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Conformal mappings and the Whitham equations

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The topic named in the first part of the title of this lecture is familiar to every student. My ultimate goal is to show how the theory of integrable equations, which has been extensively developed during the past twenty years, and the Whitham theory, which already has a ten-year history, are related to the classical problem of complex analysis. The Riemann theorem asserts that, if a domain in the complex plane has a boundary containing more than two points, then there exists a conformal mapping of this domain onto the unit disk. This is an existence theorem. Many applied sciences are engaged in constructing such conformal mappings in particular situations; moreover, these problems are related to applications in hydrodynamics, in the theory of oil-fields, and in aerodynamics. The necessity of constructing conformal mappings of special domains emerges very often.

I want to present a recent remarkable observation of Zabrodin and Wiegmann, who discovered a relation between the classical problem on conformal mappings of domains and the dispersionless Toda lattice a couple of months ago. I shall tell about the development of this observation in our joint paper (not yet published), namely, about its generalization to nonsimply connected domains and about the role which the methods of algebraic geometry play in it.

Before proceeding to the problem proper, I want to give a brief overview of the entire context in which it has arisen, in order to clarify what the Whitham equations are. Surprisingly, the same structures related to the Whitham equations arise in various fields of mathematics, not only in the theory of conformal mappings. For example, they arise in the problem of constructing n-orthogonal curvilinear coordinates, which was the central problem of differential geometry in the nineteenth century. Let $x^i(u)$ be a curvilinear coordinate system in $\mathbb{R}^n$, where x^i are the Cartesian coordinates expressed in terms of curvilinear coordinates u. Such a coordinate system is called *n-orthogonal* if all the level hypersurfaces $u_i = \text{const}$ intersect at a right angle. An example of such a coordinate system is polar coordinates. In the two-dimensional case, the problem is trivial, but starting with dimension 3, it becomes very rich. Theoretically, this problem was solved by Darboux, again at the level of an existence theorem. He proved that the local problem of constructing an n-orthogonal curvilinear coordinate system depends on $n(n-1)/2$ functions of two variables. There are fairly many particular examples of n-orthogonal coordinates system. One of such examples is elliptic coordinates. In essence, solving a given system of differential equations reduces to constructing apt coordinates, in which the system becomes trivial. This is why good coordinate systems are so important: they increase the chances of solving the equation.

The problem about n-orthogonal coordinate systems can be set in intrinsic terms as the problem of finding flat diagonal metrics $ds^2 = \sum H_i^2(u)(du)^2$. Egorov considered such metrics satisfying the additional condition $H_i^2 = \partial_i \Phi$. This is the metric symmetry condition. Such metrics are called *Darboux–Egorov metrics*. They have many special features.

This problem, the Whitham equations, and the problem about conformal mappings belong to one complex of ideas and methods. A little later, I shall tell how the problem about n-orthogonal curvilinear coordinates is related to topological quantum models of field theory.

Another theme, which has eventually united all these diverse problem, is the theory of integrable equations, commonly referred to nowadays as soliton equations. This theory emerged about 30 years ago. The best-known (and oldest) soliton equation is the Korteweg–de Vries (KdV) equation

$$u_t - \frac{3}{2}uu_x + \frac{1}{4}u_{xxx} = 0.$$

There are many other soliton equations (fortunately, having important applications) which can be integrated by the methods of soliton theory. I shall not talk about these methods; they were largely developed 10–20 years ago and continue being developed at present.

Korteweg and de Vries found the simplest solution to the KdV equation very shortly after they wrote it. This is a stationary solution $u(x,t)$ not depending on t. In this case, $u_t = 0$, and we can integrate the equation:

$$\frac{3}{4}u^2 = \frac{1}{4}u_{xx} + g_2.$$

Then, we can multiply it by u_x and integrate again:

$$\frac{1}{2}(u_x)^2 = u^3 + g_2 u + g_3.$$

The solutions to such an equation are expressed in terms of the Weierstrass function as

$$u = 2\wp(x + \text{const}; \omega_1, \omega_2);$$

the Weierstrass function $\wp$ is a doubly periodic function with periods $2\omega_1$ and $2\omega_2$ having a second-order pole at zero; i.e., $\wp = \frac{1}{x^2} + O(x)$ at zero. This solution depends on three constants; thus, we have obtained the complete set of solutions, because the equation is of the third order.

A stationary solution is constructed from an elliptic curve, i.e., a curve of genus 1. In the 1970s and in the early 1980s, in a cycle of papers by Dubrovin, Novikov, Matveev, myself, and a number of other authors, the so-called algebro-geometric methods for constructing solutions to soliton equations were developed; given a set of algebro-geometric data, they yield solutions to various nonlinear equations, including the KdV equation, the sine-Gordon equation, and other equations pertaining to this science. A solution is obtained by processing data by a machine called finite-zone integration. A solution is represented explicitly, but in terms of the Riemann theta-functions rather than in terms of elliptic functions. The algebro-geometric data set consists of a Riemann surface Γ_g of genus g with fixed points $P_1, \ldots, P_N$ and fixed local coordinates $z_1, \ldots, z_N$ in neighborhoods of these points; it also includes a fixed point on the complex multidimensional torus $J(\Gamma_g)$ being the Jacobian of this surface. A solution is constructed from such data. This makes it possible to solve very diverse equations, depending on the number of fixed points and on the classes of curves.

For the stationary solution to the KdV equation, the algebro-geometric data set includes an elliptic curve $y^2 = E^3 + g_2 E + g_3$ and a fixed point at infinity. These data play the role of integrals, for they do not change with time. But the point on the Jacobian moves. The phase space of the equation looks as follows. There is a space of integrals being curves with marked points and fixed local coordinates at these points; over each point of the space of integrals, a torus hangs. The motion on the torus is a rectilinear winding, in full accordance with the spirit of the theory of completely integrable finite-dimensional systems, i.e., with Liouville theory.

Such is the answer for soliton equations. The procedure for constructing solutions is another story. I shall not tell it now. Instead, I want to tell about what happened to this science thereafter, starting with the mid-1980s. At that time, a particular emphasis was placed on the theory of perturbations of integrable equations. Usually, we are interested not only in a specific equation but also in what happens in its neighborhood. The basic element of the perturbation theory of integrable equations is Whitham theory.

Before proceeding to Whitham theory, I want to write one formula; its various forms are encountered in all the sciences mentioned above. As I said, the description of motion for soliton equations in terms of systems of integrals and rectilinear windings of the torus is fully consistent with Liouville theory. The ultimate goal of Liouville theory is specification of action–angle variables. A Hamiltonian system is constructed from a manifold M^{2n} (phase space), a symplectic structure ω on it, and a Hamiltonian H. A Hamiltonian system is called completely integrable if, in addition to a Hamiltonian, it has n integrals in involution, for which $\{F_i, F_j\} = 0$. The compact surface levels of these integrals must be n-dimensional tori, and the motions on them must be rectilinear windings. The torus has natural coordinates, cycles. If Φ_i are the angular coordinates for the basis cycles, then the action variables are defined as the coordinates A_i canonically conjugate to the angular variables, i.e., such that the symplectic structure in these coordinates has the standard Darboux form $\omega = \sum dA_i \wedge d\Phi_i$. Selecting such coordinate systems among all coordinate systems is a separate nontrivial problem. The Liouville theorem in Arnold's setting says that we must integrate the primitive form over the basis cycles. But it is unclear how to explicitly describe this n-dimensional torus in the $2n$-dimensional manifold. Thus, this theorem also has the character of an existence theorem. All attempts to explicitly construct action–angle variables have failed. In the early 1980s, Novikov and Veselov made a remarkable observation. Analyzing the first integrable Hamiltonian equations known at that time, they discovered that the action–angle variables have the same form for all these systems. Namely, integration over a cycle on n-space is replaced by integration over a cycle on the corresponding Riemann surface, that is,

$$A_i = \oint_{a_i} Q\, dE. \tag{1}$$

Here Q is a meromorphic differential; to each Hamiltonian system, its own differential Q corresponds. These differentials may be multivalued. Nobody knew why this is so. Novikov and Veselov called these formulas analytic Poisson brackets. Their nature has been explained analytically only recently, three years ago, in my joint paper with Phong (in *Journal of Differential Geometry*). We analyzed the answers for the symplectic structures which arise in Seiberg–Witten theory for the supersymmetric Yang–Mills model and noticed that the same symplectic brackets as those describing the case of hyperelliptic curves (I should mention that everything considered by Novikov and Veselov referred to the case of hyperelliptic curves) were rediscovered by Seiberg and Witten.

Memorize formula (1), because precisely the same integral of a multivalued differential solves the problem about conformal mappings of domains.

What are the Whitham equations? Suppose that we have slightly changed (perturbed) the equation. Then the integrals of the initial equation cease to be integrals. They begin to slowly vary; as physicists say, they become adiabatic integrals. For the nonperturbed equation, a point of the phase space moves on a torus. As soon as we perturb the equation, a slow drift along the space of integrals begins. The system of equations on the moduli space of curves with

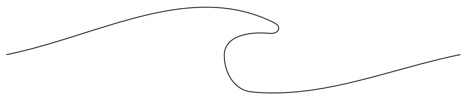

Figure 1. An overturning wave

marked points describing this motion is precisely the Whitham equations. It turns out that they are themselves integrable.

I shall not talk about algebro-geometric data any longer. I shall continue the discussion at a quite elementary level, where only curves of genus 0 are considered. The point is that the solution to the KdV equation that involves the Weierstrass function is not the simplest one; the simplest solution is a constant. In the theory of KdV equation, curves of genus 0 are trivial, and nobody was interested in this solution. But when we consider perturbations of a constant solution rather than this solution itself, the theory becomes interesting. In Whitham theory, genus 0 plays a nontrivial role. This case is called the dispersionless limit of soliton equations. It can be treated as a special case of a more general problem or considered separately.

Why “dispersionless”? The coefficients in a KdV equation are inessential, because it can be reduced to the form $u_t = uu_x + u_{xxx}$ by scale transformations. In what follows, I shall not trace the coefficients. If a solution is almost constant, we can forget about the third derivative. A good approximation is the equation $u_t = uu_x$. It is this equation that is called the dispersionless limit, because in the KdV equation, the term u_{xxx} is responsible for dispersion. The equation $u_t = uu_x$ is the simplest Whitham equation.

The KdV equation is an infinite-dimensional analogue of integrable (in the sense of Liouville) Hamiltonian systems; the equation $u_t = uu_x$ is also integrable, but in a completely different sense. Solving the equation $u_t = uu_x$ (it is called the Riemann–Hopf equation) is child’s play. Indeed, take an arbitrary function $f(\xi)$ and consider the equation $u = f(x+ut)$. This equation implicitly defines a function $u(x,t)$. This function is a solution to the equation $u_t = uu_x$. Moreover, all the solutions are obtained in this way.

This solution is commonly used as a basis for explaining the role of nonlinearity in hydrodynamics. Treating u as altitude (wave amplitude), we see that velocity of a point is proportional to its altitude. Therefore, if the function is not monotone, then the “hump” begins to outrun everything else; the wave becomes steeper and overturns (Fig. 1). At the overturning point, the third derivatives cannot be neglected, for they grow large. Hydrodynamics explains this as regularization of the behavior of the wave by dispersion (viscosity).

All Whitham equations are integrated by similar methods, which consist in writing some implicit expression for a solution.

What is the more general setting of dispersionless Lax equations? Let me remind you that constructing solutions to soliton equations is based on the Lax representation $\dot{L} = [L, A]$. For the KdV equation, we have $L = \partial^2 + u(x,t)$ (the Sturm–Liouville operator) and $A = \partial^3 + \frac{3}{2}u\partial + \frac{3}{4}u_4$. The generalizations of the KdV equation have arisen from consideration of operators with matrix coefficient and higher-order operators

$$L = \partial^n + u_{n-2}\partial^{n-2} + \cdots + u_0.$$

The Lax representation is a consistency condition for the overdetermined system of linear equations $L\psi = E\psi$, $L_2\psi = A\psi$. In general, the idea of the inverse problem method is not to start from the equation but go in the reverse direction, i.e., construct an operator and a solution from a given function ψ.

As we have agreed to begin with considering the simplest solutions to Lax equations (when u is a constant), we can solve the corresponding linear differential equation very easily. The solution is an exponential, and the eigenvalues are polynomials. Taking the eigenfunction $\psi = e^{px}$, we obtain $E(p) = p^n + u_{n-2}p^{n-2} + \cdots + u_0$ (the symbol of the corresponding differential operator). The Whitham equations are written as $\partial_i E = \{E_+^{i/n}, E\}$; here $\{f, g\} = f_p g_x - f_x g_p$ is the Poisson bracket. We shall express $u_d(X,T)$ in terms of the slow variables $X = \epsilon x$ and $T = \epsilon t$.

The subscript i is not a misprint. Each integrable equation arises as a part of the large hierarchy formed by a whole family of integrals commuting with this equation. This is in the spirit of Liouville integrability: if we have a set of integrals in involution, then each of these integrals regarded as a Hamiltonian generates its own Hamiltonian dynamics. That the integrals are in involution means that the corresponding dynamics commute.

Now, I shall explain what $E_+^{i/n}$ is. Let $E^{1/n}(p) = p + \sum v_i p^{-i}$ be the Laurent expansion. Then $E^{i/n}(p) = p^i + \cdots + O(p^{-1})$; $E_+^{i/n}$ means that only nonnegative powers of p are taken, i.e., $O(p^{-1})$ is crossed out. We obtain a polynomial whose coefficients are polynomials in u. Therefore, the result is a closed system of equations, which is the dispersionless limit of Lax equations. In the simplest case, where $E = p^2 + u$ and $i = 3$, we obtain the Riemann–Hopf equation mentioned above.

How does the general solution procedure for a dispersionless limit look like? Consider the space of pairs (Q, E), where E and Q are polynomials of forms $E = p^n + u_{n-2}p^{n-2} + \cdots + u_0$ and $Q = b_0 p + \cdots + b_{m-1}p^m$, respectively. On this space, we can introduce the Whitham coordinates $T_i = \frac{1}{i}\operatorname{res}_\infty(E^{-i/n}Q\,dE)$. The T_i so defined are functions of u and b (they linearly depend on b and polynomially on u). These T_i vanish at large i; there are precisely as many nonzero T_i as required. We can locally invert the T_i as functions of u and b

and obtain functions $u(T)$ and $b(T)$. Substituting these values $u(T)$ into E, we obtain a function $E(T)$. It turns out that $E(T)$ is a solution to an equation of dispersionless hierarchy. The polynomial Q seems to play an auxiliary role. But $Q(T)$ is also a solution to the same equation with the same Hamiltonian; namely, $\partial_i Q = \{E_+^{i/n}, Q\}$. Moreover, we always have $\{Q, E\} = 1$. The equation $\{Q, E\} = 1$ is called the *string equation*.

In the dispersionless limit, as opposed to the usual hierarchy of Lax equations, only one solution survives, since all solutions are parametrized by different higher times; the general solution satisfies a suitable string equation.

The dispersionless science had been known for several years when Dijkgraaf, Verlinde, and Witten published a paper. They considered a quite different problem, namely, classification of the topological models of field theory. Solving this problem, they obtained the very same formula in a completely different context. It became clear that, behind the dispersionless science, a very important element was hidden; now, it is known as the tau-function. The whole structure related to the dispersionless limit of the KdV equation or of the general Lax equation is coded by only one function

$$F(t) = \frac{1}{2}\,\mathrm{res}_\infty\Big(\sum_{i=1}^{\infty} T_i k^i dS\Big).$$

Here $dS = QdE = \sum_{i=1}^{\infty} T_i dk^i + O(k^{-1})$ and $k = E^{1/n}(p) = k(p) = p + O(p^{-1})$. I remind you that we deal with the case of a curve of genus 0; the marked point can be driven to infinity. The only surviving parameter is the local coordinate p. It can be verified, although this is far from being obvious, that the derivatives of the function F with respect to times T_i give all the remaining coefficients. For example, $\partial_i F = \mathrm{res}_\infty(k^i\, dS)$ and $\partial^2_{ij} F = \mathrm{res}_\infty(k^i\, d\Omega_j)$, where $\Omega_j = E_+^{j/n}$. There is the remarkable formula

$$\partial^3_{ijk} F = \sum_{q_s} \mathrm{res}_\infty\Big(\frac{d\Omega_i\, d\Omega_j\, d\Omega_k}{dQ\, dt}\Big),$$

where the summation is over the critical points of the polynomial E (such that $dE(q_s) = 0$).

Now, I return to the initial problem about conformal mappings. I shall consider only the case of domains bounded by analytic curves. Let us denote the interior domain by D and the exterior domain by $\overline{D}$. I shall be interested in schlicht conformal mappings of the exterior of the unit disk to $\overline{D}$. For reading, I recommend the book A. N. Varchenko and P. I. Etingof. *Why the Boundary of a Round Drop Becomes a Curve of Order Four* (Providence, RI: Amer. Math. Soc., 1992). It contains many beautiful particular examples of conformal mappings related to the following problem, which arises in the oil industry. Imagine that the domain under consideration is an oil-field. There are several oil wells through which the oil is pumped out. This somehow deforms the domain. The equation describing the dynamics of the domain boundary is as follows. Let Φ be a solution to the equation

$$\Delta\Phi = \sum q_i\, \delta(z - z_i)$$

with zero boundary condition $\Phi \restriction \partial D$. Then $\operatorname{grad}\Phi$ is the velocity of the boundary.

This problem is integrable in a certain sense. It turns out that the final shape of the drop does not depend on the oil pumping schedule, as it must be for commuting flows. The result depends only on the amount of oil pumped out through each oil well; the particular procedure of pumping does not matter.

The main contribution to this science was made by Richardson, who discovered an infinite set of integrals. It is these integrals that I am going to discuss next.

It is fairly easy to prove that any domain (simply connected or not) is completely determined by its harmonic moments. The harmonic moments of a domain D are defined as follows. Let $u(x, y)$ be a harmonic function. Then the corresponding harmonic moment is equal to

$$t_u = \iint_D u(x, y)\, dx\, dy.$$

When the domain changes, the harmonic moment of some function also changes. This is a local assertion. The harmonic moments are local coordinates.

It is not necessary to consider all harmonic moments; it is sufficient to take only some of the functions. For example, the set of functions

$$t_n = \iint_D z^{-n}\, dz\, d\bar{z}, \quad n \geqslant 1$$

together with the function

$$t_0 = \iint_{\overline{D}} dz\, d\bar{z},$$

where $\overline{D}$ is the exterior domain, is a local set of coordinates for a simply connected domain.

The fundamental observation made by Wiegmann and Zabrodin is a follows. Consider, in addition, the moments

$$v_n = \iint_{\overline{D}} z^n\, dz\, d\bar{z}$$

of the complement. Clearly, the functions v_n can be expressed in terms of $t_0, t_1, \dots$. It turns out that

$$\frac{\partial v_n}{\partial t_m} = \frac{\partial v_m}{\partial t_n}.$$

This means that there exists a function $F(t)$ for which $\partial_n F(t) = v_n$. It turns out that $\partial_0\,\partial_n F$ are the expansion coefficients of a schlicht function implementing a conformal mapping. We assume that this function is normalized as follows. In the complement to the unit disk, there is a coordinate w, and in $\overline{D}$, a coordinate z. We consider the mapping of the exteriors and suppose that infinity is mapped to infinity; moreover, we assume that $z = rw + O(w^{-1})$. In this case, $w(z) = r^{-1}z + \sum(\partial_0\,\partial_n F)z^{-n}$. Again, it turns out that all the conformal mappings are coded by one function. This is precisely the function which I mentioned above.

First, I want to give a new proof that locally, the coordinates t_n form a complete coordinate system. From the proof, it will be seen how this all is related to the dispersionless science.

I need the notion of the Schwarz function. Locally, a smooth curve can be specified in the form $y = f(x)$. In the complex form, this can be written as $\bar{z} = S(z)$. The function S is called the Schwarz function of the curve. For example, for the unit circle, we obtain the equation $\bar{z} = z^{-1}$.

For a real-analytic curve (without corners), the function S can be extended to a complex-analytic function in a small neighborhood of the curve.

The first assertion which I want to prove is as follows. Suppose that a contour deforms, i.e., we have a family of Schwarz functions $S(z,t)$, where t is a deformation parameter. If none of the harmonic moments t_n changes under such a deformation, then the curve is fixed, i.e., the deformation is trivial.

Assertion 1. *The* 1*-differential* $S_t(z,t)\,dz$ *is purely imaginary on the contour* ∂D*, i.e., all of its values on the vectors tangent to the contour are purely imaginary.*

This follows easily from the definition of the Schwarz function.

The next assertion uses the specifics of the coordinates t_n under consideration.

Assertion 2. *If* $\partial_t t_n = 0$*, then the holomorphic differential* $\partial_t S\,dz$ *defined in a small neighborhood of the curve can be extended to a holomorphic differential on the entire exterior.*

Before proving the second assertion, I shall explain how to derive the required result from these two assertions. Any domain $D \subset \mathbb{C}$ with coordinate z determines a closed Riemann surface. To construct it, we take another copy of the same domain with coordinate $\bar{z}$ and attach it to the given domain along the boundary. The obtained Riemann surface is called the Schottky double. Let us apply the Schwarz symmetry principle: any function analytic in the upper half-plane and real on the real axis can be analytically continued to the lower half-plane. We have a holomorphic differential in D. It can be extended to the complex conjugate, because it is purely imaginary on the boundary. As a result, we obtain a holomorphic differential on the sphere. But there are no nonzero holomorphic differentials on the sphere.

We proceed to prove the assertion that the holomorphic differential $\partial_t S\,dz$ can be extended to the entire exterior. Using the Cauchy integral, we can represent an arbitrary function on a smooth contour as the difference of a function holomorphic in the exterior domain and a function holomorphic in the interior domain. Let

$$\widehat{S}(z) = \oint_{\partial D} \frac{\partial_t S(w)\,dw}{z-w}.$$

The function $\widehat{S}(z)$ is holomorphic outside the contour, it can be extended to the boundary, and $S^+ - S^- = \partial_t S$. If the origin lies inside the domain and $|z| < |w|$, then

$$\widehat{S}(z) = \sum z^n \oint_{\partial D} \partial_t S(w) w^{-n}\,dw = \sum z^n \partial_t t_n,$$

because

$$t_n = \iint_D z^{-n}\,dz\,d\bar{z} = \oint_{\partial D} z^{-n}\bar{z}\,dz$$

by the Stokes theorem.

If the moments do not vary, then the expansion coefficients of $\widehat{S}$ at $z = 0$ are identically zero. Therefore, the function S^- is identically zero in some neighborhood of $z = 0$. But this function is analytic; hence it vanishes identically. For $\partial_t S\,dz$ to be holomorphic, one more coefficient should be zero, because we have multiplied the function by dz, and the differential has a pole of the second order.

Now, it is clear what changes when we differentiate with respect to t_n. The first assertion is valid for an arbitrary variable. The expansion coefficients are no longer identically zero; one of the coefficients is nonzero. This means, in particular, that $\partial_{t_0} S\,dz$ is a meromorphic differential with a simple pole at infinity.

When we take the double, a second pole emerges according to the symmetry principle. We obtain a differential having residue ± 1 at two points. (There is only one such differential.) This is a global property, as the Liouville theorem. An analytic function on a compact surface is constant. These two facts allow us to use global properties. The first fact makes it possible to pass from a domain with boundary to a compact surface. And the second fact, which requires special assumptions, gives an analytic continuation to a meromorphic object.

Thus, we have proved that $\partial_0 \bar{z}\, dz = \frac{dw}{w}$. Here we differentiate $\bar{z}$ at a constant z. An equivalent expression is

$$\{z(w,t_0), \bar{z}(w,t_0)\} = 1.$$

This is an assertion about the zeroth moment. The assertion about all the remaining moments is $\partial_n \bar{z}\, dz = dz^n_+$. Here the following notation is used. Let $z(w) = w + \dots$. Then $z^n(w) = w^n + \dots$. The plus sign means that we take only the positive part (a polynomial on the sphere).

You may ask why the differential has a pole at only one point, although, by the symmetry principle, it must have another pole at the symmetric point. But the functions t_n are not analytic; these are functions of both the real and imaginary parts: $t_n = x_n + iy_n$. We have

$$\frac{\partial}{\partial t_n} = \frac{\partial}{\partial x_n} - i\,\frac{\partial}{\partial y_n}.$$

Therefore,

$$\frac{\partial}{\partial x_n}\,\bar{z}\,dz = dz^n_+ - d\bar{z}^n_+$$

and

$$\frac{\partial}{\partial y_n}\,\bar{z}\,dz = i(dz^n_+ + d\bar{z}^n_+).$$

The point is that we can write down hierarchies with respect to t_n and with respect to the complex conjugate variable $\bar{t}_n$. The result is a dispersionless Toda lattice.

The following remarkable formula holds:

$$F(t) = -\frac{t_0^2}{2} + \sum_{n\geqslant 0}(n-2)(t_n v_n + \bar{t}_n \bar{v}_n).$$

This formula contains a plenty of nontrivial identities. For example, the identity $\partial_n F = v_n$ looks almost naïve. But the v_n themselves depend on t_n in a puzzling way. Substituting and differentiating these dependences, we obtain precisely v_n.

For the ellipse, the function F can be calculated explicitly:

$$F = \frac{1}{2}t_0^2 \ln t_0 - \frac{3}{4}t_0^2 - \frac{1}{2}\ln(1-4|t_2|^2) + t_0\frac{|t_1|^2 + t_1^2\bar{t}_2 + \bar{t}_1^2 t_2}{1-4t_2\bar{t}_2}.$$

This example shows how F depends on the first three moments. (The complement to the ellipse has only three nonzero moments.)

For nonsimply connected domains, the first assertion (about the derivative of the Schwarz function) remains valid. The second one relies on the summation of a geometric progression for a Cauchy integral. In the late 1980s, studying the quantization operator for boson strings, Novikov and I developed a Fourier–Laurent theory for arbitrary Riemann surfaces. The basis z^n is replaced by another basis.

The formula written above is symmetric with respect to t and v. This suggests that it makes sense to try to apply it to the old classical problem of constructing a mapping of domains from a schlicht conformal mapping of their complements. The relation between these mappings may be nontrivial. For example, the complement of the ellipse is mapped onto the complement of the disk by a simple algebraic function, while the mapping of the interior of the ellipse to the interior of the disk is an elliptic function.