# FOLIATIONS ON THE MODULI SPACE OF CURVES, VANISHING IN COHOMOLOGY, AND CALOGERO-MOSER CURVES

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Abstract. Using meromorphic differentials with real periods, we show that a certain tautological homology class on the moduli space $\mathcal{M}_g$ of smooth algebraic curves of genus $g$ vanishes. The vanishing of the entire tautological ring for degree $g-1$ and higher, part of Faber's conjecture [7], is known in both homology and Chow — it was proven by Looijenga [34], Ionel [18], and Graber-Vakil [13], and the class that we show vanishes is just one such tautological class. However, our approach, motivated by the Whitham perturbation theory of the soliton equations, is completely new, elementary in the sense that no techniques beyond elementary complex analysis are used, and also leads to a natural non-speciality conjecture, which would imply many more vanishing results and relations among tautological classes.

In the course of the proof we define and study foliations of $\mathcal{M}_g$ constructed using periods of meromorphic differentials, in a way providing for meromorphic differentials a theory similar to that developed for abelian differentials by Kontsevich and Zorich [19, 20]. In our setting we can construct local coordinates near any point of the moduli space, while for abelian differentials only coordinates along the strata with a fixed configuration of zeroes are known. The results we obtain are of independent interest for thestudy of singularities of solutions of the Whitham equations.

## 1. Introduction

In this paper we use the ideas of Whitham theory, in particular meromorphic differentials with real periods (which we used in [15] to give a new direct proof of the theorem of Diaz [4]), to construct and study certain foliations of the moduli and Teichmüller spaces of Riemann surfaces together with a meromorphic differential, and to obtain a vanishing result in the cohomology ring of the moduli spaces $\mathcal{M}_{g,n}$

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of Riemann surfaces of genus $g$ with $n$ punctures (aka algebraic curves of genus $g$ with $n$ marked points).

The vanishing result we prove is theorem 2.8 — that a certain tautological class vanishes. This result is not new — it is a corollary of Faber's [7] vanishing conjecture for the tautological ring, proven, both in cohomology and Chow, by Looijenga [34], Ionel [18], and Graber-Vakil [13]. However, our investigations lead us to a non-speciality conjecture (conjecture 3.18): the statement that the divisor of common zeros of two meromorphic differentials of the second kind (with a single double pole at a fixed point) and all periods real is never a special divisor. If true, this conjecture would imply the vanishing of many more tautological classes covered by Faber's conjecture, but would also yield further geometric vanishings, which may be new — see remark 3.19. The vanishing results are primarily algebro-geometric in nature; the constructions with meromorphic differentials that we use are in a way a generalization of those considered in Teichmüller theory for abelian differentials; and the motivation for our work comes from the Whitham perturbation theory in integrable systems — we thus give an introduction and motivation for our work from these three perspectives.

**Algebraic geometry viewpoint — the result.** The *tautological ring* of the moduli space of curves has been a central object of study in recent years. We recall that an intrinsic definition (due to Faber and Pandharipande [8]) of the tautological ring is as the minimal system of subalgebras of the Chow rings $R^*(\overline{\mathcal{M}}_{g,n}) \subset CH^*(\overline{\mathcal{M}}_{g,n})$ for all $g$ and $n$ that is closed under pushforward and pullback under all forgetful maps $\overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ and all gluing maps $\overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} \to \overline{\mathcal{M}}_{g_1+g_2,n_1+n_2}$ (one can similarly define the homology tautological rings $RH^*(\mathcal{M}_{g,n}) \subset H^*(\overline{\mathcal{M}}_{g,n})$, and this is the setup in which we work).

Much of the recent study of the tautological ring has been motivated by Faber's conjectures [7] on the tautological ring. In particular Faber conjectured that the tautological ring $R^*(\mathcal{M}_{g,n})$ looks like the homology ring of a compact complex manifold of dimension $g-2+n$, i.e. that it is a Gorenstein ring with socle in dimension $g-2+n$.

Some parts of this statement have been proven, we refer to [38] for an overview of the status. In particular the vanishing part of Faber's conjecture — that tautological classes on $\mathcal{M}_{g,n}$ of degree higher than $g-2+n+\delta_{0,n}$ vanish — has been proven in various ways in different contexts by Looijenga [34], Ionel [18], Graber-Vakil [13]. From the point of view of algebraic geometry, the main result that we prove is theorem 2.8: that the tautological class $\prod(\psi_0^2+\psi_i^2)$ on $\mathcal{M}_{g,1+g}$ vanishes. This result is an easy corollary of a general conjecture 3.18, which, if proven, would also yield further vanishing results, some of which do not seem to be immediate corollaries of Faber's vanishing conjectures.

While the tautological rings are defined to be subrings of the Chow, one can also ask similar questions for homology. In particular one can obtain bounds for the homological and homotopical dimension of $\mathcal{M}_{g,n}$ (see Mondello [36] for results and more references). One approach to proving such vanishing results was proposed by Arbarello [1] who constructed for $\mathcal{M}_g$ a stratification that is conjectured to be affine. More generally, Roth and Vakil [37] studied possible approaches to obtaining affine stratifications of $\mathcal{M}_g$. One can ask for an even stronger property: that $\mathcal{M}_g$ admits a cover by at most $g-1$ affine open sets. The existence of such covers is conjectured by Hain and Looijenga [16]; for genus up to 5 such stratifications were recently constructed by Fontanari and Looijenga [10], and covers — by Fontanari and Pascolutti [11], but no affine stratificaion, and not even a conjectural cover, is known beyond genus 5.

The approach we take here is to relax the condition of having an affine stratification, and construct what can be thought of roughly as an "affine foliation" of $\mathcal{M}_{g,1}$ (which is tangentially complex, and transverse real-analytic, i.e. each leaf is a complex submanifold, while the set of leaves is locally parameterized by real-analytic coordinates). A variation of this construction was used in our work [15] to prove Diaz' [4] bound of $g-2$ for the dimension of complete subvarieties of $\mathcal{M}_g$. In this paper we study the leaves of our foliation in more detail, describing their local structure and defining coordinates on them. This is an analog of a construction of a stratification of the moduli of curves with a holomorphic differential, and some closed leaves of our foliation correspond to spectral curves of the Calogero-Moser integrable system. Our proof, though motivated by integrable systems, only uses elementary complex analysis on Riemann surfaces. The possibility of using our techniques to try to construct complete subvarieties of $\mathcal{M}_{g,n}$ or interesting cycles representing non-zero cohomology classes remains open.

**Teichmüller theory viewpoint — the construction.** In the study of geometry and dynamics of the Teichmüller space the moduli space of pairs consisting of a Riemann surfaces $\Gamma$ (we follow the integrable systems convention for this notation) and an abelian differential $\omega$ on it has recently played a fundamental role. We denote this space $\mathcal{H}_g$ — it is the total space of the rank $g$ Hodge bundle — the complex vector bundle over $\mathcal{M}_g$, with a fiber over a point $\Gamma$ being the space $H^{1,0}(\Gamma,\mathbb{C})$ of abelian differentials on $\Gamma$ (with the 0-section removed if the differential is required to be non-zero). The space $\mathcal{H}_g$ is naturally stratified, with the open strata $\mathcal{H}_g(m) = \mathcal{H}_g(m_1, \ldots, m_n)$ for $m_1+\ldots+m_n = 2g-2$ being the loci where the multiplicities of the zeroes of the differential are equal to $(m_1, \ldots, m_n)$ (so that in particular the stratum $\mathcal{H}_g(1, \ldots, 1)$ is open in $\mathcal{H}_g$). The structure of these strata was studied by Kontsevich and Zorich in their foundational works [19, 20].

It is well-known in Teichmüller theory [6, 17] that if $p_1, \ldots, p_n$ denote the zeroes of an abelian differential $\omega$, then the periods of $\omega$ over a basis of relative homology $H_1(\Gamma \backslash \{p_1, \ldots, p_n\}, \mathbb{Z})$ give local coordinates on the stratum $\mathcal{H}_g(m)$. Among these periods, one commonly distinguishes the *absolute periods*, i.e. the integrals of $\omega$ over $H_1(\Gamma, \mathbb{Z})$, and the *relative periods*, a basis for which is given for example by $\int_{p_1}^{p_i} \omega$ for $i = 2 \ldots n$ (a more symmetric choice is to take $\int_{p}^{p_i} \omega$ for $i = 1 \ldots n$, choosing $p$ in such a way that the sum of the $n$ relative periods is zero). The foliations defined using these periods have been considered in various works, including [35]. The action of $SL(2, \mathbb{R})$ on $\mathcal{H}_g$, defined by acting on the flat structure defined on $\Gamma$ by $\omega$, has been a central object of study in recent research on dynamics and geometry of the Teichmüller space — we refer to the survey [39] for a more detailed discussion. One known difficulty in studying the space $\mathcal{H}_g$ is that is it not known how to naturally extend the local coordinates on $\mathcal{H}_g(m)$ to local coordinates on $\mathcal{H}_g$ (the difficulty is that the number of zeroes of the differential varies, and it is not clear how to account for this).

The main tool that we use in this paper are meromorphic differentials with prescribed singularities. While in our previous work [15] we used meromorphic differentials *of the third kind* — with two simple poles, in this paper we use differentials *of the second kind* — with one double pole (and thus with no residue). Generalizing the case of abelian differentials, one can consider instead of the total space of the Hodge bundle the total space of the bundle over $\mathcal{M}_{g,1}$ of meromorphic differentials with a second order pole. This means that over a point $(\Gamma, p_0) \in \mathcal{M}_{g,1}$ the fiber is the space of meromorphic differentials $\Psi$ having a double pole at $p_0$, and holomorphic on $\Gamma \setminus \{p_0\}$. Algebro-geometrically, this means that $\Psi \in H^0(\Gamma, K_\Gamma + 2p_0)$. We denote by $\Omega\mathcal{M}_{g,1}$ the total space of this bundle — it is a complex rank $g+1$ vector bundle over $\mathcal{M}_{g,1}$ (it would be interesting to try to understand a suitable action of subgroups of $SL(2, \mathbb{R})$ on $\Omega\mathcal{M}_{g,1}$, but this is beyond the scope of the current work).

Similarly to the case of abelian differentials, one can study the strata in $\Omega\mathcal{M}_{g,1}$ with prescribed configurations of zeroes, on which the absolute and relative periods (now together with extra parameters corresponding to a suitably normalized singular part of the expansion of $\Psi$ near $p_0$ — see next section for the precise definition) would serve as local coordinates. While in some ways the situation of a meromorphic differential is a generalization of that of an abelian (holomorphic) differential, there is one striking difference, which in the framework of the finite-gap theory was observed in [23] (and perhaps first, in the case of differentials of the third kind, by Maxwell in his study of electromagnetism).

Indeed, given any prescribed singular part of a meromorphic differential of the second kind at $p_0$ (or more generally, any prescribed collection of singular parts at poles, with all residues imaginary and summing to zero), on *any* Riemann surface $\Gamma$ there *exists a unique* differential of the second kind with this singular part *and all (absolute) periods real.* This is to say that within $\Omega\mathcal{M}_{g,1}$ the subset where all absolute periods are real maps 1-to-1 onto the $\mathbb{C}^*$ bundle over $\mathcal{M}_{g,1}$ corresponding to prescribing the singular part of $\Psi$. We refer to [15] for more details and discussion. This situation is in stark contrast to that of abelian differentials: all imaginary parts of absolute periods of an abelian differential are zero if and only if the differential is identically zero. In this paper, we will work with a pair of differentials with real periods, with a single double pole, where their singular parts differ by multiplying by $\sqrt{-1}$.

While absolute and relative periods give local coordinates on $\mathcal{H}_g$ and $\Omega\mathcal{M}_{g,1}$, only for the case of meromorphic differentials it appears possible to use the above to construct local coordinates on the moduli space $\mathcal{M}_{g,1}$ itself. Fixing the values of the (absolute) real periods of such a meromorphic differential defines a foliation of $\mathcal{M}_{g,1}$ (not just of the Teichmüller space, which would be much easier); the leaves of this "big" foliation are known to be smooth. Taking two linearly independent meromorphic differentials with real periods and fixing the absolute periods of both, one obtains a "small" foliation. Motivated by interpreting an everywhere dense set of the leaves of the "small" foliation as normalizations of spectral curves of the Calogero-Moser system (see below), we make conjecture 3.14, that *all* small leaves are in fact smooth. It turns out that potential singularities of the small leaves are controlled by the common zeroes of the two differentials, and, if true, this conjecture would imply that the divisor of common zeroes is nonspecial, conjecture 3.18 (one could attempt to apply similar techniques to study singularities of plane curves; this is currently under investigation). While we cannot prove either conjecture, we use our description of the tangent spaces to the leaves of foliations, and their intersections with the common zero sets, together with the results of [15] on degeneration of differentials with real periods, to obtain a bound on the number of common zeroes, by a degeneration transverse to the leaves. This implies the vanishing result for the homology classes these loci represent.

**Remark 1.1.** Along the way of our study of meromorphic differentials we use local real-analytic coordinates $s_{i,j}$ (introduced in [31], see formula 9 below) near *any* point of $\mathcal{M}_{g,1}$, irrespective of the configuration of zeroes of the meromorphic differential. It appears that this construction can also be made to work for abelian differentials, thus solving the problem of constructing local coordinates on $\mathcal{H}_g$ near a point on any stratum $\mathcal{H}_g(m)$ (not just along the stratum, but on the whole space).

**Remark 1.2.** It would be interesting to see if our methods can be applied to study the local structure of the strata $\mathcal{H}_g(m)$ and their degenerations, or perhaps of the leaves of the corresponding foliation by absolute periods of abelian differentials. While algebro-geometrically one can describe the local deformation space to $\mathcal{H}_g(m)$, it seems that the tangent spaces to the leaves of the foliations, and the local structure near the boundary of $\mathcal{H}_g(m)$ where the zeroes come together are more subtle and have not yet been completely explored — see [21] for recent results in this direction.

**Integrable systems viewpoint — the motivation.** Meromorphic differentials with real periods, on which our approach is based, are central to various constructions in Whitham theory, are one of the main ingredients in string topology, and were used in our previous work [15] to give a proof of the Diaz' bound. From the point of view of integrable systems what we do can be interpreted as studying singularities of exact solutions of the Whitham equations. Indeed, these solutions are defined (implicitly) by an explicit set of functions on the moduli space of curves with two differentials. The set of such functions, called "Whitham times", is degenerate exactly on the loci where these differentials have common zeros. Therefore, our study of the tangent spaces to the leaves of foliations at these loci is important for the further study of possible types of singularities of solutions of the Whitham equations. This problem is of its own interest and we plan to address it elsewhere.

**The structure of the paper, and of the proof.** In section 2 we review (following our earlier work [15]) the constructions of meromorphic differentials with real periods. We then construct (real-analytic) loci $\mathcal{D}_n^\nu \subset \mathcal{M}_{g,n+1}$ obtained by prescribing the multiplicities of common zeroes at $p_1, \ldots, p_n$ of a pair of $\mathbb{R}$-linearly-independent differentials $\Psi_1$ and $\Psi_2$ of the second kind with double poles at $p_0$ and all periods real. In theorem 2.6 we compute the cohomology classes represented by these loci, and our main vanishing result, theorem 2.8, proven in section 4, is that one of these loci is empty, thus giving a zero homology class.

We believe that much more is actually true, and make conjectures about the precise geometry of the situation. In section 3 we define, extending the ideas of [15], two real-analytic foliations of the moduli spaces, with smooth complex leaves denoted $\mathcal{L}$ and $\mathcal{L}'$, corresponding to fixing (absolute) periods of $\Psi_1$ and $\Psi_2$, respectively. The main results are propositions 3.5 and 3.8, computing the tangent spaces to the leaves of the foliation in terms of certain differentials on a cut Riemann surface (equivalently, on the universal cover), in a way reminiscent of Rauch's variational formula, and proving that the loci $\mathcal{D}_n^\nu$, if non-empty, intersect any leaf $\mathcal{L}$ (or $\mathcal{L}'$) transversely.

The leaves $\mathcal{L}$ and $\mathcal{L}'$ of the two foliations are smooth, but we also make conjecture 3.14: that the leaves of the intersection of the two foliations $\mathcal{S} := \mathcal{L} \cap \mathcal{L}'$ are also smooth (and also transverse to the common zero sets $\mathcal{D}_n^\nu$ of the differentials). It turns out that this conjectured smoothness is equivalent to conjecture 3.18, that the divisor of common zeroes of $\Psi_1$ and $\Psi_2$ is never special. While we do not prove either conjecture, in section 4 we use the description of the tangent space to prove, by degeneration, that the differentials $\Psi_1$ and $\Psi_2$ cannot have $g$ common zeroes, which implies the vanishing result 2.8. The description of the behavior under degeneration of the zeroes of meromorphic differentials with real periods that appears here may be of independent interest.

In section 5 we review the theory of the Calogero-Moser integrable system, for motivation, and obtaining new results that are not directly used in the proof of our vanishing theorem. The main result we prove in this section is proposition 5.9, showing that a certain dense set of algebraic leaves of the small foliation arise precisely as the loci of normalizations of spectral curves of the Calogero-Moser system.

## 2. Meromorphic differentials of the second kind, and loci representing products of $\psi$ classes

In this section we recall the construction of meromorphic differentials with prescribed singularities and all periods being real. From the point of view of integrable systems, these constructions are motivated by the Whitham perturbation theory of integrable systems, see [24, 25, 31, 32]. We follow the notation of [15], where many of these constructions were used, specializing from the case of general meromorphic differentials to differentials of the second kind. Algebro-geometrically, this means we are considering sections of $H^0(\Gamma, K_\Gamma + 2p_0)$ with prescribed jet of the singularity at $p_0$, with all periods real.

**Definition 2.1.** Let $\mathcal{M}_{g,1+n}(1, 0, \ldots, 0)$ denote the moduli space of smooth algebraic curves $\Gamma$ of genus $g$ with labeled marked distinct points $p_0, \ldots, p_n \in \Gamma$, together with a 1-jet of a local coordinate at $p_0$. We recall that explicitly choosing a 1-jet of a local coordinate near $p_0$ means on a small neighborhood $U \supset p_0$ choosing a coordinate $z : U \to \mathbb{C}$ such that $z(p_0) = 0$, and identifying two local coordinates $z$ and $z'$ if $z' = z + O(z^2)$. Notice this identification is equivalent to identifying if $\frac{dz'}{z'^2} = \frac{dz}{z^2} + O(1)dz$, and thus the chosen 1-jet of a local coordinate determines uniquely a "singular part with a double pole", which is to say a local form $dz/z^2$ of a meromorphic differential near $p_0$.

Algebro-geometrically such a 1-jet of a local coordinate can be thought of as a non-zero cotangent vector, $z \in T^*_{p_0}\Gamma \setminus \{0\}$, and the space $\mathcal{M}_{g,1+n}(1, 0, \ldots, 0)$ is then the total space of the cotangent bundle to $\mathcal{M}_{g,1+n}$ at the first point, with the zero-section removed.

**Definition 2.2.** Similarly, define $\mathcal{M}_{g,1+n}(2, 0, \ldots, 0)$ by fixing a 2-jet of a local coordinate: identifying two local coordinates $z$ and $z'$ if $z' = z + O(z^3)$. The space $\mathcal{M}_{g,1+n}(2, 0, \ldots, 0)$ is then the total space of an affine bundle of complex dimension one over $\mathcal{M}_{g,1+n}(1, 0, \ldots, 0)$. This is just a way of saying that a given 2-jet of a local coordinate defines a 1-jet of a coordinate, and that for a given 1-jet of a local coordinate there are $\mathbb{C}$ different choices for its extension to a 2-jet (in fact the bundle of 2-jets is an extension of the cotangent bundle by the cotangent bundle, but we will not need this).

As it was pointed out in [23] (see [15] for details), for *any* Riemann surface with marked points, and for *any collection of singular parts* at these marked points, with all residues imaginary, and summing to zero, there exists a *unique* meromorphic differential with these singular parts and all periods real. Since for differentials of the second kind there can be no residue at the only pole, we get

**Definition 2.3.** For any $(\Gamma, p_0, \ldots, p_n, z) \in \mathcal{M}_{g,1+n}(1, 0, \ldots, 0)$ let $\Psi_1, \Psi_2 \in H^0(\Gamma, K_\Gamma + 2p_0)$ be the unique differentials of the second kind such that all their periods are real and their singular parts at $p_0$ are $\frac{dz}{z^2}$ and $i\frac{dz}{z^2}$, respectively.

The existence and uniqueness mean that $\Psi_i$ are two real-analytic nowhere vanishing (single-valued) sections of the total space of the bundle of differentials of the second kind over $\mathcal{M}_{g,1+n}(1, 0, \ldots, 0)$. In what follows, for technical reasons we will need to work primarily on $\mathcal{M}_{g,1+n}(2, 0, \ldots, 0)$, and by abuse of notation we will also denote $\Psi_i$ the corresponding differentials of the second kind over it. We will now use $\Psi_1$ and $\Psi_2$ to define the main tools of our study.

**Definition 2.4.** For $\nu = \{\nu_1, \ldots, \nu_n\} \in (\mathbb{Z}_+)^{\times n}$ we let $E := \sum \nu_i p_i$ be the corresponding divisor on $\Gamma$. We then denote $\widehat{\mathcal{D}}_n^\nu \subset \mathcal{M}_{g,1+n}(2, 0, \ldots, 0)$ the locus where $\Psi_1$ and $\Psi_2$ both vanish at each $p_j$ to order at least $\nu_j$ (i.e. $\Psi_j \in H^0(K_\Gamma + 2p_0 - E)$), and let $\widehat{\mathcal{D}}^\nu \subset \mathcal{M}_{g,1}(2)$ be the image of $\widehat{\mathcal{D}}_n^\nu$ under the forgetful map. In words, $\widehat{\mathcal{D}}^\nu$ is the locus where $\Psi_1$ and $\Psi_2$ have at least $n$ *distinct* common zeros on $\Gamma$ of common multiplicities $\nu_1, \ldots, \nu_n$.

Note that the space of meromorphic differentials of the second kind, with a second order pole at $p_0$ and all periods real, is a two-dimensional real vector space, with an $\mathbb{R}$-basis given by $\Psi_1$ and $\Psi_2$ (to express any other such differential as their linear combination, it is sufficient to express the singular part as an $\mathbb{R}$-linear combination of $dz/z^2$ and $idz/z^2$). Thus if $\Psi_1$ and $\Psi_2$ both vanish on $E$, so does any differential with a double pole at $p_0$ and all periods real.

It follows that the defining conditions for $\widehat{\mathcal{D}}_n^\nu$ are independent of the choice of the local coordinate at $p_0$, and thus $\widehat{\mathcal{D}}_n^\nu$ is the preimage of some locus $\mathcal{D}_n^\nu \subset \mathcal{M}_{g,1+n}$. Similarly $\widehat{\mathcal{D}}^\nu$ is the preimage of some locus $\mathcal{D}^\nu \subset \mathcal{M}_{g,1}$. To simplify notation we will drop the $\nu$ if all multiplicities are equal to one.

**Remark 2.5.** It is natural to ask what the dimensions of these loci of common zeroes are. Since $\Psi_j$ depend on the moduli real-analytically, all of the loci we defined are real-analytic and not complex, and thus though we know the defining equations, we have no a priori lower or upper bounds on their dimensions. Below we will in fact show that $\widehat{\mathcal{D}}_n$ for $n < g$ has expected dimension (real codimension $2n$), while $\widehat{\mathcal{D}}_g$ is empty, contrary to expected codimension.

The loci $\mathcal{D}_n^\nu$, while defined by real-analytic conditions, still represent cohomology classes, which we will now compute. Instead of the Deligne-Mumford compactification, it is easier for us to work on $\mathcal{C}_g^{1+n}$, the $(n+1)$'th fiberwise product of the universal curve (so the fiber of the map $\mathcal{C}_g^{1+n}\to\mathcal{M}_g$ over $\Gamma\in\mathcal{M}_g$ is $\Gamma^{\times(1+n)}$). This is to say that on $\mathcal{C}_g^{1+n}$ the marked points are allowed to coincide. Following [7],[34], we denote by $K_i$ the class of the universal cotangent bundle to $\mathcal{C}_g^{1+n}$ at $p_i$ — this is the analog of the class $\psi_i$ on $\overline{\mathcal{M}}_{g,1+n}$. It is easier to work with classes on $\mathcal{C}_g^{1+n}$ in particular because by definition the pullback of $K_i$ from $\mathcal{C}_g^k$ to $\mathcal{C}_g^{k+1}$ under the forgetful map is equal to $K_i$, see [7]. We also denote by $D_{ij}\subset\mathcal{C}_g^k$ the divisor where the points $p_i$ and $p_j$ coincide, and we finally note that as a set we have $\mathcal{M}_{g,2}=\mathcal{C}_g^2$, though the objects parameterized by these are different.

We note that our differentials $\Psi_i$ are in fact defined over the bundle of jets of the universal coordinate over $\mathcal{M}_{g,1}=\mathcal{C}_g^1$, and thus their common zero loci $\mathcal{D}_n^\nu$ are well-defined on $\mathcal{C}_g^{1+n}$ (where they are closed).

**Theorem 2.6.** *The cohomology class of the locus $\mathcal{D}_n^\nu\subset\mathcal{C}_g^{1+n}$ is*

$$[\mathcal{D}_n^\nu]=\prod_{i=1}^n(K_0^2+K_i^2+2K_0D_{0i})^{\nu_i}.$$

*Proof.* Let us first handle the simplest case — the locus $\mathcal{D}_1$, where on $\mathcal{C}_g^2$ (which we consider as the partial compactification of $\mathcal{M}_{g,1+1}$ where the two points are allowed to coincide) the two differentials $\Psi_1$ and $\Psi_2$ with double poles at $p_0$ have a common zero at $p_1$. Since our differentials $\Psi_i$ and their zero loci are well-defined on $\mathcal{M}_{g,2}(1,0)$, we will work there, using the classes $\psi_0$ and $\psi_1$ and $\delta_{01}$ instead of $K_0,K_1,D_{01}$. By definition $\Psi_i$ are defined as sections of the bundle of meromorphic differentials over $\mathcal{M}_{g,1+1}(1,0)$, which is a $\mathbb{C}^*$-bundle over $\mathcal{M}_{g,1+1}$. The homology ring of the total space of a $\mathbb{C}^*$ bundle is the quotient of the homology ring of $\mathcal{M}_{g,1+1}$ by the Chern class of this bundle, which is to say by the class $\psi_0$.

Considered over $\mathcal{M}_{g,1+1}(1,0)$, the value of the differential $\Psi_i$ at $p_1$ is the value of a holomorphic differential on $\Gamma$ at $p_1$, that is a section of $\psi_1$, and thus the class of the locus $\widehat{\mathcal{D}}_1\subset\mathcal{M}_{g,1+1}(1,0)$ is equal to $\psi_1^2$. Since $\widehat{\mathcal{D}}_1$ and $\psi_1^2$ on $\mathcal{M}_{g,1+1}(1,0)$ are both pullbacks from $\mathcal{M}_{g,1+1}$, it follows that the class of the locus $\mathcal{D}_1\subset\mathcal{M}_{g,1+1}$, in the quotient of the cohomology ring by the class $\psi_0$, is equal to $\psi_1^2$. Since by [2] it is known that all cohomology classes on $\mathcal{C}_g^2$ in complex codimension 2 are tautological, it follows that there we must have

$$[\mathcal{D}_1]=\psi_1^2+a\psi_0^2+b\psi_0\psi_1+c\psi_0\delta_{01}\qquad(1)$$

for some $a, b, c \in \mathbb{C}$, and we now need to compute the coefficients.

It follows from the reciprocity law that $\mathcal{D}_1$ is symmetric in $p_0$ and $p_1$. To prove this, we let $\Xi$ be a meromorphic differential of the second type, with a second order pole at $p_1$ and holomorphic elsewhere, with all periods real. We denote by $F_i(p) = \int_{p_1}^p \Psi_i$ the integrals of the meromorphic differentials — these are meromorphic functions on $\Gamma$ cut along a basis for cycles in $H_1(\Gamma, \mathbb{Z})$. By the reciprocity law for differentials of the second kind (see [14, p.241]) it then follows that the sum of the residues $2\pi i \sum \mathrm{Res}(F_i \Xi)$ is equal to some linear combination of periods of $\Psi_i$ and $\Xi$. Note, however, that since $\Psi_i$ vanishes at $p_1$, the function $F_i$ will have a second order zero at $p_1$, and thus the meromorphic differential $F_i\Xi$ will not have a residue at $p_1$. It thus follows that the residue of $F_i\Xi$ at $p_0$ must be purely imaginary. However, locally near $p_0$ we have $F_1 = -dz/z + O(1)$ and $F_2 = -idz/z + O(1)$, and thus for the residues we have $\mathrm{Res}_{p_0}(F_1\Xi) = \Xi(p_0)$ and $\mathrm{Res}_{p_0}(F_2\Xi) = -i\Xi(p_0)$. For both of these to be purely imaginary then implies $\Xi(p_0) = 0$, i.e. any differential with a double pole at $p_1$ and all periods real vanishes at $p_0$. Therefore the class $[\mathcal{D}_1]$ must be symmetric in $p_0$ and $p_1$, which implies $a = 1$ in (1) — note that $\psi_0\delta_{01} = \psi_1\delta_{01}$.

Note now that if the points $p_0$ and $p_1$ collide on $\mathcal{C}_g^2$, which is to say that on the universal family over the partial compactification $\mathcal{M}_{g,1+1}$ they go onto a new rational component, the differentials $\Psi_1$ and $\Psi_2$, being just $dz$ and $idz$ on the sphere, do not vanish at $p_1$. Thus we must have

$$0 = [\mathcal{D}_1 \cap \delta_{01}] = (\psi_0^2 + \psi_1^2 + b\psi_0\psi_1 + c\psi_0\delta_{01})\delta_{01} = (2 + b - c)\psi_0^2\delta_{01}$$

where we have used the identity $\delta_{01}^2 = -\psi_0\delta_{01} = -\psi_1\delta_{01}$ from [7]. It thus follows that $c = 2 + b$, so that rewriting the class in terms of $K_i$ instead of $\psi_i$, and denoting the coefficient $b_g$ for the class in genus $g$ we have

$$[\mathcal{D}_1] = \psi_0^2 + \psi_1^2 + b_g\psi_0\psi_1 + (2+b_g)\psi_0\delta_{01} = K_0^2 + K_1^2 + b_g K_0 K_1 + (2+b_g)K_0 D_{01}$$

A direct way to compute the last unknown coefficient $b_g$ would be to compute the pushforward of $[\mathcal{D}_1]$ from $\mathcal{C}_g^2$ to $\mathcal{M}_g$ — it would be a multiple of the fundamental class, and the coefficient would give $b_g$. Geometrically, computing this pushforward is equivalent to computing the number of pairs of points $p_0, p_1$ on a generic curve $\Gamma \in \mathcal{M}_g$ such that $\Psi_1$ and $\Psi_2$ with double poles at $p_0$ and all periods real vanish at $p_1$. Instead of doing this computation directly, we use a degeneration argument.

In [15] we showed that on a nodal curve the limit of $\Psi_i$ is identically zero on all irreducible components of the normalization except the one containing $p_0$, while on the component containing $p_0$ in the limit $\Psi_i$ is simply equal to the differential of the second kind with real periods on the normalization of that component. We now consider the degeneration of the locus $\mathcal{D}_1$ as we approach the boundary component $\delta_{irr} \subset \mathcal{M}_{g,1}$. In the open part of $\delta_{irr}$ the points $p_0$ and $p_1$, while not necessarily distinct themselves, as we are working on the universal curve $\mathcal{C}_g^2$, do not approach the node. By the results of [15] such a nodal curve lies in the closure of $\mathcal{D}_1$ if and only if the differentials $\Psi_1$ and $\Psi_2$ on its genus $g-1$ normalization have a common zero

We thus have, possibly up to classes supported on the loci $D_{02}$, $D_{03}$, $D_{12}$, $D_{13}$ (where the points $p_0$ or $p_1$ approach the node), $\mathcal{D}_1^{(g)}|_{\delta_{irr}} = \rho^*(\mathcal{D}_1^{(g-1)})$, where $\rho : \delta_{irr} \to \mathcal{M}_{g,1}$ forgets the node. Since the tautological classes of dimension two are independent on $\mathcal{C}_g^2$ (for example, this can be easily seen from Faber's formularium [7]), this in particular implies that the coefficients of $K_0K_1$ should be the same on both sides — recall that unlike the $\psi$ classes, where the pullback includes the diagonals, the pullback of $K_i$ from $\mathcal{M}_{g,n}$ to $\mathcal{M}_{g,n+1}$ is equal to $K_i$. Thus we must have $b_g = b_{g-1} =: b$. On the other hand, the same argument applies to the pushforward

$$\pi_*([\mathcal{D}_1]) = \kappa_1 + ((2g-2)b + 2 + b)K_0$$

under the map forgetting $p_1$. This pushforward restricted to the open part of $\delta_{irr}$ must again be equal to the pullback of the pushforward of $[\mathcal{D}_1]$ from $\mathcal{M}_{g-1,1}$. Thus we must have $(2g-1)b = (2g-3)b$, which finally implies $b = 0$.

For many simple common zeroes, the class of the locus $\mathcal{D}_n^{1,\dots,1}$, being the intersection of the pullbacks of $\mathcal{D}_1$ to $\mathcal{C}_g^n$ from various $\mathcal{C}_g^2$ by maps forgetting all points except $p_0$ and $p_i$, is given by the product $\prod_{i=1}^n (K_0^2 + K_i^2 + 2K_0D_{0i})$.

**Remark 2.7.** A more elaborate version of the degeneration argument above, tracking the number of common zeroes under a degeneration where they may approach a node, will turn out to be a key step in proving our main vanishing result, theorem 2.8, stating that $\mathcal{D}_g$ is empty.

For the case of common zeroes of higher multiplicity, we would like to treat a multiple zero as a limit of simple zeroes coming together. Formally this means that we claim that the class $\mathcal{D}_{n+1}^{\nu_1,\dots,\nu_{n+1}}$ intersected with the diagonal $D_{n,n+1} \subset \mathcal{C}_g^{1+1+n}$ is equal to the class $\mathcal{D}_n^{\nu_1,\dots,\nu_n+\nu_{n+1}}$, when we identify $\mathcal{C}_g^{1+n}$ with the diagonal $D_{n,n+1} \subset \mathcal{C}_g^{1+1+n}$.

Once this claim is proven, the general formula for the class follows immediately by intersecting the formula for $\mathcal{D}_n^{1,\dots,1}$ with various diagonals (which amounts to some of the factors now becoming equal). To prove the claim we would need to study the local structure of the strata $\mathcal{D}_n^\nu$. In the next section we will describe explicitly the tangent spaces to these loci. In particular we will describe the tangent space to a leaf of a foliation (which we will define) at a point on each $\mathcal{D}_n^\nu$, and from this description the above claim will follow. □

Our study of the infinitesimal geometry of the foliations will lead to some precise conjectures, while the main result we will prove is the following

**Theorem 2.8.** *The locus $\mathcal{D}_g \subset \mathcal{C}_g^{1+g}$ is empty, i.e. $\Psi_1$ and $\Psi_2$ cannot have $g$ distinct common zeroes*

By theorem 2.6 we know the cohomology class represented by $\mathcal{D}_g$, and thus obtain

**Corollary 2.9.** *The class $\prod_{i=1}^{g}(K_0^2+K_i^2+2K_0D_{0i})$ is zero in cohomology (or real cobordisms) of $\mathcal{C}_g^{1+g}$.*

We note that the above class is of dimension $2g$, the lowest dimension where the tautological classes vanish by Faber's conjecture.

## 3. The geometry of the foliations on $\mathcal{M}_g$

In this section we recall the local coordinates on the Teichmüller (or moduli) spaces defined using meromorphic differentials with real periods. We then define two natural foliations of the moduli space, the "big" and the "small", obtained by fixing the absolute periods (i.e. integrals over $H_1(\Gamma,\mathbb{Z})$) of either one of or both $\Psi_1, \Psi_2$. We note that it is a priori easy to construct foliations of the Teichmüller space: it is homeomorphic to an open ball, admits many non-constant holomorphic functions, and the level sets of such a function give a foliation. However, it is much harder to construct a foliation of the moduli space or its compactification.

The foliations we define are real-analytic, but their leaves are complex (so the foliations are real-analytic "in the transverse direction"). We then proceed to describe the tangent spaces to the leaves of these foliations, which will be needed in the following sections. The construction of these foliations, while similar to the foliations in the moduli spaces $\mathcal{H}_g$ of curves with abelian differentials, is motivated by the Whitham perturbation theory of integrable systems [24, 25, 31, 32]. In [15] we used the "big" foliation (in that paper we had a situation with only one meromorphic differential with real periods), and we refer to that paper for more details.

**Definition 3.1.** For any $a_1, \ldots, a_g, b_1, \ldots, b_g \in \mathbb{R}$ the leaf $\mathcal{L}_{\underline{a},\underline{b}}$ (which we denote $\mathcal{L}$ if $\underline{a}, \underline{b}$ are not important) of the foliation of the Teichmüller space $\mathcal{T}_{g,1}(2)$ is defined to be the locus where the integrals of $\Psi_1$ over the chosen basis of cycles $A_1, \ldots, A_g, B_1, \ldots, B_g$ are equal to $a_1, \ldots, a_g, b_1, \ldots b_g$. Note that such a leaf $\mathcal{L}_{\underline{a},\underline{b}}$ is defined by saying that there exists a differential with prescribed periods (the reality is implied) — these are holomorphic conditions, and thus the leaf is a complex submanifold $\mathcal{L}_{\underline{a},\underline{b}} \subset \mathcal{T}_{g,1}(2)$. If a different basis of $H_1(\Gamma, \mathbb{Z})$ is chosen, the periods of $\Psi_1$ over the basis are still fixed along a leaf (though numerically different). Thus a leaf $\mathcal{L}$ is invariant under the action of the mapping class group, and so descends to a "tangentially complex" leaf $\mathcal{L}$ of a foliation on $\mathcal{M}_{g,1}(2)$.

Since the absolute and relative periods of $\Psi_1$ give local coordinates at any point of the moduli space, all leaves of $\mathcal{L}$ — the level sets of a subset of these coordinates — are smooth. We will now describe the tangent space to the leaves explicitly: this construction is one of the key elements of the Whitham theory, and can also be thought of in terms of the deformation theory of Riemann surfaces. The construction is analytic, giving locally a map from the tangent space to a certain space of differentials, and gives a local description of the tangent space. It would be interesting to relate this construction to the Rauch variational formulas: the leaves of the foliations are the level sets of periods, and thus the variation of the periods along their tangent spaces is zero.

**Definition 3.2.** Let $\Gamma^* := \Gamma^{cut}$ be the (closed) result of cutting $\Gamma$ along the standard basis of cycles, i.e this is a $2g$-gon, identifying the sides of which pairwise one gets $\Gamma$. Let then $\widehat{\mathcal{T}}$ be the space of differentials on $\Gamma^*$ that are holomorphic on the interior of $\Gamma^* \setminus \{p_0\}$, have at most a simple pole at $p_0$, are continuous on $\Gamma^*$, and such that the "jump" of the differential on any cut cycle $c \in \{a_1, \ldots a_g, b_1, \ldots, b_g\}$ (the difference of its values on the two corresponding sides of the $2g$-gon) is equal to $r_c^1 \Psi_1 + r_c^2 \Psi_2$ for some real numbers $r_c^1, r_c^2$.

We now define the $\mathbb{R}$-linear homomorphism

$$\widehat{\tau} : T_X(\mathcal{M}_{g,1}(2)) \longmapsto \widehat{\mathcal{T}}, \qquad (X = (\Gamma, p_0, z) \in \mathcal{M}_{g,1}(2)). \tag{2}$$

(We note that while all of the previos constructions work given a 1-jet of the local coordinate, for defining this map we need the full 2-jet of $z$.) The construction of this map in full generality is discussed in [25, 31]. For the applications in this paper we will be primarily concerned with the case when no common zeroes of $\Psi_1$ and $\Psi_2$ are multiple.

To define the map $\widehat{\tau}$, we proceed as follows. Since $\Gamma^*$ is simply connected, one can choose globally on $\Gamma^*$ a branch of the integral $F_1 :=$

$\int \Psi_1$. To fix the constant of integration, we use the 2-jet of a local coordinate: locally near $p_0$ we must have $F_1 = z^{-1} + a + O(z)$, and we require $a = 0$. Note that this indeed depends precisely on the 2-jet of the local coordinate: if $z' = z + O(z^2) = z + cz^2 + \ldots$ is an equivalent 1-jet, then

$$(z')^{-1} = z^{-1}(1 + cz + \ldots)^{-1} = z^{-1}(1 - cz + \ldots) = z^{-1} - c + \ldots$$

and thus the condition for the constant term of $F_1$ to still be zero in the $z'$ coordinate is equivalent to $c = 0$, which is to say that $z$ and $z'$ define the same 2-jet of the loca coordinate.

The function $F_1$ is then uniquely defined (given a point in $\mathcal{M}_{g,1}(2)$) and defines a local coordinate on $\Gamma^*$ away from $p_0$ and the locus where its derivative vanishes, i.e away from the zeros $p_1, \ldots, p_{2g}$ of $\Psi_1$. Since $\Psi_1$ is not an exact differential, it has non-zero periods along the cycles, and thus $F_1$ is not a well-defined function on $\Gamma$ — it has "jumps" along the cut cycles (while in the definition of $\widehat{\mathcal{T}}$ above the jumps were differentials, here the jumps are functions along the cuts!). Indeed, the jumps of $F_1$ are constant along a cycle, and equal to the corresponding period of $\Psi_1$.

Thus on the universal cut curve over $\mathcal{T}_{g,1}(2)$ (or locally on the family of $\Gamma^*$ over $\mathcal{M}_{g,1}(2)$) the function $F_1$ is non-degenerate (its gradient is non-zero) away from the zeros of $\Psi_1$. It thus makes sense to consider, locally on the universal cut curve, the variation of the function $F_2$ in the direction of some vector $v \in T_X(\mathcal{M}_{g,1}(2))$ along the level set of $F_1$. Geometrically this means that to lift $v$ to a tangent vector to $\mathcal{C}^*_{g,1+1}(2,0)$ (the total space of the family of all $\Gamma^*$) we will locally choose a section of $\mathcal{C}^*_{g,1+1}(2,0) \to \mathcal{M}_{g,1}(2)$ given by a level set of $F_1$. Technically this means the following: let the $x$'s be local coordinates on $\mathcal{M}_{g,1}(2)$; then on the universal cut curve the local coordinates near some point $((\Gamma, p_0, z), p)$ (where $(\Gamma, p_0, z) \in \mathcal{M}_{g,1}(2)$ and $p \in \Gamma^*$) are the $x$'s together with the value of $F_1$. Consider now in the universal cut curve over $\mathcal{M}_{g,1}(2)$ the level set $\{x \mid F_1(x) = F_1(p)\}$. It locally maps isomorphically to $\mathcal{M}_{g,1}(2)$, and the inverse map is given by $P_{F_1}(x)$ such that $F_1(P_{F_1}) \equiv F_1(p)$. Corresponding to a tangent vector $v \in T_X(\mathcal{M}_{g,1}(2))$, which we think of as a variation $x \mapsto x + \varepsilon v$, we can compute the variation $F_2(x + \varepsilon v, P_{F_1}(x + \varepsilon v)) - F_2(x, p)$, and this is what we want. As this is the partial derivative of $F_2$ for $F_1$ fixed, we write it as $\partial_v F_2 = \partial_v F_2(F_1)$.

This derivative is holomorphic on $\mathcal{M}_{g,1+1}(2,0)$ away from $p_0$, the cuts, and from the points $p_s$, where it acquires simple poles, as $F_1$ ceases to be a local coordinate on $\Gamma^*$ there. It thus follows that $\Omega_v := \partial_v F_2(F_1)\Psi_1$ has no poles at $p_s$ and defines a holomorphic differential on $\Gamma^*$ away from $p_0$ where it may have a simple pole. If the vector $v$

is tangent to a leaf $\mathcal{L}$, i.e it does not change the periods of $\Psi_1$, then the values of $\Omega_v$ on the two sides of the cuts differ by $r\Psi_1$, where $r$ is the derivative along $v$ of the corresponding period of $\Psi_2$ (which is by definition real). If $v$ changes the periods of $\Psi_1$ then as shown in [25] the additional jump of the derivative on the cut is of the form $r_2\Psi_2$. We thus define the map (2) by setting $\widehat{\tau}(v) := \Omega_v$, which by definition is an element of $\widehat{\mathcal{T}}$.

By definition the image under $\widehat{\tau}$ of the tangent space to a leaf $T_X(\mathcal{L})$ consists of differentials in $\widehat{\mathcal{T}}$ with jumps proportional to $\Psi_1$ only. Denoting the space of such differentials by $\mathcal{T} \subset \widehat{\mathcal{T}}$, we get the map

$$\tau : T_X(\mathcal{L}) \longmapsto \mathcal{T}. \tag{3}$$

**Remark 3.3.** The image under the map $\widehat{\tau}$ of the vector field $\partial_t$ corresponding to changing the 2-jet for a fixed 1-jet of the local coordinate, $z \to z + cz^2$ is the holomorphic differential

$$\widehat{\tau}(\partial_t) = i\Psi_2 + \Psi_1 \tag{4}$$

A direct dimension count shows that the real dimension of $\widehat{\mathcal{T}}$ is equal to $6g$, the same as the dimension of $\mathcal{M}_{g,1}(2)$. One thus wonders whether $\widehat{\tau}$ may be an isomorphism. Generically this is indeed the case:

**Lemma 3.4** ([31])**.** *On the open set $\mathcal{M}_{g,1}(2)\backslash\widehat{\mathcal{D}}_1$, where $\Psi_1$ and $\Psi_2$ have no common zeros, the map $\widehat{\tau} : T_X(\mathcal{M}_{g,1}(2)) \to \widehat{\mathcal{T}}$ is an isomorphism.*

(our more detailed analysis below will also yield this as a corollary). Thus one is led to study the behavior of the map $\widehat{\tau}$ when $\Psi_1$ and $\Psi_2$ have common zeroes, and this description is the main result of this section. The loci $\mathcal{D}_n^\nu$ arise naturally as the degeneracy loci of the map $\tau$.

The most important case for us is that of $\nu = (1, \ldots, 1)$, in which case the result is the following

**Proposition 3.5.** *If all zeroes of $\Psi_1$ on $\Gamma$ are simple, and $X \in \mathcal{D}_n \setminus \mathcal{D}_{n+1}$ (so $\Psi_1$ and $\Psi_2$ have exactly $n$ common simple zeros), the image of the map $\widehat{\tau}$ is equal to*

$$\widehat{\mathcal{T}}_n := \widehat{\tau}(T_X(\mathcal{M}_{g,1}(2))) = \{\Omega \in \widehat{\mathcal{T}} \mid \Omega(p_1) = \ldots = \Omega(p_n) = 0\}. \tag{5}$$

*Proof.* Recall the notation $F_1(p) := \int^p \Psi_1$ that we introduced for the abelian integral of $\Psi_1$ in the definition of the map $\widehat{\tau}$, and let $\phi_i := F_1(p_i)$ (these are local coordinates on the leaf $\mathcal{L}$, as discussed in [15]). Since by assumption $p_i$ is a simple zero of $\Psi_1$, the difference $F_1(p) - \phi_i$ vanishes to precisely the second order for $p$ approaching $p_i$, and thus $z_i(p) :=$

$\sqrt{F_1(p)-\phi_i}$ gives a local coordinate on $\Gamma$ near $p_i$. By definition in this coordinate we locally have $\Psi_1 = dF_1 = 2z_i dz_i$. Let us now consider the Taylor expansion of $F_2(p) := \int^p \Psi_2$ near $p_i$ in coordinate $z_i$, denoting the coefficients

$$F_2(z_i) = \alpha_{i,0} + \alpha_{i,1} z_i + \alpha_{i,2} z_i^2 + \dots . \tag{6}$$

In this notation, the conditions for $\Psi_2 = dF_2$ to vanish at the points $p_i$ (i.e. at $z_i = 0$) for $1 \le i \le n$ are then

$$\alpha_{i,1} = 0, \quad i = 1, \dots, n. \tag{7}$$

Consider now a tangent vector $v \in T_X(\mathcal{M}_{g,1}(2))$, and compute the partial derivative of $F_2$ for $F_1$ fixed (as in the definition of $\widehat{\tau}$). Notice that the derivative of $z_i$ for fixed $F_1$ is given by $\partial_v z_i(F_1) = -(\partial_v \phi_i) z_i^{-1}/2$. Therefore,

(8)

$$\partial_v F_2 = \frac{\Omega_v}{\Psi_1} = -\frac{1}{2}(\partial_v \phi_i)\,\alpha_{i,1} z_i^{-1} + (\partial_v \phi_i' - (\partial_v \phi_i)\,\alpha_{i,2}) + (\partial_v \alpha_{i,1} - \frac{3}{2}(\partial_v \phi_i)\,\alpha_{i,3}) z_i + \dots$$

Since this expression is regular near $z_i = 0$ (i.e. near the zero $p_i$ of $\Psi_1$), it follows that $\Omega_v(p_i) = 0$. By definition the differential $\Omega_v$ is the image in $\widehat{\mathcal{T}}$ of $v \in T_X(\mathcal{M}_{g,1}(2))$, thus we have proven that $\widehat{\tau}(T_X(\mathcal{M}_{g,1}(2)))$ is contained in the subspace $\widehat{\mathcal{T}}_n$ of $\widehat{\mathcal{T}}$ consisting of differentials vanishing at $p_1, \dots, p_n$.

To prove that the image is equal to this subspace, it is sufficient to show that vanishing conditions are linearly independent and then compute the dimension of the kernel of $\widehat{\tau}$. The linear independence of equations (5) follows from the following stronger result (indeed, the independence of the vanishing conditions on $\widehat{\mathcal{T}}$ means that the matrix of values of a basis of $\widehat{\mathcal{T}}$ at $p_1 \dots p_n$ has rank $n$; this would follow if the matrix of values of a basis of $\mathcal{T}$ at $p_1 \dots p_{2g}$ has full rank):

**Lemma 3.6.** *If a differential $\Omega \in \mathcal{T}$ vanishes at all zeros of $\Psi_1$, i.e. $\Omega(p_1) = \dots = \Omega(p_{2g}) = 0$, then $\Omega = 0$.*

*Proof.* Indeed, if $\Omega$ vanishes at all zeroes of $\Psi_1$, then the ratio $f := \Omega/\Psi_1$ is a holomorphic function on $\Gamma^*$ with all jumps on cut cycles being real numbers (since the jumps of $\Omega$ are real multiples of $\Psi_1$). Then $df$ is a holomorphic differential on $\Gamma$ with all periods real, in which case it must be zero, so that $f$ is a constant. Thus $\Omega$ is a constant multiple of $\Psi_1$. Since by definition of $\mathcal{T}$ the differential $\Omega \in \mathcal{T}$ cannot have a double pole at $p_0$, as $\Psi_1$ does, we have $f(p_0) = 0$, and thus this constant multiple is zero. □

We now describe the kernel of the map $\widehat{\tau}$ on $T_X(\mathcal{M}_{g,1}(2))$ explicitly, and thus in particular will see that its complex dimension is equal to $n$. By definition a tangent vector $v \in \operatorname{Ker}\widehat{\tau}$ does not change the periods of $\Psi_1$ or $\Psi_2$ (this means that $v$ is a tangent vector to a leaf of the "small" foliation, which we define below).

As we know from [15, 31], the critical values $\phi_i$ of $F_1$ are local coordinates on a leaf $\mathcal{L}$. Thus any tangent vector $v$ to $\mathcal{L}$ is a real linear combination of $(\operatorname{Re}\partial_{\phi_i})$ and $(\operatorname{Im}\partial_{\phi_i})$. From the leading term of (8) we see that if $\partial_v\phi_j \neq 0$ for some $j > n$, then $\partial_v F_2(p_j) \neq 0$. Conversely, if for all $j > n$ we have $\partial_v\phi_j = 0$, then $\partial_v F_2$ would be regular at all the points $p_1 \ldots, p_{2g}$. Thus by definition the differential $\Omega_v$ would vanish at $p_1, \ldots, p_{2g}$, which by Lemma 3.6 would imply $\Omega_v = 0$ and thus $v \in \operatorname{Ker}\widehat{\tau}$, i.e the kernel $\operatorname{Ker}\widehat{\tau}$ as a $\mathbb{C}$-vector space is spanned by the vector fields $\partial_{\phi_i}$, $1 \leq i \leq n$. □

The description of $\widehat{\tau}$ in full generality, for $\Psi_1$ and $\Psi_2$ having common zeroes of arbitrary multiplicity, is given by

**Proposition 3.7.** *For $X = (\Gamma, p_0, p_1, \ldots, p_n) \in \mathcal{M}_{g,1+n}$ if $\Psi_1$ and $\Psi_2$ have exactly $n$ common zeroes at $p_1, \ldots, p_n$, with multiplicity $\nu_i$ at $p_i$ (i.e if $X \in \mathcal{D}_n^\nu$ and $X$ does not lie in any "more degenerate" locus $\mathcal{D}$), the image $\widehat{\tau}(T_X(\mathcal{M}_{g,1}(2))) =: \widehat{\mathcal{T}}_n^\nu$ is the subspace $\widehat{\mathcal{T}}_n^\nu \subset \widehat{\mathcal{T}}$ consisting of differentials $\Omega \in \widehat{\mathcal{T}}$ vanishing at each $p_i$ to order at least $\nu_i$.*

*Idea of the proof.* Since we will not use this result in full generality, we do not give a complete proof. The idea is similar to that of the case of simple zeroes above. Indeed, note that $F_1(p) - \phi_i$ vanishes to order $\nu_i + 1$ near $p_i$, and thus a coordinate $z_i$ near $p_i$ can be chosen such that on a universal cut curve we have

$$F_1(p, x) = z_i^{\nu_i+1} + \sum_{i=0}^{\nu_i-1} s_{i,j}(x) z_i^j, \tag{9}$$

where $x = (\Gamma, p_0, k) \in \mathcal{M}_{g,1}(2)$ and $z_i = z_i(x, p)$ for $p \in \Gamma^*$ are local functions on the universal cut curve, and $s_{i,j}(x)$ are local holomorphic functions on the leaf $\mathcal{L}$ through $\Gamma$. We then compute for any $v \in T_X(\mathcal{M}_{g,1}(2))$

$$\partial_v F_2(F_1) = \sum_{k=0}^{\infty} (\partial_v \alpha_{i,k}) z_i^k + (\partial_v z_i) \sum_{k=1}^{\infty} k\alpha_{i,k} z_i^{k-1} \tag{10}$$

If $X$ is in $\widehat{\mathcal{D}}_n^\nu$, then by definition the first $\nu_i$ terms in the Taylor expansion vanish, i.e we have

$$\alpha_{i,k} = 0, \quad i = 1, \ldots, n; \; k = 1, \ldots, \nu_i \tag{11}$$

Therefore, $\partial_v F_2(F_1)$ is regular at the points $p_i$, $i = 1, \dots, n$ and the differential $\Omega_v$ vanishes at these points to order at least $\nu_j$, i.e $\Omega_v \in \widehat{T}_n^\nu$, and the result follows. □

As the first consequence of the description of the tangent space to the leaves, we show that the loci $\widehat{\mathcal{D}}_n^\nu$, if non-empty, are of expected dimension; the proof is by computing the tangent spaces to $\widehat{\mathcal{D}}_n^\nu$. As before, we will give a detailed proof for the case of simple zeroes, and then give the corresponding (here, technically much more involved) statement in full generality.

**Proposition 3.8.** *The locus $\mathcal{D}_n$, if non-empty, has expected dimension, i.e. its real codimension is equal to $2n$. Moreover, on an open subset $\mathcal{D}_n^* \subset \mathcal{D}_n$, the homomorphism (2) restricts to an isomorphism $\widehat{\tau} : T_X(\mathcal{D}_n^*) \simeq \widehat{\mathcal{T}}_n$, with $\widehat{\mathcal{T}}_n$ given by (5).*

*Proof.* We first prove that the subset $\widehat{\mathcal{D}}_n^* \subset \widehat{\mathcal{D}}_n$, where the coefficients $\alpha_{1,3}, \dots, \alpha_{n,3}$ in (6) are all non-zero, if non-empty, is of expected dimension, and then show that $\widehat{\mathcal{D}}_n^*$ is open within $\widehat{\mathcal{D}}_n$.

The defining equations for $\widehat{\mathcal{D}}_n$ are given in expansion (6) by conditions (7), i.e. by $\alpha_{1,1} = \dots = \alpha_{n,1} = 0$. For any $v \in \operatorname{Ker} \widehat{\tau}$ we have $\partial_v F_2 \equiv 0$. Substituting the expression for $\partial_v F_2$ from (8), this yields

$$\partial_v \alpha_{i,1} = (\partial_v \phi_i) \alpha_{i,3}, \quad i = 1, \dots, n. \tag{12}$$

We know that $\operatorname{Ker} \widehat{\tau}$ is spanned by $\partial_{\phi_j}$, $1 \le j \le n$. Thus the $n \times n$ matrix of derivatives $\partial_{\phi_j} \alpha_{i,1}$ is diagonal with diagonal entries being $\alpha_{i,3}$, and thus if we have $\alpha_{1,3} \cdot \ldots \cdot \alpha_{n,3} \neq 0$, this matrix is non-degenerate (since $\phi_i$ are part of a local coordinate system). Then the gradients of the defining equations (7) are linearly independent, and thus the locus defined by these equations is locally smooth of expected codimension.

It thus remains to prove that the locus where $\alpha_{1,3} \cdot \ldots \cdot \alpha_{n,3} \neq 0$ is open within $\widehat{\mathcal{D}}_n$, i.e. that it cannot happen that some $\alpha_{i,3}$ vanishes generically. Indeed, if we had $\alpha_{i,3} \equiv 0$ in a neighborhood of $X$, from the vanishing of the further terms in expansion (6) for $1 \le i \le n$ we would get

$$\partial_{\phi_i} \alpha_{i,2m-1} = \frac{2m+1}{2} \alpha_{i,2m+1}. \tag{13}$$

Thus if $\alpha_{i,3}$ were to vanish identically in a neighborhood, so would $\alpha_{i,2m+1}$ for all $m$. If this were the case, $F_2$ locally on $\Gamma$ near $p_i$ would be an even function of $z_i$. Since by definition $F_1(z_i) = \phi_i + z_i^2$, for any $q_1 \neq p_i$ in the neighborhood of $p_i$ there exists a unique point $q_2 \neq p_i$ such that $F_1(q_1) = F_1(q_2)$. We can thus consider $q_2$ as a function of

$q_1$ and analytically continue it along any path in $\Gamma$ not containing any zeroes $p_i$ of $\Psi_1$ (i.e on the locus on $\Gamma$ where $F_1$ gives a local coordinate). The above vanishing would imply that $F_2(q_1) = F_2(q_2)$ identically in the neighborhood of $p_i$, and thus that this also holds for the analytic continuation. On the other hand, let us continue $q_2(q_1)$ along a path where $q_1$ approaches some zero $p_k$ of $\Psi_1$ for $k > n$. If $\Psi_1$ vanishes at $p_k$ to order $\nu$, the map $F_1$ locally near $p_k$ is of degree $\nu + 1$, i.e there are $\nu+1$ points near $p_k$ with the same value of $F_1$. Since we know from the above that $F_2$ must take the same value at all of these points (equal to $F_2(q_2)$), it means that $F_2$ must also have the same values at these $\nu + 1$ points, and thus $\Psi_2 = dF_2$ must also have a zero of order $\nu$ at $p_k$. Thus $\Psi_2$ vanishes at all zeroes of $\Psi_1$, which is impossible, and we have arrived at a contradiction. □

**Remark 3.9.** For future use, we now describe explicitly the inverse map for the isomorphism $\widehat{\tau} : (T_X(\mathcal{D}_n^*)) \to \widehat{\mathcal{T}}_n$. From the leading term in the expansion (8) we get

$$\partial_v \phi_i = -\frac{\Omega_v}{\Psi_2}(p_i), \quad i > n. \tag{14}$$

By definition, the differential $\Omega_v$ vanishes at $p_1, \ldots, p_n$. Hence, its expansion has the form

$$\Omega_v = (2\beta_{i,2} z_i + 3\beta_{i,3} z_i^2 + \ldots)\, dz_i \tag{15}$$

Comparing the coefficients of $z_i$ in (8) and (15) and using $\partial_v \alpha_{i,1} = 0$ (since $v$ is tangent to $\widehat{\mathcal{D}}_n^{1,\ldots,1,*}$) gives

$$\partial_v \phi_i = -\frac{\beta_{i,3}}{\alpha_{i,3}}, \quad 1 \le i \le n. \tag{16}$$

We now give the corresponding expression in the case of common zeroes of higher multiplicity — in this case it is technically more involved.

**Proposition 3.10.** *The locus $\mathcal{D}^\nu$, if non-empty, has expected dimension, i.e. its real codimension is equal to $2d_\nu := 2\sum(2\nu_i - 1)$. Moreover, on an open subset $\mathcal{D}_n^{\nu *} \subset \mathcal{D}_n^\nu$ (where for any $1 \le i \le n$ we have $\alpha_{i,j} \ne 0$ for some $\nu_i + 1 < j \le 2\nu_i - 1$ for $\alpha$ defined by (11)), the homomorphism (2) restricts to an isomorphism $\widehat{\tau} : T_X(\mathcal{D}_n^{\nu *}) \simeq \widehat{\mathcal{T}}_n^\nu \subset \widehat{\mathcal{T}}$, where $\widehat{\mathcal{T}}_n^\nu$ is the subspace of differentials $\Omega \in \widehat{\mathcal{T}}$ such that for any $1 \le i \le n$ there exist constants $c_i^1$, $c_i^2 \in \mathbb{C}$ such that the differential*

$$\Omega + c_i^1 \Psi_1 + c_i^2 \Psi_2 \tag{17}$$

*vanishes at $p_i$ to order at least $2\nu_i + 1$.*

Note that if all multiplicities $\nu_i$ are equal to 1, then on an open set where $\Psi_1$ and $\Psi_2$ have sufficiently generic expansions near $p_i$, we can use the second and third order terms of their expansions to cancel the corresponding terms in the expansion of $\Omega$. Thus for $\nu_i = 1$ condition (17) on an open set is equivalent simply to saying that $\Omega(p_i) = 0$, so that we get proposition 3.8 as a special case for $\nu = (1, \dots, 1)$.

As we will not need this more general version (proposition 3.8 suffices to prove our vanishing result), we do not give the details of the proof here. The method is again similar to the case of simple zeroes: one uses $z_i := \sqrt[\nu_i+1]{F_1(p) - \phi_i}$ as the local coordinate so that one gets $\partial_v z_i = -(\partial_v \phi_i)\frac{z_i^{-\nu_i}}{\nu_i+1}$, and more elaborate, but similar, computations of the series for $\partial_v F_2$ yield the desired result.

We will now overlap the foliations corresponding to the differentials $\Psi_1$ and $\Psi_2$.

**Definition 3.11.** Similarly to the above, we define $\mathcal{L}'$ to be the foliation of $\mathcal{M}_{g,1}(2)$ obtained by fixing the periods of the differential $\Psi_2$ (the leaves are of course again smooth tangentially complex and have expected codimension), and then let $\mathcal{S} := \mathcal{L} \cap \mathcal{L}'$ be the leaves of the "small" foliation — the intersections of the leaves of the two "big" foliations.

**Remark 3.12.** For any point $(\Gamma, p_0)$ the projection to $\mathcal{M}_{g,1}$ of the leaf $\mathcal{S} \subset \mathcal{M}_{g,1}(2)$ passing through $(\Gamma, p_0, z)$ is independent of $z$. Indeed, for a different choice of the local coordinate, $\Psi_1$ and $\Psi_2$ would be replaced by some $\mathbb{R}$-linear combinations, and the periods of such linear combinations would still be constant along a leaf $\mathcal{S}$. The resulting well-defined "small foliation" on $\mathcal{M}_{g,1}$ is a natural object from the point of view of algebraic geometry, worth further investigation.

**Remark 3.13.** The leaves of analogous "small" foliations, which can be defined for a moduli space of algebraic curves with a pair of differentials having poles of arbitrary but fixed order are central to several theories with distinct goals and origins. As shown in [31, 32], they provide a general solution to the Seiberg-Witten ansatz in the theory of $N = 2$ supersymmetric gauge theories. It was also shown that the Jacobian bundle over each leaf is the phase space of a completely integrable Hamiltonian system.

As discussed above, the leaves $\mathcal{L}$ and $\mathcal{L}'$ of the big foliations are smooth. Of course their intersection, the leaves of the small foliation $\mathcal{S} = \mathcal{L} \cap \mathcal{L}'$ a priori might have singularities. However, we make the following

**Conjecture 3.14** (Structural conjecture)**.** *All leaves $\mathcal{S}$ of the small foliation are smooth, and transverse to the loci $\widehat{\mathcal{D}}^\nu$, i.e. for any leaf $\mathcal{S}$ the intersection $\widehat{\mathcal{D}}^\nu \cap \mathcal{S}$, if non-empty, has expected codimension in $\mathcal{S}$ (complex codimension $2d_\nu$).*

The motivation for this conjecture comes from interpreting the leaves $\mathcal{S}$ as perturbations of the loci of spectral curves of the Calogero-Moser curves, see section 5. If true, the conjecture would immediately yield the following

**Corollary 3.15.** *For any set of multiplicities $\nu$ such that $d_\nu \geq g$ the locus $\mathcal{D}_n^\nu$ is empty. By theorem 2.6 the corresponding class*

$$\prod_{i=1}^{n}(K_0^2 + K_i^2 + 2K_0D_{0i})^{\nu_i}$$

*vanishes in homology of $\mathcal{C}_g^{1+n}$.*

*Proof.* Indeed, the leaves of the small foliation $\mathcal{S} \subset \mathcal{M}_{g,1}(2)$ have complex dimension $3g-3+1+2-2g = g$. Moreover, note that the loci $\widehat{\mathcal{D}}^\nu$ and the leaves $\mathcal{S}$ are in fact both well-defined on $\mathcal{M}_{g,1}(1)$. Considered on $\mathcal{M}_{g,1}(2)$, their intersection is the preimage of their intersection on $\mathcal{M}_{g,1}(1)$. Therefore, if the intersection $\widehat{\mathcal{D}}^\nu \cap \mathcal{S} \subset \mathcal{M}_{g,1}(2)$ is not empty, it has dimension at least 1. On the other hand, by the proposition above, the codimension of a non-empty intersection $\widehat{\mathcal{D}}^\nu \cap \mathcal{S}$ is equal to $d_\nu$, so that the dimension of this intersection is $g - d_\nu$. For this to be at least 1 we must then have $d_\nu < g$. □

We will now use the above description of the tangent spaces to the foliations to determine what properties of $\Psi_1$ and $\Psi_2$ would correspond to the leaves $\mathcal{S}$ being smooth.

Notationally, the constructions for $\mathcal{L}'$ are completely the same as for $\mathcal{L}$, and we denote them the same with an apostrophe added. Thus the image under $\widehat{\tau}$ of the tangent space to a "small" leaf $\mathcal{S}$ will lie in the intersection of the images of the maps $\tau$ and $\tau'$. Note, however, that if the jumps of a differential on $\Gamma^*$ are proportional to both $\Psi_1$ and $\Psi_2$, the jumps must be simply equal to zero. Thus the intersection $\mathcal{T} \cap \mathcal{T}'$ consists of differentials that are defined on $\Gamma$, and since such a differential cannot have a single simple pole, it is holomorphic. Thus we get the following restriction

$$\tau_c : T_X(\mathcal{S}) \longmapsto H^0(\Gamma, K_\Gamma) = \mathcal{T} \cap \mathcal{T}' \tag{18}$$

Note that while the maps $\widehat{\tau}$ and $\tau$ are only $\mathbb{R}$-linear, *the map $\tau_c$ is in fact $\mathbb{C}$-linear.*

Proposition 3.5 identifies the tangent space to $\mathcal{M}_{g,1}(2)$ and its image under the map $\tau$. Note that while the map $\tau$ changes rank on the loci $\mathcal{D}_n^\nu$, the moduli space is smooth, and we thus know that the dimension of its tangent space is the same at all points. We will now use the local description of the tangent spaces and of the map $\tau$ (rather its restriction $\tau_c$) to identify the loci where the leaves $\mathcal{S}$ of the small foliation may potentially have singularities.

**Definition 3.16.** We let $\Sigma \subset \mathcal{M}_{g,1}(2)$ be the locus where the divisor $E$ (of common zeroes of $\Psi_1$ and $\Psi_2$) is special, i.e. where $H^0(\Gamma, K_\Gamma - E) > g - \deg E$ or equivalently $H^0(\Gamma, E) > 1$ (in particular if $\Psi_1$ and $\Psi_2$ have no common zeroes, or only one simple common zero, this is not a point of $\Sigma$).

**Lemma 3.17.** *The singular locus of a leaf $\mathcal{S}$ is equal to the intersection $\mathcal{S} \cap \Sigma$.*

*Proof.* The image of the tangent space to $\mathcal{S}$ at a point $X = (G, p_0) \in \widehat{\mathcal{D}}_n^\nu \cap \mathcal{S}$ under the map $\tau_c$ given by (18), being the intersection of the tangent spaces to $\mathcal{L}$ and $\mathcal{L}'$ computed in proposition 3.5, is equal to $H^0(\Gamma, K_\Gamma - E)$. Since the complex dimension of $\mathcal{S} \subset \mathcal{M}_{g,1}(2)$ is equal to $g$, the leaf $\mathcal{S}$ is smooth whenever $\Psi_1$ and $\Psi_2$ have no common zeroes, since in that case the tangent space to it is $H^0(\Gamma, K_\Gamma)$, with the map $\tau_c$ being an isomorphism.

In general the image $\tau_c(T_X(\mathcal{S}))$ has dimension $h^0(\Gamma, K_\Gamma - E)$, while the kernel of $\tau_c$ on $T_X(\mathcal{S})$ always has dimension $\deg E$. Thus for $X$ to be a smooth point of $\mathcal{S}$ we must have $g = \dim T_X(\mathcal{S}) = H^0(\Gamma, K_\Gamma - E) + \deg E$, which is equivalent to the divisor $E$ being non-special. □

From the above description we see that the tangent space $T_X(\widehat{\mathcal{D}}_n \cap \mathcal{S})$ is isomorphic to $H^0(\Gamma, K_\Gamma - E)$, and thus away from $\Sigma$ has dimension $g - \deg E$, as expected (this also implies that a leaf $\mathcal{S}$ away from $\mathcal{S} \cap \Sigma$ intersects the locus $\widehat{\mathcal{D}}_n$ transversely). If conjecture 3.14 holds, by the lemma above each leaf $\mathcal{S}$ would have an empty intersection with $\Sigma$. Since the small leaves foliate $\mathcal{M}_{g,1}$, the structural conjecture implies (and is in fact equivalent to) the following

**Conjecture 3.18** (Non-speciality conjecture)**.** *The divisor $E$ is never special on $\Gamma$, i.e. the locus $\Sigma$ is empty.*

Notice that the differential $\Psi_1 + i\Psi_2$ is holomorphic, vanishing at all common zeroes of $\Psi_1$ and $\Psi_2$. Thus no matter what the divisor of common zeroes $E$ is, we have $h^0(\Gamma, K_\Gamma - E) > 0$. However, for $\deg E \geq g$, the expected dimension of this space is zero, and thus we see that if $\deg E \geq g$, the divisor $E$ must be special. The non-speciality conjecture would thus imply $\deg E < g$, and in particular would imply that $\mathcal{D}_n^\nu \subset \mathcal{M}_{g,1+n}$ is empty if $\sum \nu_i > g$. We note that from Faber's vanishing conjecture, one would expect this for $g \le d_\nu = \sum_{i=1}^n (2\nu_i - 1)$. Our vanishing result, theorem 2.8, which we prove in the next section without using the conjecture, would thus be a direct corollary of the non-speciality conjecture in the case of all multiplicities equal to 1, where these two bounds agree.

**Remark 3.19.** The non-speciality conjecture would imply many more vanishing results, for some of which it is not a priori clear if the vanishing follows by Faber's conjecture. For example if $F \subset \mathcal{M}_{g,3}(2,0,0)$ is the locus of $(\Gamma, p_0, p_1, p_2)$ where $p_1 + p_2$ is a special divisor (which is thus a hyperelliptic component), then the non-speciality also implies $F(\psi_0^2 + \psi_2^2)(\psi_0^2 + \psi_3^2) = 0$. Computing the class of $F$, and of more general special loci of this kind, is thus a natural problem. Recently some computations of the strata in $\mathcal{H}_g$ were done in [21], and it would be interesting to see if the techniques of that work could be used to obtain further vanishing results.

## 4. The main vanishing result: proof by degeneration

In this section we use the infinitesimal description of the tangent spaces to $\mathcal{L}$ and to $\mathcal{D}$ developed above to prove our main vanishing result, the statement that the locus $\mathcal{D}_g$ is empty.

*Proof of theorem 2.8.* The idea of proving that $\Psi_1$ and $\Psi_2$ cannot have $g$ distinct common zeroes is to use a degeneration argument to eventually apply an induction in genus. We first deal with the base case of induction: that for $g = 1$ the differentials $\Psi_1$ and $\Psi_2$ cannot have a common zero on an elliptic curve. This is straightforward: If $\Psi_1$ and $\Psi_2$ have a common zero, their linear combination $\Psi_1 + i\Psi_2$ also vanishes at the same point. However, a holomorphic differential is nowhere zero on an elliptic curve, and we have a contradiction.

Now assume that $\mathcal{D}_g$ is non-empty for some genus $g$. For the case of simple zeroes, on an open subset of $\widehat{\mathcal{D}}_g^* \subset \widehat{\mathcal{D}}_g$ condition (17) simply means that $\Omega$ vanishes at $p_i$. From propositions 3.5 and 3.8 it follows that for $X \in \widehat{\mathcal{D}}_g^*$ the map $\hat{\tau} : T_X(\mathcal{D}_g^*) \to \widehat{\mathcal{T}}_g$ is an isomorphism, while $\widehat{\tau}(T_X(\mathcal{M}_{g,1}(2))) = \widehat{\mathcal{T}}_g$ is the same space.

We further compute $\widehat{\tau}(T_X(\widehat{\mathcal{D}}_g \cap \mathcal{L})) = \widehat{\mathcal{T}}_g \cap \mathcal{T} = \mathcal{T}_g$ to be the space of differentials with jumps proportional to $\Psi_1$ and vanishing at $p_1, \ldots, p_g$. From lemma 3.6 it follows that $\mathcal{T}_g \subset \mathcal{T}$ has complex codimension $g$, equal to the codimension of $\mathcal{L} \subset \mathcal{M}_{g,1}(2)$. Thus the complex codimension of $\mathcal{L} \cap \widehat{\mathcal{D}}_g \subset \widehat{\mathcal{D}}_g$ is equal to $g$. Since the codimensions of $\mathcal{L}$ and of

$\mathcal{D}_g$ in $\mathcal{M}_{g,1}(2)$ are both $g$, their intersection is codimension $2g$ (and dimension $g$). For dimension reasons it thus follows from the above that $T_X(\mathcal{D}_g)/T_X(\mathcal{L}\cap\mathcal{D}_g) = T_X(\mathcal{M}_{g,1}(2))/T_X(\mathcal{L})$. Since the right-hand-side is the local parameter space for the leaves of the big foliation near $X$, this means that $\mathcal{D}_g$ must intersect all big leaves near $X$. Since the argument can be applied near every point of $\mathcal{D}_g$, we obtain the following

**Lemma 4.1.** *The locus $\mathcal{D}_g$, if non-empty, must intersect every leaf $\mathcal{L}$ of the "big" foliation by absolute periods of $\Psi_1$.*

The idea is now to consider a family of leaves of $\mathcal{L}$ where a suitable period of $\Psi_1$ approaches zero, and to guarantee that in the limit we would have a nodal curve, still with many common zeroes of $\Psi_1$ and $\Psi_2$. To find which period to shrink, we use the specifics of the situation of a meromorphic differentials with one double pole and real periods.

We consider the structure on the surface similar to that defined by a holomorphic differential. Indeed, recall that the imaginary part of the abelian integral $f_1(p) := \operatorname{Im} F_1 = \operatorname{Im}\int^p \Psi_1$ is a well-defined global real-valued function on the Riemann surface $\Gamma\setminus\{p_0\}$, and taking value $\infty$ at $p_0$ (it does not have jumps as all periods of $\Psi_1$ are real). Then any level set $C_c := \{f_1(p) = c\}$ for $c$ an arbitrary real number is a (real) curve on $\Gamma$, passing — or we could say approaching in the limits at plus and minus infinity — the point $p_0$. Away from any zeroes of $\Psi_1$ lying on it, the (locally defined) real part of $F_1$ gives a local coordinate on $C_c$ — in particular, $C_c$ is smooth away from any zeroes of $\Psi_1$ that lie on it. Moreover, note that the real part of $F_1$ has to be monotonous along $C_c$, and we will choose orientation on each $C_c$ so that $\operatorname{Re} F_1$ is monotonically increasing (so far this construction is the same as for holomorphic differentials, except that all our level sets pass through $p_0$). For a generic value of $c$ the curve $C_c$ will avoid any of the finitely many zeroes of $\Psi_1$ and will be smooth. The set of curves $C_c$ for $c\in\mathbb{R}$ fills out $\Gamma\setminus\{p_0\}$: any point $p\in\Gamma$ lies on $C_c$ for some $c$, and $C_{c_1}\cap C_{c_2} = \{p_0\}$ for $c_1\neq c_2$.

Recall that the absolute and relative periods of $\Psi_1$ give local coordinates at any point of $\mathcal{M}_{g,1}(2)$; the relative periods gives local coordinates on a leaf $\mathcal{L}$; recall also from propositions 3.5 and 3.8 that local coordinates on $\mathcal{D}_g\cap\mathcal{L}$ are given by those relative periods of $\Psi_1$ that correspond to the zeroes of $\Psi_1$ that are not zeroes of $\Psi_2$. In particular we could choose a generic point $X\in\mathcal{D}_g\cap\mathcal{L}$ such that $p_1,\dots,p_g$ are the common zeroes of $\Psi_1$ and $\Psi_2$, and $q_1,\dots,q_g$ are the zeroes of $\Psi_1$ that are not zeroes of $\Psi_2$; we can ensure that $q_1,\dots,q_g$ are all distinct. Moreover, we can choose $X$ in such a way that $f_1(q_1)$ is not equal to the value of $f_1$ at any other zero of $\Psi_1$. Notice that all of the above conditions are open, and we can thus consider a small open neighborhood of $X$ where all of them will be satisfied.

Consider now the level set $C_c$ containing the zero $q_1$ of $\Psi_1$ — it will be a self-intersecting curve, starting from $p_0$, passing more than once through $q_1$, and returning to $p_0$. Consider the loop $\gamma$ formed on $\Gamma$ by the curve $C_c$ from the first time it passes through $q_1$ until the next. Since $f_1$ is constant on $\gamma$, and the real part of $F_1$ is monotonous increasing on this loop, it follows that $\int_\gamma \Psi_1$ is a positive real number. In particular, $\gamma$ is a homologically non-trivial loop on $\Gamma$. Since the choice of $\gamma$ does not require choosing a basis for the homology or the fundamental group of $\Gamma$, the class $\gamma$ is well-defined for any $X \in \mathcal{M}_{g,1}(2)$ with labeled zeroes of $\Psi_1$, in particular for any $X \in \mathcal{D}_g$.

We will now perturb a small neighborhood of $X$ in $\mathcal{D}_g \cap \mathcal{L}$ to a small neighborhood of some point $X_t$ lying in the intersection of $\mathcal{D}_g$ with another leaf $\mathcal{L}_t$ of the big foliation. More precisely, consider a family of leaves $\mathcal{L}_t$ of the big foliation, parameterized by $t \in (0, 1]$, determined by the condition that on $\mathcal{L}_t$ all the periods of $\Psi_1$ are the same as on $X$, except for the period over $\gamma$, which is multiplied by $t$ (thus $\mathcal{L}_1 = \mathcal{L}$). By the lemma we can also choose a family $X_t \subset \mathcal{M}_{g,1}(2)$ such that $X_t \in \mathcal{L}_t \cap \mathcal{D}_g$ and $X_1 = X$. Moreover, for $\Psi_1$ and $\Psi_2$ to have an extra common zero is one complex condition (that $\Psi_2$ is zero at some zero of $\Psi_1$), and thus we can choose a family $X_t$ such that the number of common zeroes of $\Psi_1$ and $\Psi_2$ is constant in the family, and moreover the multiplicities of all zeroes of $\Psi_1$ are constant in the family, so that the above construction works on any $X_t$: we thus have a well-defined family of zeroes $q_1(t) \in \Gamma_t$ such that $q_1(1) = q_1 \in \Gamma$, and a well-defined family of level sets of $f_1$ on each $\Gamma_t$, passing through $q_1(t)$. By continuity the homology class of $\gamma_t$ must then be constant and thus equal to $\gamma$.

Consider now the limit of $X_t$ as $t \to 0$. Notice that the locus $\mathcal{D}_g \subset \mathcal{M}_{g,1}(2)$ by construction is a preimage of a locus in $\mathcal{M}_{g,1}$, and thus in the limit as $t \to 0$ the Riemann surfaces $(\Gamma_t, p_0)$ must converge to some point in $(\Gamma_0, p_0) \in \overline{\mathcal{M}}_{g,1}$. In the limit $t \to 0$, by construction the integral of $\Psi_1$ over $\gamma$ approaches zero. However, since the real part of $F_1$ is monotonically increasing along $C_c \supset \gamma$, this integral can only be zero if the path $\gamma$ is pinched to zero. Thus the curve $\Gamma_0$ must be degenerate, and the zero $q_1(0)$ of $\Psi_1$ on $\Gamma_0$ must coincide with the node of $\Gamma_0$.

Moreover, notice that since all the other (absolute) periods of $\Psi_1$ on $\Gamma_0$, except the one over $\gamma$, were preserved, no other degeneration is possible: the normalization of $\Gamma_0$ at $q_1(0)$ is a smooth curve. Since a non-trivial homology element $\gamma$ was pinched, the normalization of $\Gamma_0$ at $q_1(0)$ is smooth, of genus $g-1$. Recall from [15] that the differentials of the second kind with real periods cannot develop singularities in the limit, and thus the lifts of $\Psi_1$ and $\Psi_2$ to the normalization of $\Gamma_0$ (at $q_1(0)$) are simply the corresponding differentials with real periods on this normalization. Any zero $p_i(t)$ or $q_i(t)$ that does not approach $q_1(0)$ in the limit remains a zero of the lifting of $\Psi_1$ to the normalization of $\Gamma_0$ (common with $\Psi_2$ if it is the limit of $p_i(t)$), while the multiplicity of any zero of $\Psi_1$ approaching the node $q_1(0)$ in the limit $t \to 0$ decreases by one under the normalization (blowup map). In genus $g-1$ the differential $\Psi_1$ has $2g-2$ zeroes, and thus exactly two zeroes of $\Psi_1$ on $\Gamma_t$ must approach the node $q_1(0)$ in the limit. One of these zeroes is of course $q_1(0)$, and it thus follows that at most one common zero $p_i(t)$ of $\Psi_1$ and $\Psi_2$ may disappear in the limit. Thus on the normalization of $\Gamma_0$ the differentials $\Psi_1$ and $\Psi_2$ will have at least $g-1$ common zeroes. By induction in $g$ it follows that the locus $\mathcal{D}_g$ is empty for any genus $g$. □

## 5. The Calogero-Moser integrable system and leaves of the foliation

In general the leaves $\mathcal{L}$, $\mathcal{S}$ are not algebraic, and could be everywhere dense in the moduli space. In this section we will show that a certain everywhere dense set of leaves of the small foliation actually arise as loci of normalizations of spectral curves of the Calogero-Moser system. Thus our foliations provide a perturbation theory for the Calogero-Moser integrable system. Algebro-geometrically the Calogero-Moser spectral curves are certain ramified covers of an elliptic curve; we will use coordinates on these loci related to the branch points of such a cover, similar to the Lyashko-Looijenga coordinates for the Hurwitz spaces of covers of $\mathbb{P}^1$. While not used the proof of our vanishing result above, this description is of independent interest for studying the geometry of the Calogero-Moser system, and also motivates the conjectures we made.

*Teichmüller theory viewpoint: Calogero-Moser loci as leaves of the foliation.*

**Definition 5.1.** We define the Calogero-Moser locus $\mathcal{K}_g \subset \mathcal{M}_{g,1}$ to be the locus of all $(\Gamma, p_0)$ for which there exist two linearly independent differentials of the second kind $\Phi_1, \Phi_2 \in H^0(\Gamma, K_\Gamma + 2p_0)$ with all periods *integer*.

Since the space of differentials of the second kind (sections of $K_\Gamma + 2p_0$) with real periods has real dimension 2, such $\Phi_1, \Phi_2$ then give an $\mathbb{R}$-basis of it. Thus for any choice of the local coordinate $(\Gamma, p_0, z) \in \mathcal{M}_{g,1}(2)$ the corresponding differentials $\Psi_1, \Psi_2$ with real periods can be written as $\Psi_i = r_{i,1}\Phi_1 + r_{i,2}\Phi_2$, with some real coefficients $r_{i,j}$. It thus follows that the following gives an alternative definition of the Calogero-Moser locus.

**Definition 5.2.** The Calogero-Moser locus $\widehat{\mathcal{K}}_g \subset \mathcal{M}_{g,1}(2)$ is the union of all leaves of the small foliation $\mathcal{S}$ for which there exist real numbers $r_{1,1}, r_{1,2}, r_{2,1}, r_{2,2}$ such that all (real by definition) periods of $\Psi_i$ lie in $r_{i,1}\mathbb{Z} + r_{i,2}\mathbb{Z}$. This $\widehat{\mathcal{K}}_g$ is then the preimage of the locus $\mathcal{K}_g$ as defined above.

If $\Phi_1$ and $\Phi_2$ have integer periods, then of course any $\mathbb{Z}$-linear combination of them will also have integer periods, and thus on $(\Gamma, p_0) \in \widehat{\mathcal{K}}_g$ there exists a whole lattice of differentials with integer periods. For a given Calogero-Moser curve $(\Gamma, p_0) \in \mathcal{K}_g$ we choose $\Phi_1, \Phi_2$ generating this lattice over $\mathbb{Z}$, and let

$$\tau := \frac{\Phi_2}{\Phi_1}(p_0) \tag{19}$$

(which means taking the ratio of the singular parts of $\Phi_2$ and $\Phi_1$ at $p_0$). Since $\Phi_i$ are $\mathbb{R}$-linearly independent, $\mathrm{Im}\tau \neq 0$, and by swapping $\Phi_1$ and $\Phi_2$ if necessary we may assume that $\mathrm{Im}\,\tau > 0$. From the definition of $\tau$ it follows that $dz := \Phi_2 - \tau\Phi_1$ is a holomorphic differential on $\Gamma$ with all periods lying in the lattice $\mathbb{Z} + \tau\mathbb{Z}$. Thus integrating it defines a holomorphic map to an elliptic curve

$$z : \Gamma \to E := \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}; \qquad p \mapsto z(p) := \int_{p_0}^{p} dz \tag{20}$$

Note that $\tau$ depends on the choice of generators $\Phi_i$ in the lattice of differentials with integer periods, but the corresponding elliptic curve $E$, and the map $z$ do not (so $\tau$ is defined up to a $PSL(2, \mathbb{Z})$ action).

**Definition 5.3.** For $N \in \mathbb{Z}$ we denote $\mathcal{K}_{g,N} \subset \mathcal{K}_g$ the locus of those curves for which the degree of the map $z : \Gamma \to E$ is equal to $N$. We will see that $\mathcal{K}_{g,N}$ is non-empty only if $N \geq g$.

Since the degree depends continuously on a curve in $\mathcal{K}_g$, being an integer it is locally constant on $\mathcal{K}_g$, and thus each $\mathcal{K}_{g,N}$ is a union of some connected components of $\mathcal{K}_g$. Analytically, $N$ can be computed as

$$N=\sum_{k=1}^{g}\left(\int_{a_k}\Phi_2\int_{b_k}\Phi_1-\int_{b_k}\Phi_2\int_{a_k}\Phi_1\right)=2\pi i\,\mathrm{res}_{p_0}\left(F_1\Phi_2\right). \tag{21}$$

In what follows we will always assume that the generators $\Phi_1,\Phi_2$ of the lattice of differentials with integral periods are fixed, or equivalently we consider a fixed elliptic curve $E$ with a fixed basis of cycles. We denote $\mathcal{K}_{g,N}^{\tau}\subset\mathcal{K}_{g,N}\subset\mathcal{K}_g\subset\mathcal{M}_{g,1}(2)$ the locus of curves in $\mathcal{K}_{g,N}$ mapping to this fixed $E$. By construction $\mathcal{K}_{g,N}^{\tau}$ is a union of some leaves of the small foliation.

*Integrable systems viewpoint: spectral curves of the Calogero-Moser integrable system.*

The reason we call the above the loci of Calogero-Moser curves is that they are actually spectral curves of the Calogero-Moser integrable system. We will now prove this statement. In particular, it will follow that the transcendental construction of the Calogero-Moser locus above is in fact algebraic — that Calogero-Moser curves are a special class of covers of elliptic curves. We start by recalling the necessary facts from the theory of the elliptic Calogero-Moser system.

**Definition 5.4.** The elliptic Calogero-Moser (CM) system introduced in [3] is a system of $N$ particles on an elliptic curve $E$ with pairwise interactions. The phase space of this system is

$$\mathcal{P}_N:=(\mathbb{C}\times E)^{\times N}\setminus\{\text{diagonals in } E\}=\{q_i,\dots,q_N\in\mathbb{C},x_1,\dots,x_N\in E,x_i\neq x_j\}, \tag{22}$$

where we think of the variables $x_i$ as the positions of the particles, and of $q_i$ as their momenta, elements of the trivial tangent space to $E$, denoting $q_i=\dot{x}_i$ (the dot denotes the time derivative). The Calogero-Moser Hamiltonian is the function $H_2:\mathcal{P}_N\to\mathbb{C}$ defined as

$$H_2:=\frac{1}{2}\sum_{i=1}^{N}q_i^2-2\sum_{i\neq j}\wp(x_i-x_j), \tag{23}$$

In [22] the second-named author showed that the equations of motion of the elliptic CM system admit Lax representation with "elliptic spectral parameter $z$". This is to say that the Hamiltonian equations of motion for the CM system are equivalent to the matrix-valued differential equation $\dot{L}=[L,M]$, where $L=L(z)$ and $M=M(z)$ are certain $N\times N$ matrices depending on the point $z\in E$. Explicitly, the entries of the matrix $L$ are given by

$$L_{ii}(t,z)=\frac{1}{2}\,q_i,\quad L_{ij}=F(x_i-x_j,z),\quad i\neq j, \tag{24}$$

where $F$ is defined as

$$F(x,z) := \frac{\sigma(z-x)}{\sigma(z)\sigma(x)} e^{\zeta(z)x}, \tag{25}$$

with $\zeta$ and $\sigma$ the standard Weierstrass functions. The entries of the matrix $M(z)$ are given by

$$M_{ij}(z) := \left(\wp(z) - 2\sum_{j\neq i} \wp(x_i - x_j)\right)\delta_{ij} - 2(1-\delta_{ij})F'(x_i - x_j, z). \tag{26}$$

**Definition 5.5.** The spectral curve $\Gamma_{cm}$ of the CM system is the normalization at the point $(k,z)=(\infty,0)$ of the closure in $\mathbb{P}^1 \times E$ of the affine curve given in $\mathbb{C} \times (E \setminus \{0\})$ by

$$R(k,z) = \det(kI + L(z)) =: k^N + \sum_{i=1}^{N} r_i(z) k^{N-i} = 0, \tag{27}$$

where $I$ is the identity matrix, and we simply denoted the coefficient of $k^{N-i}$ in the expansion of the determinant by $r_i$.

The normalized spectral curve $\widetilde{\Gamma}_{cm}$ is a smooth algebraic curve that is the normalization of $\Gamma_{cm}$ (note that $\Gamma_{cm}$ might have singularities in the affine part).

From the Riemann-Hurwitz formula it follows that the arithmetic genus of $\Gamma_{cm}$ is equal to $N$; thus the genus of its normalization $\widetilde{\Gamma}_{cm}$ is less than $N$ if and only if $\Gamma_{cm}$ is singular.

From the explicit formulas (24) for $L$ one sees that each $r_i$ is a meromorphic function on $E$ with a pole of order $i$ at $z=0$. As shown in [22], near $z=0$ the characteristic polynomial $R(k,z)$ admits a factorization of the form

$$R(k,z) = \prod_{i=1}^{N} (k + a_i z^{-1} + h_i + O(z)), \tag{28}$$

with $a_1 = 1-N$ and $a_i = 1$ for $i > 1$. This implies that the closure in $\mathbb{P}^1 \times E$ of the affine curve in $\mathbb{C} \times (E \setminus 0)$ given by equation (27) is obtained by adding one point $(\infty, 0)$. Among the $N$ branches of this closure passing through $(\infty,0)$, there are $N-1$ branches tangent to each other and one branch transverse to them. After blowing up the point $(\infty,0)$ once, we get a smooth point $p_0$ corresponding to the transverse branch, and a singular point, where generically the $N-1$ branches passing through it are transverse. Thus generically the second blowup at this singular point would give a smooth algebraic curve, which is then equal to $\Gamma_{cm}$, abd obtained from the affine curve (27) by adding $N$ smooth points.

The family of CM curves parameterized by equations of the form (28) is $N$-dimensional. From the explicit parametrization of the family obtained in [5] it follows that the parameter space for this family is in fact $\mathbb{C}^N$. Indeed, in [5] it is shown that

(29)
$$R(k,z) = f(k - \zeta(z), z), \quad \text{where } f(\phi, z) = \frac{1}{\sigma(z)}\, \sigma\left(z + \frac{\partial}{\partial\phi}\right) H(\phi)$$

for $H(\phi) = \phi^N + \sum_{i=0}^{N-1} I_i \phi^i$ some polynomial in $\phi$. Writing out this formula explicitly yields

$$f(\phi, z) = \frac{1}{\sigma(z)} \sum_{n=0}^{N} \frac{1}{n!}\, \partial_z^n \sigma(z) \left(\frac{\partial}{\partial\phi}\right)^n H(\phi). \tag{30}$$

The coefficients $I_i$ of the polynomial $H(\phi)$ then give the coordinates on the space $\mathbb{C}^N$ parameterizing the CM curves.

The construction of CM curves was crucial for the identification of the theory of the CM system and the theory of the elliptic solutions of the Kadomtsev-Petviashvili (KP) equation established in [22]. This identification is based on the following result:

**Lemma 5.6.** *The equation*

$$\left(\partial_t - \partial_x^2 + u(x,t)\right) \psi(x,t) = 0 \tag{31}$$

*with elliptic potential (i.e. u(x,t) is an elliptic function of the variable x) has a meromorphic in $x$ solution $\psi$ if and only if $u$ is of the form*

$$u = 2 \sum_{i=1}^{N} \wp(x - x_i(t)) \tag{32}$$

*with poles $x_i(t)$ satisfying the equations of motion of the CM system.*

**Remark 5.7.** In [22] a slightly weaker form of the lemma was proven. Namely, its assertion was proved under the assumption that equation (32) has a family of *double-Bloch* solutions (i.e. meromorphic solutions with monodromy $\psi(x + \omega_\alpha, t) = w_\alpha \psi(x,t)$, where $\omega_a$ are periods of the elliptic curve and $w_a$ are constants.) This weaker version is sufficient for our further purposes, but for completeness we included above the strongest form of the lemma, proven in [27] (see [28] for details).

*Spectral curves for the elliptic CM system are the small leaves.*

We now relate the spectral curves of the CM system to the leaves of the foliation.

**Lemma 5.8.** *Any normalized spectral curve $\widetilde{\Gamma}_{cm}$ of the Calogero-Moser system lies in the Calogero-Moser locus $\mathcal{K}_{g,N}^{\tau}$.*

*Proof.* We define the constants $c_1, c_2$ by the identities

$$\int_0^1 (\wp(z) - c_1)\, dz = 0, \quad \int_0^\tau (\wp(z) - c_2) dz = 0. \tag{33}$$

Then Riemann's bilinear relations imply

$$\int_0^\tau (\wp(z) - c_1)\, dz = 2\pi i, \quad \int_0^1 (\wp(z) - c_2) dz = \frac{2\pi i}{\tau}. \tag{34}$$

The differentials $\Phi_1, \Phi_2$ on $\widetilde{\Gamma}_{cm}$ are then the pullbacks of the differentials on $\Gamma_{cm}$ defined explicitly by

$$\Phi_1 := \frac{1}{2\pi i}\left(dk - (\wp(z) - c_1) dz\right), \qquad \Phi_2 := \frac{\tau}{2\pi i}\left(dk - (\wp(z) - c_2) dz\right). \tag{35}$$

Indeed, from (28) it follows that $\Phi_i$ on $\Gamma_{cm}$ has a single second order pole at $p_0$, and is holomorphic elsewhere (one only needs to check the other points in the preimage of $(\infty, 0)$ under normalization). To see that all periods of $\Phi_i$ are integral, first note that the periods on $\Gamma_{cm}$ of the exact differential $dk$ (where $k$ is the $\mathbb{P}^1$ coordinate, i.e. a meromorphic function on $\Gamma_{cm}$) are of course zero. Since the differential $(\wp(z) - c_i)dz$ on $\Gamma_{cm}$ is a pullback of a differential on $E$, its period over a cycle in $\Gamma_{cm}$ is equal to a period over the image of this cycle on $E$. Thus periods of $(\wp(z) - c_i)dz$ on $\Gamma_{cm}$ are integer linear combinations of its integrals from 0 to 1 and from 0 to $\tau$ on $E$, which are integral by (33),(34).

For generic values of the free parameters $I_i$ of the CM system, the spectral curve $\Gamma_{cm}$ is smooth, of genus $N$, admitting a degree $N$ cover $\Gamma_{cm} \to E$; thus we would have $\widetilde{\Gamma}_{cm} = \Gamma_{cm} \in \mathcal{K}_{N,N}$. If $\Gamma_{cm}$ is singular, the pullbacks of $\Phi_i$ to $\widetilde{\Gamma}_{cm}$ still have a unique double pole at the preimage of the smooth point $p_0 \in \Gamma_{cm}$, and their periods on $\widetilde{\Gamma}_{cm}$, being a subset of their periods on $\Gamma_{cm}$, will still be integral. Thus we must then have $(\widetilde{\Gamma}_{cm}, p_0) \in \mathcal{K}_{g,N}$, clearly with $g < N$. □

We will now show that the two definitions of the Calogero-Moser curves coincide, i.e. that any curve in $\mathcal{K}^\tau_{g,N}$ arises as the normalization of a spectral curve of the CM system, and thus as a cover of an elliptic curve.

**Proposition 5.9.** *The locus $\mathcal{K}_g$ is equal to the locus of smooth genus $g$ curves that are normalizations of some spectral curve of the Calogero-Moser system. In fact the locus $\mathcal{K}_{g,N}$ is the locus of curves that are normalizations $\widetilde{\Gamma}_{cm}$ of spectral curves $\Gamma_{cm}$ of the $N$-particle Calogero-Moser system (i.e. of curves given by (27)).*

*Proof.* One direction of the proposition is lemma 5.8. The proof of the converse statement, that any curve $(\Gamma, p_0)$ with associated $\Phi_1, \Phi_2$ with integer periods lies in $\mathcal{K}_{g,N}^\tau$, essentially reduces to the statement that the general construction of algebro-geometric solutions of the KP equation proposed in [29, 30] in the case when $(\Gamma, p_0) \in \mathcal{K}_{g,N}^\tau$ leads to elliptic solutions.

To explain this, we recall the definition of the Baker-Akhiezer function and related constructions.

**Definition 5.10.** For $(\Gamma, p_0, z) \in \mathcal{M}_{g,1}(2)$, and a generic set of $g$ points $\gamma_1, \ldots, \gamma_g \in \Gamma$, the Baker-Akhiezer function $\psi(x,t,p) : \mathbb{C} \times \mathbb{C} \times \Gamma \to \mathbb{C}$ is the unique function meromorphic on $\Gamma \setminus \{p_0\}$, with simple poles at $\gamma_i$, and having an essential (exponential) singularity at $p_0$, such that in a neighborhood of $p_0$ it admits an expression of the form

$$\psi(x,t,z) = e^{xz^{-1}+tz^{-2}} \left(1 + \sum_{s=1}^{\infty} \xi_s(x,t) z^{-s}\right) \tag{36}$$

where $\xi_s$ are some holomorphic functions $\mathbb{C} \times \mathbb{C} \to \mathbb{C}$.

In [29] the following explicit expression for the Baker-Akhiezer function was obtained:

$$\psi(x,t,p) = \frac{\theta(A(p) + Ux + Vt + Z_0)\, \theta(A(p_0) + Z_0)}{\theta(A(p_0) + Ux + Vt + Z_0)\, \theta(A(p) + Z_0)}\, e^{x\int^p \Omega_2 + t\int^p \Omega_3}, \tag{37}$$

where $A : \Gamma \hookrightarrow J(\Gamma)$ is the Abel-Jacobi embedding of the curve into the Jacobian, and $U$ and $V$ are the vectors of $B$-periods of the normalized (i.e. with all $A$-periods zero) differentials $\Omega_2$ and $\Omega_3$, with poles at $p_0$ of second and third order, respectively. The vector $Z_0$ is the image of the divisor $\gamma_1 + \ldots + \gamma_g$ under the Abel map.

As shown in [29], the Baker-Akhiezer function satisfies partial differential equation (31) with the potential $u(x,t)$ given explicitly as

$$u(x,t) = 2\partial_x^2 \ln \theta(Ux + Vt + Z_0). \tag{38}$$

Now let us show that the curves $\Gamma$ with marked point $p_0$ in $\mathcal{K}_g$ can be characterized by the property that the vector $U$ in (38) spans an elliptic curve in the Jacobian (i.e. $\mathbb{C}U \subset J(\Gamma)$ is closed).

Recall that $U$ is the vector of $B$-periods of the normalized meromorphic differential $\Omega_2$ above. Note that the differentials $\Omega_2 - \Phi_1$ and $\tau\Omega_2 - \Phi_2$ are holomorphic. Since all the $A$-periods of $\Omega_2$ are zero, the $A$-periods of these two differentials are all integer, and thus both $\Omega_2 - \Phi_1$ and $\tau\Omega_2 - \Phi_2$ are integer linear combinations of a basis $\omega_1 \ldots \omega_g$ of holomorphic differentials on $\Gamma$ dual to the $A$-cycles. Thus the vector

$U$ of $B$-periods of $\Omega_2$ is a sum of the integer vector of $B$-periods of $\Phi_1$ and of some integral linear combination of $B$-periods of $\omega_i$. Thus $U \sim 0 \in \mathbb{C}^g/\mathbb{Z}^g + \tau_\Gamma \mathbb{Z}^g$ (where $\tau_\Gamma$ is the period matrix of $\Gamma$). Similarly it follows that $\tau U$ also lies in the lattice of periods of $\Gamma$, and thus the vector $U$ spans an elliptic curve in $J(\Gamma)$.

Let $N$ be the degree of the restriction of the theta function from $J(\Gamma)$ to any translate of the elliptic curve $E$ generated by $U$. Since any algebraic function on an elliptic curve can be expressed in terms of the elliptic $\sigma$ function, we can write

$$\theta(\tau_\Gamma, Ux + Vt + Z_0) = f(t, Z_0) \prod_{i=1}^{N} \sigma(x - x_i(t)) \tag{39}$$

where $f$ is a suitable non-vanishing holomorphic function. Substituting this expression into (38) implies that $u$ is of the form (32). The Baker-Akhiezer function is a meromorphic function of $x$. From lemma 5.6 it then follows that $q_i(t)$ in (39) satisfy the equations of motion of the CM system.

**Remark 5.11.** For the last statement a weaker version of lemma 5.6 suffices, because from the definition of the Baker-Akhiezer function it follows that it has monodromy given by

$$\psi(x+1, t, p) = e^{2\pi i F_1(p)} \psi(x, t, p), \qquad \psi(x+\tau, t, p) = e^{2\pi i F_2(p)} \psi(x, t, p). \tag{40}$$

Such monodromy was called in [33] the double-Bloch property. Geometrically, it means that as a function of $x$, $\psi$ is a (meromorphic) section of a certain bundle on the elliptic curve $E = \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$.

Thus, starting from a curve $\Gamma \in \mathcal{K}_{g,N}$, we have recovered a solution of the CM system, and can use equation (27) to define a Calogero-Moser curve $\Gamma_{cm}$. In order to show that its normalization $\widetilde{\Gamma}_{cm}$ coincides with the original curve it is enough to check that

$$\psi(x, t, p) = \sum_{i=1}^{N} c_i(t, p) F(x - x_i(t), z) e^{kx + k^2 t} \tag{41}$$

satisfy all the defining properties of the Baker-Akhiezer function on $\widetilde{\Gamma}_{cm}$. Here the functions $c_i$ are coordinates of the vector $C = (c_1, \ldots, c_N)$ satisfying

$$(L(t, z) + k)C = 0, \quad \partial_t C = M(t, z)C\,. \tag{42}$$

This verification is straightforward, and the proof is thus complete. □

Since we have constructed all curves in $\mathcal{K}_g$ as normalizations of spectral curves of the Calogero-Moser system, and normalizing can only reduce the arithmetic genus, we get in particular

**Corollary 5.12.** *The locus $\mathcal{K}^{\tau}_{g,N}$ is empty if $g > N$.*

**Remark 5.13.** A further investigation into singularities of the Calogero-Moser curves is of independent interest, and motivated our conjectures. To start with, one easily sees that in the first nontrivial case $N = g$, i.e. on $\mathcal{K}_{g,g}$, the differentials $\Phi_1$ and $\Phi_2$ cannot have common zeroes. Indeed, if for some point $p \in \Gamma_{cm}$ we have $\Phi_1(p) = \Phi_2(p) = 0$, then also $dk(p) = dz(p) = 0$, as these two differentials are linear combinations of $\Phi_1$ and $\Phi_2$. However, if both $dk$ and $dz$ vanish at a point of the closure in $\mathbb{P}^1 \times E$ of the affine curve, this point is singular, while we assumed the curve to be smooth. Following this line of thought, one would expect the common zeroes of $\Phi_1$ and $\Phi_2$ on Calogero-Moser curves to be closely related to the singularities of the curve. For the two simplest possible classes of singularities — nodes and simple cusps — the situation is as follows: the differentials $\Psi_1$ and $\Psi_2$ (or equivalently $dk$ and $dz$) do not have a zero at a point of $\widetilde{\Gamma}_{cm}$ that is a preimage of a node on $\Gamma_{cm}$, and have a simple common zero at a preimage of a cusp (and a multiple common zero at a preimage of any more complicated singularity).

Note that the loci $\mathcal{K}_{g,N}$ become dense in $\mathcal{M}_{g,1}$ as $N \to \infty$. An open set of $\mathcal{K}_{g,N}$ correspond to singular CM curves having $N - g$ nodes. The intersection of $\mathcal{K}_{g,N}$ with $\mathcal{D}_n$ corresponds to the CM curves with $n$ simple cusps and $N - g - n$ nodes. Our main result, that $\mathcal{D}_g$ is empty, for the case of CM curves can be formulated as follows:

**Corollary 5.14.** *Let $\Gamma_{cm}$ be an $N$-particle CM curve (i.e. given by equation (29)), whose only singularities are $n$ simple cusps and $k$ nodes. Then we have the bound*

$$2n + k < N. \tag{43}$$

Note that the family of $N$-particle Calogero-Moser curves is $N$ dimensional. Hence, (43) is the expected inequality, and the statement that (43) holds is in striking contrast with the case of plane curves, where it is known that the similar inequality is false.

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