

Integrable chains on algebraic curves

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Abstract

The discrete Lax operators with the spectral parameter on an algebraic curve are defined. A hierarchy of commuting flows on the space of such operators is constructed. It is shown that these flows are linearized by the spectral transform and can be explicitly solved in terms of the theta-functions of the spectral curves. The Hamiltonian theory of the corresponding systems is analyzed. The new type of completely integrable Hamiltonian systems associated with the space of rank $r = 2$ discrete Lax operators on a *variable* base curve is found.

1 Introduction.

The main goal of this work is to construct discrete analogs of the the zero-curvature equations on an algebraic curve introduced in [1], and identified later in [2] with infinite-dimensional field analogs of the Hitchin system [3]. The starting point was an attempt to find the general setting for the difference-differential equations introduced in the recent work [4] by S.Novikov and the author. These equations were used for the construction of high rank solutions of the 2D Toda lattice equations simultaneously with the construction of commuting high rank difference operators.

Almost all (1+1)-soliton equations admit zero-curvature representation ([5])

$$\partial_t L - \partial_x M + [L, M] = 0, \quad (1.1)$$

where $L(x, t, z)$ and $M(x, t, z)$ are *rational* matrix functions of the *spectral* parameter z . The discrete analog of (1.1) is the equation

$$\partial_t L_n = M_{n+1} L_n - L_n M_n, \quad (1.2)$$

where, as before, $L_n = L_n(t, z)$ and $M_n = M_n(t, z)$ are rational functions of the spectral parameter. In both the cases poles of L and M are fixed. The singular parts of L and M at

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the poles are dynamical variables. Their number equals the number of equations equivalent to (1.1) or (1.2), respectively.

The Riemann-Roch theorem implies that naive direct generalization of equations (1.1, 1.2) for matrix functions, which are meromorphic on an algebraic curve Γ of genus $g > 0$, leads to an over-determined system of equations (see details in [2]). In [1] it was found for the continuous case, that if the matrix functions L and M have moving poles with special dependence on x and t besides fixed poles, then equation (1.1) is a well-defined system on the space of singular parts of L and M at fixed poles. In [2] it was shown that a part of these equations can be used to express M in terms of L . After that the zero-curvature equations can be seen as a hierarchy of commuting flows on the space of admissible matrix-valued meromorphic functions $L(x, z)$, $z \in \Gamma$. The admissible meromorphic matrix-functions on a smooth genus g algebraic curve were identified with x -connections on x -parametric families $\mathcal{V}(x)$ of stable rank r and degree rg holomorphic vector bundles on Γ . In the stationary case, the factor-space $\mathcal{L}^{\mathcal{K}}/SL_r$ of x -independent connections $L(z)$ with the divisor of poles equivalent to the canonical divisor \mathcal{K} is isomorphic to the phase space of the Hitchin system. The latter is the cotangent space $T^*(\mathcal{M})$ of the moduli space \mathcal{M} of stable rank r and degree rg holomorphic vector-bundles on Γ .

A discrete analog of x -parametric family of vector bundles is a sequence of vector bundles $\mathcal{V}_n \in \mathcal{M}$. The discrete analog of a meromorphic x -connection with the pole divisor D_+ is a chain $L_n(z)$ of meromorphic homomorphisms $L_n \in H^0(\text{Hom}(\mathcal{V}_{n+1}, \mathcal{V}_n(D_+)))$. It is assumed that L_n is almost everywhere invertible and the inverse homomorphism has a fixed divisor of poles D_- , i.e. $L_n^{-1} \in H^0(\text{Hom}(\mathcal{V}_n, \mathcal{V}_{n+1}(D_-)))$. In the next section we show the algebraic integrability of the space $\mathcal{L}_N^{D_+, D_-}$ of periodic chains, considered modulo gauge transformations $L'_n = g_{n+1}L_n g_n$, $g_n \in GL_r$. Namely, we show that an open set of the factor-space $\mathcal{L}_N^{D_+, D_-}/GL_r^N$ is isomorphic to an open set of Jacobian bundle over the space \mathcal{S}^{D_+, D_-} of the spectral curves $\hat{\Gamma}$. The spectral curves are defined by the characteristic equation for the monodromy operator $T = L_N L_{N-1} \cdots L_1$, which represents them as an r -sheet cover of the base curve $\pi : \hat{\Gamma} \rightarrow \Gamma$. The spectral transform identifies an open set of the *restricted* chains, corresponding to the sequences of bundles \mathcal{V}_n with fixed determinant, with an open set of the bundle over \mathcal{S}^{D_+, D_-} with fibers $J_C(\hat{\Gamma})$, which are preimages in $J(\hat{\Gamma})$ of some point of $J(\Gamma)$.

In Section 3 a hierarchy of gauge invariant commuting Lax equations (1.2) is constructed. For periodic chains these equations are linearized by the spectral transform and can be explicitly solved in terms of the Riemann theta-function of the spectral curve.

One of the most general approaches to the Hamiltonian theory of the soliton equations, admitting the zero-curvature representation with the spectral parameter on the rational or an elliptic curve, is based on the classical r -matrix description of the Poisson structure of the corresponding phase space. Essentially there are two types of the Poisson structures reflecting Lie algebraic or Lie group nature of the auxiliary linear problem. The first type usually is referred to as linear brackets and the second one as quadratic or Sklyanin brackets (see the book [6], survey [7] and references therein).

The alternative approach to the Hamiltonian theory of the soliton equations was developed in [8, 9]. It is based on the existence of some universal two-form defined on a space

of meromorphic matrix-functions. A direct and simple corollary of the definition of this form is that its contraction by the vector-field, defined by zero-curvature equation, is an *exact* one-form. Therefore, whenever this form is non-degenerate the corresponding system is the Hamiltonian system. In Section 4 it is shown that the group version of our approach is applicable to the case of the Lax chains on algebraic curves of arbitrary genus (see [10] for the genus zero case). It is shown that for the cases of the rational and elliptic curves the corresponding form ω restricted to the symplectic leaves $\mathcal{P}_* \subset \mathcal{L}_N^{D+, D-} / GL_r^N$ is induced by the Sklyanin symplectic form on the space of the monodromy matrices. For $g > 1$ the form ω is degenerate on \mathcal{P}_* . That does not allow us to treat Lax equations (1.2) for $g > 1$ within the framework of conventional Hamiltonian theory. At the same time the corresponding system has new very unusual features. The space \mathcal{P}_* is equipped by g -parametric *family* of two-forms ω_{dz} , parameterized by the holomorphic differentials dz on Γ . For all of them the fibers $J_C(\hat{\Gamma})$ of the spectral bundle are maximum isotropic subspaces. For each of the flows defined by the Lax equations on \mathcal{P}_* , the contraction $i_{\partial_t} \omega_{dz}$ is an exact form δH^{dz} . Although each of the forms ω_{dz} is degenerate on \mathcal{P}_* , their family is *non-degenerate*.

In a certain sense the state-of-art described above is dual to that in the theory of bi-Hamiltonian systems. Usually the bi-Hamiltonian structure is defined on the Poisson manifold equipped by a family of compatible brackets. The vector-fields that are Hamiltonian with respect to one bracket are Hamiltonian with respect to the other ones, but correspond to different Hamiltonians. The drastic difference between the bi-Hamiltonian systems and the systems \mathcal{P}_* of restricted chains is in the nature of the symplectic leaves. For the bi-Hamiltonian systems usually they are *globally* defined as levels of single-valued action-type variables. For \mathcal{P}_* the form ω_{dz} becomes non-degenerate on levels of *multi-valued* angle-type functions.

It is worth to understand if there exists the general Hamiltonian-type setting, in which these characteristic features of \mathcal{P}_* for $g > 1$ provide the basis for something that might be the notion of *super-integrable* systems. It is also possible, that there is no need for the new setting. The results of the last section provide some evidence that the Lax chains on the fixed base curve Γ might be "extended" to the conventional completely integrable Hamiltonian system. Namely, we show that for the rank $r = 2$ the space of the periodic Lax chains with *variable base curve* Γ is the Poisson manifold with leaves $\hat{\mathcal{P}}_\Delta$, corresponding to the chains (modulo gauge equivalence) with fixed determinant Δ of the monodromy matrix T , and with the fixed regular eigenvalue w of T at the punctures P_k^\pm . The universal form defines the structure of a completely integrable system on $\hat{\mathcal{P}}_\Delta$. The Hamiltonians of the Lax equations on $\hat{\mathcal{P}}_\Delta$ are in involution. They are given by the formula

$$H_f = \sum_{q=\hat{\mathcal{P}}_k^\pm} \text{res}_q (f \ln w) d \ln \Delta,$$

where f is a meromorphic function on Γ with poles at the punctures P_k^\pm . The common level of all the integrals H_f is identified with the Prim variety of the corresponding spectral curve.

2 Algebraic integrability of chains

Let Γ be a smooth genus g algebraic curve. According to [11], a generic stable, rank r and degree rg holomorphic vector bundle \mathcal{V} on Γ is parameterized by a set of rg distinct points γ_s on Γ , and a set of r -dimensional vectors $\alpha_s = (\alpha_s^i)$, considered modulo scalar factor $\alpha_s \rightarrow \lambda_s \alpha_s$ and a common gauge transformation $\alpha_s^i \rightarrow g_j^i \alpha_s^j$, $g \in GL_r$, i.e. by a point of the factor-space

$$\mathcal{M}_0 = S^{rg} (\Gamma \times CP^{r-1}) / GL_r$$

In [1, 12] the data $(\gamma, \alpha) = (\gamma_s, \alpha_s)$, $s = 1, \dots, rg$, $i = 1, \dots, r$, were called the Tyurin parameters.

Let D_{\pm} be two effective divisors on Γ of the same degree \mathcal{D} . Throughout the paper, if it is not stated otherwise, it is assumed that all the points of the divisors $D_{\pm} = \sum_{k=1}^{\mathcal{D}} P_k^{\pm}$ have multiplicity 1, $P_k^{\pm} \neq P_m^{\pm}, k \neq m$. For any sequence of the Tyurin parameters $(\gamma(n), \alpha(n))$ we introduce the space $\mathcal{L}_{\gamma(n), \alpha(n)}^{D_+, D_-}$ of meromorphic matrix functions $L_n(q), q \in \Gamma$, such that:

1⁰. L_n is holomorphic except at the points γ_s , and at the points P_i^+ of D_+ , where it has at most simple poles;

2⁰. the singular coefficient $L_s(n)$ of the Laurent expansion of L_n at γ_s

$$L_n(z) = \frac{L_s(n)}{z - z_s} + O(1), \quad z_s = z(\gamma_s), \quad (2.1)$$

is a rank 1 matrix of the form

$$L_s(n) = \beta_s(n) \alpha_s^T(n) \longleftrightarrow L_s^{ij}(n) = \beta_s^i(n) \alpha_s^j(n), \quad (2.2)$$

where $\beta_s(n)$ is a vector, and z is a local coordinate in the neighborhood of γ_s ;

3⁰. the vector $\alpha_s^T(n+1)$ is a left null-vector of the evaluation of L_n at $\gamma_s(n+1)$, i.e.

$$\alpha_s(n+1) L_n(\gamma_s(n+1)) = 0; \quad (2.3)$$

4⁰. the determinant of $L_n(q)$ has simple poles at the points $P_k^+, \gamma_s(n)$, and simple zeros at the points $P_k^-, \gamma_s(n+1)$.

The last condition implies the following constraint for the equivalence classes of the divisors

$$[D_+] - [D_-] = \sum_s [\gamma_s(n+1) - \gamma_s(n)] \in J(\Gamma), \quad (2.4)$$

where $J(\Gamma)$ is the Jacobian of Γ . If $2N > g(r+1)$, then the Riemann-Roch theorem and simple counting of the number of the constraints (2.1)-(2.3) imply that the functional dimension of $\mathcal{L}_{\gamma(n), \alpha(n)}^{D_+, D_-}$ (its dimension as the space of functions of the discrete variable n) equals $2\mathcal{D}(r-1) - gr^2 + g + r^2$.

The geometric interpretation of $\mathcal{L}_{\gamma(n), \alpha(n)}^{D_+, D_-}$ is as follows. In the neighborhood of γ_s the space of local sections of the vector bundle $V_{\gamma, \alpha}$, corresponding to (γ, α) , is the space \mathcal{F}_s of meromorphic functions having a simple pole at γ_s of the form

$$f(z) = \frac{\lambda_s \alpha_s^T}{z - z(\gamma_s)} + O(1), \quad \lambda_s \in C. \quad (2.5)$$

Therefore, if \mathcal{V}_n is a sequence of the vector bundles on Γ , corresponding to the sequence of the Tyurin parameters $(\gamma(n), \alpha(n))$, then L_n can be seen as a homomorphism of the vector bundle \mathcal{V}_{n+1} to the vector bundle $\mathcal{V}_n(D_+)$, obtained from \mathcal{V}_n with the help of simple Hecke modification at the punctures P_k^+ , i.e.

$$L_n \in \text{Hom}(\mathcal{V}_{n+1}, \mathcal{V}_n(D_+)). \quad (2.6)$$

These homomorphisms are invertible almost everywhere. The inverse matrix-functions are the homomorphisms of the vector bundles

$$L_n^{-1} \in \text{Hom}(\mathcal{V}_n, \mathcal{V}_{n+1}(D_-)). \quad (2.7)$$

The total space \mathcal{L}^{D_+, D_-} of the chains corresponding to all the sequences of the Tyurin parameters is a bundle over the space of sequences of holomorphic vector bundles

$$\mathcal{L}^{D_+, D_-} \longmapsto \prod_n \mathcal{M}_0 \equiv \{\gamma(n), \alpha(n)\}. \quad (2.8)$$

The fibers of this bundle are just the spaces $\mathcal{L}_{\gamma(n), \alpha(n)}^{D_+, D_-}$. Our next goal is to show algebraic integrability of the total space $\mathcal{L}_N^{D_+, D_-}$ of the N -periodic chains, $L_n = L_{n+N}$. Equation (2.4) implies that the periodicity of chains requires the following constraint on the equivalence classes of the divisors D_{\pm} :

$$N([D_+] - [D_-]) = 0 \in J(\Gamma), \quad (2.9)$$

which will be always assumed below. The dimension of $\mathcal{L}_N^{D_+, D_-}$ equals

$$\dim \mathcal{L}_N^{D_+, D_-} = 2N\mathcal{D}(r-1) + Nr^2 + g. \quad (2.10)$$

The last term in the sum corresponds to the dimension of the equivalence class of the set $\gamma(0)$, which defines the equivalence classes of all the divisors $\gamma(n)$ with the help of (2.4).

Let $L_n \in \mathcal{L}_N^{D_+, D_-}$ be a periodic chain. Then the Floque-Bloch solutions of the equation

$$\psi_{n+1} = L_n \psi_n \quad (2.11)$$

are solutions that are eigenfunctions for the monodromy operator

$$T_n \psi_n = \psi_{n+N} = w \psi_n, \quad T_n = L_{n+N-1} \cdots L_{n+1} L_n. \quad (2.12)$$

The monodromy matrix $T_n(q)$ belongs to the space of the Lax matrices introduced in [2], $T_n \in \mathcal{L}^{ND_+}$, $T_n^{-1} \in \mathcal{L}^{ND_-}$. The Floque-Bloch solutions are parameterized by the points $Q = (w, q)$, $q \in \Gamma$, of the so-called spectral curve $\hat{\Gamma}$ defined by the characteristic equation

$$R(w, q) = \det(w \cdot 1 - T_{n_0}(q)) = w^r + \sum_{i=0}^{r-1} r_i(q) w^i = 0. \quad (2.13)$$

From (2.1)-(2.3) it follows that the monodromy matrix T_{n_0} has poles at the punctures P_k^+ and the points $\gamma_s(n_0)$. Therefore, *a priori* the coefficients $r_i(q)$ of (2.13) might have poles only at the same set of points. The characteristic equation is n_0 -independent, because

$T_{n_0+1} = L_{n_0+N}T_{n_0}L_{n_0}^{-1} = L_{n_0}T_{n_0}L_{n_0}^{-1}$. Hence, $r_i(q)$ are meromorphic functions on Γ with the poles only at the punctures P_k^+ . Equation (2.13) defines an affine part of the spectral curve. Let us consider its compactification over the punctures P_k^+ . By definition L_n and its determinant have simple poles at P_k^+ . Hence, its residue at this point has rank 1, i.e. the Laurent expansion of L_n at P_k^+ has the form

$$L_n = \frac{h_k(n)p_k^T(n)}{z - z(P_k^+)} + O(1), \quad (2.14)$$

where z is a local coordinate in the neighborhood of P_k^+ , and $h_k(n), p_k(n)$ are r -dimensional vectors. These vectors are defined up to the gauge transformation

$$h_k(n) \mapsto c_{k,n}q_k(n), \quad h_k(n) \mapsto c_{k,n}^{-1}q_k(n), \quad (2.15)$$

where $c_{k,n}$ is a constant. The leading term of the Laurent expansion of the monodromy matrix equals

$$T_{n_0} = \prod_{n=n_0}^{n_0+N-1} \left(p_k^T(n+1)h_k(n) \right) \frac{h_k(n_0+N-1)p_k^T(n_0)}{(z - z(P_k^+))^N} + O\left(\left(z - z(P_k^+)\right)^{-N+1}\right). \quad (2.16)$$

Equation (2.16) implies that in the neighborhood of P_k^+ one of the roots of the characteristic equation has the form

$$w = (z - z(P_k^+))^{-N} \left(c_k^+ + O(z - z(P_k^+)) \right), \quad c_k^+ = \prod_{n=0}^{N-1} \left(p_k^T(n+1)h_k(n) \right). \quad (2.17)$$

The corresponding compactification point of $\hat{\Gamma}$ is smooth, and will be denoted by \hat{P}_k^+ . The determinant of T_{n_0} has the pole of order N at P_k^+ . Therefore, in the general position all the other branches of $w(z)$ are regular at P_k^+ . The coefficients $r_i(z)$ are the elementary symmetric polynomials of the branches of $w(z)$. Hence, all of them have poles at P_k^+ of order N . Note, that the coefficient $r_0(z) = \det T_{n_0}$ has zero of order N at P_k^- .

The same arguments applied to L_n^{-1} show that over the puncture P_k^- there is one point of $\hat{\Gamma}$ denoted by \hat{P}_k^- in the neighborhood of which w has zero of order N , i.e.

$$w = (z - z(P_k^-))^N \left(c_k^- + O(z - z(P_k^-)) \right). \quad (2.18)$$

Let us fix a normalization of the Floque-Bloch solution by the condition that the sum of coordinates ψ_0^i of the vector ψ_0 equals 1,

$$\sum_{i=1}^r \psi_0^i = 1. \quad (2.19)$$

Then, the corresponding Floque-Bloch solution $\psi_n(Q)$ is well-defined for each point Q of $\hat{\Gamma}$.

Theorem 2.1 *The vector-function $\psi_n(Q)$ is a meromorphic vector-function on $\hat{\Gamma}$, such that: (i) outside the punctures \hat{P}_k^\pm (which are the points of $\hat{\Gamma}$ situated on marked sheets over P_k^\pm) the divisor $\hat{\gamma}$ of its poles $\hat{\gamma}_\sigma$ is n -independent; (ii) at the punctures \hat{P}_k^+ and \hat{P}_k^- the vector-function $\psi_n(Q)$ has poles and zeros of the order n , respectively; (iii) in the general position, when $\hat{\Gamma}$ is smooth, the number of these poles equals $\hat{g} + r - 1$, where \hat{g} is the genus of $\hat{\Gamma}$.*

Proof. The coordinates of the vector-function $\psi_0(Q)$ are rational expressions in w and the entries of T_0 . Hence, it is a meromorphic function on $\hat{\Gamma}$. Let $\hat{\gamma}_\sigma$ be a set of the poles of ψ_0 . In order to show that $\hat{\gamma}_\sigma$ are the only poles of $\psi_n = L_{n-1} \cdots L_0 \psi_0$ outside the preimages on $\hat{\Gamma}$ of the punctures P_k^+ , it is enough to prove by induction that ψ_n at all the preimages $\gamma_s^i(n)$ of the points $\gamma_s(n)$ satisfies the equation $\alpha_s(n) \psi_n(\gamma_s^i(n)) = 0$. The step of the induction is a direct corollary of (2.3). The initial statement of the induction follows from the equation $T_0 \psi_0 = w \psi_0$. Indeed, w is regular at $\gamma_s^i(0)$. Therefore, the left hand side of the equation has to be regular at these points as well. The monodromy matrix T_0 has a simple pole at $\gamma_s(0)$ of the form $m_s \alpha_s^T(0)$, where m_s is some r -dimensional vector. Hence, $\alpha_s(0) \psi_0(\gamma_s^i(0)) = 0$. The proof of (ii) is based on the following statement.

Lemma 2.1 *Let \tilde{L}_n be a formal series of the form*

$$\tilde{L}_n = h(n)p(n)^T \lambda^{-1} + \sum_{i=0}^{\infty} \chi_i(n) \lambda^i \quad (2.20)$$

where $q(n), p(n)$ are vectors and $\chi_i(n)$ are matrices. Then the equations

$$\phi_{n+1} = \tilde{L}_n \phi_n, \quad \phi_{n+1}^* \tilde{L}_n = \phi_n^* \quad (2.21)$$

where ϕ_n and ϕ_n^* are r -dimensional vectors and co-vectors over the field of the Laurent series in the variable λ , have $(r-1)$ -dimensional spaces of solutions of the form

$$\Phi_n = \sum_{i=0}^{\infty} \tilde{\xi}_i(n) \lambda^i, \quad \Phi_n^* = \sum_{i=0}^{\infty} \tilde{\xi}_i^*(n) \lambda^i. \quad (2.22)$$

The equations (2.20) have unique formal solutions of the form

$$\phi_n = \lambda^{-n} \left(\sum_{i=0}^{\infty} \xi_i(n) \lambda^i \right), \quad \phi_n^* = \lambda^n \left(\sum_{i=0}^{\infty} \xi_i^*(n) \lambda^i \right), \quad (2.23)$$

such that

$$(\Phi_n^* \phi_n) = (\phi_n^* \Phi_n) = 0, \quad (\phi_n^* \phi_n) = 1, \quad (2.24)$$

and normalized by the conditions

$$\chi_0(0) = q(-1), \quad \sum_{j=1}^i \chi_j^j(0) = 0, \quad i > 0. \quad (2.25)$$

For the proof of the lemma it is enough to substitute the formal series (2.22) or (2.23) in (2.21) and use recurrent relations for the coefficients of the Laurent series.

The subspaces of the solutions of equation (2.21) of the form (2.22) are invariant under the monodromy operator. Therefore, equation (2.17) and the uniqueness of the formal solutions (2.23) imply that the Laurent expansion of the Floque-Bloch solution ψ_n in the neighborhood of \hat{P}_k^+ has the form (2.23), where $\lambda = (z - z(P_k^+))$. Hence, ψ_n has the pole of order n at \hat{P}_k^+ . The same arguments used for the equation $\psi_n = L_n^{-1} \psi_{n+1}$ imply that ψ_n has zero of order n at the punctures \hat{P}_k^- .

Let $S^{D+,D-}$ be a space of the meromorphic functions $r_i(z)$ on Γ with poles of order N at the punctures P_k^+ , and such that r_0 has zeros of order N at the punctures P_k^- . The Riemann-Roch theorem implies that $S^{D+,D-}$ is of dimension

$$\dim S^{D+,D-} = N\mathcal{D}(r-1) - (g-1)(r-1) + 1. \quad (2.26)$$

The characteristic equation (2.13) defines a map $\mathcal{L}_N^{D+,D-} \rightarrow S^{D+,D-}$. Usual arguments show that this map on an open set is surjective. These arguments are based on solution of *the inverse spectral problem*, which reconstruct L_n , modulo gauge equivalence

$$L'_n = g_{n+1}L_n g_n^{-1}, \quad g_n \in GL_r, \quad (2.27)$$

from a generic set of spectral data: a smooth curve $\widehat{\Gamma}$ defined by $\{r_i\} \in S^{D+,D-}$, and a point of the Jacobian $J(\widehat{\Gamma})$, i.e. the equivalence class $[\widehat{\gamma}]$ of degree $\widehat{g} + r - 1$ divisor $\widehat{\gamma}$ on $\widehat{\Gamma}$. Here \widehat{g} is the genus of $\widehat{\Gamma}$.

For a generic point of $S^{D+,D-}$ the corresponding spectral curve $\widehat{\Gamma}$ is smooth. Its genus \widehat{g} can be found with the help of the Riemann-Hurwitz formula $2\widehat{g} - 2 = 2r(g-1) + \deg \nu$, where ν is the divisor on Γ , which is projection of the branch points of $\widehat{\Gamma}$ over Γ . The branch points are zeros on $\widehat{\Gamma}$ of the function $\partial_w R(w, z)$. This function has the poles of order $N(r-1)$ on the marked sheet over P_k^+ , and poles of order N on all the other sheets. The numbers of poles and zeros of a meromorphic function are equal. Therefore, $\deg \nu = 2N\mathcal{D}(r-1)$ and we obtain

$$\widehat{g} = N\mathcal{D}(r-1) + r(g-1) + 1. \quad (2.28)$$

Moreover, the product of $\partial_k R$ on all the sheets of $\widehat{\Gamma}$ is a meromorphic function on Γ . Its divisor of zeros coincides with ν and the divisor of poles is $N(r-1)D_+$. Therefore, these divisors are equivalent, i.e. in the Jacobian $J(\Gamma)$ of Γ we have the equality

$$[\nu] = 2N(r-1)[D_+] \in J(\Gamma). \quad (2.29)$$

The degree of the divisor $\widehat{\gamma}$ of the poles of ψ_0 can be found in the usual way. Let $\Psi_n(q)$, $q \in \Gamma$, be a matrix with columns $\psi_n(Q^i)$, where $Q^i = (w_i(q), q)$ are preimages of q on $\widehat{\Gamma}$

$$\Psi_n(q) = \{\psi_n(Q^1), \dots, \psi_n(Q^r)\}. \quad (2.30)$$

This matrix depends on ordering of the roots $w_i(q)$ of (2.13), but the function $F(q) = \det^2 \Psi_0(q)$ is independent of this. Therefore, F is a meromorphic function on Γ . Its divisor of poles equals $2\pi_*(\widehat{\gamma})$, where $\pi : \widehat{\Gamma} \rightarrow \Gamma$ is the projection. In the general position, when the branch points of $\widehat{\Gamma}$ over Γ are simple, the function F has simple zeros at the images of the branch points, and double zeros at the points $\gamma_s(0)$, because evaluations of ψ_0 at preimages of γ_s span the subspace orthogonal to $\alpha_s(0)$. Therefore, the zero divisor of F is $\nu + 2\gamma(0)$, where $\gamma(0) = \gamma_1(0) + \dots + \gamma_{rg}(0)$, and we obtain the equality for equivalence classes of the divisors

$$2[\pi_*(\widehat{\gamma})] = [\nu] + 2[\gamma(0)] = 2[\gamma(0)] + 2N(r-1)[D_+], \quad (2.31)$$

which implies

$$\deg \widehat{\gamma} = \deg \nu/2 + rg = \widehat{g} + r - 1. \quad (2.32)$$

The theorem is proven.

Let us fix a point P_0 on Γ , and let Ψ_n be the matrix defined by (2.30) for $q = P_0$. Normalization (2.19) implies that Ψ_0 leaves the co-vector $e_0 = (1, \dots, 1)$ invariant, i.e.

$$e_0 \Psi_0 = e_0. \quad (2.33)$$

The spectral curve $\hat{\Gamma}$ and the pole divisor $\hat{\gamma}$ are invariant under the gauge transformation $L_n \rightarrow \Psi_{n+1}^{-1} L_n \Psi_n$, $\psi_n \rightarrow \Psi_n^{-1} \psi$, but the matrix Ψ_n gets transformed to the identity $\Psi_n = I$. Let $F = \text{diag}(f_1, \dots, f_r)$ be a diagonal matrix, then the gauge transformation

$$L_n \rightarrow F L_n F^{-1}, \quad \psi_n(Q) \rightarrow f^{-1}(Q) F \psi_n, \quad \text{where } f(Q) = \sum_{i=1}^r f_i \psi_i(Q), \quad (2.34)$$

which preserves the normalization (2.19) and the equality $\Psi_n = I$, changes $\hat{\gamma}$ to an equivalent divisor $\hat{\gamma}'$ of zeros of the meromorphic function $f(Q)$. The gauge transformation of L_n by a permutation matrix corresponds to the permutation of preimages $P_0^i \in \hat{\Gamma}$ of $P_0 \in \Gamma$, which was used to define Ψ_0 .

A matrix g with different eigenvalues has representation of the form $g = \Psi_0 F$, where Ψ_0 satisfy (2.33) and F is a diagonal matrix. That representation is unique up to conjugation by a permutation matrix. Therefore, the correspondence described above $L_n \rightarrow \{\hat{\Gamma}, \hat{\gamma}, \Psi_n\}$ descends to a map

$$\mathcal{L}_N^{D+, D-} / GL_r^N \longmapsto \{\hat{\Gamma}, [\hat{\gamma}]\}, \quad (2.35)$$

which is well-defined on an open set of $\mathcal{L}_N^{D+, D-} / GL_r^N$.

According to the Riemann-Roch theorem for each smooth genus \hat{g} algebraic curve $\hat{\Gamma}$ with fixed points $q^1, \dots, q^r, \hat{P}_k^\pm$, and for each nonspecial degree $\hat{g} + r - 1$ effective divisor $\hat{\gamma}$, there is a unique meromorphic function $\psi_n^i(Q), Q \in \hat{\Gamma}$, such that: ψ_n^i has poles of order n at P_k^+ , and zeros of order n at P_k^- ; outside these points it has divisor of poles in $\hat{\gamma}$; ψ_n^i is normalized by the conditions $\psi_n^i(q^j) = \delta_i^j$. Let $\psi_n(Q)$ be a meromorphic vector-function with the coordinates $\psi_n^i(Q)$. Note, that it satisfies (2.19).

Let $\hat{\Gamma}$ be a smooth algebraic curve that is an r -fold branch cover of Γ $\pi : \hat{\Gamma} \rightarrow \Gamma$. Then for each point $q \in \Gamma$ we define the matrix $\Psi_n(q)$ with the help of (2.30). It depends on a choice of order of the sheets of the cover π , but the matrix function

$$\tilde{L}_n(q) = \Psi_{n+1}(q) \Psi_n^{-1}(q), \quad (2.36)$$

is independent of the choice, and therefore, is a meromorphic matrix function on Γ . It has simple poles at $P_k^+ \in D$ and is holomorphic at the points of the branch divisor ν . By reversing the arguments used for the proof of (2.32), we get that the degree of the zero divisor γ of $\det \Psi_n$ equals rg . In general position the zeros $\gamma_s(n)$ are simple. The expansion of \tilde{L}_n at $\gamma_s(n)$ satisfies constraints (2.2, 2.3), where $\alpha_s(n)$ is a unique (up to multiplication) vector orthogonal to the vector-columns of $\Psi_n(\gamma_s(n))$. The determinant of Ψ_n has zeros of order n at the points P_k^- . Hence, the determinant of L_n has simple zeros at P_k^- . Therefore, $\tilde{L}_n \in \mathcal{L}^{D+, D-}$.

If $\hat{\Gamma}$ is defined by equation (2.13), where (r_j) corresponds to a generic point of the space S^{D_+, D_-} , and the points \hat{P}_k^\pm used in the definition of ψ_n are the punctures, at which $w(Q)$ has poles and zeros of order N , then the uniqueness of ψ_n implies

$$W\psi_{n+N} = w\psi_n, \quad W^{ij} = w(q^i)\delta^{ij}. \quad (2.37)$$

From (2.37) it follows that $\tilde{L}_n = W\tilde{L}_{n+N}W^{-1}$, and the gauge equivalent chain $L_n = W^{(n+1)/N}\tilde{L}_nW^{-n/N}$ is N -periodic, $L_n = L_{n+N}$. If the points P_0^i used for normalization of ψ_j are preimages of $P_0 \in \Gamma$, then L , given by (2.36), is diagonal at $q = P_0$, and the correspondence $\{\hat{\Gamma}, \hat{\gamma}\} \rightarrow L$ descends to a map

$$\{\hat{\Gamma}, [\hat{\gamma}]\} \rightarrow \mathcal{L}_N^{D_+, D_-} / GL_r^N, \quad (2.38)$$

which is well-defined on an open set of the Jacobian bundle over S^{D_+, D_-} , where it is inverse to (2.35).

Restricted chains. Let us introduce subspaces $\mathcal{L}_{N, C, \Delta}^{D_+, D_-} \subset \mathcal{L}_N^{D_+, D_-}$ of the Lax chains with *fixed* equivalence classes of the divisors of Tyurin parameters

$$[\gamma(n)] = C + n([D_+] - [D_-]) \in J(\Gamma), \quad (2.39)$$

and with *fixed* determinant $\det T = \Delta = r_0(q)$. The subspace of the corresponding spectral curves will be denoted by $\mathcal{S}_\Delta \in S^{D_+, D_-}$. The points of \mathcal{S}_Δ are sets of functions $r_i(q)$, $i = 1, \dots, r-1$, with the poles of order N at P_k^+ . From equation (2.31) it follows that for the restricted chains the equivalence class $[\hat{\gamma}] \in J(\hat{\Gamma})$ of the poles of the Floque-Bloch solutions belongs to the abelian subvariety

$$J_C(\hat{\Gamma}) = \pi_*^{-1}(C + N(r-1)[D_+]/2), \quad \pi_* : J(\hat{\Gamma}) \mapsto J(\Gamma). \quad (2.40)$$

Corollary 2.1 *The correspondence*

$$\mathcal{L}_{N, C, \Delta}^{D_+, D_-} / GL_r^N \leftrightarrow \{\hat{\Gamma} \in \mathcal{S}_\Delta, [\hat{\gamma}] \in J_C(\hat{\Gamma})\} \quad (2.41)$$

is one-to one on the open sets.

3 Lax equations

Our next goal is to construct a hierarchy of commuting flows on an open set of \mathcal{L}^{D_+, D_-} . Let us identify the tangent space $T_L(\mathcal{L}^{D_+, D_-})$ at the point $L = \{L_n\}$ with the space of meromorphic matrix functions spanned by derivatives $\partial_\tau L_n|_{\tau=0}$ of all the one-parametric deformations $L_n(q, \tau) \in \mathcal{L}^{D_+, D_-}$ of L_n . Let us show that the latter space can be identified with the space of matrix functions $l_n(q)$ on Γ such that:

1⁰. l_n has simple poles at the punctures P_k^+ of the form

$$l_n = \frac{\dot{h}_k(n)p_k^T(n) + h_k(n)\dot{p}_k^T(n)}{z - z(P_k^+)} + O(1), \quad (3.1)$$

where $\dot{h}_k(n), \dot{p}_k(n)$ are vectors, defined up to the gauge transformation

$$\dot{h}_k(n) \longmapsto \dot{h}_k(n) + \tilde{c}_{n,k} h_k(n), \quad \dot{p}_k(n) \longmapsto \dot{p}_k(n) - \tilde{c}_{n,k} p_k(n). \quad (3.2)$$

The vectors $h_k(n), p_k(n)$ are defined by the expansion (2.14) of L_n .

2⁰. l_n has double poles at the points γ_s , where it has the expansion of the form

$$l_n = \dot{z}_s(n) \frac{\beta_s(n) \alpha_s^T(n)}{(z - z_s(n))^2} + \frac{\dot{\beta}_s(n) \alpha_s^T(n) + \beta_s(n) \dot{\alpha}_s^T(n)}{z - z_s(n)} + O(1). \quad (3.3)$$

Here $\dot{z}_s(n)$ is a constant, and $\dot{\alpha}_s(n), \dot{\beta}_s(n)$ are certain vectors. The vectors $\alpha_s(n), \beta_s(n)$ are defined by L_n .

3⁰. In addition it is required that the following equation holds:

$$\alpha_s(n+1) l_n(\gamma_s(n+1)) + \dot{\alpha}_s(n+1) L_n(\gamma(n+1)) + \dot{z}_s \alpha_s(n+1) L'_n(\gamma_s(n+1)) = 0, \quad (3.4)$$

where $L' = \partial_z L(z)$.

4⁰. The function

$$\text{Tr} \left(l_n L_n^{-1} \right) = O(1), \quad z \rightarrow P_k^- \quad (3.5)$$

is regular at the punctures P_k^- .

The constraints (1⁰–4⁰) can be easily checked for a tangent vector $\partial_\tau L|_{\tau=0}$, if we identify $(\dot{z}_s, \dot{\alpha}_s, \dot{\beta}_s, \dot{p}_k, \dot{q}_k)$ with the derivatives

$$\dot{z}_s = \partial_\tau z(\gamma_s(\tau)), \quad \dot{\alpha}_s = \partial_\tau \alpha_s(\tau), \quad \dot{\beta}_s = \partial_\tau \beta_s(\tau), \quad \dot{h}_k = \partial_\tau h_k(\tau), \quad \dot{p}_k = \partial_\tau p_k(\tau). \quad (3.6)$$

Direct counting of the number of the constraints shows that the space of matrix functions, which satisfy (3.1-3.5), equals the dimension of $\mathcal{L}^{D+, D-}$. Therefore, on an open set these relations are necessary and sufficient conditions for l_n to be a tangent vector.

Lemma 3.1 *Let M_n be a meromorphic matrix function on Γ with poles at P_k^+ and with simple poles at $\gamma_s(n)$ of the form:*

$$M_n = \frac{\mu_s(n) \alpha_s^T(n)}{z - z_s(n)} + m_s(n) + O(z - z_s), \quad z_s(n) = z(\gamma_s(n)), \quad (3.7)$$

where $\mu_s(n)$ is a vector. Then the matrix-function $M_{n+1} L_n - L_n M_n$ is a tangent vector to $\mathcal{L}^{D+, D-}$ at L_n , if and only if it has the form (3.1) in the neighborhood of P_k^+ .

Proof. It is straightforward to check that, if we define $\dot{z}_s(n), \dot{\alpha}_s(n)$ by the formulae (2.7), (2.8) in [2], i.e.

$$\dot{z}_s(n) = -\alpha_s^T(n) \mu_s(n), \quad z_s = z(\gamma_s), \quad (3.8)$$

$$\dot{\alpha}_s^T(n) = -\alpha_s^T(n) m_s(n), \quad (3.9)$$

then $M_{n+1}L_n - L_nM_n$ satisfies the constraints (3.3) and (3.4). The constraint (3.5) is also satisfied, because $\text{Tr}((M_{n+1}L_n - L_nM_n)L_n^{-1}) = \text{Tr}(M_{n+1} - M_n)$, and M_n is regular at P_k^- .

The Lemma directly implies, that the Lax equation $\partial_t L_n = M_{n+1}L_n - L_nM_n$ is a well-defined system on an open set of $\mathcal{L}^{D+, D-}$, whenever we can define $M_n = M_n(L)$, as a function of L , that satisfies the conditions of Lemma 3.1. Our next goal is to define a set of such functions $M_n^{(\pm k, l)}(L)$, parameterized by the puncture P_k^\pm and a non-negative integer $l \in Z_+$.

Let us fix a point $P_0 \in \Gamma$ and local coordinates z in the neighborhoods of the punctures P_k^+ . Let $\phi_n^{(k)}, \phi_n^{(*, k)}$ be the formal solutions in the neighborhood of P_k^+ of equation (2.11) and the dual equation $\psi_{n+1}^* L_n = \psi_n^*$, which have the form (2.23), where $\lambda = z - z(P_k^+)$. From Lemma 2.1, applied to the inverse chain L_n^{-1} , it follows that equation (2.11) and its dual in the neighborhood of P_k^- have formal solutions $\phi_n^{(-k)}, \phi_n^{(*, -k)}$ of the form

$$\phi_n^- = \lambda^n \left(\sum_{i=0}^{\infty} \xi_i^-(n) \lambda^i \right), \quad \phi_n^{*, -} = \lambda^{-n} \left(\sum_{i=0}^{\infty} \xi_i^{*, -}(n) \lambda^i \right). \quad (3.10)$$

From the Riemann-Roch theorem (see details in [2]) it follows that there is a unique matrix function $M_n^{(\pm k, l)}$ such that:

- (i) it has the form (3.7) at the points γ_s ;
- (ii) outside of the divisor γ it has pole at the point P_k^\pm , only, where

$$M_n^{(\pm k, l)} = \left(z - z(P_k^\pm) \right)^{-l} \phi_n^{(\pm k)} \phi_n^{(*, \pm k)} + O(1); \quad (3.11)$$

- (iii) $M_n^{(\pm k, l)}$ is normalized by the condition $M_n^{(\pm k, l)}(P_0) = 0$.

Note, that although $\phi_n^{(\pm k)}$ and $\phi_n^{(*, \pm k)}$ are formal series, the constraint (3.11) involves only a finite number of their coefficients, and therefore, is well-defined.

Theorem 3.1 *The equations*

$$\partial_a L_n = M_{n+1}^a L_n - L_n M_n^a, \quad \partial_a = \partial / \partial t_a, \quad a = (\pm k, l), \quad (3.12)$$

define a hierarchy of commuting flows on an open set of $\mathcal{L}^{D+, D-}$, which descends to the commuting hierarchy on an open set of $\mathcal{L}_N^{D+, D-} / GL_r^N$.

Equation (3.11) implies that M_n^a satisfies the conditions of Lemma 3.1. Therefore, the right hand side of (3.12) is a tangent vector to $\mathcal{L}^{D+, D-}$ at the point L . Hence, (3.12) is a well-defined dynamical system on an open set of $\mathcal{L}^{D+, D-}$. Commutativity of flows (3.12) is equivalent to the equation

$$\partial_b M_n^a - \partial_a M_n^b + [M_n^a, M_n^b] = 0. \quad (3.13)$$

As shown in the proof of Theorem 2.1 of [2], this equation holds if its left hand side is regular at the points P_k^\pm . From the uniqueness of the formal solutions $\phi_n^{(\pm k)}, \phi_n^{(*, \pm k)}$ it follows that

$$\partial_a \phi_n^{(\pm k)} = M_n^a \phi_n^{(\pm k)} - f^{(\pm k, a)} \phi_n^{(\pm k)}, \quad (3.14)$$

$$-\partial_a \phi_n^{(*, \pm k)} = M_n^a \phi_n^{(*, \pm k)} - f^{(\pm k, a)} \phi_n^{(*, \pm k)}, \quad (3.15)$$

where $f^{(\pm k, a)}$ are scalar functions. The left hand side of (3.14) is regular at all the punctures P_m^\pm . Vanishing of the singular terms of the right hand side of these equations implies that for $a = (\pm m, l)$

$$f^{(\pm k, a)} = \delta_{m, k} (z - z(P_k^\pm))^{-l} + O(1). \quad (3.16)$$

Equations (3.14, 3.15) and standard arguments used in KP theory (see details in [2]) imply, that the left hand side of (3.13) is regular at P_k^\pm . By definition, the matrix functions M_n^a are periodic, if L_n are periodic. The matrices M_n^a under the transformations (2.27) get transformed to $\tilde{M}_n^a = g_n M_n^a g_n^{-1}$. Therefore, the flows (3.12) are well-defined on $\mathcal{L}_N^{D+, D-}$. The theorem thus is proven.

In general the flows (3.12) do not preserve the leaves of the foliation $\mathcal{L}_{N, C, \Delta}^{D+, D-} \subset \mathcal{L}_N^{D+, D-}$. Let us find their linear combinations for which the subspaces of the restricted chains are invariant. Let f be a meromorphic function on Γ with poles only at the punctures P_k^\pm . Then we define

$$M_n^f = \sum_a c_a^f M_n^a, \quad (3.17)$$

where c_a^f are the coefficients of the singular part of the Laurent expansion

$$f = \sum_{l>0} c_{(\pm k, l)}^f (z - z(P_k^\pm))^{-l} + O(1). \quad (3.18)$$

Theorem 3.2 *The equations*

$$\partial_f L_n = M_{n+1}^f L_n - L_n M_n^f, \quad \partial_f = \partial / \partial t_f, \quad (3.19)$$

define a hierarchy of commuting flows on an open set of $\mathcal{L}_{N, C, \Delta}^{D+, D-}$, which descends to the commuting hierarchy on an open set of $\mathcal{L}_{N, C, \Delta}^{D+, D-} / GL_r^N$.

Proof. The flows (3.19) are linear combinations with constant coefficients of the basic flows (3.12). Therefore, they are well-defined and commute with each other. Let us consider the common Floque-Bloch solution $\hat{\psi}_n(t, Q)$ of (2.11) and the equation

$$\partial_f \hat{\psi}_n = M_n^f \hat{\psi}_n, \quad (3.20)$$

normalized by the condition (2.19) at $t = 0$. Then equations (3.14-3.18) imply that in the neighborhood of P_k^\pm the function $\hat{\psi}_0(t, Q)$ has the form

$$\hat{\psi}_0 = e^{t_f f(z)} O(1). \quad (3.21)$$

Standard arguments, used in the construction the Baker-Akhiezer functions, imply that outside the punctures \hat{P}_k^\pm the functions $\hat{\psi}_n$ has *time-independent* poles at the pole divisor $\hat{\gamma}(0)$ of $\hat{\psi}_0$. The pole divisor $\hat{\gamma}(t_f)$ of $\psi_n(t_f, Q)$ is the divisor of zeros of the function $F(t, Q) = \sum_i \psi_0^i(t, Q)$, where ψ_0^i are the coordinates of ψ_0 . The function $\tilde{F}(t, q) = \prod_j F(t, Q^j(q))$, where Q^j are the preimages of q on $\hat{\Gamma}$, is meromorphic on Γ outside P_k^\pm . Equation (3.21) implies that $\tilde{F}(t_f, q) e^{-t_f f(q)}$ is a meromorphic function on Γ . It has poles at the divisor $\pi_*(\hat{\gamma}(0))$ and zeros at the divisor $\pi_*(\hat{\gamma}(t_f))$. Therefore, these divisors are equivalent and the Theorem is thus proven.

4 Hamiltonian approach

In this section we apply the general algebraic approach to the Hamiltonian theory of the Lax equations proposed in [8, 9], and developed in [13], to the Lax equations for *periodic* chains on the algebraic curves. As it was mentioned in the introduction, this approach is based on the existence of two universal two-forms on a space of meromorphic matrix-function. They can be traced back to the fact that there are two basic algebraic structures on a space of *operators* (see details in [10]). The first one is the Lie algebra structure defined by the commutator of operators. The second one is the Lie group structure. The discrete Lax equations or chains correspond to the Lie group structure.

The entries of $L_n(q) \in \mathcal{L}_N^{D+, D-}$ can be regarded as functions on $\mathcal{L}_N^{D+, D-}$. Therefore, L_n by itself can be seen as matrix-valued function and its external derivative δL_n as a matrix-valued one-form on $\mathcal{L}_N^{D+, D-}$. The matrix Ψ_n (2.30) with columns formed by the canonically normalized Floque-Bloch solutions $\psi_n(Q^i)$ of (2.11) can also be regarded as a matrix function on $\mathcal{L}_N^{D+, D-}$ (modulo permutation of the columns). Hence, its differential $\delta \Psi_n$ is a matrix-valued one-form on $\mathcal{L}_N^{D+, D-}$. Let us define a two-form $\Omega(z)$ on $\mathcal{L}_N^{D+, D-}$ with values in the space of meromorphic functions on Γ by the formula

$$\Omega(z) = \sum_{n=0}^{N-1} \text{Tr} \left(\Psi_{n+1}^{-1} \delta L_n \wedge \delta \Psi_n \right). \quad (4.1)$$

It can be also represented in the form

$$\Omega(z) = \text{Tr} \left(\Psi_N^{-1} \delta T \wedge \delta \Psi_0 \right) = \text{Tr} \left(\Psi_0^{-1} T^{-1} \delta T \wedge \delta \Psi_0 \right) = \text{Tr} \left(W^{-1} \Psi_0^{-1} \delta T \wedge \delta \Psi_0 \right), \quad (4.2)$$

where $W = \text{diag}(w_i(z))$ is a diagonal matrix, whose diagonal entries are the eigenvalues of the monodromy matrix $T = T_0$

Fix a meromorphic differential dz on Γ with poles at a set of points q_m . Then the formula

$$\omega = -\frac{1}{2} \sum_{q \in I} \text{res}_q \Omega dz, \quad I = \{\gamma_s, P_k^\pm, q_m\} \quad (4.3)$$

defines a scalar-valued two-form on $\mathcal{L}_N^{D+, D-}$. This form depends on a choice of the normalization of Ψ_n . A change of the normalization corresponds to the transformation $\Psi'_n = \Psi_n V$, where $V = V(z)$ is a diagonal matrix, which might depend on a point z of Γ . The corresponding transformation of Ω has the form:

$$\Omega' = \Omega + \delta (\text{Tr} (\ln W v)), \quad v = \delta V V^{-1}. \quad (4.4)$$

Here we use the standard formula for a variation of the eigenvalues

$$\Psi_0^{-1} \delta T \Psi_0 = \delta W + \Psi_0^{-1} \delta \Psi_0 W - W \Psi_0^{-1} \delta \Psi_0, \quad (4.5)$$

and the equation $\delta v = v \wedge v = 0$, which is valid because v is diagonal.

Let $\mathcal{X}^{D+, D-}$ be a subspace of the chains $\mathcal{L}_N^{D+, D-}$ such, that the restriction of $\delta(\ln w) dz$ to $\mathcal{X}^{D+, D-}$ is a differential *holomorphic* at all the preimages on $\hat{\Gamma}$ of the punctures P_k^\pm .

Lemma 4.1 *The two-form ω , defined by (4.3) and restricted to $\mathcal{X}^{D+,D-} \subset \mathcal{L}_N^{D+,D-}$, is independent of the choice of normalization of the Floque-Bloch solutions, and is gauge invariant, i.e. it descends to a form on $\mathcal{P} = \mathcal{X}^{D+,D-} / GL_r^N$.*

The proof of the lemma is almost identical to the proof of Lemma 2.4 in [2].

By definition, a vector field ∂_t on a symplectic manifold is Hamiltonian, if the contraction $i_{\partial_t}\omega(X) = \omega(\partial_t, X)$ of the symplectic form is an exact one-form $dH(X)$. The function H is the Hamiltonian, corresponding to the vector field ∂_t . The proof of the following theorem is almost identical to the proof of Theorem 4.2 in [2].

Theorem 4.1 *Let ∂_a be the vector fields corresponding to the Lax equations (3.12). Then the contraction of ω , defined by (4.3) and restricted to \mathcal{P} , equals*

$$i_{\partial_a}\omega = \delta H_a, \quad (4.6)$$

where

$$H_{(\pm k, l)} = \text{res}_{P_k^\pm} \left(z - z(P_k^\pm) \right)^{-l} (\ln w) dz. \quad (4.7)$$

The theorem implies that Lax equations (3.12) are Hamiltonian whenever the form ω is non-degenerate. In order to analyze this problem we first find the Darboux variables for ω .

Theorem 4.2 *Let $L_n \in \mathcal{L}_N^{D+,D-}$ be a periodic chain, and let $\hat{\gamma}_s$ be the poles of the normalized (2.19) Floque-Bloch solution ψ_n . Then the two-form ω defined by (4.3) is equal to*

$$\omega = \sum_{s=1}^{\hat{g}+r-1} \delta \ln w(\hat{\gamma}_s) \wedge \delta z(\hat{\gamma}_s). \quad (4.8)$$

The proof of the Theorem is analogous to the proof of the Theorem 4.3 in [2] and equation (5.7) in [15]. The meaning of the right hand side of the formula (4.8) is as follows. By definition, the spectral curve is equipped with the meromorphic function $w(Q)$. The pull-back of the Abelian integral $z(q) = \int^q dz$ on Γ is a multi-valued holomorphic function on $\hat{\Gamma}$. The evaluations $w(\hat{\gamma}_s)$, $z(\hat{\gamma}_s)$ at the points $\hat{\gamma}_s$ define functions on the space $\mathcal{L}_N^{D+,D-}$, and the wedge product of their external differentials is a two-form on $\mathcal{L}_N^{D+,D-}$. (Note, that differential $\delta z(\hat{\gamma}_s)$ of the multi-valued function $z(\hat{g}_s)$ is single-valued, because the periods of dz are constants).

From (4.8) it follows that ω can be represented in the form:

$$\omega = \sum_{k=1}^{\hat{g}} \delta A_k \wedge \delta \varphi_k, \quad (4.9)$$

where φ_k are the coordinates on $J(\hat{\Gamma})$, corresponding to a choice of a - and b -cycles on $\hat{\Gamma}$ with the canonical matrix of intersections, and

$$A_k = \oint_{a_k} (\ln w) dz. \quad (4.10)$$

Note, that the external differential δA_k of the multi-valued function A_k is single-valued, because all the periods of dz are fixed.

The spectral map (2.35) identifies an open set of $\mathcal{L}_{N,C,\Delta}^{D_+,D_-}/GL_r^N$ with an open set of the Jacobian bundle over $\mathcal{S}_\Delta \subset \mathcal{S}^{D_+,D_-}$, i.e.

$$\mathcal{L}_{N,C,\Delta}^{D_+,D_-}/GL_r^N \longmapsto \mathcal{S}_\Delta. \quad (4.11)$$

From (4.9) it follows, that the form ω can be non-degenerate only if the base and the fibers of the bundle (4.11), restricted to \mathcal{P} , have the same dimension.

First, let us consider the case of the chains on the rational curve (see details in [10]). The basic example is the chain corresponding to the Toda lattice, in which L_n has the form

$$L_n = \begin{pmatrix} 0 & 1 \\ c_n & z + v_n \end{pmatrix}. \quad (4.12)$$

For $g = 0$ equations (2.26) and (2.28) imply that the space of the spectral curves and their Jacobians are of dimensions $N\mathcal{D}(r-1) + r - 1$ and $N\mathcal{D}(r-1) - r + 1$, respectively. The differential dz has double pole at the infinity. Therefore, the subspace $\mathcal{X}^{D_+,D_-} \subset \mathcal{L}_N^{D_+,D_-}$ is defined by the constraint that the eigenvalues of the monodromy matrix are fixed up to the order $O(z^{-2})$. The number of corresponding equations on \mathcal{S}_Δ is $2r - 1$. Let \mathcal{P}_* be a subspace of \mathcal{P} corresponding to the restricted chains. For $g = 0$ that means that the determinant of the monodromy matrix is fixed. Then $\dim \mathcal{P}_* = 2 \dim J(\hat{\Gamma})$ and arguments identical to that used at the end of Section 4 in [2] prove, that the form ω is non-degenerate on \mathcal{P}_* .

Consider now the case $g > 0$. Let dz be a holomorphic differential on Γ . Then, for each branch of $w = w_i(z)$ the differential $\delta(\ln w) dz$ is always holomorphic at P_k^\pm . Hence, $\mathcal{P} = \mathcal{L}_N^{D_+,D_-}/GL_r^N$. Recall, that

$$\dim \mathcal{S}_\Delta = (r-1)(N\mathcal{D} - g + 1), \quad \dim J(\hat{\Gamma}) = (r-1)(N\mathcal{D} + g - 1) + g. \quad (4.13)$$

Therefore, for $g > 0$ the form ω is degenerate on \mathcal{P} . For $g = 1$ the space \mathcal{P} is a Poisson manifold with the symplectic leaves, which are factor-spaces

$$\mathcal{P}_* = \mathcal{L}_{N,C,\Delta}^{D_+,D_-}/GL_r^N \quad (4.14)$$

of the restricted chains. In that case $\dim \mathcal{S}_\Delta = \dim J_C(\hat{\Gamma}) = N\mathcal{D}(r-1)$. As in the genus zero case, the arguments identical to that used at the end of Section 4 in [2] prove that the form ω is non-degenerate on \mathcal{P}_* .

Corollary 4.1 *For $g = 0$ and $g = 1$ the form ω defined by (4.3) descends to the symplectic form on \mathcal{P}_* , which coincides with the pull-back of the Sklyanin symplectic structure restricted to the space of the monodromy operators. The Lax equations (3.19) are Hamiltonian with the Hamiltonians*

$$H_f = \sum_{q=\hat{P}_k^\pm} \text{res}_q(\ln w) f dz. \quad (4.15)$$

The Hamiltonians H_f are in involution $\{H_f, H_h\} = 0$.

The first statement is a direct corollary of equation (4.8) and the Proposition 3.33 in [14]. The second statement is a corollary of equation (4.7) and the definition of H_f . Finally, involutivity of the Hamiltonians follows from the commutativity of the corresponding Lax flows.

Now we are in the position to discuss the case $g > 1$ mentioned in the Introduction. The space \mathcal{P}_* is equipped by g -parametric *family* of two-forms ω_{dz} , parameterized by the holomorphic differentials dz on Γ . For all of them the fibers $J_C(\hat{\Gamma})$ of the spectral bundle are maximum isotropic subspaces. For each vector-field ∂_f defined by (3.19) the equation $i_{\partial_f}\omega_{dz} = \delta H_f^{dz}$ holds.

Equation (4.9) implies, that each of the forms ω_{dz} is degenerate on \mathcal{P}_* . Let us describe the kernel of ω_{dz} . According to Theorem 3.2, the tangent vectors to $J_C(\hat{\Gamma})$ are parameterized by the space $\mathcal{A}(\Gamma, P_{\pm})$ of meromorphic functions f on Γ with the poles at P_k^{\pm} modulo the following equivalence relation. The function f is equivalent to f_1 , if there is a meromorphic function $F \in \mathcal{A}(\hat{\Gamma}, \hat{P}_k^{\pm})$ on $\hat{\Gamma}$ with the poles at \hat{P}_k^{\pm} , such that in the neighborhoods of these punctures the function $\pi^*(f - f_1) - F$ is regular. Let $K_{dz} \subset \mathcal{A}(\Gamma, P_{\pm})$ be the subspace of functions such that there is a meromorphic function \tilde{F} on $\hat{\Gamma}$ with poles at \hat{P}_k^{\pm} and at the preimages $\pi^*(q_s)$, $dz(q_s) = 0$ of the zero-divisor of dz , and such that $f - \tilde{F}$ is regular at \hat{P}_k^{\pm} . Then, from equations (4.6) and (4.15) it follows that: $f \in K_{dz} \mapsto i_{\partial_f}\omega_{dz} = 0$. Let \mathcal{K}_{dz} be the factor-space of K_{dz} modulo the equivalence relation. Then the Riemann gap theorem implies that in the general position \mathcal{K}_{dz} is of dimension $2(g-1)(r-1)$, which equals the dimension of the kernel of ω_{dz} . Therefore, the kernel of ω_{dz} is isomorphic to \mathcal{K}_{dz} . Using this isomorphism, it is easy to show that the intersection of all the kernels of the forms ω_{dz} is empty, and thus the family of these forms is *non-degenerate*.

5 Variable base curves

Until now it has been always assumed that the base curve is *fixed*. Let \mathcal{M}_{Δ} be the space of smooth genus g algebraic curves Γ with the fixed meromorphic function Δ , having poles and zeros of order N at punctures $P_k^{\pm}, k = 1, \dots, \mathcal{D}$. For simplicity, we will assume that the punctures P_k^{\pm} are distinct. The space \mathcal{M}_{Δ} is of dimension $\dim \mathcal{M}_{\Delta} = 2(\mathcal{D} + g - 1)$. The total space $\hat{\mathcal{L}}_{N, \Delta}$ of all the restricted chains corresponding to these data and the trivial equivalence class $C = 0 \in J(\Gamma)$ can be regarded as the bundle over \mathcal{M}_{Δ} with the fibers $\mathcal{L}_{N, 0, \Delta}^{D+, D-} = \mathcal{L}_{N, 0, \Delta}^{D+, D-}(\Gamma)$. By definition, the curve Γ corresponding to a point $(\Gamma, \Delta) \in \mathcal{M}_{\Delta}$ is equipped by the meromorphic differential $dz = d \ln \Delta$. The function Δ defines local coordinate everywhere on Γ except at zeros of its differential. Let ω_{Δ} be the form defined by (4.3), where $dz = d \ln \Delta$ and the variations of L_n and Ψ_n are taken with fixed Δ , i.e.

$$\omega_{\Delta} = -\frac{1}{2} \sum_{q \in I} \text{res}_q \sum_{n=0}^{N-1} \text{Tr} \left(\Psi_{n+1}^{-1}(\Delta) \delta L_n(\Delta) \wedge \delta \Psi_n(\Delta) \right) d \ln \Delta, \quad I = \{\gamma_s, P_k^{\pm}\}. \quad (5.1)$$

Then ω_{Δ} is well-defined on leaves $\hat{\mathcal{X}}_{\Delta}$ of the foliation on $\hat{\mathcal{L}}_{N, \Delta}$ defined by the condition: the differential $\delta \ln w(\Delta) d \ln \Delta$ restricted to $\hat{\mathcal{X}}_{\Delta}$ is *holomorphic* at the punctures P_k^{\pm} . This condition is equivalent to the following constraints. In the neighborhood of P_k^{\pm} there are

$(r - 1)$ regular branches w_i^\pm of the multi-valued function w , defined by the characteristic equation (2.13):

$$w_i^\pm = c_i^\pm + O(\Delta^{\mp 1}), \quad i = 1, \dots, r - 1. \quad (5.2)$$

The leaves $\hat{\mathcal{X}}_\Delta$ are defined by $2\mathcal{D}(r - 1)$ constraints:

$$\delta c_i^\pm = 0 \longmapsto c_i^\pm = \text{const}_i^\pm. \quad (5.3)$$

Note, that the differential $\delta \ln w(\Delta) d \ln \Delta$ is regular at P_k^\pm for the singular branches of w , because the coefficients c^\pm of the expansions (2.17) and (2.18) are also fixed due to the equation $c^\pm \prod_{i=1}^{r-1} c_i^\pm = 1$.

The factor-space $\hat{\mathcal{P}}_\Delta = \hat{\mathcal{X}}_\Delta / GL_r^N$ is of dimension $\dim \hat{\mathcal{P}}_\Delta = 2N\mathcal{D}(r - 1) - 2\mathcal{D}(r - 1) + 2(\mathcal{D} + g - 1)$. The space $\hat{\mathcal{S}}_\Delta^0 \subset \hat{\mathcal{S}}_\Delta$ of the corresponding spectral curves is of dimension $\dim \hat{\mathcal{S}}_\Delta^0 = (r - 1)(N\mathcal{D} - g + 1) - 2\mathcal{D}(r - 1) + 2(\mathcal{D} + g - 1)$. The second and the third terms in the last formulae are equal to the number of the constraints (5.3) and the dimension of \mathcal{M}_Δ , respectively.

For the case $r = 2$ the last formulae imply the match of the dimensions $\dim \hat{\mathcal{P}}_\Delta = 2 \dim \hat{\mathcal{S}}_\Delta^0$. For $r = 2$ the spectral curves are two-sheet cover of the base curves, and the fiber of the spectral bundle is the Prim variety $J_0(\hat{\Gamma}) = J_{Prim}(\hat{\Gamma})$.

Theorem 5.1 *For $r = 2$ the form ω_Δ defined by (5.1), and restricted to $\hat{\mathcal{P}}_\Delta$ is non-degenerate. If $\hat{\gamma}_s$ are the poles of the normalized Floque-Bloch solution ψ_n , then*

$$\omega_\Delta = \sum_{s=1}^{\hat{g}+r-1} \delta \ln w(\hat{\gamma}_s) \wedge \delta \ln \Delta(\hat{\gamma}_s) = \sum_{s=1}^{\hat{g}+r-1} \delta \ln w(\hat{\gamma}_s) \wedge \delta \ln w(\hat{\gamma}_s^\sigma), \quad (5.4)$$

where $\sigma : \hat{\Gamma} \rightarrow \hat{\Gamma}$ is the involution, which permutes the sheets of $\hat{\Gamma}$ over Γ .

For every function $f \in \mathcal{A}(\Gamma, P_k^\pm)$ the Lax equations (3.19) are Hamiltonian with the Hamiltonians

$$H_f = \sum_{q=\hat{P}_k^\pm} \text{res}_q (f \ln w) d \ln \Delta.$$

The Hamiltonians H_f are in involution. Their common level sets are fibers $J_{Prim}(\hat{\Gamma})$ of the spectral map.

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