathbf{t}, \bar{\mathbf{t}}) = 0 \quad \text{at } t_k + \bar{t}_k = 0. \tag{2.44}$$

This is the necessary condition which should be obeyed by the tau function of the 2D Toda lattice in order to provide a solution to the C-Toda hierarchy. We conjecture that this condition implies

$$(\partial_{t_j} + \partial_{\bar{t}_j}) \log \tau(x, \mathbf{t}, \bar{\mathbf{t}}) = 0 \quad \text{at } t_k + \bar{t}_k = 0 \tag{2.45}$$

for all $j \geq 1$, i.e., $\partial_{y_j} \log \tau = 0$. We stress that in general this condition does not mean that the tau function is independent of y_j . (In particular, the second derivative $\partial_{y_j}^2 \log \tau$ may be nonzero.) However, an important particular case is the reduction to the 1D Toda lattice, where the solutions do not depend on the variables y_j . Then, the conditions (2.45) are obviously satisfied and we see that any solution of the 1D Toda hierarchy solves the constrained Toda hierarchy.

2.3 Tau function of the C-Toda hierarchy

The wave functions of the C-Toda hierarchy can be expressed through the tau function $\tau = \tau^T$ of the 2D Toda hierarchy according to Formula (2.30). However, one may ask whether there exists a tau function τ^C of the C-Toda hierarchy which depends on the time variables $T_j = \frac{1}{2}(t_j - \bar{t}_j) = t_j$ only. (Hereafter, because at $t_j + \bar{t}_j = 0$ we have $T_j = t_j$, we use the notation t_j for the time variables T_j .) Below, we show that the answer is in the affirmative.

Theorem 2.1 *There exists a function $\tau^C = \tau^C(x, \mathbf{t})$ such that*

$$\psi(x, \mathbf{t}; k) = e^{\frac{1}{2}\varphi(x, \mathbf{t})} \sqrt{\chi^2(x, \mathbf{t}; k) - \chi^2(x - \eta, \mathbf{t}; k)}, \quad k \rightarrow \infty, \tag{2.46}$$

$$\psi(x, \mathbf{t}; k^{-1}) = e^{\frac{1}{2}\varphi(x, \mathbf{t})} \sqrt{\bar{\chi}^2(x, \mathbf{t}; k) - \bar{\chi}^2(x + \eta, \mathbf{t}; k)}, \quad k \rightarrow \infty, \tag{2.47}$$

where

$$\chi(x, \mathbf{t}; k) = k^{x/\eta} e^{\xi(\mathbf{t}, k) - \frac{1}{2}\varphi(x, \mathbf{t})} \frac{\tau^C(x, \mathbf{t} - [k^{-1}])}{\tau^C(x, \mathbf{t})}, \tag{2.48}$$

$$\bar{\chi}(x, \mathbf{t}; k) = k^{-x/\eta} e^{-\xi(\mathbf{t}, k)} \frac{\tau^C(x + \eta, \mathbf{t} + [k^{-1}])}{\tau^C(x, \mathbf{t})}, \tag{2.49}$$

$$\varphi(x, \mathbf{t}) = \log \left(\frac{\tau^C(x + \eta, \mathbf{t})}{\tau^C(x, \mathbf{t})} \right)^2. \tag{2.50}$$

Definition 2.1 The function $\tau^C = \tau^C(x, \mathbf{t})$ is called the tau function of the C-Toda hierarchy.

Proof of Theorem 2.1 The starting point of the proof is the bilinear relation (2.42):

$$\left(\oint_{C_\infty} - \oint_{C_0} \right) \psi(x, \mathbf{t}, -\mathbf{t}; k) \psi(x', \mathbf{t}', -\mathbf{t}'; k^{-1}) \frac{dk}{2\pi i k} = 0, \quad x - x' \in \eta\mathbb{Z}. \tag{2.51}$$

We can represent the wave functions in the form

$$\begin{aligned} \psi(x, \mathbf{t}, -\mathbf{t}; k) &= k^{x/\eta} e^{\xi(\mathbf{t}, k)} w(x, \mathbf{t}; k), \quad k \rightarrow \infty, \\ \psi(x, \mathbf{t}, -\mathbf{t}; k^{-1}) &= k^{-x/\eta} e^{-\xi(\mathbf{t}, k)} \bar{w}(x, \mathbf{t}; k), \quad k \rightarrow \infty, \end{aligned} \tag{2.52}$$

then the bilinear relation can be written as

$$\begin{aligned} &\oint_{C_\infty} k^{n-1} e^{\xi(\mathbf{t}-\mathbf{t}', k)} w(x, \mathbf{t}; k) \bar{w}(x-n\eta, \mathbf{t}'; k) dk \\ &= \oint_{C_0} k^{n-1} e^{-\xi(\mathbf{t}-\mathbf{t}', k^{-1})} \bar{w}(x, \mathbf{t}; k^{-1}) w(x-n\eta, \mathbf{t}'; k^{-1}) dk. \end{aligned} \tag{2.53}$$

One can always normalize the functions $w(x, \mathbf{t}; k)$, $\bar{w}(x, \mathbf{t}; k)$ in the following way:

$$w(x, \mathbf{t}; \infty) = 1, \quad \bar{w}(x, \mathbf{t}; \infty) = r(x, \mathbf{t}) = e^{\varphi(x, \mathbf{t})}. \tag{2.54}$$

Now, choosing $\mathbf{t} - \mathbf{t}'$ and n in some special ways, one is able to obtain different relations for the functions $w(x, \mathbf{t}; k)$, $\bar{w}(x, \mathbf{t}; k)$ with certain shifts of the variables.

1. $\mathbf{t} - \mathbf{t}' = [a^{-1}]$, $n = 1$. In this case, $e^{\xi(\mathbf{t}-\mathbf{t}', k)} = \frac{a}{a-k}$ and the bilinear relation acquires the form

$$\begin{aligned} &\oint_{C_\infty} \frac{a}{a-k} w(x, \mathbf{t}; k) \bar{w}(x-\eta, \mathbf{t} - [a^{-1}]; k) dk \\ &= \oint_{C_0} \left(1 - \frac{1}{ka} \right) \bar{w}(x, \mathbf{t}; k^{-1}) w(x-\eta, \mathbf{t} - [a^{-1}]; k^{-1}) dk. \end{aligned}$$

The residue calculus yields

$$w(x, \mathbf{t}; a) \bar{w}(x - \eta, \mathbf{t} - [a^{-1}]; a) = r(x - \eta, \mathbf{t} - [a^{-1}]) - a^{-2} r(x, \mathbf{t}). \tag{2.55}$$

2. $\mathbf{t} - \mathbf{t}' = [a^{-1}] + [b^{-1}]$, $n = 2$. In this case, the bilinear relation acquires the form

$$\oint_{C_\infty} \frac{abk}{(a-k)(b-k)} w(x, \mathbf{t}; k) \bar{w}(x-2\eta, \mathbf{t} - [a^{-1}] - [b^{-1}]; k) dk$$

$$= \oint_{C_0} k \left(1 - \frac{1}{ka}\right) \left(1 - \frac{1}{kb}\right) \bar{w}(x, \mathbf{t}; k^{-1}) w(x-2\eta, \mathbf{t} - [a^{-1}] - [b^{-1}]; k^{-1}) dk.$$

The residue calculus yields

$$c \frac{ab}{a-b} \left(aw(x, \mathbf{t}; a) \bar{w}(x-2\eta, \mathbf{t} - [a^{-1}] - [b^{-1}]; a) - bw(x, \mathbf{t}; b) \bar{w}(x-2\eta, \mathbf{t} - [a^{-1}] - [b^{-1}]; b)\right)$$

$$= abr(x-2\eta, \mathbf{t} - [a^{-1}] - [b^{-1}]) - (ab)^{-1}r(x, \mathbf{t}). \tag{2.56}$$

3. $\mathbf{t} - \mathbf{t}' = [a^{-1}] - [b^{-1}]$, $n = 0$. In this case,

$$\oint_{C_\infty} k^{-1} \frac{a(b-k)}{b(a-k)} w(x, \mathbf{t}; k) \bar{w}(x, \mathbf{t} - [a^{-1}] + [b^{-1}]; k) dk$$

$$= \oint_{C_0} k^{-1} \frac{k-a^{-1}}{k-b^{-1}} \bar{w}(x, \mathbf{t}; k^{-1}) w(x, \mathbf{t} - [a^{-1}] + [b^{-1}]; k^{-1}) dk$$

and residue calculus yields

$$c \left(1 - \frac{a}{b}\right) w(x, \mathbf{t}; a) \bar{w}(x, \mathbf{t} - [a^{-1}] + [b^{-1}]; a) - \left(1 - \frac{b}{a}\right) \bar{w}(x, \mathbf{t}; b) w(x, \mathbf{t} - [a^{-1}] + [b^{-1}]; b)$$

$$= \frac{b}{a} r(x, \mathbf{t}) - \frac{a}{b} r(x, \mathbf{t} - [a^{-1}] + [b^{-1}]). \tag{2.57}$$

Expressing \bar{w} through w with the help of (2.55), we can represent the other two relations, (2.56) and (2.57), as a system of two equations for two “variables”

$$X_a = \frac{w(x-\eta, \mathbf{t} - [b^{-1}]; a)}{w(x, \mathbf{t}; a)}, \quad X_b = \frac{w(x-\eta, \mathbf{t} - [a^{-1}]; b)}{w(x, \mathbf{t}; b)}. \tag{2.58}$$

The system has the form

$$\left\{ \begin{aligned} & \frac{ab}{a-b} \left[ag(x-\eta, \mathbf{t}-[b^{-1}]; a)X_a^{-1} - bg(x-\eta, \mathbf{t}-[a^{-1}]; b)X_b^{-1} \right] \\ & = abr(x-2\eta, \mathbf{t}-[a^{-1}]-[b^{-1}]) - (ab)^{-1}r(x, \mathbf{t}) \\ & \left(1-\frac{a}{b}\right)g(x, \mathbf{t}; a)X_a - \left(1-\frac{b}{a}\right)g(x, \mathbf{t}; b)X_b \\ & = \frac{b}{a}r(x-\eta, \mathbf{t}-[b^{-1}]) - \frac{a}{b}r(x-\eta, \mathbf{t}-[a^{-1}]), \end{aligned} \right. \tag{2.59}$$

where

$$g(x, \mathbf{t}; z) = r(x - \eta, \mathbf{t} - [z^{-1}]) - z^{-2}r(x, \mathbf{t}). \tag{2.60}$$

Next, we take the product of the left-hand sides of the two equations (2.59) and equate it to the product of the right-hand sides. After some transformations, we obtain the remarkable relation

$$\left(\frac{X_a}{X_b}\right)^2 = \frac{w^2(x, \mathbf{t}; b)w^2(x - \eta, \mathbf{t} - [b^{-1}]; a)}{w^2(x, \mathbf{t}; a)w^2(x - \eta, \mathbf{t} - [a^{-1}]; b)} = \frac{g(x, \mathbf{t}; b)g(x - \eta, \mathbf{t} - [b^{-1}]; a)}{g(x, \mathbf{t}; a)g(x - \eta, \mathbf{t} - [a^{-1}]; b)} \tag{2.61}$$

which implies that

$$w_0(x, \mathbf{t}; z) := w(x, \mathbf{t}; z)g^{-1/2}(x, \mathbf{t}; z)$$

obeys the relation

$$\frac{w_0(x, \mathbf{t}; b)w_0(x - \eta, \mathbf{t} - [b^{-1}]; a)}{w_0(x, \mathbf{t}; a)w_0(x - \eta, \mathbf{t} - [a^{-1}]; b)} = 1. \tag{2.62}$$

It follows from this relation that there exists a function $\tau^C(x, \mathbf{t})$ such that

$$w_0(x, \mathbf{t}; z) = \frac{\tau^C(x - \eta, \mathbf{t} - [z^{-1}])}{\tau^C(x, \mathbf{t})}. \tag{2.63}$$

The proof is almost literally a repetition of the proof of a similar statement for the CKP hierarchy presented in [24].

Therefore, we have

$$w(x, \mathbf{t}; k) = g^{1/2}(x, \mathbf{t}; k) \frac{\tau^C(x - \eta, \mathbf{t} - [k^{-1}])}{\tau^C(x, \mathbf{t})} \tag{2.64}$$

with $g(x, \mathbf{t}; k)$ as in (2.60). The normalization of w implies that

$$1 = w(x, \mathbf{t}, \infty) = r^{1/2}(x - \eta, \mathbf{t}) \frac{\tau^C(x - \eta, \mathbf{t})}{\tau^C(x, \mathbf{t})},$$

whence

$$r(x, \mathbf{t}) = \left(\frac{\tau^C(x + \eta, \mathbf{t})}{\tau^C(x, \mathbf{t})} \right)^2. \tag{2.65}$$

On the other hand, we know that

$$r(x, \mathbf{t}) = \frac{\tau^T(x + \eta, \mathbf{t}, -\mathbf{t})}{\tau^T(x, \mathbf{t}, -\mathbf{t})}, \tag{2.66}$$

where τ^T is the tau function of the 2D Toda lattice hierarchy. This implies the following relation between the two tau functions:

$$\tau^T(x, \mathbf{t}, -\mathbf{t}) = C(\mathbf{t})(\tau^C(x, \mathbf{t}))^2, \tag{2.67}$$

where $C(\mathbf{t})$ is a quasi-constant in x (i.e., it is an η -periodic function of x) depending on \mathbf{t} . Below, we shall see that in fact C does not depend on \mathbf{t} .

Finally, we conclude that the factor $w(x, \mathbf{t}; k)$ which enters the $k \rightarrow \infty$ asymptotics of the wave function $\psi(x, \mathbf{t}, -\mathbf{t}; k)$ (see (2.52)) is expressed through the tau function as follows:

$$w(x, \mathbf{t}; k) = \left[1 - k^{-2} \left(\frac{\tau^C(x + \eta, \mathbf{t})\tau^C(x - \eta, \mathbf{t} - [k^{-1}])}{\tau^C(x, \mathbf{t})\tau^C(x, \mathbf{t} - [k^{-1}])} \right)^2 \right]^{1/2} \frac{\tau^C(x, \mathbf{t} - [k^{-1}])}{\tau^C(x, \mathbf{t})}. \tag{2.68}$$

The function $\bar{w}(x, \mathbf{t}; k)$ which enters the $k \rightarrow 0$ asymptotics of the function $\psi(x, \mathbf{t}, -\mathbf{t}; k)$ can be found from the relation (2.55) which reads

$$w(x, \mathbf{t}; k)\bar{w}(x - \eta, \mathbf{t} - [k^{-1}]; k) = g(x, \mathbf{t}; k).$$

After a simple algebra, we obtain

$$\begin{aligned} \bar{w}(x, \mathbf{t}; k) &= \left[1 - k^{-2} \left(\frac{\tau^C(x, \mathbf{t})\tau^C(x + 2\eta, \mathbf{t} + [k^{-1}])}{\tau^C(x + \eta, \mathbf{t})\tau^C(x + \eta, \mathbf{t} + [k^{-1}])} \right)^2 \right]^{1/2} \\ &\quad \times \frac{\tau^C(x + \eta, \mathbf{t})}{\tau^C(x, \mathbf{t})} \frac{\tau^C(x + \eta, \mathbf{t} + [k^{-1}])}{\tau^C(x, \mathbf{t})}. \end{aligned} \tag{2.69}$$

We can represent Eqs. (2.68) and (2.69) in a more suggestive form. Introduce modified “wave functions” $\chi, \bar{\chi}$ which are connected with τ^C in the same way as ψ ,

ψ^* are connected with τ^T :

$$\chi(x, \mathbf{t}; k) = k^{x/\eta} e^{\xi(\mathbf{t},k) - \frac{1}{2}\varphi(x,\mathbf{t})} \frac{\tau^C(x, \mathbf{t} - [k^{-1}])}{\tau^C(x, \mathbf{t})}, \tag{2.70}$$

$$\bar{\chi}(x, \mathbf{t}; k) = k^{-x/\eta} e^{-\xi(\mathbf{t},k)} \frac{\tau^C(x + \eta, \mathbf{t} + [k^{-1}])}{\tau^C(x, \mathbf{t})}, \tag{2.71}$$

where

$$e^{\varphi(x,\mathbf{t})} = r(x, \mathbf{t}) = \left(\frac{\tau^C(x + \eta, \mathbf{t})}{\tau^C(x, \mathbf{t})} \right)^2. \tag{2.72}$$

Recalling (2.65), it is easy to check that Formulas (2.68) and (2.69) are equivalent to

$$\psi(x, \mathbf{t}; k) = e^{\frac{1}{2}\varphi(x,\mathbf{t})} \sqrt{\chi^2(x, \mathbf{t}; k) - \chi^2(x - \eta, \mathbf{t}; k)}, \quad k \rightarrow \infty, \tag{2.73}$$

$$\psi(x, \mathbf{t}; k^{-1}) = e^{\frac{1}{2}\varphi(x,\mathbf{t})} \sqrt{\bar{\chi}^2(x, \mathbf{t}; k) - \bar{\chi}^2(x + \eta, \mathbf{t}; k)}, \quad k \rightarrow \infty. \tag{2.74}$$

These formulas resemble the corresponding formula for the CKP hierarchy (see [24]), with the x -derivative substituted by the difference. \square

We already proved relation (2.67) between τ^C and τ^T . Now, we are going to prove that $C(\mathbf{t}) = C$ is a quasi-constant in x which does not depend on the times, so that τ^C is essentially the square root of τ^T (restricted to the submanifold $\mathbf{t} + \bar{\mathbf{t}} = 0$ and satisfying the ‘‘turning points’’ condition (2.45)).

Theorem 2.2 *The tau functions τ^C and τ^T are related as $\tau^T = C(\tau^C)^2$, where C is a quasi-constant in x , i.e., the tau function of the C -Toda hierarchy is essentially square root of the 2D Toda lattice tau function.*

Proof First of all, we recall that together with (2.68) alternative formulas for $w(x, \mathbf{t}; k)$ through τ^T hold:

$$w(x, \mathbf{t}; k) = \frac{\tau^T(x, \mathbf{t} - [k^{-1}], -\mathbf{t})}{\tau^T(x, \mathbf{t}, -\mathbf{t})} = \frac{\tau^T(x, \mathbf{t}, -\mathbf{t} + [k^{-1}])}{\tau^T(x, \mathbf{t}, -\mathbf{t})} \tag{2.75}$$

(the second equality is due to (2.43)). Substituting them into (2.68) and taking square of both sides, we obtain the relation

$$\begin{aligned} & \left(\frac{\tau^C(x, \mathbf{t} - [k^{-1}])}{\tau^C(x, \mathbf{t})} \right)^2 - k^{-2} \left(\frac{\tau^C(x + \eta, \mathbf{t})\tau^C(x - \eta, \mathbf{t} - [k^{-1}])}{\tau^C(x, \mathbf{t})\tau^C(x, \mathbf{t})} \right)^2 \\ &= \frac{\tau^T(x + \eta, \mathbf{t}, -\mathbf{t})}{\tau^T(x, \mathbf{t}, -\mathbf{t})} \left[\frac{\tau^T(x, \mathbf{t} - [k^{-1}], -\mathbf{t})\tau^T(x, \mathbf{t}, -\mathbf{t} + [k^{-1}])}{\tau^T(x + \eta, \mathbf{t}, -\mathbf{t})\tau^T(x, \mathbf{t}, -\mathbf{t})} \right]. \end{aligned}$$

Now, we are going to use the Hirota–Miwa equation (2.27) for $a = b = k$ which we rewrite in the form

$$\begin{aligned} & \frac{\tau^T(x, \mathbf{t} - [k^{-1}], -\mathbf{t})\tau^T(x, \mathbf{t}, -\mathbf{t} + [k^{-1}])}{\tau^T(x + \eta, \mathbf{t}, -\mathbf{t})\tau^T(x, \mathbf{t}, -\mathbf{t})} \\ &= \frac{\tau^T(x, \mathbf{t} - [k^{-1}], -\mathbf{t} + [k^{-1}])}{\tau^T(x + \eta, \mathbf{t}, -\mathbf{t})} - k^{-2} \frac{\tau^T(x - \eta, \mathbf{t} - [k^{-1}], -\mathbf{t} + [k^{-1}])}{\tau^T(x + \eta, \mathbf{t}, -\mathbf{t})}. \end{aligned}$$

Substituting the right-hand side instead of the brackets [...] in the previous relation, we get

$$\begin{aligned} & \left(\frac{\tau^C(x, \mathbf{t} - [k^{-1}])}{\tau^C(x, \mathbf{t})} \right)^2 - k^{-2} \left(\frac{\tau^C(x + \eta, \mathbf{t})\tau^C(x - \eta, \mathbf{t} - [k^{-1}])}{\tau^C(x, \mathbf{t})\tau^C(x, \mathbf{t})} \right)^2 \\ &= \frac{\tau^T(x, \mathbf{t} - [k^{-1}], -\mathbf{t} + [k^{-1}])}{\tau^T(x, \mathbf{t}, -\mathbf{t})} - k^{-2} \frac{\tau^T(x - \eta, \mathbf{t} - [k^{-1}], -\mathbf{t} + [k^{-1}])\tau^T(x + \eta, \mathbf{t}, -\mathbf{t})}{\tau^T(x, \mathbf{t}, -\mathbf{t})\tau^T(x, \mathbf{t}, -\mathbf{t})}. \end{aligned}$$

Plugging here (2.67), we obtain

$$\left(\frac{C(\mathbf{t} - [k^{-1}])}{C(\mathbf{t})} - 1 \right) \left[\left(\frac{\tau^C(x, \mathbf{t} - [k^{-1}])}{\tau^C(x + \eta, \mathbf{t})} \right)^2 - k^{-2} \left(\frac{\tau^C(x - \eta, \mathbf{t} - [k^{-1}])}{\tau^C(x, \mathbf{t})} \right)^2 \right] = 0.$$

Since the factor in the square brackets is nonzero, we conclude that $C(\mathbf{t} - [k^{-1}]) - C(\mathbf{t}) \equiv 0$ as a power series in k . This implies that $C(\mathbf{t})$ does not depend on \mathbf{t} and, therefore, $\tau^C = \sqrt{\tau^T}$. □

3 Turning points of Ruijsenaars–Schneider model

3.1 Elliptic Ruijsenaars–Schneider model

Here, we collect the main facts on the elliptic Ruijsenaars–Schneider system [15] following the paper [16].

The N -particle elliptic Ruijsenaars–Schneider system (a relativistic extension of the Calogero–Moser system) is a completely integrable model. The canonical Poisson brackets between coordinates and momenta are $\{x_i, p_j\} = \delta_{ij}$. The integrals of motion in involution have the form

$$I_k = \sum_{I \subset \{1, \dots, N\}, |I|=k} \exp\left(\sum_{i \in I} p_i\right) \prod_{i \in I, j \notin I} \frac{\sigma(x_i - x_j + \eta)}{\sigma(x_i - x_j)}, \quad k = 1, \dots, N, \tag{3.1}$$

where $\sigma(x)$ is the Weierstrass σ -function and η is a parameter which has a meaning of the inverse velocity of light. The σ -function with quasi-periods $2\omega_1, 2\omega_2$ such that

$\text{Im}(\omega_2/\omega_1) > 0$ is defined as

$$\sigma(x) = \sigma(x|\omega_1, \omega_2) = x \prod_{s \neq 0} \left(1 - \frac{x}{s}\right) e^{\frac{x}{s} + \frac{x^2}{2s^2}},$$

$$s = 2\omega_1 m_1 + 2\omega_2 m_2 \quad \text{with integer } m_1, m_2.$$

It is connected with the Weierstrass ζ - and \wp -functions by the formulas $\zeta(x) = \sigma'(x)/\sigma(x)$, $\wp(x) = -\zeta'(x) = -\partial_x^2 \log \sigma(x)$. Important particular cases of (3.1) are

$$I_1 = H_1 = \sum_i e^{p_i} \prod_{j \neq i} \frac{\sigma(x_i - x_j + \eta)}{\sigma(x_i - x_j)} \tag{3.2}$$

which is the Hamiltonian H_1 of the chiral Ruijsenaars–Schneider model and

$$I_N = \exp\left(\sum_{i=1}^N p_i\right). \tag{3.3}$$

It is natural to put $I_0 = 1$. In comparison with the paper [16], our formulas differ by the canonical transformation

$$e^{p_i} \rightarrow e^{p_i} \prod_{j \neq i} \left(\frac{\sigma(x_i - x_j + \eta)}{\sigma(x_i - x_j - \eta)}\right)^{1/2}, \quad x_i \rightarrow x_i,$$

which allows one to eliminate square roots in [16].

Let us denote the time variable of the Hamiltonian flow with the Hamiltonian $H = I_1$ by t_1 . The velocities of the particles are

$$\dot{x}_i = \frac{\partial H_1}{\partial p_i} = e^{p_i} \prod_{j \neq i} \frac{\sigma(x_i - x_j + \eta)}{\sigma(x_i - x_j)}, \tag{3.4}$$

where dot means the t_1 -derivative. The Hamiltonian equations $\dot{p}_i = -\partial H_1/\partial x_i$ are equivalent to the following equations of motion:

$$\begin{aligned} \ddot{x}_i &= -\sum_{k \neq i} \dot{x}_i \dot{x}_k \left(\zeta(x_i - x_k + \eta) + \zeta(x_i - x_k - \eta) - 2\zeta(x_i - x_k)\right) \\ &= \sum_{k \neq i} \dot{x}_i \dot{x}_k \frac{\wp'(x_i - x_k)}{\wp(\eta) - \wp(x_i - x_k)}. \end{aligned} \tag{3.5}$$

One can also introduce integrals of motion I_{-k} as

$$I_{-k} = I_N^{-1} I_{N-k} = \sum_{I \subset \{1, \dots, N\}, |I|=k} \exp\left(-\sum_{i \in I} p_i\right) \prod_{i \in I, j \notin I} \frac{\sigma(x_i - x_j - \eta)}{\sigma(x_i - x_j)}. \tag{3.6}$$

In particular,

$$I_{-1} = \sum_i e^{-p_i} \prod_{j \neq i} \frac{\sigma(x_i - x_j - \eta)}{\sigma(x_i - x_j)}. \tag{3.7}$$

It can be easily verified that equations of motion in the time \bar{t}_1 corresponding to the Hamiltonian $\bar{H}_1 = \sigma^2(\eta)I_{-1}$ are the same as (3.5):

$$\overset{\circ}{\ddot{x}}_i = - \sum_{k \neq i} \overset{\circ}{x}_i \overset{\circ}{x}_k \left(\zeta(x_i - x_k + \eta) + \zeta(x_i - x_k - \eta) - 2\zeta(x_i - x_k) \right). \tag{3.8}$$

Here and below, \circ means the \bar{t}_1 -derivative. The velocity $\overset{\circ}{x}_i$ is given by

$$\overset{\circ}{x}_i = \frac{\partial \bar{H}_1}{\partial p_i} = -\sigma^2(\eta)e^{-p_i} \prod_{j \neq i} \frac{\sigma(x_i - x_j - \eta)}{\sigma(x_i - x_j)}. \tag{3.9}$$

Multiplying (3.4) and (3.9), we obtain the important relation between \dot{x}_i and $\overset{\circ}{x}_i$:

$$\dot{x}_i \overset{\circ}{x}_i = -\sigma^2(\eta) \prod_{k \neq i} \frac{\sigma(x_i - x_k + \eta)\sigma(x_i - x_k - \eta)}{\sigma^2(x_i - x_k)} \tag{3.10}$$

(see [17,18]). The physical Hamiltonian of the Ruijsenaars–Schneider model is $H = H_1 + \bar{H}_1$.

3.2 The Ruijsenaars–Schneider model from the 2D Toda lattice

In the paper [17] (see also the review [11]), it was shown that the Ruijsenaars–Schneider dynamics is the same as dynamics of poles of elliptic solutions to the 2D Toda equation in the Toda times t_1, \bar{t}_1 . Later, in [18], this observation was extended to a complete isomorphism between the elliptic Ruijsenaars–Schneider model (with higher Hamiltonian flows) and elliptic solutions to the whole 2D Toda lattice hierarchy.

In terms of the tau function, the 2D Toda equation (the first equation of the hierarchy) reads

$$\partial_t \partial_{\bar{t}} \log \tau(x) = - \frac{\tau(x + \eta)\tau(x - \eta)}{\tau^2(x)}, \tag{3.11}$$

where $t = t_1, \bar{t} = \bar{t}_1$. The tau function for elliptic solutions of the 2D Toda lattice hierarchy has the form

$$\tau(x, \mathbf{t}, \bar{\mathbf{t}}) = \exp\left(- \sum_{k \geq 1} k t_k \bar{t}_k\right) \prod_{i=1}^N \sigma(x - x_i(\mathbf{t}, \bar{\mathbf{t}})). \tag{3.12}$$

The zeros x_i of the tau function are poles of the solution. They are assumed to be all distinct.

One can see that the relation (3.10) is a consequence of the 2D Toda equation. Indeed, (3.10) is obtained from (3.11) with the tau function (3.12) by equating the coefficients at the highest (second order) poles at $x = x_i$ of both sides.

3.3 The Lax matrix and the spectral curve

The equations of motion of the Ruijsenaars–Schneider model admit the Lax representation. The Lax matrix depends on a spectral parameter λ and has the form [17,18]

$$L_{ij}(\lambda) = e^{-(x_i-x_j)\zeta(\lambda)} \dot{x}_i \frac{\sigma(x_i - x_j - \eta + \lambda)}{\sigma(\lambda)\sigma(x_i - x_j - \eta)}, \quad i, j = 1, \dots, N. \quad (3.13)$$

The characteristic polynomial of the Lax matrix is the generating function of the integrals of motion (3.1):

$$\det(zI - L(\lambda)) = \sum_{n=0}^N \frac{\sigma(\lambda - n\eta)}{\sigma(\lambda)\sigma^n(\eta)} I_n z^{N-n} \quad (3.14)$$

(here I is the unity matrix).

The characteristic equation

$$R(z, \lambda) := \det(zI - L(\lambda)) = 0 \quad (3.15)$$

defines a Riemann surface $\tilde{\Gamma}$ which is an N -sheet covering of the λ -plane. Any point of it is $P = (z, \lambda)$, where z, λ are connected by Eq. (3.15). There are N points of the curve above each point λ . It is easy to see from the right-hand side of (3.14) that the Riemann surface $\tilde{\Gamma}$ is invariant under the simultaneous transformations

$$\lambda \mapsto \lambda + 2\omega_\alpha, \quad z \mapsto e^{-2\zeta(\omega_\alpha)\eta} z. \quad (3.16)$$

The factor of $\tilde{\Gamma}$ over the transformations (3.16) is an algebraic curve Γ which covers the elliptic curve with periods $2\omega_\alpha$. It is the spectral curve of the Ruijsenaars–Schneider model. The points $P_\infty = (\infty, 0)$ and $P_0 = (0, N\eta)$ are special. They are marked points of the algebraic curve, where the Baker–Akhiezer function for the elliptic solutions of the 2D Toda lattice hierarchy has essential singularities.

Let us note that the Lax matrix has the form of the elliptic Cauchy matrix times diagonal matrices from the left and from the right. The explicit form of determinant of the elliptic Cauchy matrix is known:

$$\det_{1 \leq i, j \leq N} \left(\frac{\sigma(x_i - y_j + \lambda)}{\sigma(\lambda)\sigma(x_i - y_j)} \right) = \frac{\sigma\left(\lambda + \sum_{i=1}^N (x_i - y_i)\right) \prod_{i < j} \sigma(x_i - x_j)\sigma(y_j - y_i)}{\sigma(\lambda) \prod_{i, j} \sigma(x_i - y_j)}. \quad (3.17)$$

This allows one to obtain an explicit expression for the matrix inverse to the $L(\lambda)$:

$$(L^T(\lambda))_{ij}^{-1} = e^{(x_i-x_j)\zeta(\lambda)} \dot{x}_i^{-1} \frac{\sigma(x_i-x_j-\eta+N\eta-\lambda)\sigma^2(\eta)}{\sigma(N\eta-\lambda)\sigma(x_i-x_j-\eta)} \times \prod_{k \neq i} \frac{\sigma(x_i-x_k-\eta)}{\sigma(x_i-x_k)} \prod_{m \neq i} \frac{\sigma(x_j-x_m+\eta)}{\sigma(x_j-x_m)}. \tag{3.18}$$

Here, L^T is the transposed matrix.

3.4 Turning points

Turning points of the Ruijsenaars–Schneider model are defined by the conditions

$$\dot{x}_i + \overset{\circ}{x}_i = 0 \quad \text{or} \quad (\partial_{t_1} + \partial_{\bar{t}_1})x_i = 0, \quad i = 1, \dots, N. \tag{3.19}$$

They mean that the velocities of all particles in the physical Ruijsenaars–Schneider model with the Hamiltonian $H = H_1 + \bar{H}_1$ are equal to zero. From Eq. (3.10), we see that this is equivalent to

$$\begin{aligned} \dot{x}_i &= \sigma(\eta) \prod_{k \neq i} \frac{(\sigma(x_i-x_k+\eta)\sigma(x_i-x_k-\eta))^{1/2}}{\sigma(x_i-x_k)} \\ &= \sigma^N(\eta) \prod_{k \neq i} \sqrt{\wp(\eta) - \wp(x_i-x_k)} \end{aligned} \tag{3.20}$$

or

$$e^{p_i} = \sigma(\eta) \prod_{j \neq i} \left(\frac{\sigma(x_i-x_j-\eta)}{\sigma(x_i-x_j+\eta)} \right)^{1/2}. \tag{3.21}$$

The turning points form an N -dimensional submanifold $\mathcal{T} \subset \mathcal{P}$ of the $2N$ -dimensional phase space \mathcal{P} .

Proposition 3.1 *The Hamiltonian flow $\partial_{T_1} = \partial_{t_1} - \partial_{\bar{t}_1}$ with the Hamiltonian $\bar{H} = H_1 - \bar{H}_1$ preserves the submanifold \mathcal{T} .*

Proof The corresponding time variable will be denoted as $T_1 = \frac{1}{2}(t_1 - \bar{t}_1)$. We have

$$*_i \frac{\partial \bar{H}}{\partial p_i} = 2\sigma(\eta) \prod_{k \neq i} \frac{(\sigma(x_i-x_k+\eta)\sigma(x_i-x_k-\eta))^{1/2}}{\sigma(x_i-x_k)} \quad \text{on } \mathcal{T}, \tag{3.22}$$

where star means the T_1 -derivative. Taking the T_1 -derivative of (3.21), we get

$$*_i p_i = \frac{1}{2} \sum_{j \neq i} (*_i x_i - *_j x_j) \left(\zeta(x_i-x_j-\eta) - \zeta(x_i-x_j+\eta) \right). \tag{3.23}$$

At the same time,

$$\begin{aligned}
 {}^*P_i = & -\frac{\partial \tilde{H}}{\partial x_i} = -e^{p_i} \prod_{j \neq i} \frac{\sigma(x_i - x_j + \eta)}{\sigma(x_i - x_j)} \sum_{l \neq i} \left(\zeta(x_i - x_l + \eta) - \zeta(x_i - x_l) \right) \\
 & + \sigma^2(\eta) e^{-p_i} \prod_{j \neq i} \frac{\sigma(x_i - x_j - \eta)}{\sigma(x_i - x_j)} \sum_{l \neq i} \left(\zeta(x_i - x_l - \eta) - \zeta(x_i - x_l) \right) \\
 & + \sum_{l \neq i} e^{p_l} \prod_{j \neq l} \frac{\sigma(x_l - x_j + \eta)}{\sigma(x_l - x_j)} \left(\zeta(x_l - x_i + \eta) - \zeta(x_l - x_i) \right) \\
 & - \sigma^2(\eta) \sum_{l \neq i} e^{-p_l} \prod_{j \neq l} \frac{\sigma(x_l - x_j - \eta)}{\sigma(x_l - x_j)} \left(\zeta(x_l - x_i - \eta) - \zeta(x_l - x_i) \right).
 \end{aligned}$$

Plugging here the turning point condition (3.21) and using (3.22), we obtain (3.23). This means that the submanifold \mathcal{T} is indeed invariant under the T_1 -flow. \square

Now, we are going to prove that for any turning point the spectral curve Γ admits a holomorphic involution.

Theorem 3.1 *For any turning point, the spectral curve Γ admits the holomorphic involution*

$$\iota : (z, \lambda) \rightarrow (z^{-1}, N\eta - \lambda). \tag{3.24}$$

Proof Substituting (3.20) into (3.13) and (3.18), we see that

$$(L^T(\lambda))^{-1} = UL(N\eta - \lambda)U^{-1}, \tag{3.25}$$

where $U = \text{diag}(U_1, \dots, U_N)$ is the diagonal matrix with

$$U_i = e^{x_i(\zeta(\lambda) + \zeta(N\eta - \lambda))} \prod_{k \neq i} \frac{\sigma(x_i - x_k)}{\sigma(x_i - x_k + \eta)}. \tag{3.26}$$

Therefore, the spectral curve (3.15) has the holomorphic involution (3.24). \square

Note that the involution interchanges the two marked points: $\iota P_\infty = P_0, \iota P_0 = P_\infty$. The following proposition characterizes fixed points of the involution.

Proposition 3.2 *The involution ι has two fixed points for even N and four fixed points for odd N .*

Proof The fixed points may lie above points λ_* such that $\lambda_* = N\eta - \lambda_*$ modulo the lattice with periods $2\omega_\alpha$, i.e., $\lambda_* = \frac{1}{2}N\eta - \omega$, where ω is either 0 or one of the three half-periods. Substituting this into the equation of the spectral curve (3.14) and taking into account that for turning points it holds $I_k = I_{N-k}$, we conclude that for even N the fixed points are $(\pm 1, \frac{1}{2}N\eta)$, while for odd N the fixed points are $(1, \frac{1}{2}N\eta)$ and three points $(-e^{-\zeta(\omega)\eta}, \frac{1}{2}N\eta - \omega)$ for the three half-periods ω . \square

We have shown that from the condition on the turning points it follows that the spectral curve has a holomorphic involution with fixed points. Now, we are going to prove the inverse statement: The involution of the curve (which can be not necessarily the spectral curve of the Ruijsenaars–Schneider model) having fixed points implies the turning points condition for zeros of the tau function corresponding to the algebraic geometrical solution constructed from the curve according to the general construction of quasi-periodic (algebraic geometrical) solutions [27,28]. Quasi-periodic solutions to the Toda lattice equation were constructed in [29]. The algebraic geometrical data include an algebraic curve Γ of genus g with two marked points P_0, P_∞ , local parameters near the marked points and an effective divisor \mathcal{D} of degree g on Γ . Algebraic geometrical solutions of the constrained Toda hierarchy were recently constructed in [30].

Theorem 3.2 *Let Γ be an algebraic curve with holomorphic involution ι which has fixed points and two marked points P_∞, P_0 such that $P_0 = \iota P_\infty$. Let k^{-1} be a local parameter in the vicinity of P_∞ ($k^{-1}(P_\infty) = 0$), we assume that the local parameter in the vicinity of P_0 is k ($k(P_0) = 0$), so that $\iota(k) = k^{-1}$. Besides, we fix an effective divisor \mathcal{D} of degree g on Γ such that*

$$\mathcal{D} + \iota\mathcal{D} = \mathcal{K} + P_0 + P_\infty, \tag{3.27}$$

where \mathcal{K} is the canonical class. Then, zeros of the tau function of the solution to the 2D Toda lattice constructed from these algebraic geometrical data satisfy the turning points condition.

Proof Let $\psi(x; P) = \psi(x, t, \bar{t}; P)$ be the Baker–Akhiezer function on the curve Γ (P is a point on Γ). It has simple poles at the points of the divisor \mathcal{D} . Its behavior in the vicinity of the marked points is

$$\psi(x; P) = \begin{cases} k^{x/\eta} e^{kt} \left(1 + \sum_{s \geq 1} \xi_s(x) k^{-s} \right), & P \rightarrow P_\infty \quad (k \rightarrow \infty), \\ e^{\varphi(x)} k^{x/\eta} e^{k^{-1}\bar{t}} \left(1 + \sum_{s \geq 1} \chi_s(x) k^s \right), & P \rightarrow P_0 \quad (k \rightarrow 0). \end{cases} \tag{3.28}$$

The function $\varphi(x)$ is expressed through the tau function as in (2.28). The Baker–Akhiezer function satisfies the linear equation

$$\partial_t \psi(x; P) = \psi(x + \eta; P) + v(x) \psi(x; P), \tag{3.29}$$

where

$$v(x) = \partial_t \log \frac{\tau(x + \eta)}{\tau(x)} = \dot{\varphi}(x). \tag{3.30}$$

Substituting (3.28) into (2.13), we obtain, in the limit $k \rightarrow \infty$:

$$v(x) = \xi_1(x) - \xi_1(x + \eta), \quad \xi_1(x) = -\partial_t \log \tau(x). \tag{3.31}$$

The dual Baker–Akhiezer function $\psi^*(x; P)$ satisfies the equation

$$-\partial_t \psi^*(x; P) = \psi^*(x - \eta; P) + v(x)\psi^*(x; P). \tag{3.32}$$

Its behavior in the vicinity of the marked points is

$$\psi^*(x; P) = \begin{cases} k^{-x/\eta} e^{-kt} \left(1 + \sum_{s \geq 1} \xi_s^*(x) k^{-s} \right), & P \rightarrow P_\infty \quad (k \rightarrow \infty), \\ e^{-\varphi(x)} k^{-x/\eta} e^{-k^{-1}\bar{t}} \left(1 + \sum_{s \geq 1} \chi_s^*(x) k^s \right), & P \rightarrow P_0 \quad (k \rightarrow 0). \end{cases} \tag{3.33}$$

Substituting (3.33) into (3.29), we obtain $v(x) = \xi_1^*(x) - \xi_1^*(x + \eta)$. Comparing with (3.31), we conclude that

$$\xi_1^*(x) = -\xi_1(x + \eta). \tag{3.34}$$

On the curve with involution such that $P_0 = \iota P_\infty$, we can consider the function

$$\psi^\iota(x; P) = \psi(x; \iota P). \tag{3.35}$$

The condition (3.27) imposed on the divisor \mathcal{D} and the behavior of ψ^ι near the marked points imply (due to uniqueness of the Baker–Akhiezer function) that we can identify

$$\psi^*(x, t, \bar{t}; P) = e^{-\varphi(x)} \psi^\iota(x, t, \bar{t}; P) \Big|_{t+\bar{t}=0}, \tag{3.36}$$

whence

$$\chi_s(x) = \xi_s^*(x) \tag{3.37}$$

and the behavior of the function ψ^ι near P_∞ is

$$\psi^\iota(x; P) = e^{\varphi(x)} k^{-x/\eta} e^{k\bar{t}} \left(1 + \sum_{s \geq 1} \xi_s^*(x) k^{-s} \right), \quad k \rightarrow \infty. \tag{3.38}$$

Substituting this into the linear equation (2.13) as $k \rightarrow \infty$, we obtain, in the order k^{-1} :

$$\dot{\xi}_1^*(x) = e^{\varphi(x+\eta) - \varphi(x)}. \tag{3.39}$$

Equation (3.34) allows one to rewrite this relation as

$$\dot{\xi}_1(x + \eta) = -e^{\varphi(x+\eta)-\varphi(x)}, \tag{3.40}$$

or, using (2.28) and (3.31),

$$\partial_t^2 \log \tau(x) = \frac{\tau(x + \eta)\tau(x - \eta)}{\tau^2(x)} \quad \text{at } t + \bar{t} = 0. \tag{3.41}$$

This is the turning points condition in terms of the tau function. Writing it as

$$\frac{\ddot{\tau}(x)}{\tau(x)} - \left(\frac{\dot{\tau}(x)}{\tau(x)}\right)^2 = \frac{\tau(x + \eta)\tau(x - \eta)}{\tau^2(x)} \tag{3.42}$$

and comparing the leading singularities of both sides at $x = x_i$, where x_i is any zero of the tau function, we obtain the turning points condition (3.20). □

Comparing (3.41) with the 2D Toda Eq. (3.11), we can represent it in the form

$$(\partial_{t_1} + \partial_{\bar{t}_1})\partial_{t_1} \log \tau(x) = 0 \quad \text{or} \quad (\partial_{t_1} + \partial_{\bar{t}_1})\xi_1(x) = 0 \text{ at } t_1 + \bar{t}_1 = 0 \tag{3.43}$$

which agrees with (2.44). Moreover, from (3.36) it follows that the solution constructed according to the data from Theorem 3.2 is a solution to the whole C-Toda hierarchy. Indeed, (3.36) is equivalent to (2.41), which, being written in terms of the tau function, yields (2.43) and (2.45).

4 Conclusion

The main result of this paper is introduction of a new integrable hierarchy which we have called the constrained Toda hierarchy or simply C-Toda hierarchy. It is obtained from the 2D Toda lattice by imposing a constraint on the two Lax operators of the latter. The constraint is invariant with respect to only a “half” of the hierarchical time flows, so the other half of the time variables should be “frozen” (fixed to zero values). The story is to much extent analogous to the way in which the CKP hierarchy is obtained from the KP hierarchy. The analogy also manifests itself in the construction of the tau function of the C-Toda hierarchy.

A related result concerns elliptic solutions to the C-Toda hierarchy and their relation with the elliptic Ruijsenaars–Schneider model. We have shown that zeros of the tau function of the elliptic solutions move as Ruijsenaars–Schneider particles restricted to a half-dimensional submanifold in the phase space corresponding to *turning points*. In this respect, too, the situation is analogous to the CKP case, where the dynamics of poles of elliptic solutions is the Calogero–Moser dynamics restricted to the submanifold of turning points, i.e., points with zero momenta, as is shown in [24].

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