MOSCOW MATHEMATICAL JOURNAL
Volume 2, Number 4, October–December 2002, Pages 717–752

ISOMONODROMY EQUATIONS ON ALGEBRAIC CURVES, CANONICAL TRANSFORMATIONS AND WHITHAM EQUATIONS

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Dedicated to Yu. I. Manin on the occasion of his 65th birthday

Abstract. We construct the Hamiltonian theory of isomonodromy equations for meromorphic connections with irregular singularities on algebraic curves. We obtain an explicit formula for the symplectic structure on the space of monodromy and Stokes matrices. From these we derive Whitham equations for the isomonodromy equations. It is shown that they provide a flat connection on the space of spectral curves of Hitchin systems.

2000 Math. Subj. Class. 14, 79.

Key words and phrases. Algebraic curves, meromorphic connections, monodromy matrices, Hamiltonian theory, symplectic form.

1. Introduction

The goal of this paper is multi-fold. Our first objective is to construct isomonodromy equations for meromorphic connections with irregular and regular singularities on algebraic curves. The isomonodromy equations for linear systems with irregular singularities on rational curves generalizing Schlesinger's equations [30] were introduced by Jimbo, Miwa and Ueno [13]. A particular case of these equations was considered earlier by Flaschka and Newell [5] in connection with the theory of self-similar solutions of the mKdV equation. Fuchsian systems on higher genus Riemann surfaces were considered in [12]. The case of linear systems with one irregular singularity on an elliptic curve was treated in [28]. The recent burst of interest to isomonodromy equations for linear systems with regular singularities on higher genus Riemann surfaces is due to their connections with the classical limit of Knizhnik–Zamolodchikov–Bernard equations for correlation functions of the Wess–Zumino–Witten–Novikov theory. In the case of rational and elliptic curves these connections were revealed in [29], [9], [14]. The general case was considered in [27], where a more complete list of references can be found.

Received February 18, 2002; in revised form April 20, 2002.

Supported in part by National Science Foundation under the grant DMS-01-04621 and by CRDF Award RP1-2102.

The conventional modern approach to the theory of isomonodromy equations is based on their representation in the form of compatible non-autonomous Hamiltonian systems, that can be identified with the Hamiltonian reduction of some free Hamiltonian theory. This approach presents an almost exhaustive geometric description of the system, but it requires solving the corresponding moment map equations in order to get an explicit form of the equations or their Lax representation. The moment map equations are differential equations on an algebraic curve. They have been solved explicitly only in very few cases [27].

As in [30], [13], the starting point of our approach is the Lax representation of isomonodromy equations. In the next section the space of meromorphic connections on stable, rank r, and degree rg holomorphic vector bundles on an algebraic curve Γ with the poles divisor $D=\sum_m (h_m+1)P_m$ is identified with orbits $\mathcal{A}^D/\mathrm{SL}_r$ of the adjoint action of SL_r on a certain subspace $\mathcal{A}^D$ of meromorphic matrix-valued differentials on Γ. A characteristic property of $\widetilde{L}\in\mathcal{A}^D$ is that all its additional singularities at points $\gamma_s\notin D$ are of the form $d\Phi\Phi^{-1}$, where Φ is holomorphic. We show that an open set of $\mathcal{A}^D$ corresponding to the case when all additional poles of $\widetilde{L}$ are simple, is parameterized by the data

$$\begin{gathered}\widetilde{L}_m, \quad (\gamma_s,\,\kappa_s), \quad L_{s0}=\beta_s\alpha_s^T,\\ \sum_{s=0}^{rg} L_{s0}+\sum_m \operatorname{res}_{P_m}\widetilde{L}_m=0,\end{gathered} \tag{1.1}$$

where $\widetilde{L}_m$ is the singular part of $\widetilde{L}$ at P_m, $(\gamma_s,\,\kappa_s)$ is a point of the bundle of scalar affine connections on Γ, and L_{s0} is a *rank* 1 matrix such that $\operatorname{Tr} L_{s0}=1$. We identify matrices L_{s0} with pairs of r-dimensional vectors $\alpha_s=(\alpha_s^i)$, $\beta_s=(\beta_s^i)$, considered modulo the transformation $\alpha_s\mapsto\lambda_s\alpha_s$, $\beta_s\mapsto\lambda_s^{-1}\beta_s$, and such that $(\alpha_s^T\beta_s)=1$.

From the definition of $\widetilde{L}\in\mathcal{A}^D$ it follows that the equation

$$d\Psi=\widetilde{L}\Psi \tag{1.2}$$

has a multi-valued holomorphic solution on $\Gamma\setminus D$. Let us fix a point $Q\in\Gamma$. Then, the analytical continuation of Ψ, normalized by the condition $\Psi(Q)=1$, defines a representation of the fundamental group $\pi_1(\Gamma\setminus D;\,Q)\to\mathrm{GL}_r$. The Stokes matrices and the so-called exponents at irregular singularities P_m, $h_m>0$, can be defined purely locally, as in the case of genus $g=0$, if a local coordinate in the neighborhood of P_m is fixed.

The Stokes data and the exponents at P_m depend only on the h_m-jet of the local coordinate, and therefore, we identify the space of isomonodromy deformations of the linear system (1.2) with the moduli space $\mathcal{M}_{g,1}(h)$ of smooth genus g algebraic curves with a puncture Q and with fixed h_m-jets of local coordinate at punctures P_m. Here and below the *isomonodromy deformations are those preserving the monodromy representation, the Stokes matrices, and the exponents.* For brevity, we call all these data *monodromy data*.

It is necessary to emphasize that for $g=0$ our definition of the deformation space is equivalent to the traditional one. According to [13], [5], the isomonodromy deformations of $\widetilde{L}$ are parameterized by the positions of poles and by the exponents at irregular singularities. In this setting, the local coordinates at the poles are always fixed, and are defined by the global coordinate on the complex plane. It is easy to show that the deformations of the exponents, corresponding to gauge invariant equations for $\widetilde{L}$, can be identified with deformations corresponding to changes of the local coordinate.

A change of the normalization point Q and a gauge transformation $\widetilde{L}' = g\widetilde{L}g^{-1}$, $g \in \mathrm{GL}_r$ correspond to conjugation of the monodromy data by a constant matrix. Hence, the space of isomonodromy deformations of meromorphic connections $\mathcal{A}^D/\mathrm{SL}_r$ is the moduli space $\mathcal{M}_g(h)$ of curves with h_m-jets of local coordinates at P_m. For the connections with regular singularities $h_m \equiv 0$ at N points the deformation space is just $\mathcal{M}_{g,N}$. We consider the space $\mathcal{A}(h)$ of all admissible meromorphic differentials with fixed multiplicities $(h_m + 1)$ at the punctures and the corresponding quotient space of meromorphic connections as bundles

$$\mathcal{A}(h) \longrightarrow \mathcal{M}_{g,1}(h), \quad \mathcal{A}(h)/\mathrm{SL}_r \longrightarrow \mathcal{M}_g(h). \tag{1.3}$$

In Section 4 we derive the Lax representation for a full hierarchy of isomonodromy equations. We show that the Lax equations are equivalent to a system of well-defined compatible evolution equations on the space of dynamical variables which are the parameters (1.1). Naïvely, the Lax representation

$$\partial_{T_a}\widetilde{L} = [M_a, \widetilde{L}] - dM_a \tag{1.4}$$

of the isomonodromy equations is just a coordinate-dependent way of saying that M_a are the coefficients of a flat connection on the space of linear systems (1.2) defined by the monodromy data. In order to give sense to (1.4), it is necessary first to express M_a as a function of $\widetilde{L}$, and then to show, that the Lax equation is equivalent to a well-defined system of differential equations for $\widetilde{L}$. *A priori* the last statement is not obvious, because (1.4) has to be fulfilled identically on Γ, and the space of $\widetilde{L}$ is finite-dimensional. For example, for $g > 0$ it is impossible to define the isomonodromy deformations for matrix-valued differentials with poles only at D. The presence of extra poles γ_s, which become dynamical variables, is a key element, which allows us to overcome that difficulty in defining the isomonodromy equations on higher genus algebraic curves. Exactly the same idea was used in our earlier work [21], where an explicit parameterization of Hitchin systems [11] was obtained, and where infinite-dimensional field generalizations of Hitchin systems were proposed.

In Section 5 we show that the approach to the Hamiltonian theory of soliton equations proposed in [23], [24], [19] is also applicable to the case of isomonodromy deformations. The key element of this approach is a definition of the universal two-form which is expressed in terms of the Lax operator and its eigenvectors. The proof of the fact that the contraction of this form by the vector-field defined by a Lax equation is an exact one-form is very general and does not rely on any specific form of the Lax operator. It provides a direct way to show that the Lax equations are Hamiltonian on suitable subspaces, and at the same time allows to identify the corresponding Hamiltonians.

It turns out that the universal two-form on a space of meromorphic connections is defined identically to the case of isospectral equations if we replace eigenvectors by a solution of equation (1.2). More precisely, let $\mathcal{P}_0$ be a subspace of $\mathcal{A}(h)$ with fixed exponents at the punctures, and let ψ_m be the formal local solutions of (1.2) at P_m (see (3.6) below). Then the formula

$$\omega = -\frac{1}{2}\sum_{s=1}^{rg} \operatorname{res}_{\gamma_s} \operatorname{Tr}(\psi^{-1}\delta \widetilde{L} \wedge \delta\psi) - \frac{1}{2}\sum_{P_m} \operatorname{res}_{P_m} \operatorname{Tr}(\psi_m^{-1}\delta\widetilde{L} \wedge \delta\psi_m) \tag{1.5}$$

defines a closed, nondegenerate differential two-form on the quotient space $\mathcal{P} = \mathcal{P}_0/\mathrm{SL}_r$. The Lax equations restricted to $\mathcal{P}_0$ descend to a system of commuting flows which are Hamiltonian with respect to the symplectic structure defined by ω.

We show that ω can be written in terms of the parameters (1.1) as

$$\omega = \sum_{s=1}^{rg}\left(\delta\kappa_s \wedge \delta z_s + \sum_{i=1}^{r} \delta\beta_s^i \wedge \delta\alpha_s^i\right) + \sum_m \omega_m, \tag{1.6}$$

where ω_m is the canonical symplectic structure on an orbit $\widetilde{\mathcal{O}}_m$ of the adjoint action of the group of invertible formal holomorphic matrix functions on the space of singular parts of meromorphic matrix differentials in a formal disc with the pole of order h_m. (A set of orbits $\widetilde{\mathcal{O}}_m$ corresponds to the set of fixed exponents.)

A remarkable property of the symplectic structure for isospectral equations defined in terms of the Lax operator is that it provides, under quite general assumptions, a straightforward way of construction of action-angle type variables (see examples in [19], [20], [23]–[26]). In Section 6 we show that in the case of isomonodromy equations almost the same arguments lead to an expression of the symplectic form ω in terms of the monodromy data.

For example, the monodromy data corresponding to a meromorphic connection on an elliptic curve are just a pair of matrices A and B, considered modulo mutual conjugation. The monodromy matrix around the puncture is equal to

$$J = B^{-1}A^{-1}BA. \tag{1.7}$$

Symplectic leaves $\mathcal{P}$ are defined by a choice of the orbit for J. Therefore, they can be seen as level sets of the invariants $\operatorname{Tr} J^k$. We show that the symplectic form on $\mathcal{P}$ defined by ω is equal to the restriction to $\mathcal{P}$ of the two-form

$$\chi(A,\,B) = \operatorname{Tr}\big[B^{-1}\delta B \wedge \delta A A^{-1} - A^{-1}\delta A \wedge \delta B B^{-1} + \delta J J^{-1} \wedge B^{-1}A^{-1}\delta(AB)\big]. \tag{1.8}$$

The expression for ω on symplectic leaves in the space of conjugacy classes of representation of the fundamental group of a genus g Riemann surface with one puncture is given by the formula (6.9). In a different form this result was obtained in [8]. An r-matrix representation of the Poisson structure on the space of flat connections on Riemann surfaces with boundaries was found in [6].

To the best of the author's knowledge, the general closed expression for the symplectic structure on orbits of the adjoint action of SL_r on the space of monodromy matrices A_i, B_i and Stokes matrices, given by Theorem 6.1, is new. Even in the genus 0 case, the Poisson structure on the space of Stokes matrices corresponding to meromorphic connections with one irregular singularity of order 2 and one regular singularity was found only recently [1]. The Poisson structure was identified with that of the Poisson–Lie group G^* dual to $G = \mathrm{GL}_r$. In [2] this result was generalized to G-valued Stokes matrices for an arbitrary simple Lie group, and very interesting connections with the theory of Weyl quantum groups were found there. The Poisson structure on the space of Stokes matrices corresponding to a skew-symmetric meromorphic connection with one regular and one irregular order 2 singularity was found earlier in [34]. The Poisson structure for (2×2) Stokes matrices corresponding to meromorphic connections with one irregular singularity of order 4 was obtained in [5].

To some extent, our main result of Section 6 is preliminary. The general expression for ω in terms of the monodromy data came out of the blue, as a result of straightforward computations. It seems important to find its interpretation in terms of the Poisson–Lie group theory.

The last goal of this paper is to establish connections between solutions of isomonodromy equations on algebraic curves and solutions of Hitchin systems. It is well-known that solutions of the Schlesinger equations can be treated, after proper rescaling, as "modulation" of solutions of the Garnier system [7]. An attempt to revisit this connection in light of the Whitham theory [4], [16], [18], [17] was made in [32], but the heuristic arguments used in [32] do not allow to derive the modulation equations in a closed form.

The problem which we address in Section 7 is as follows. The space of meromorphic ε-connection with fixed multiplicities $h = \{h_m\}$ of poles is the space of orbits of the adjoint action of SL_r on the space $\mathcal{A}_\varepsilon(h)$ of meromorphic differentials $\widetilde{L}_\varepsilon$ such that $\varepsilon^{-1} L_\varepsilon \in \mathcal{A}(h)$. They are parameterized by the data (1.1) such that $\operatorname{Tr} L_{s0} = \varepsilon$. The family of meromorphic ε-connections defined for $\varepsilon \neq 0$ extends to a smooth family over the whole disc. The central fiber over $\varepsilon = 0$ parameterizes the space $\mathcal{L}$ of Lax matrices on algebraic curves introduced in [21]. The orbits of the adjoint action of SL_r on a subspace of $\mathcal{L}$, corresponding to a *fixed* algebraic curve Γ, and fixed singular parts of the eigenvalues of $\widetilde{L}_m$ can be identified with the phase space of the generalized Hitchin system.

In order to get a smooth at $\varepsilon = 0$ family of isomonodromy equations for ε-connections, it is necessary to rescale the coordinates T_a on $\mathcal{M}_{g,1}(h)$. More precisely, if we define the coordinates $t_a = \varepsilon^{-1} T_a$, then the deformations of $\widetilde{L}_\varepsilon$ that preserve the monodromy data associated with a solution of the equation

$$\varepsilon \, d\psi = \widetilde{L}_\varepsilon \psi \tag{1.9}$$

are described by the equations

$$\partial_{t_a} \widetilde{L}_\varepsilon - \varepsilon \, dM_a + [\widetilde{L}_\varepsilon, \, M_a] = 0. \tag{1.10}$$

The equations (1.10) are Hamiltonian and the corresponding Hamiltonians do converge to certain *quadratic* Hamiltonians of the Hitchin system, as $\varepsilon \to 0$. Therefore, locally solutions of (1.10) converge to solutions of the Hitchin system. At the same time a global behaviour of solutions of the isomonodromy and isospectral flows is quite different. The monodromy data preserved by (1.10) vanish in the limit $\varepsilon \to 0$. The space of integrals of the Hitchin system can be regarded as the space $\mathcal{S}$ of so-called spectral curves. It is of dimension which is only half of the dimension of the space of monodromy data. For $\widetilde{L}_0 \in \mathcal{L} = \mathcal{A}_0(h)$ the time-independent spectral curve is defined by the characteristic equation

$$\det(\tilde{k} - \widetilde{L}) = 0. \tag{1.11}$$

The spectral curve $\widehat{\Gamma}$ is r-fold branch cover of the initial algebraic curve Γ. The equations of motion for the Hitchin system are linearized on the Jacobian of $\widehat{\Gamma}$.

In Section 7 we apply ideas of the multi-scale perturbation theory to construct asymptotic solutions of the isomonodromy equations using solutions of the Hitchin system. In this approach the leading term of the approximation describes the motion which is, up to the first order, the original fast motion on the Jacobian, combined with a slow drift on the moduli space of spectral curves. We obtain an explicit form of the Whitham equations describing that slow drift. They imply that *the real parts of the periods of the differential* $\tilde{k}$ on $\widehat{\Gamma}$ *are preserved along a slow drift*. We would like to emphasize that the correspondence

$$(\widehat{\Gamma} \in \mathcal{S}) \longmapsto \operatorname{Re} \oint_c \tilde{k}, \quad c \in H_1(\widetilde{\Gamma}), \tag{1.12}$$

defines a flat *real* connection on the moduli space of spectral curves, considered as a bundle over $\mathcal{M}_g(h)$. To some extent, our result provides evidence for the assumption that this connection is a residual of the flat connection on $\mathcal{A}_\varepsilon(h)$ defined by the monodromy data in the limit $\varepsilon \to 0$. It would be quite interesting to find a more geometric interpretation of that residual correspondence.

2. Meromorphic connections

Let V be a stable, rank r, and degree rg holomorphic vector bundle on a smooth genus g algebraic curve Γ. Then the dimension of the space of its holomorphic sections is $r = \dim H^0(\Gamma, V)$. Let $\sigma_1, \ldots, \sigma_r$ be a basis of this space. The vectors $\sigma_i(\gamma)$ are linear independent at the fiber of V over a generic point $\gamma \in \Gamma$, and are linearly dependent

$$\sum_{i=1}^{r} \alpha_s^i \sigma_i(\gamma_s) = 0 \tag{2.1}$$

at zeros γ_s of the corresponding section of the determinant bundle associated to V. For a generic V these zeros are simple, i. e., the number of distinct points γ_s is equal to $rg = \deg V$, and the vectors $\alpha_s = (\alpha_s^i)$ of the linear dependence (2.1) are uniquely defined up to a multiplication. A change of the basis σ_i corresponds to a linear transformation $\alpha_s' = g^T \alpha_s$. Hence, an open set $\mathcal{M} \subset \widehat{\mathcal{M}}$ of the moduli space of vector bundles is parameterized by points of the quotient space

$$\mathcal{M} = \mathcal{M}_0/\mathrm{SL}_r, \quad \mathcal{M}_0 \subset S^{rg}(\Gamma \times \mathbb{C}P^{r-1}), \tag{2.2}$$

where SL_r acts diagonally on the symmetric power of $\mathbb{C}P^{r-1}$. In [22], [15] the parameters (γ_s, α_s) were called Tyurin parameters.

Let $(\gamma, \alpha) = \{\gamma_s, \alpha_s\}$ be a point of the symmetric product $X = S^{rg}(\Gamma \times \mathbb{C}P^{r-1})$. Throughout the paper it is assumed that the points $\gamma_s \in \Gamma$ are distinct, $\gamma_s \neq \gamma_k$. The vector bundle $V_{\gamma,\alpha}$ corresponding to (γ, α) under the inverse to the Tyurin map is described in terms of Hecke modification of the trivial bundle. In this description, the space of local sections of the vector bundle $V_{\gamma,\alpha}$ is identified with the space $\mathcal{F}_s$

of meromorphic (row)vector-functions in the neighborhood of γ_s that have simple pole at γ_s of the form

$$f^T(z) = \frac{\lambda_s \alpha_s^T}{z - z(\gamma_s)} + O(1), \qquad \lambda_s \in C. \tag{2.3}$$

Our next goal is to describe in similar terms the space of meromorphic connections on $V_{\gamma,\alpha}$. Let $D = \sum_m (h_m + 1) P_m$ be an effective divisor on Γ that does not intersect γ. Then we define the space $\mathcal{A}^D_{\gamma,\alpha}$ of meromorphic matrix valued differentials $\widetilde{L} = L(z)dz$ on Γ such that:

1^0. $\widetilde{L}$ is holomorphic everywhere except for the points γ_s, where it has at most simple poles, and for the points P_m of D, where it has poles of degree not greater than $(h_m + 1)$;

2^0. the singular term of the expansion

$$\widetilde{L} = \left(\frac{L_{s0}}{z - z_s} + L_{s1} + L_{s2}(z - z_s) + O((z - z_s)^2) \right) dz, \qquad z_s = z(\gamma_s), \tag{2.4}$$

is a rank 1 matrix of the form

$$L_{s0} = \beta_s \alpha_s^T \ \longleftrightarrow \ L_{s0}^{ij} = \beta_s^i \alpha_s^j, \tag{2.5}$$

where β_s is a vector. The trace of the residue of $\widetilde{L}$ at γ_s equals 1:

$$\operatorname{res}_{g_s} \operatorname{Tr} \widetilde{L} = 1 \ \longmapsto \ \alpha_s^T \beta_s = \operatorname{tr} L_{s0} = 1; \tag{2.6}$$

3^0. α_s^T is a left eigenvector of the matrix L_{s1}, i. e.,

$$\alpha_s^T L_{s1} = \kappa_s \alpha_s^T. \tag{2.7}$$

Note that the condition (3^0) is well-defined, although expansion (2.4) itself does depend on the choice of a local coordinate z in the neighborhood of γ_s. Under a change of local coordinate $w = w(z)$ the eigenvalue κ_s in (2.7) gets transformed to κ_s', where

$$\kappa_s = \kappa_s' w'(z_s) - \frac{w''(z_s)}{2w'(z_s)}. \tag{2.8}$$

Therefore, the pair (γ_s, κ_s) is a well-defined point of a total space of the bundle $C^{\mathrm{aff}}(\Gamma)$ of scalar affine connections on Γ.

The sum of all residues of a meromorphic differential equals zero. Therefore,

$$\sum_{P_m \in D} \operatorname{res}_{P_m} \operatorname{Tr} \widetilde{L} = -rg. \tag{2.9}$$

Hence, in what follows we always assume that $\deg D = N > 0$. The Riemann-Roch theorem implies that for a generic degree N divisor D and a generic set of Tyurin parameters (γ, α) the space $\mathcal{A}^D_{\gamma,\alpha}$ is of dimension

$$\dim \mathcal{A}^D_{\gamma,\alpha} = r^2(N + rg + g - 1) - r^2 g(r-1) - rg - rg(r-1) = r^2(N + g - 1). \tag{2.10}$$

The first term is the dimension of the space of meromorphic differentials on Γ with the pole divisor $D + \gamma$. The consecutive terms count the numbers of constraints (2.4)–(2.7). A key characterization of these constraints is the following.

Lemma 2.1. *A meromorphic matrix-function L in the neighborhood U of γ_s with a pole at γ_s satisfies constraints (2.4)–(2.7) if and only if it is of the form*

$$\widetilde{L} = d\Phi_s(z)\Phi_s^{-1}(z) + \Phi_s(z)\widetilde{L}_s(z)\Phi_s^{-1}(z), \tag{2.11}$$

where $\widetilde{L}_s$ and Φ_s are holomorphic in U, and $\det \Phi_s$ *has at most simple zero at γ_s.*

The proof is almost identical to that of Lemma 2.1 in [21].

The constraints (2.4)–(2.7) imply that the space $\mathcal{F}_s$ is invariant under the adjoint action of the operator $(\partial_z - L)$, i. e.,

$$(f^T \in \mathcal{F}_s) \implies \big(f^T(\partial_z - L) = -\big(\partial_z f^T + f^T L\big) \in \mathcal{F}_s\big). \tag{2.12}$$

Therefore, for a generic set of Tyurin parameters (γ, α) the quotient space $\mathcal{A}^D_{\gamma,\alpha}/\mathrm{SL}_r$ corresponding to the gauge transformations

$$\widetilde{L} \longmapsto g\widetilde{L}g^{-1}, \quad g \in \mathrm{SL}_r, \tag{2.13}$$

can be identified with the space of meromorphic connections on $V_{\gamma,\alpha}$ that have poles at P_m of degree not greater than $h_m + 1$.

The explicit parameterization of an open set of the phase space of the Hitchin systems proposed in [21] can be easily extended to the case under consideration. Consider first an open set of Tyurin parameters such that the dimension of the space $\mathcal{F}_{\gamma,\alpha}$ of meromorphic (row)vector-functions on Γ with simple poles at γ_s of the form (2.3) equals r. Then, as shown in [21], the matrix α_s^i is of rank r. We call (γ, α) a nonspecial set of Tyurin parameters if they additionally satisfy the following constraint: there is a subset of $(r+1)$ indices $s_1, \ldots, s_{r+1}$ such that all minors of $(r+1)\times r$ matrix $\alpha_{s_j}^i$ are nondegenerate. The action of the gauge group on the space of nonspecial sets of Tyurin parameters $\mathcal{M}_0$ is free. We also assume that the corresponding points γ_s do not coincide with the points P_m.

By definition, the singular part $\widetilde{L}_m$ of a meromorphic differential $\widetilde{L}$ is an equivalence class of meromorphic differentials in the neighborhood of P_m considered modulo holomorphic differentials.

Lemma 2.2. *Let $\mathcal{A}^D$ be the affine bundle over $\mathcal{M}_0$ with fibers $\mathcal{A}^D_{\gamma,\alpha}$. Then the map*

$$\widetilde{L} \in \mathcal{A}^D \longmapsto \{\alpha_s, \beta_s, \gamma_s, \kappa_s, \widetilde{L}_m\}, \tag{2.14}$$

is a bijective correspondence between points of the bundle $\mathcal{A}^D$ over $\mathcal{M}_0$ and sets of data (2.14) subject to the constraints $(\alpha_s^T\beta_s) = 1$, and

$$\sum_{s=1}^{rg} \beta_s\alpha_s^T + \sum_{P_m\in D'} \mathrm{res}_{P_m} \widetilde{L}_m = 0, \tag{2.15}$$

modulo gauge transformations

$$\alpha_s \mapsto \lambda_s\alpha_s, \quad \beta_s \mapsto \lambda_s^{-1}\beta_s. \tag{2.16}$$

Recall that we consider the pairs (γ_s, κ_s) as points of the bundle $C^{\mathrm{aff}}(\Gamma)$.

Example. Let Γ be a hyperelliptic curve defined by the equation

$$y^2 = R(x) = x^{2g+1} + \sum_{i=0}^{2g} u_i x^i. \tag{2.17}$$

The parameterization of connections on Γ with simple pole at the infinity is almost identical to the parameterization of the Hitchin systems on Γ proposed in [21]. A set of points γ_s on Γ is a set of pairs $(y_s,\, x_s)$ such that

$$y_s^2 = R(x_s). \tag{2.18}$$

A meromorphic differential on Γ with residues $(\beta_s\alpha_s^T)$ at γ_s and a simple pole at the infinity is of the form

$$L\,\frac{dx}{2y} = \left(\sum_{i=0}^{g-1} L_i x^i + \sum_{s=1}^{rg} (\beta_s\alpha_s^T)\,\frac{y+y_s}{x-x_s}\right)\frac{dx}{2y}, \tag{2.19}$$

where L_i is a set of arbitrary matrices. Constraints (2.7) are a system of linear equations defining L_i:

$$\sum_{i=0}^{g} \alpha_n^T L_i x_k^i + \sum_{s\neq n} (\alpha_n^T\beta_s)\alpha_s^T\,\frac{y_n+y_s}{x_n-x_s} = \kappa_n\alpha_n^T, \qquad n = 1,\, \ldots,\, rg, \tag{2.20}$$

in terms of data $\{\gamma_s,\, \kappa_s,\, \alpha_s,\, \beta_s\}$, where $(\alpha_s,\, \beta_s)$ are arbitrary vectors such that $\alpha_s^T\beta_s = 1$.

For $g > 1$, correspondence (2.14) descends to a system of local coordinates on $\mathcal{A}^D/\mathrm{SL}_r$. Consider the open set of $\mathcal{M}_0$ consisting of such elements that the vectors α_j, $j = 1,\, \ldots,\, r$, are linearly independent and all coefficients of the expansion of α_{r+1} in this basis do not vanish,

$$\alpha_{r+1} = \sum_{s=1}^{r} c_j\alpha_j, \qquad c_j \neq 0. \tag{2.21}$$

Then for each point of this open set there exists a unique matrix $W \in \mathrm{GL}_r$ such that $\alpha_j^T W$ is proportional to the basis vector e_j with coordinates $e_j^i = \delta_j^i$, and $\alpha_{r+1}^T W$ is proportional to the vector $e_0 = \sum_j e_j$. Using the global gauge transformation defined by W,

$$b_s = W^{-1}\beta_s, \qquad a_s = W^T\alpha_s, \tag{2.22}$$

and the part of local transformations

$$a_s \mapsto \lambda_s a_s; \qquad a_s \mapsto \lambda_s^{-1} b_s, \tag{2.23}$$

for $s = 1,\, \ldots,\, r+1$, we see that on the open set of $\mathcal{M}_0$ each equivalence class has a representation of the form $(a_s,\, b_s)$ such that

$$a_i = e_i, \;\; i = 1,\, \ldots,\, r; \qquad a_{r+1} = e_0. \tag{2.24}$$

This representation is unique up to local transformations (2.23) for $s = r{+}2,\, \ldots,\, rg$.

In the gauge (2.24) equation (2.15) can be easily solved for $b_1,\, \ldots,\, b_{r+1}$. Using (2.24), we get

$$b_j^i + b_{r+1}^i = -\sum_{s=r+2}^{rg} b_s^i a_s^j - \sum_m \operatorname{res} \widetilde{L}_m^{ij}. \tag{2.25}$$

The condition $a_j^T b_j = 1$ for $a_j = e_j$ implies $b_j^j = 1$. Hence,

$$b_{r+1}^i = -1 - \sum_{s=r+2}^{rg} b_s^i a_s^i - \sum_m \operatorname{res} \widetilde{L}_m^{ij}. \tag{2.26}$$

Note, that constraint (2.9) implies $a_{r+1}^T b_{r+1} = 1$.

Sets of vectors a_s, b_s, $a_s^T b_s = 1$, $r+1 < s \le rg$, modulo transformations (2.23), points $\{\gamma_s, \kappa_s\} \in S^{rg}(C^{\mathrm{aff}}(\Gamma))$, and sets $\widetilde{L}_m$, satisfying (2.9), provide a parameterization of an open set of the bundle $\mathcal{A}^D/\mathrm{SL}_r$ over $\mathcal{M} = \mathcal{M}_0/\mathrm{SL}_r$. The dimension of this bundle equals

$$\dim \mathcal{A}^D/\mathrm{SL}_r = r^2(N + 2g - 2) + 1. \tag{2.27}$$

In the same way, taking various subsets of $(r+1)$ indices we obtain charts of local coordinates which cover $\mathcal{A}^D/\mathrm{SL}_r$.

3. Monodromy data

Our next goal is to introduce monodromy data corresponding to $\widetilde{L} \in \mathcal{A}_{\gamma,\alpha}^D$ along the lines of their definition in the zero genus case. From Lemma 2.1 it follows that the equation

$$d\Psi = \widetilde{L}\Psi \tag{3.1}$$

has multi-valued holomorphic solutions on $\Gamma \setminus \{P_m\}$, $P_m \in D$. Let Q be a point on Γ. Then the normalization

$$\Psi(Q) = 1 \tag{3.2}$$

defines Ψ uniquely in the neighborhood of Q. Analytic continuation of Ψ along cycles in $\Gamma \setminus \{P_m\}$ defines the monodromy representation

$$\mu\colon \pi_1(\Gamma \setminus \{P_m\}; Q) \longrightarrow \mathrm{GL}_r. \tag{3.3}$$

It is well-known that for connections with simple poles the correspondence $\widetilde{L} \to \mu$ is an injection, and that the inverse map is defined on an open set of the space of representations. For connections with poles of higher order additional so-called Stokes data are needed. Their construction is local, and here we mainly follow [31].

Lemma 3.1. *Let $\widetilde{L}$ be a formal Laurent series*

$$\widetilde{L} = \sum_{i=-h}^{\infty} L_s w^{s-1} dw \tag{3.4}$$

such that the leading coefficient has the form

$$L_{-h} = \Phi K \Phi^{-1}, \quad K = \operatorname{diag}(k_1, \ldots, k_r), \quad \begin{cases} k_i - k_j \neq 0, & h > 0, \\ k_i - k_j \notin Z, & h = 0, \end{cases} \quad i \neq j. \tag{3.5}$$

Then equation (3.1) has a unique formal solution

$$\psi = \Phi\left(1 + \sum_{s=1}^{\infty} \xi_s w^s\right) \exp\left(\sum_{i=-h}^{\infty} K_i \int w^{i-1} dw\right), \tag{3.6}$$

where K_i are diagonal matrices, $K_{-h} = K$, and the matrices ξ_s have zero diagonals, $\xi_s^{ii} = 0$.

Substitution of (3.6) into (3.1) gives a system of equations, which for $h > 0$ are of the form

$$K_s + [K, \xi_{s+h}] = R_s(\xi_1, \dots, \xi_{s+h-1}, K_{-h+1}, \dots, K_s), \quad s > -h, \tag{3.7}$$

where R_s are some explicit expressions. They recursively determine the off-diagonal part of ξ_s and the diagonal matrix K_s. For $h = 0$ ψ is constructed in the similar way.

The central result of [31] can be formulated as follows. Let (3.4) be the Laurent expansion of a meromorphic differential in a punctured disk U holomorphic in $\widehat{U} = U \setminus 0$. Let V be a sector of $\widehat{U}$ which for any pair (i, j) contains only one ray such that

$$\mathrm{Re}(k_i - k_j)w^{-h} = 0. \tag{3.8}$$

Then there exists a holomorphic in V solution Ψ_V of (3.1) such that the formal solution (3.6) is an asymptotic series for Ψ_V. The asymptotic is uniform in any closed subsector of V.

The punctured disk can be covered by a set of sectors $V_1, \dots, V_{2h+1}$ which satisfy the constraint described above, and such that the sectors V_ν and $V_{\nu+1}$ do intersect each other. On their intersection the solutions $\Psi_\nu = \Psi_{V_\nu}$ and $\Psi_{\nu+1} = \Psi_{V_{\nu+1}}$ satisfy the relation

$$\Psi_{\nu+1} = \Psi_\nu S_\nu, \quad \nu = 1, \dots, 2h. \tag{3.9}$$

Stokes matrices S_ν are constant matrices. For each S_ν there exists a unique permutation of indices under which S_ν gets transformed to an upper triangular matrix with the diagonal elements equal to 1.

The last property of the Stokes matrices follows from a more precise statement which we will use in Section 6. Namely, if w tends to 0 in the intersection of V_ν and $V_{\nu+1}$, then the following limit exists and equals

$$\lim_{w\to 0} \exp(Kw^{-h})\, S_\nu \exp(-Kw^{-h}) = 1. \tag{3.10}$$

For any pair $(i \neq j)$ the left-hand side of (3.8) has a definite sign in $V_\nu \cap V_{\nu+1}$. Therefore, if this sign is positive, then (3.10) implies $S_\nu^{ij} = 0$.

Let us fix a local coordinate w_m in a neighborhood of P_m, $w_m(P_m) = 0$, and paths c_m connecting Q with P_m. In the neighborhood of P_m we also fix a set of sectors $V_\nu^{(m)}$ described above, and always assume that the path c_m in the neighborhood of P_m belongs to the first sector $V_1^{(m)}$. Then the Laurent expansion of $\widetilde{L} \in \mathcal{A}^D$ at P_m in this coordinate, defines the diagonal matrices $K_i^{(m)}$, the Stokes' matrices $S_\nu^{(m)}$, and the transition matrix G_m, which connects Ψ and $\Psi_1^{(m)}$

$$\Psi = \Psi_1^{(m)} G_m. \tag{3.11}$$

In each of the sectors $V_\nu^{(m)}$ we have

$$\Psi = \Psi_\nu^{(m)} g_\nu^{(m)}, \tag{3.12}$$

where

$$g_1^{(m)} = G_m, \quad g_{\nu+1}^{(m)} = (S_1^{(m)} S_2^{(m)} \cdots S_\nu^{(m)})^{-1} G_m, \quad \nu = 1, \dots, 2h_m. \tag{3.13}$$

The monodromy μ_m around P_m is equal to

$$\mu_m = (g_1^{(m)})^{-1} e^{2\pi i K_0^{(m)}} g_{2h_m+1}^{(m)} = G_m^{-1} e^{2\pi i K_0^{(m)}} (S_1^{(m)} S_2^{(m)} \cdots S_{2h_m}^{(m)})^{-1} G_m. \quad (3.14)$$

If we choose a basis a_j, b_j of cycles on Γ with the canonical matrix of intersections, then we denote the monodromy matrices along the cycles by A_j, B_j, $j = 1, \ldots, g$.

Lemma 3.2. *The correspondence*

$$\widetilde{L} \in \mathcal{A}^D \longmapsto \{K_i^{(m)}, S_\nu^{(m)}, G_m, A_j, B_j\}, \quad (3.15)$$
$$-h_m \le i \le 0, \qquad \nu = 1, \ldots, 2h_m,$$

where the transition and the Stokes matrices are considered modulo transformations

$$G_m \longmapsto W_m G_m, \qquad S_\nu^{(m)} \longmapsto W_m S_\nu^{(m)} W_m^{-1}, \qquad W_m = \operatorname{diag}(W_{m,i}), \quad (3.16)$$

is an injection.

An important remark. The definition of the full set of the Stokes data requires a choice of the local coordinate in a neighborhood of the puncture. However, the data (3.15) depend only on the h_m-jets of the local coordinates, because it contains only diagonal matrices K_i with indices $i \le 0$. We define an h-jet $[w]_h$ to be an equivalence class of w, with w' and w equivalent if

$$w' = w + O(w^{h+1}). \quad (3.17)$$

Proof. Suppose that $\widetilde{L}$ and $\widetilde{L}_1$ have the same data (3.15) modulo (3.16). Then solutions Ψ and Ψ_1 of the corresponding systems (3.1) have the same monodromy along each cycle on $\Gamma \setminus \{P_m\}$. Therefore, $\phi = \Psi_1 \Psi^{-1}$ is a single-valued meromorphic matrix function on $\Gamma \setminus \{P_m\}$. From (3.6) and (3.14) it follows that ϕ is bounded in the neighborhood of P_m. Hence, ϕ is a meromorphic function on Γ and is holomorphic at the points P_m. The function Ψ is invertible everywhere except for the poles γ_s of $\widetilde{L}$. Equation (3.1) implies that vector rows of the residue of $\Psi_1 \Psi^{-1}$ at γ_s has the form (2.3). The assumption that (γ, α) are nonspecial Tyurin parameters implies that ϕ is a constant matrix. Then, from the normalization (3.2) it follows that $\Psi_1 = \Psi$ and $\widetilde{L} = \widetilde{L}_1$.

Simple counting shows that $\mathcal{A}^D$ and the space of data (3.15) modulo transformations (3.16) have the same dimension. Therefore, the map (3.15) is a bijective correspondence between $\mathcal{A}^D$ and an open set of the data. □

4. Isomonodromy deformations

Our next goal is to construct differential equations describing deformations of $L \in \mathcal{A}^D(\Gamma)$ which preserve the full set of data (3.15). For brevity we call them isomonodromy deformations. As it was mentioned above, in order to define the data (3.15) it is necessary to fix a normalization point $Q \in \Gamma$, a basis of a_i, b_i cycles, paths c_m connecting Q with P_m, and a set of h_m-jets of local coordinates in neighborhoods of the punctures P_m.

Let $h = \{h_m, \sum_m (h_m + 1) = N\}$ be a set of nonnegative integers. Then we denote the moduli space of smooth genus g algebraic curves with a puncture $Q \in \Gamma$, and fixed h_m-jets of local coordinates w_m in neighborhoods of punctures P_m by

$\mathcal{M}_{g,1}(h)$. The space $\mathcal{A}(h)$ of admissible meromorphic differentials on algebraic curves with fixed multiplicities (h_m+1) of the poles can be seen as the total space of the bundle

$$\mathcal{A}(h) \longrightarrow \mathcal{M}_{g,1}(h) = \{\Gamma,\, P_m,\, [w_m],\, Q\} \tag{4.1}$$

with fibers $\mathcal{A}^D(\Gamma)$, $D = \sum_m (h_m+1)P_m$. Here and below $[w_m]$ stands for the h_m-jet of w_m. The space $\mathcal{M}_{g,1}(h)$ is of dimension

$$\dim \mathcal{M}_{g,1}(h) = 3g - 2 + N. \tag{4.2}$$

An explicit form of the isomonodromy equations depends on a choice of coordinates on $\mathcal{M}_{g,1}(h)$. Their Lax representation requires in addition some sort of connection on the universal curve $\mathcal{N}_g(h)$ which is the total space of the bundle

$$\mathcal{N}_g(h) \longrightarrow \mathcal{M}_{g,1}(h). \tag{4.3}$$

The fiber of the bundle over a point $(\Gamma,\, P_m,\, [w_m],\, Q)$ is the curve Γ.

The following construction solves two problems simultaneously. It goes back to the theory of Whitham equations [16], [18]. Details can be found in [23], [24]. First of all, locally we can replace the moduli space of algebraic curves by the Teichmuller space of marked algebraic curves, i. e., by smooth algebraic curves with fixed basis $\{a_i,\, b_i\}$ of cycles, and paths c_m between Q and punctures, which do not intersect the cycles. Let us fix a set of integers r_m, $\sum_m r_m = 0$. Then, for any set of local coordinates w_m at P_m, there is a unique meromorphic differential dE which in the neighborhood of P_m is of the form

$$dE = d(w_m^{-h_m} + r_m \log w_m + O(w_m)), \tag{4.4}$$

and is normalized by the condition

$$\oint_{a_i} dE = 0. \tag{4.5}$$

The differential dE depends only on the h_m-jets of local coordinates w_m. The zero divisor of dE has degree $2g-2+N$. Let $\mathcal{M}^0_{g,1}(h)$ be an open set of $\mathcal{M}_{g,1}(h)$ such that the corresponding differential dE has simple zeros $q_s \neq Q$,

$$dE(q_k) = 0, \quad k = 1,\, \ldots,\, 2g-2+N. \tag{4.6}$$

The Abelian integral

$$E(q) = \int_Q^q dE \tag{4.7}$$

is single-valued on the cover $\widehat{\Gamma}^*$ of $\Gamma \setminus \{P_m\}$ generated by shifts along the cycles b_i and shifts along the cycles c'_m around the punctures P_m. We regard the curve Γ with cuts along the cycles a_i and the paths c_m as a marked sheet of Γ^*. The critical values

$$T_k = E(q_k) \tag{4.8}$$

of E on this sheet, and the b-periods of dE,

$$T_{b_i} = \oint_{b_i} dE, \tag{4.9}$$

provide a system of local coordinates on $\mathcal{M}^0_{g,1}(h)$ (see details in [23]). The Abelian integral E defines a local coordinate on $\widehat{\Gamma}^*$ everywhere except for the preimages $\hat{q}^*_s$ of the critical points q_s. Therefore, (E, T_k, T_{b_i}) can be seen as a system of local coordinates on an open set of the total space of the bundle $\widehat{\mathcal{N}}^*_g(h)$ over $\mathcal{M}_{g,1}(h)$ with fibers $\widehat{\Gamma}^*$.

Let $\widetilde{L}(\tau) \in \mathcal{A}(h)$ be a one-parameter family of admissible differentials. Its projection under (4.1) defines a path $T_a(\tau)$ in $\mathcal{M}_{g,1}(h)$. Here and below $\{a\}$ stands for both types of indices, i. e., $T_a = \{T_k, T_{b_i}\}$. We regard the family $\widetilde{L}(\tau)$ as a family of one-forms

$$\widetilde{L}(\tau) = L(E;\, T_a(\tau))\, dE, \tag{4.10}$$

where L is a function of the variable E on $\widehat{\Gamma}^*(\tau)$ which is meromorphic everywhere except for $\hat{q}^*_s(\tau)$, and such that

$$L(E + T_{b_j};\, T_a) = L(E;\, T_a), \qquad L(E + 2\pi i r_m;\, T_a) = L(E;\, T_a). \tag{4.11}$$

In the same way the corresponding solution Ψ of equation (3.1) can be seen as a multi-valued function $\Psi(E;\, T_a)$ of the variable E which is holomorphic everywhere except for $\hat{q}^*_s$, and such that

$$\Psi(E + T_{b_j};\, T_a) = \Psi(E;\, T_a) B_j, \qquad \Psi(E + 2\pi i r_m;\, T_a) = \Psi(E;\, T_a)\mu_m. \tag{4.12}$$

Let Γ^* be the cover of Γ generated by shifts along b_i-cycles, and $\hat{q}_s$, $\widehat{P}_m$, $\hat{\gamma}_s$ be preimages on Γ^* of the corresponding points on Γ.

Lemma 4.1. *A one-parameter family of meromorphic connections*

$$\widetilde{L}(\tau) \in \mathcal{A}^{D(\tau)}_{\gamma(\tau),\alpha(\tau)}(\Gamma(\tau))$$

is an isomonodromy family if and only if the logarithmic derivative of the corresponding solution Ψ of (3.1)

$$M(E,\, \tau) = \partial_\tau \Psi(E,\, \tau)\Psi^{-1}(E,\, \tau) \tag{4.13}$$

is single-valued on $\Gamma^*(\tau)$ as a function of E, and

(i) *it equals zero at Q, and is holomorphic everywhere except for the points $\hat{\gamma}_s$, $\hat{q}_k$, where it has at most simple poles,*

(ii) *the vector rows of M in a neighborhood of $\hat{\gamma}_s$ have the form (2.3),*

(iii) *the singular part of M at $\hat{q}_k(\tau)$ equals*

$$M(E,\, \tau) = -\partial_\tau E(\hat{q}_k)\, L(E,\, \tau) + O(1), \qquad E \to E(\hat{q}_k), \tag{4.14}$$

(iv) *M satisfy the following monodromy properties*

$$M(E + T_{b_j};\, T_a) = M(E;\, T_a) - (\partial_\tau T_{b_j})\, L(E;\, T_a), \tag{4.15}$$

Proof. The same arguments as in the proof of Lemma 3.2 show that if the Stokes data do not depend on τ, then M is holomorphic at the punctures $\widehat{P}_m$. The matrix M is single-valued on Γ^* because monodromies A_j also do not depend on τ. Unlike the previous case, M is single-valued only on Γ^*, and acquires additional poles at $\hat{q}_k$, because E is multivalued on Γ, and is not a local coordinate at the critical points $\hat{q}_k$.

At the points $\hat{q}_k = \hat{q}_k(\tau)$ $(E - E(\hat{q}_k))^{1/2}$ is a local coordinate. Recall that $E(\hat{q}_k)$ equals T_k plus an integer linear combination of T_{b_j} which depends on the branch of E corresponding to $\hat{q}_k$. The matrix function Ψ is holomorphic in a neighborhood of $\hat{q}_k$. Therefore, its expansion at $\hat{q}_k$ is of the form

$$\Psi = \phi_0(\tau) + \phi_1(\tau)(E - E(\hat{q}_k))^{1/2} + O(E - E(\hat{q}_k)), \quad \hat{q}_k = \hat{q}_k(\tau). \tag{4.16}$$

Then

$$M = -\partial_\tau E(\hat{q}_k)\, \frac{\phi_1\phi_0^{-1}}{2\sqrt{E - E(\hat{q}_k)}} + O(1). \tag{4.17}$$

The logarithmic differential of Ψ is of the form

$$d\Psi\Psi^{-1} = L\, dE = \frac{\phi_1\phi_0^{-1}\, dE}{2\sqrt{E - E(\hat{q}_k)}} + O(1)\, dE. \tag{4.18}$$

Equations (4.17) and (4.18) imply (4.14). Equation (4.15) follows directly from (3.1) and (4.12), and the lemma is proved. □

Let us now introduce basic functions M_a corresponding to the isomonodromy deformations along coordinates T_a. Simple dimension counting proves the following statement.

Lemma 4.2. *If (γ, α) is a nonspecial set of Tyurin parameters, then for each $\widetilde{L} \in \mathcal{A}^D_{\gamma,\alpha}(\Gamma)$ there is a unique meromorphic function M_k on Γ such that:*

(i) *M_k is holomorphic everywhere except for the points γ_s, and for the point q_k,*

(ii) *the vector rows of M_k at γ_s are of the form (2.3),*

(iii) *at the point q_k the singular part of M_k is of the form $M_k = -L + O(1)$,*

(iv) *$M(Q) = 0$.*

Let us denote Γ with a cut along the cycle a_i by Γ_i^*.

Lemma 4.3. *If (γ, α) is a non-special set of Tyurin parameters, then for each $\widetilde{L} \in \mathcal{A}^D_{\gamma,\alpha}(\Gamma)$ there is a unique function M_{b_i} on Γ_i^* such that:*

(i) *M_{b_i} is holomorphic everywhere except for the points γ_s, where the vector rows of M_k are of the form (2.3),*

(ii) *M_{b_i} can be extended as a continuous function to the closure of Γ_i^*, and its boundary values $M_{b_i}^{\pm}$ on the two sides of the cut satisfy the relation $M_{b_i}^+ - M_{b_i}^- = -L$,*

(iii) *$M_{b_i}(Q) = 0$.*

A meromorphic matrix function on Γ_i^* which satisfies the boundary condition (ii) can be represented by a Cauchy type integral over the cycle. The difference of any two such functions is a meromorphic function on Γ. Therefore, once again the proof of existence and uniqueness of the function with prescribed analytical properties is reduced to the Riemann–Roch theorem.

If we keep the same notation for the pullback of M_a on Γ^*, then the logariphimic derivative $M = \partial_\tau\Psi\Psi^{-1}$ in Lemma 4.1 can be written as

$$M = \sum_a (\partial_\tau T_a)\, M_a. \tag{4.19}$$

Now we are in a position to define a hierarchy of differential equations which describe the isomonodromy deformations. In the neighborhood of γ_s the Laurent expansions of $L = \widetilde{L}/dE$ and M_a are of the form

$$L = \frac{\beta_s \alpha_s^T}{E - e_s} + L_{s1} + L_{s2}(E - e_s) + O((E - e_s)^2), \quad e_s = E(\gamma_s), \tag{4.20}$$

$$M_a = \frac{m_s^a \alpha_s^T}{E - e_s} + M_{s1}^a + M_{s2}^a(E - e_s) + O((E - e_s)^2), \tag{4.21}$$

where m_s^a are vectors.

Theorem 4.1. *The Lax equations*

$$\partial_a \widetilde{L} - dM_a + [\widetilde{L}, M_a] = 0 \tag{4.22}$$

define a hierarchy of commuting flows on $\mathcal{A}(h)$ which preserve the extended set of monodromy data (3.15)*. They are equivalent to the equations*

$$\partial_a e_s = -\alpha_s^T m_s^a, \quad e_s = E(\gamma_s), \tag{4.23}$$

$$\partial_a \alpha_s^T = -\alpha_s^T M_{s1}^a - \lambda_s \alpha_s^T, \tag{4.24}$$

$$\partial_a \beta_s = M_{s1}^a \beta_s - (L_{s1} - \kappa_s) m_s^a + \lambda_s \beta_s, \tag{4.25}$$

$$\partial_a \kappa_s = \alpha_s^T (M_{s2}^a - L_{s2}) \beta_s, \tag{4.26}$$

$$\partial_a \widetilde{L}_m = [M_a, \widetilde{L}_m]_+, \tag{4.27}$$

where λ_s are scalar functions, and $[M_a, \widetilde{L}_m]_+$ denotes the singular part of $[M_a, \widetilde{L}_m]$ at P_m.

Note that if $h_m = 0$, then the right-hand side of (4.27) is just $[M_a(q_k), \widetilde{L}_m]$.

Proof. First, let us show that the left-hand side of (4.22), which we denote by ϕ, is a single-valued meromorphic function on Γ which is holomorphic everywhere except for the points γ_s and P_m. Indeed, for $M_a = M_k$ it is single-valued by the definition of M_k, but may have a pole at q_k. Taking the derivative of the Laurent expansion of L at q_k, we see that $\partial_a L$ acquires pole at q_k of the form $\partial_a L = -dL/dE + O(1)$. Hence, the singular part of $\partial_a \widetilde{L}$ is just $-dL$, which cancels with the singular part of dM_k. From (4.27) it follows that $[M_k, \widetilde{L}]$ is regular at q_k. Almost identical arguments show that for $M_a = M_{b_i}$ the matrix differentials dM_a and $\partial_a \widetilde{L}$ have the same monodromy properties along the cycle b_i, and therefore, ϕ is single-valued on Γ.

Equations (4.23)–(4.25) and (4.27) are equivalent to the condition that ϕ is a holomorphic matrix differential on Γ. Then equation (4.26) is equivalent to the condition $\alpha_s^T \phi(\gamma_s) = 0$. This gives us a system of $r^2 g$ linear equations for ϕ. As it is shown in [21], for nonspecial sets of Tyurin parameters these equations are linearly independent, and therefore, imply $\phi = 0$.

The matrix functions M_a are uniquely defined by $\widetilde{L}$. Hence equations (4.23)–(4.27) are a closed system of differential equations on the space of parameters

$(e_s,\, \kappa_s,\, \alpha_s,\, \beta_s,\, \widetilde{L}_m)$. Compatibility of the equations for different indices a is equivalent to the equation

$$\partial_a M_b - \partial_b M_a + [M_b,\, M_a] = 0. \tag{4.28}$$

In order to prove (4.28) we first check that the left-hand side of the equation is a single-valued meromorphic matrix function which is holomorphic everywhere except for γ_s. Then, from equations (4.23)–(4.25) it follows that at γ_s this function has at most a simple pole of the form (2.3). For nonspecial sets of Tyurin parameters the last condition implies that the left-hand side of (4.28) is a constant matrix function on Γ. It equals zero due to the normalization $M_a(0) = 0$. Recall that at the marked point $E(Q) = 0$.

From the Lax representation of equations (4.23)–(4.27) it follows that if $\widetilde{L}$ is a solution of these equations and Ψ is the normalized solution of (3.1), then $\partial_a \Psi \Psi^{-1} = M_a$. Lemma 4.1 implies the isomonodromy property of the flow, and the theorem is proved. □

Example 1. The Schlesinger equations [30]

$$\partial_i A_j = \frac{[A_i,\, A_j]}{t_i - t_j}, \quad i \neq j, \tag{4.29}$$

$$\partial_i A_i = \sum_{j \neq i} \frac{[A_j,\, A_i]}{t_i - t_j} \tag{4.30}$$

describe the isomonodromy deformations of a meromorphic connection with regular singularities

$$L\, dz = \sum_i \frac{A_i}{z - t_i}\, dz \tag{4.31}$$

on the rational curve. In the conventional approach, the coordinates t_i of punctures on the complex plane are considered as coordinates on the space of rational curves with punctures. In our approach, which also works for higher genus case, we use the function $E = \sum_i \ln(z - t_i)$ to parameterize points of the complex plane. Critical values $T_k(t)$ of E,

$$T_k = \sum_i \ln(q_k - t_i), \tag{4.32}$$

locally define t_i uniquely up to a common shift $t_i \to t_i + c$. The critical points q_k are roots of the equation

$$E'(q_k) = \sum_i \frac{1}{q_k - t_i} = 0, \quad E'(z) = \partial_z E(z). \tag{4.33}$$

Note that (4.33) implies

$$E'(z) = \sum_i \frac{1}{z - t_i} = \frac{\prod_k (z - q_k)}{\prod_i (z - t_i)}. \tag{4.34}$$

From (4.33) it follows that

$$\partial_i T_k = -\frac{1}{q_k - t_i}. \tag{4.35}$$

According to Lemma 4.2, the matrix $M_k(z)$ corresponding to the variable T_k has the only pole at q_k which coincides with the singular part of $-L/E_z$. Hence

$$M_k = -\frac{\mathrm{res}_{q_k}(L/E')}{z-q_k} = -\frac{1}{z-q_k}\left(\sum_i \frac{A_i}{q_k-t_i}\right)\frac{\prod_j(q_k-t_j)}{\prod_{s\neq k}(q_k-q_s)}. \tag{4.36}$$

From equation (4.27) we obtain that isomonodromy deformations of L with respect to new coordinates are of the form

$$\partial_{T_k}A_j = -\frac{1}{t_j-q_k}\left(\sum_i \frac{[A_i,\,A_j]}{q_k-t_i}\right)\frac{\prod_j(q_k-t_j)}{\prod_{s\neq k}(q_k-q_s)}. \tag{4.37}$$

It is instructive to check directly that equations (4.37) are equivalent to (4.29). For $i\neq j$ we have

$$\partial_i A_j = \sum_k (\partial_i T_k)\partial_{T_k}A_j = -\sum_k \mathrm{res}_{z=q_k}\frac{[L(z),\,A_j]\prod_{s\neq i,j}(z-t_s)}{\prod_k(z-q_k)} \tag{4.38}$$

The expression in the right-hand side of (4.38) has poles at t_i and q_k. Hence, $\partial_i A_j$ equals the residue of this expression at $z=t_i$,

$$\partial_i A_j = [A_i,\,A_j]\frac{\prod_{s\neq i,j}(t_i-t_s)}{\prod_k(t_i-q_k)}. \tag{4.39}$$

Equation (4.34) implies

$$1 = \mathrm{res}_{t_i}\,E'(z) = \frac{\prod_k(t_i-q_k)}{\prod_{s\neq i}(t_i-t_s)}. \tag{4.40}$$

Therefore, the last factor in (4.39) equals $1/(t_i-t_j)$, and we obtain equation (4.29). Equation (4.30) can be replaced by the equation $\sum_i \partial_i A_j = 0$. Therefore, it suffices to check that $\sum_i \partial_i T_k = 0$. The last equation follows from (4.33) and (4.35).

Example 2. The Painleve-II equation

$$u_{xx} - xu - 2u^3 = \nu \tag{4.41}$$

describes an isomonodromy deformation of the rational connection

$$L = Az^2 + Bz + C + Dz^{-1}, \tag{4.42}$$

where

$$A = -4i\sigma_3, \quad B = -4u\sigma_2, \quad C = -(2iu^2+x)\sigma_3 - 2u_x\sigma_1, \quad D = \nu\sigma_2, \tag{4.43}$$

and σ_i are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4.44}$$

We would like to stress once again that the conventional definition of the isomonodromy deformations of rational connections with irregular singularities are those preserving monodromy, transition, and Stokes matrices. Exponents K_i are considered as parameters of the deformation (see [5], [13]).

In this example we show that the same equations can be seen as equations describing deformations over the space of jets in local coordinate which preserve the full set of data (3.15), including exponents. Let us consider the isomonodromy deformation of L corresponding to the deformation of z defined by the function

$$E(z) = \frac{4}{3}\, z^3 + xz + \ln z. \tag{4.45}$$

The critical points q_k are roots of the equation

$$4q_k^2 + x + q_k^{-1} = 0 \implies E'(z) = 4z^{-1} \prod_{k=1}^{3} (z - q_k). \tag{4.46}$$

As above, the Lax matrix M_k corresponding to the coordinate $T_k = E(q_k)$ equals

$$M_k = -\frac{\operatorname{res}_{q_k}(L/E')}{z - q_k}. \tag{4.47}$$

As is (4.35), we obtain that $\partial_x T_k = q_k$. Therefore, if Ψ is a solution of (3.1), then

$$\partial_x \Psi(E) = M\Psi(E), \quad M = -\sum_k \frac{q_k \operatorname{res}_{q_k}(L/E')}{z - q_k}. \tag{4.48}$$

In our notation we skip indication to an explicit dependence of functions on x, but keep track of the variable, which is considered fixed with respect to x.

The matrix M in (4.48) equals

$$M = -\sum_k \operatorname{res}_{q_k} F(q) = \operatorname{res}_z F(q) + \operatorname{res}_\infty F(q), \quad F = \frac{qL(q)}{E'(q)(z - q)}. \tag{4.49}$$

The residue at $q = z$ equals

$$\operatorname{res}_z F(q) = -zL(z)/E'(z). \tag{4.50}$$

Expansion of $F(q)$ at $q = \infty$ is of the form

$$F = -\frac{1}{4}(Aq^2 + Bq + C + Dq^{-1}) \left(\sum_{s=0}^{\infty} zq^{-1} \right) q^{-2}(1 + O(q^{-2})). \tag{4.51}$$

Therefore,

$$\operatorname{res}_\infty F(q) = \frac{1}{4}(Az + B). \tag{4.52}$$

The derivatives with fixed values of E and z are related to each other by the chain rule

$$\partial_x \Psi(x, z) = \partial_x \Psi(x, E(z)) + \frac{d\Psi}{dE}\, \partial_x E(x, z) = \partial_x \Psi(x, E(z)) + \frac{L(x, z)}{E'(x, z)}\, z. \tag{4.53}$$

Equations (4.48)–(4.53) imply

$$\partial_x \Psi(x, z) = (Az + B)\Psi(x, z). \tag{4.54}$$

The compatibility condition of (3.1) and (4.54) gives the well-known Lax representation for (4.41).

5. Hamiltonian approach

In this section we show that the general algebraic approach to the Hamiltonian theory of Lax equations proposed in [23], [24], [19] is also applicable to isomonodromy equations. Since the arguments here are very close to those of the author's earlier work [21], except for slight modifications, we shall be brief.

The entries of $\widetilde{L} \in \mathcal{A}(h)$ can be regarded as functions on $\mathcal{A}(h)$ with values in the space of meromorphic differentials on Γ. Therefore, $\widetilde{L}$ can be seen itself as a matrix-valued function and its external derivative $\delta\widetilde{L}$ is a matrix-valued one-form on $\mathcal{A}(h)$. The formal solutions ψ_m of the form (3.6) corresponding to the expansion of L to the punctures P_m can be also regarded as matrix functions on $\mathcal{A}(h)$ defined modulo permutation of columns and the transformation

$$\psi'_m = \psi_m f_m, \tag{5.1}$$

where f_m is a diagonal matrix. Hence, its differential $\delta\psi_m$ is a one-form on $\mathcal{A}^D$ with values in the space of formal series of the form (3.6). In the same way we consider the differentials $\delta K_i^{(m)}$ of exponents in (3.6).

Let $\mathcal{P}_0$ be a subspace of $\mathcal{A}(h)$ such that the restriction to $\mathcal{P}_0$ of the differentials $\delta K_i^{(m)}$ of the exponents in (3.6) vanishes for $i \leq 0$, i. e.,

$$\delta K_i^{(m)}\big|_{\mathcal{P}_0} = 0, \quad i \leq 0. \tag{5.2}$$

In other words, $\mathcal{P}_0$ is a subspace of $\mathcal{A}(h)$ such that for $\widetilde{L} \in \mathcal{P}_0$ the singular parts $\widetilde{L}_m$ of $\widetilde{L}$ at the punctures are points of a fixed set of orbits $\widetilde{\mathcal{O}}_m$ of the adjoint action of $\mathrm{GL}_r^+(w)$ on the space of equivalence classes of meromorphic differentials at P_m, modulo holomorphic differentials. Here $\mathrm{GL}_r^+(w)$ is the group of invertible, holomorphic in a neighborhood of P_m matrix functions.

We define a scalar-valued two-form on $\mathcal{P}_0$ by the formula

$$\omega = -\frac{1}{2}\left(\sum_{s=1}^{rg} \operatorname{res}_{\gamma_s} \widetilde{\Omega} + \sum_{P_m} \operatorname{res}_{P_m} \widetilde{\Omega}\right), \tag{5.3}$$

where

$$\widetilde{\Omega} = \mathrm{Tr}(\psi^{-1}\delta\widetilde{L} \wedge \delta\psi), \tag{5.4}$$

and ψ in a neighborhood of γ_s is a solution of (3.1), and in a neighborhood of the puncture $\psi = \psi_m$ it is the formal solution (3.6).

Let us check that ω is well-defined. Indeed, if $\psi' = \psi g$ is another solution of (3.1) in a neighborhood of γ_s, then

$$\widetilde{\Omega}' = \widetilde{\Omega} + \mathrm{Tr}[(\psi^{-1}\delta\widetilde{L}\psi) \wedge \delta g g^{-1}]. \tag{5.5}$$

Taking the external derivative of (3.1) we obtain the equalities

$$\delta\, d\psi = \delta\widetilde{L}\psi + \widetilde{L}\delta\psi, \quad -\delta\, d\psi^{-1} = \psi^{-1}\delta\widetilde{L} + \delta\psi^{-1}\widetilde{L}. \tag{5.6}$$

They imply

$$\psi^{-1}\delta\widetilde{L}\psi = d(\psi^{-1}\delta\psi). \tag{5.7}$$

Therefore,

$$\widetilde{\Omega}' = \widetilde{\Omega} + \mathrm{Tr}[d(\psi^{-1}\delta\psi) \wedge \delta g g^{-1}]. \tag{5.8}$$

The matrix g is a constant matrix in the neighborhood of γ_s. Therefore, the second term in (5.5) is a full differential of a meromorphic function and does not contribute to the residue.

Consider now the residues of $\widetilde{\Omega}$ at the puncture P_m. Essential singularities of ψ_m and ψ_m^{-1} mutually cancel. Therefore, $\widetilde{\Omega}$ is a formal meromorphic differential in the neighborhood of P_m, and its residue at P_m is well-defined. It does not depend on permutation of columns of ψ_m. Under transformation (5.1) it gets an additional term

$$\operatorname{Tr}\big[(\psi_m^{-1}\delta\widetilde{L}\psi_m)\wedge\delta f_m f_m^{-1}\big] = \operatorname{Tr}\big[d(\psi_m^{-1}\delta\psi_m)\wedge\delta f_m f_m^{-1}\big]. \tag{5.9}$$

The matrix f_m is diagonal, and therefore commutes with $K^{(m)}$. Constraints (5.2) imply that (5.9) is a holomorphic differential, and therefore has zero residue.

Theorem 5.1. *The two-form ω defined by (5.3) is gauge invariant and descends to a closed, nondegenerate form on $\mathcal{P} = \mathcal{P}_0/\mathrm{SL}_r$. Under correspondence (2.14) it takes the form*

$$\omega = \sum_{s=1}^{rg}\left(\delta\kappa_s\wedge\delta z_s + \sum_{i=1}^{r}\delta\beta_s^i\wedge\delta\alpha_s^i\right) + \sum_m \omega_m, \tag{5.10}$$

where ω_m is the canonical symplectic structure on an orbit $\widetilde{\mathcal{O}}_m$.

The isomonodromy equations (4.22) are Hamiltonian with respect to the symplectic structure defined by ω. The Hamiltonians H_a are equal to

$$H_k = -\frac{1}{2}\operatorname{res}_{q_s}\operatorname{Tr}(\widetilde{L}^2/dE), \qquad H_{b_i} = -\frac{1}{2}\oint_{a_i}\operatorname{Tr}(\widetilde{L}^2/dE) \tag{5.11}$$

Note that κ_s as an affine connection and $z_s = z(\gamma_s)$ depend on a choice of the local coordinate z in some neighborhood of γ_s, the first term in (5.10) being independent of this choice.

Recall that the tangent space to $\widetilde{\mathcal{O}}_m$ at $\widetilde{L}_m$ is isomorphic to $\mathfrak{sl}_r^+/\mathfrak{sl}_r^+(\widetilde{L}_m)$, where $\mathfrak{sl}_r^+(\widetilde{L}_m)$ is the subalgebra of traceless matrix functions ξ which are holomorphic in a neighborhood of P_m, and such that $[\widetilde{L}_m, \xi]$ is holomorphic at P_m. The symplectic structure on $\widetilde{\mathcal{O}}_m$ is defined by the formula (see details in [21])

$$\omega_m = \operatorname{res}_{P_m}\operatorname{Tr}(\widetilde{L}_m\,[\xi,\,\eta]). \tag{5.12}$$

Proof. It is easy to check directly that under the gauge transformation

$$L' = g^{-1}Lg, \qquad \psi' = g^{-1}\psi \tag{5.13}$$

$\widetilde{\Omega}$ gets transformed to $\widetilde{\Omega}' = \widetilde{\Omega} + F$, where

$$F = \operatorname{Tr}\big(\psi^{-1}[\widetilde{L},\,\delta h]\wedge\delta\psi - [\widetilde{L},\,\delta h]\wedge\delta h - \delta\widetilde{L}\wedge\delta h\big), \qquad \delta h = \delta g g^{-1}. \tag{5.14}$$

Using (5.6), we obtain

$$\operatorname{Tr}\big(\psi^{-1}[\widetilde{L},\,\delta h]\wedge\delta\psi\big) = \operatorname{Tr}\big(\delta h\wedge\delta\widetilde{L} - d(\psi^{-1}\delta h\wedge\delta\psi)\big). \tag{5.15}$$

The last term in (5.15) is holomorphic in neighborhoods of γ_s and P_m. The rest of F is a global meromorphic differential on Γ with the only poles at γ_s and P_m. Therefore, the sum of all residues of F vanishes. Hence, ω is gauge invariant. Arguments needed to complete the proof of (5.10) are identical to those of the proof of Theorem 4.1 in [21]. From (5.10) it follows that ω is closed. It descends to a non-degenerate form on $\mathcal{P}_m$, because ω_m is nondegenerate on $\widetilde{O}_m$, and the first term in (5.10) equals

$$\omega_0 = \sum_{s=1}^{rg} \delta\kappa_s \wedge \delta z_s + \sum_{s=r+1}^{rg} \delta b_s^T \wedge \delta a_s, \quad g > 1, \quad z_s = z(\gamma_s), \tag{5.16}$$

where a_s, b_s are local coordinates on $\mathcal{M}$ defined by (2.22).

Our next goal is to show that the isomonodromy equations are Hamiltonian with respect to the symplectic form ω. By definition a vector field ∂_a on a symplectic manifold is Hamiltonian if the contraction $i_{\partial_a}\omega(X) = \omega(X, \partial_a)$ of the symplectic form is an exact one-form $dH_a(X)$. The function H_a is the Hamiltonian corresponding to the vector field ∂_a.

For each $\widetilde{L} \in \mathcal{A}^D_{\gamma,\alpha}$ let us define meromorphic differentials $d\Omega_a = d\Omega_a(L)$. The differential $d\Omega_k$ is a unique meromorphic differential on Γ whose pole is of the form

$$d\Omega_k = -dL + 0(1), \quad \widetilde{L} = L\, dE, \tag{5.17}$$

at q_k, and which is holomorphic everywhere else and satisfies the equations

$$\alpha_s^T\, d\Omega_a(\gamma_s) = 0. \tag{5.18}$$

The differential $d\Omega_{b_i}$ is a unique holomorphic differential on Γ_i^* satisfying (5.18) and continuous on the closure of Γ_i^*. Its boundary values on the two sides of the cut along the a_i-cycle satisfy the relation

$$d\Omega^+_{b_i} - d\Omega^-_{b_i} = -dL. \tag{5.19}$$

Lemma 5.1. *The evaluations of one-forms δL and $\delta\psi$ on the vector field ∂_a defined by the Lax equation* (4.22) *equal*

$$\delta\widetilde{L}(\partial_a) = \partial_a L - d\Omega_a = [M_a, \widetilde{L}] + dM_a - d\Omega_a, \tag{5.20}$$

$$\delta\psi(\partial_a) = M_a\psi + \phi_a, \tag{5.21}$$

where ϕ_a is a solution of the equation

$$d\phi_a = \widetilde{L}\phi_a - d\Omega_a\psi. \tag{5.22}$$

Proof. The right-hand sides of (5.20), (5.21) are not equal to the derivatives of $\widetilde{L}$ and ψ, since by definition δ is the external differential on a fiber $\mathcal{A}^D(\Gamma)$, but not on the total space of the bundle $\mathcal{A}(h)$. In other words, if I_k are coordinates on the space $\mathcal{A}^D(\Gamma)$ on a fixed curve with punctures, then

$$\delta\widetilde{L} = (\partial\widetilde{L}/\partial I_k)\delta I_k \implies \delta\widetilde{L}(\partial_a) = (\partial\widetilde{L}/\partial I_k)\partial_a I_k. \tag{5.23}$$

The data (2.14) are coordinates on $\mathcal{A}^D(\Gamma)$. From equations (4.23)–(4.27) it follows that the difference Φ of both sides of (5.20) is a holomorphic differential on Γ such that $\alpha_s^T\Phi(\gamma_s) = 0$. For nonspecial sets of Tyurin parameters the last equation implies $\Phi \equiv 0$. Evaluation of (5.6) at ∂_a, and equation (5.20) imply (5.21) directly. $\square$

From (5.20)–(5.22) it follows that the evaluation $\widetilde{\Omega}(\partial_a)$ of the matrix valued two-form $\widetilde{\Omega}$ given by (5.4) equals

$$\widetilde{\Omega}(\partial_a) = \mathrm{Tr}\big(\psi^{-1}\delta L(M_a\psi + \phi_a) - \psi^{-1}([M_a, \widetilde{L}] + dM_a - d\Omega_a)\delta\psi\big). \tag{5.24}$$

From (5.6) and (5.7) it follows that

$$\widetilde{\Omega}(\partial_a) = \mathrm{Tr}\big(M_a\delta\widetilde{L} + \delta\widetilde{L}M_a - \psi^{-1}\, d\Omega_a\delta\psi - d(\psi^{-1}M_a\delta\psi) - d(\delta\psi^{-1}\phi_a)\big). \tag{5.25}$$

The last two terms in (5.25) are differentials of meromorphic functions in the neighborhoods of γ_s and P_m. Therefore, their residues at these points equal zero. From (5.18) it follows that the third term is holomorphic at γ_s. It is holomorphic also at P_m. For M_k the first two terms are meromorphic on Γ with poles at γ_s, P_m and with a pole at the critical point q_k. Hence,

$$i_{\partial_k}\omega = \frac{1}{2}\,\mathrm{res}_{q_k}\,\mathrm{Tr}(\delta\widetilde{L}M_k + M_k\delta\widetilde{L}). \tag{5.26}$$

By definition, the matrix M_k in a neighborhood of q_k has the form $M_k = -\widetilde{L}/dE + O(1)$. That implies (5.11) for $T_a = T_k$. In the similar way we prove that $\omega(\partial_{T_{b_i}})$ equals to the external differential of H_{b_i}, and therefore, the theorem is proved. □

The basic flows constructed above easily allow one to describe isomonodromy equations corresponding to various subspaces of $\mathcal{M}_{g,1}(h)$, and to various changes of coordinates. Let $T_a = T_a(\tau)$ depend on a variable τ, and let $z = z(E, T_a)$ be a local coordinate along $\Gamma(T_a(\tau))$. Then the matrix function M which defines an isomonodromic deformation of $\widetilde{L}$ in the τ-direction equals

$$M = \sum_a (\partial_\tau T_a)M_a(z) + \frac{\widetilde{L}}{dE}\,\partial_\tau E(z) \tag{5.27}$$

Let us consider the following instructive example.

Isomonodromy equations on a fixed algebraic curve. A variation of the coordinates T_a introduced above changes simultaneously a curve, punctures and jets of local coordinates. In these coordinates it is hard to identify variations that preserve Γ. For such deformations it is more convenient to use a more traditional setting.

If $z = z_m$ is a local coordinate on Γ in an open domain U_m, then the variables $t_m = z(P_m)$ are local coordinates on the space of punctures $P_m \in U_m$. Let $\widetilde{L} \in \mathcal{A}^D_{\gamma,\alpha}(\Gamma)$ be an admissible meromorphic differential on Γ with regular singularities at P_m, i. e., in U_m it is of the form

$$\widetilde{L} = \left(\frac{L_m}{z - t_m} + O(1)\right) dz, \tag{5.28}$$

and corresponds to a nonspecial set of Tyurin parameters (γ, α).

From (5.27) it follows that $M^{(m)}$ corresponding to the coordinate t_m can be defined as the unique meromorphic matrix function on Γ such that:

(i) $M^{(m)}$ is holomorphic on Γ everywhere except for γ_s and for the point P_m;

(ii) the rows of $M^{(m)}$ at γ_s are of the form (2.3);

(iii) in the neighborhood of P_m the matrix $M^{(m)}$ is of the form

$$M^{(m)} = -\frac{L_m}{z - t_m} + O(1), \tag{5.29}$$

and is normalized by the condition $M^{(m)}(Q) = 0$.

Corollary 5.1. *The Lax equations*

$$\partial_{t_m} \widetilde{L} - dM^{(m)} + [\widetilde{L},\, M^{(m)}] = 0 \tag{5.30}$$

describe isomonodromy deformations of $\widetilde{L}$ with respect to the variables t_m. They descend to the Hamiltonian equations on $\mathcal{P}$ with the Hamiltonians

$$H^{(m)} = -\frac{1}{2}\operatorname{res}_{P_m} \operatorname{Tr}(\widetilde{L}^2/dz). \tag{5.31}$$

A proof of the last statement is almost identical to that of Theorem 5.1. The differential $d\Omega_a$ in (5.20)–(5.25) has to be changed by the differential $d\Omega^{(m)}$. The latter has the only pole at P_m, where

$$\Omega^{(m)} = -\frac{L_m}{z - t_m} + 0(1). \tag{5.32}$$

It is normalized by the same condition (5.18). As a result of that change the only term in (5.25) having nontrivial sum of residues at γ_s and P_m, is the third. It has nontrivial residue at P_m which can be easily found using (3.1).

Elliptic Schlesinger equations. Let $L\,dz$ be a meromorphic connection on an elliptic curve $\Gamma = C/\{2n\omega_1,\, 2m\omega_2\}$ with simple poles at punctures $z = t_m$. In this example we denote the parameters γ_s and κ_s by q_s and p_s, respectively.

In the gauge $\alpha_s = e_s$, $e_s^j = \delta_s^j$ the j-th column of the matrix L^{ij} has poles only at the points q_j and punctures t_m. Equation (2.7) implies $L^{ji}(q_j) = 0$, $i \neq j$. From equations (2.5), (2.7) it follows that L^{jj} at q_j has the expansion $L^{jj}(z) = (z - q_j)^{-1} + p_j + O(z - q_j)$. An elliptic function with these properties is uniquely determined by its residues L_m^{ij} at the punctures t_m, and can be written in terms of the Weierstrass ζ-function as follows:

$$L^{ii}(z) = p_i + \sum_m L_m^{ii}\big(\zeta(z - t_m) - \zeta(z - q_i) - \zeta(q_i - t_m)\big), \qquad \sum_m L_m^{ii} = -1, \tag{5.33}$$

$$L^{ij}(z) = \sum_m L_m^{ij}\big(\zeta(z - t_m) - \zeta(z - q_j) - \zeta(q_i - t_m) + \zeta(q_i - q_j)\big), \qquad i \neq j. \tag{5.34}$$

The Poisson brackets are defined by the standard formulae

$$\{p_i,\, q_j\} = \delta_{ij}, \qquad \{L_m^{ij},\, L_k^{ls}\} = \delta_{mk}\big(-\delta_{jl}L_m^{is} + \delta_{is}L_m^{lj}\big). \tag{5.35}$$

The elliptic Schelesinger equations are generated by the Hamiltonians

$$\begin{aligned} H^{(m)} = {} & -\sum_i p_i L_m^{ii} + \sum_i \sum_{k\neq m} L_m^{ii}L_k^{ii}(\zeta(t_m - t_k) - \zeta(t_m - q_i) - \zeta(q_i - t_k)) \\ & - \sum_{i\neq j} L_m^{ij}L_m^{ji}(\zeta(q_j - q_i) - \zeta(t_m - q_i) - \zeta(q_j - t_m)) \\ & - \sum_{k\neq m}\sum_{i\neq j} L_m^{ij}L_k^{ji}(\zeta(t_m - t_k) - \zeta(t_m - q_i) - \zeta(q_j - t_k) - \zeta(q_i - q_j)). \end{aligned} \tag{5.36}$$

Example 3. As an example of isomonodromy equations corresponding to deformations of algebraic curves, we consider a meromorphic connection on an elliptic curve $\Gamma = C/\{n,\, m\tau\}$ with one puncture, which, without loss of generality, we put at $z = 0$. In the framework of the Hamiltonian reduction approach this example was considered in [27].

We use the same gauge as in the previous example. Let us assume that the residue of $\widetilde{L}$ at $z = 0$ is of the form $-(1+h)+f$, where $1+h$ is a scalar matrix, and f is a matrix of rank one: $f^{ij} = a^i b^j$. As it was mentioned above, the equations $\alpha_i = e_i$ fix the gauge up to transformations by diagonal matrices. We can use these transformations to make $a^i = b^i$. The corresponding momentum is given then by the collection $(a^i)^2$ and we fix it to have values $(a^i)^2 = h$. Then, using the same arguments as before, we see that the matrix L can be written as

$$\begin{aligned} L^{ij} &= h\frac{\sigma(z+q_i-q_j)\,\sigma(z-q_i)\,\sigma(q_j)}{\sigma(z)\,\sigma(z-q_j)\,\sigma(q_i-q_j)\,\sigma(q_i)}, \quad i \neq j; \\ L^{ii} &= p_i + \zeta(z-q_i) - \zeta(z) + \zeta(q_i), \end{aligned} \tag{5.37}$$

where $\sigma(z) = \sigma(z \,|\, 1,\, \tau)$ is the Weierstrass σ-function.

According to Theorem 5.1, the isomonodromy deformation of L with respect to the module τ of the elliptic curve is generated by the Hamiltonian

$$H = -\frac{1}{2}\int_0^1 \operatorname{Tr} L^2\, dz. \tag{5.38}$$

The addition formula for the σ-function implies

$$\int_0^1 L^{ij}L^{ji}\, dz = h^2 \int_0^1 (\wp(z) - \wp(q_i-q_j))\, dz = h^2(2\eta_1 - \wp(q_i-q_j)), \quad i \neq j. \tag{5.39}$$

Here and below $\eta_1 = \zeta(1/2)$, $\eta_2 = \zeta(\tau/2)$. The formula

$$(\zeta(z-q_i) - \zeta(z) + \zeta(q_i))^2 = \wp(z-q_i) + \wp(z) + \wp(q_i), \tag{5.40}$$

and the monodromy property $\sigma(z+1) = -\sigma(z)e^{2\eta_1(z-1/2)}$ of the σ-function imply

$$\int_0^1 (L^{ii})\, dz = p_i^2 + \wp(q_i) + 2\eta_1 + 2p_i(\zeta(q_i) - 2\eta_1 q_i). \tag{5.41}$$

The $(p,\, q)$-independent term in H which is proportional to $\eta_1(\tau)$ does not effect the equations of motion. Therefore, the Hamiltonian generating the isomonodromy equations for $p_i = p_i(\tau)$, $q_i = q_i(\tau)$ equals

$$-4\pi i H = \sum_i \big(p_n^2 + 2p_n(\zeta(q_n) - 2\eta_1 q_n) + \wp(q_n)\big) - h^2 \sum_{n\neq m} \wp(q_n - q_m). \tag{5.42}$$

The equations of motion are

$$q_{n,\tau} = -\frac{1}{2\pi i}(p_n + \zeta(q_n) - 2\eta_1 q_n), \tag{5.43}$$

$$p_{n,\tau} = \frac{1}{4\pi i}\bigg(-2p_n(\wp(q_n) + 2\eta_1) + \wp'(q_n) - h^2 \sum_{n\neq m} \wp'(q_n - q_m)\bigg). \tag{5.44}$$

Equation (5.43) implies

$$q_{n,\tau\tau} = -\frac{1}{2\pi i}(p_{n,\tau} - q_{n,\tau}(\wp(q_n) + 2\eta_1) + \chi(q_n)), \tag{5.45}$$

where

$$\chi(z) = \chi(z;\,\tau) = \partial_\tau(\zeta(z\,|\,1,\,\tau) - 2\eta_1(\tau)z)). \tag{5.46}$$

The function $\xi = \zeta(z\,|\,1,\,\tau) - 2\eta_1(\tau)z$ has the following monodromy properties: $\xi(z+1) = \xi(z)$, $\xi(z+\tau) = \xi(z) - 2\pi i$. Therefore, $\chi(z)$ is an entire function of z such that $\chi(z+1) = \chi(z)$, $\chi(z+\tau) = \chi(z) - \partial_z\xi(z) = \chi(z) + (\wp(z) + 2\eta_1)$. These analytic properties imply the following expression for χ in terms of the Weierstrass functions:

$$\chi(z) = -\frac{1}{4\pi i}(2(\zeta(z) - 2\eta_1 z)(\wp(z) + 2\eta_1) + \wp'(z)). \tag{5.47}$$

From (5.43)–(5.47) we get

$$q_{n,\tau\tau} = -\frac{h}{8\pi^2}\sum_{n\neq m}\wp'(q_n - q_m\,|\,1,\,\tau). \tag{5.48}$$

For $r = 2$ equation (5.48) for the variable $u = q_1 - q_2$ is a particular case of the Painlevé VI equation (see details in [27] and [10]). It is to be said that although equations (5.48) do coincide with those obtained in [27], the Hamiltonian (5.42) has a new intriguing form.

6. Canonical transformations

In the previous section the symplectic form ω, initially defined by formula (5.3), was then expressed in terms of dynamical variables (2.14). As a result it was identified with the canonical symplectic structure on the space of meromorphic connections. The main goal of this section is to express ω in terms of monodromy data (3.15).

Note first that the sum in (5.3) is taken over all poles of $\widetilde{L}$. It is not equal to zero, because the solutions of (3.1) used in (5.3), (5.4) are formal local solutions in the neighborhoods of punctures. Consider now the differential Ω_0 given by the same formula as $\widetilde{\Omega}$ in (5.4), i. e.,

$$\Omega_0 = \mathrm{Tr}(\Psi^{-1}\delta\widetilde{L}\wedge\delta\Psi), \tag{6.1}$$

but where Ψ is a (global) multi-valued holomorphic solution of (3.1) on $\Gamma\setminus\{P_m\}$. The differential Ω_0 is single-valued on Γ with cuts along cycles $(a_k,\,b_k)$ and paths c_m between the marked point Q and the punctures P_m. Therefore,

$$\sum_{s=1}^{rg}\mathrm{res}_{\gamma_s}\,\Omega_0 = \frac{1}{2\pi i}\oint_{\mathcal{L}}\Omega_0 - \frac{1}{2\pi i}\oint_{\mathcal{C}}\Omega_0, \tag{6.2}$$

where $\mathcal{L} = \prod_{k=1}^{g}(a_k b_k a_k^{-1} b_k^{-1})$, and $\mathcal{C} = \prod_m C_m$ are loops in $\Gamma\setminus\{P_m\}$ (see Fig. 1).

If $\Psi(Q) = 1$ at the initial point, then the monodromy of Ψ along the loop $aba^{-1}b^{-1}$ is equal to

$$J(A,\,B) = B^{-1}A^{-1}BA, \tag{6.3}$$

where A, B are the monodromies corresponding to the cycles a and b. The monodromy of Ψ along b segment of the loop is $A^{-1}BA = BJ$. From (5.8) it follows that the sum of integrals of Ω_0 along the a and a^{-1} segments of the loop is equal to

$$\begin{aligned} I_1 &= -\operatorname{Tr}(A^{-1}\delta A \wedge \delta(BJ)J^{-1}B^{-1}) \\ &= -\operatorname{Tr}\big[A^{-1}\delta A \wedge \delta BB^{-1} + B^{-1}A^{-1}(\delta A)B \wedge \delta JJ^{-1}\big]. \end{aligned} \tag{6.4}$$

The monodromy of Ψ along the a^{-1} segment of the loop is $A^{-1}J$. Therefore, the sum of integrals of Ω_0 along b and b^{-1} segments of the loop is equal to

$$\begin{aligned} I_2 &= -\operatorname{Tr}\big[(A^{-1}B^{-1}\delta(BA) - A^{-1}\delta A) \wedge \delta(A^{-1}J)J^{-1}A\big] \\ &= -\operatorname{Tr}[-B^{-1}\delta B \wedge \delta AA^{-1} + B^{-1}\delta B \wedge \delta JJ^{-1}]. \end{aligned} \tag{6.5}$$

The sum $\chi = I_1 + I_2$ equals

$$\chi(A, B) = \operatorname{Tr}\big[B^{-1}\delta B\wedge\delta AA^{-1} - A^{-1}\delta A\wedge\delta BB^{-1} + \delta JJ^{-1}\wedge B^{-1}A^{-1}\delta(AB)\big]. \tag{6.6}$$

Due to analytical continuation, the solution Ψ on the segment of the loop $\mathcal{L}$ differs from the normalized solution Ψ_0 used in the previous formulae by the factor

$$H_1 = 1; \quad H_k = J_{k-1}J_{k-2}\cdots J_1, \ \ k > 1; \quad J_s = J(A_s, B_s). \tag{6.7}$$

From (5.8) it follows that the integral of Ω_0 over $(a_k b_k a_k^{-1} b_k^{-1})$ under the transformation $\Psi = \Psi_0 H_k$ acquires an additional term

$$\operatorname{Tr}(J_k^{-1}\delta J_k \wedge \delta H_k H_k^{-1}). \tag{6.8}$$

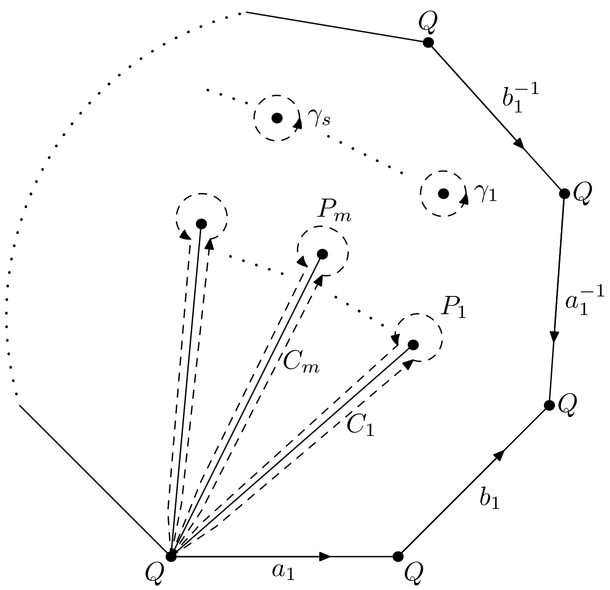

FIGURE 1.

Let us denote the integral of Ω_0 over $\mathcal{L}$ by

$$\omega_1(\boldsymbol{A},\,\boldsymbol{B}) := \oint_{\mathcal{L}} \Omega_0 = \sum_{k=1}^{g} \bigl[\chi(A_k,\,B_k) + \operatorname{Tr}(J_k^{-1}\delta J_k \wedge \delta H_k H_k^{-1})\bigr]. \tag{6.9}$$

It is a two-form on the space of sets of matrices $\boldsymbol{A} = \{A_k\}$, $\boldsymbol{B} = \{B_k\}$.

Next we compute the integral of Ω_0 along the cycle C_m, which goes along one side of the cut c_m, then goes around P_m along a small circle c'_m, and finally goes back along the other side of c_m (see Fig. 1).

Consider first the integral of Ω_0 around the puncture. We split the circle c' into $2h+1$ arcs c_ν which lie in the sectors V_ν (here and below we skip, for brevity, the index m of the puncture). Recall that in each of the sectors the formal solution ψ of (3.1) given by Lemma 3.1 is an asymptotic series for the holomorphic function $\Psi_\nu = \Psi g_\nu^{-1}$. Let Ω_ν be given by the same formula as for Ω_0 with Ψ replaced by Ψ_ν. Then,

$$\int_{c_\nu} \Omega_0 = \int_{c_\nu} \Omega_\nu + \operatorname{Tr}\biggl[\int_{c_\nu} d(\Psi_\nu^{-1}\delta\Psi_\nu) \wedge \delta g_\nu g_\nu^{-1}\biggr]. \tag{6.10}$$

The form $\widetilde{\Omega}$ defined by (5.4), where ψ is the formal solution (3.1), gives an asymptotic series for Ω_ν in V_ν. Therefore, as c' shrinks to the puncture,

$$\lim_{c'\to P} \sum_\nu \int_{c_\nu} \Omega_\nu = (2\pi i)\operatorname{res}_P \widetilde{\Omega}. \tag{6.11}$$

The sum of second terms in (6.10) equals

$$\begin{aligned} &\operatorname{Tr}\bigl(\Psi_{2h+1}^{-1}(p)\delta\Psi_{2h+1}(p) \wedge \delta g_{2h+1} g_{2h+1}^{-1} - \Psi_1^{-1}(p)\delta\Psi_1(p) \wedge \delta g_1 g_1^{-1}\bigr) \\ &+ \sum_{\nu=1}^{2h} \operatorname{Tr}\bigl(\Psi_\nu^{-1}(p_\nu)\delta\Psi_\nu(p_\nu) \wedge \delta g_\nu g_\nu^{-1} - \Psi_{\nu+1}^{-1}(p_\nu)\delta\Psi_{\nu+1}(p_\nu) \wedge \delta g_{\nu+1} g_{\nu+1}^{-1}\bigr), \end{aligned} \tag{6.12}$$

where $p_\nu \in V_\nu \cap V_{\nu+1}$ is the common endpoint of the arcs c_ν and $c_{\nu+1}$. The point p is the intersection point of the cut c and the circle c'. We assume that the cut tends to the puncture in the intersection $V_1 \cap V_{2h+1}$.

The matrices Ψ_{2h+1} and Ψ_1 are connected by the relation $\Psi_{2h+1} = \Psi_1 e^{2\pi i K_0}$. Recall that the monodromy μ along the whole path C is $\mu = g_1^{-1} e^{2\pi i K_0} g_{2h+1}$. Therefore, the first two terms in (6.12) give

$$\begin{aligned} &\operatorname{Tr}\bigl[\Psi_1^{-1}\delta\Psi_1 \wedge (e^{2\pi i K_0}\delta g_{2h+1} g_{2h+1} e^{-2\pi i K_0} - \delta g_1 g_1^{-1})\bigr] \\ &\qquad = \operatorname{Tr}\bigl(\Psi_1^{-1}\delta\Psi_1 \wedge g_1 \delta\mu\mu^{-1} g_1^{-1}\bigr). \end{aligned} \tag{6.13}$$

Boundary values of Ψ on the two sides of the cut c between Q and P satisfy the relation $\Psi^+ = \Psi^-\mu$. Therefore, the sum of integrals of Ω_0 along the first and the last segments of the path C equals

$$\begin{aligned} &-\operatorname{Tr}\bigl(\Psi^{-1}(p)\delta\Psi(p) \wedge \delta\mu\mu^{-1}\bigr) \\ &\qquad = \operatorname{Tr}\bigl(\delta\mu\mu^{-1} \wedge g_1^{-1}\delta g_1 + \delta\mu\mu^{-1} \wedge g_1^{-1}\Psi_1^{-1}(p)\delta\Psi_1(p) g_1\bigr). \end{aligned} \tag{6.14}$$

Here we use the relation $\Psi(p) = \Psi_1(p) g_1$. The sum of (6.13) and (6.14) is equal to

$$I_3 = \operatorname{Tr}(\delta\mu\mu^{-1} \wedge g_1^{-1}\delta g_1). \tag{6.15}$$

Recall that $\Psi_{\nu+1} = \Psi_\nu S_\nu$, where the Stokes matrix S_ν equals $S_\nu = g_\nu g_{\nu+1}^{-1}$. Therefore, the terms of the sum in (6.12) are equal to

$$\operatorname{Tr}\bigl[-S_\nu^{-1}\delta S_\nu \wedge \delta g_{\nu+1} g_{\nu+1}^{-1} + \Psi_\nu^{-1}(p_\nu)\delta\Psi_\nu(p_\nu) \wedge (\delta g_\nu g_\nu^{-1} - S_\nu \delta g_{\nu+1} g_{\nu+1}^{-1} S_\nu^{-1})\bigr] \\ = \operatorname{Tr}\bigl(-\delta S_\nu S_\nu^{-1} \wedge \delta g_\nu g_\nu^{-1} + \Psi_\nu^{-1}\delta\Psi_\nu \wedge \delta S_\nu S_\nu^{-1}\bigr).$$

In the sector V_ν we have $\Psi_\nu e^{-Kw^{-h}} = O(1)$, where K is the leading exponent K_{-h}. Therefore, (3.10) implies

$$\lim_{p_\nu \to P} \operatorname{Tr}(\Psi_\nu^{-1}\delta\Psi_\nu \wedge \delta S_\nu S_\nu^{-1}) = 0. \tag{6.16}$$

Hence, the second term in (6.12) tends to

$$I_4 = \sum_{\nu=1}^{2h} \operatorname{Tr}(-\delta S_\nu S_\nu^{-1} \wedge \delta g_\nu g_\nu^{-1}), \tag{6.17}$$

as c' shrinks to P. Let us denote the sum of (6.15) and (6.17) by

$$\sigma(S,\, G,\, K_0) = \operatorname{Tr}\left(\delta\mu\mu^{-1} \wedge G^{-1}\delta G - \sum_{\nu=1}^{2h} \delta S_\nu S_\nu^{-1} \wedge \delta g_\nu g_\nu^{-1}\right), \tag{6.18}$$

where $S = \{S_\nu\}$, and matrices μ, g_ν are given by (3.13), (3.14). The integral of Ω_0 along C equals $\sigma(S,\, G,\, K_0)$ under the assumption that $\Psi = \Psi_0$, where $\Psi_0 = 1$ at the initial point of the cycle. Due to analytic continuation along the path $C = \prod_m C_m$, the initial value for the cycle C_m equals

$$F_1 = 1; \qquad F_m = \mu_{m-1}\mu_{m-2}\cdots\mu_1, \qquad m > 1. \tag{6.19}$$

From (5.8) it follows that the integral of Ω_0 along the segment C_m of the path C acquires under the transformation $\Psi = \Psi_0 F_m$ the additional term

$$\operatorname{Tr}(\mu_m^{-1}\delta\mu_m \wedge \delta F_m F_m^{-1}). \tag{6.20}$$

Let us define a family of two-forms on the space of sets of matrices $\boldsymbol{S} = \{S_\nu^{(m)}\}$, $\boldsymbol{G} = \{G^{(m)}\}$ parameterized by a set $\boldsymbol{K}_0 = \{K_0^{(m)}\}$ of diagonal matrices:

$$\omega_2(\boldsymbol{S},\, \boldsymbol{G} \,|\, \boldsymbol{K}_0) := \oint_{\mathcal{C}} \Omega_0 - 2\pi i \sum_m \operatorname{res}_{P_m} \widetilde{\Omega} \\ = \sum_m \bigl[\sigma(S^{(m)},\, G_m,\, K_0^{(m)}) + \operatorname{Tr}(\mu_m^{-1}\delta\mu_m \wedge \delta F_m F_m^{-1})\bigr]. \tag{6.21}$$

Summarizing we obtain the following statement.

Theorem 6.1. *The symplectic form ω defined by (5.3) is equal to*

$$\omega = \frac{1}{4\pi i}\bigl[\omega_2(\boldsymbol{S},\, \boldsymbol{G} \,|\, \boldsymbol{K}_0) - \omega_1(\boldsymbol{A},\, \boldsymbol{B})\bigr], \tag{6.22}$$

where ω_1 and ω_2 are given by (6.9) and (6.21), respectively.

It would be quite interesting to check directly that formula (6.22) defines a symplectic structure on orbits of the adjoint action of SL_r on the space of the sets $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{S}, \boldsymbol{G})$ of matrices, which satisfy the only relation

$$\overleftarrow{\prod_k}(B_k^{-1}A_k^{-1}B_kA_k) = \overleftarrow{\prod_m}\mu_m. \tag{6.23}$$

The factors in (6.23) are ordered so that the indices increase from right to left.

Example. Let us consider the case of meromorphic connections on the rational curve with one irregular singularity of order 2 and one regular singularity. Without loss of generality, we assume that $\widetilde{L} = L\,dz$ has irregular singularity at $z = 0$ and regular singularity at $z = \infty$, i. e.,

$$L = l_{-1}z^{-2} + l_0z^{-1}. \tag{6.24}$$

Let us fix a gauge in which $l_1 = K_0^\infty$ is a diagonal matrix. Then the monodromy matrix at the infinity is $\mu_\infty = \exp(2\pi i K_0^\infty)$. Recall that we always assume that the exponents are fixed. The monodromy data at $z = 0$ are two Stokes matrices S_1, S_2, transition matrix G, and the exponents K_1, K_0. The monodromy matrix at $z = 0$ equals

$$\mu_0 = G^{-1}e^{2\pi K_0}S_2^{-1}S_1^{-1}G = \mu_\infty^{-1}\,. \tag{6.25}$$

The transition matrices to the first and to the second sectors at $z = 0$ equal

$$g_1 = G, \quad g_2 = S_1^{-1}G. \tag{6.26}$$

Substitution of (6.26) into (6.18) implies

$$4\pi i\,\omega = -\operatorname{Tr}\bigl(\delta S_1S_1^{-1}\wedge\delta GG^{-1} + \delta S_2S_2^{-1}\wedge\delta(S_1^{-1}G)G^{-1}S_1\bigr). \tag{6.27}$$

Using skew-symmetry of the wedge product and (6.25), we can rewrite the last term as

$$\begin{aligned}\operatorname{Tr}\bigl(\delta S_2S_2^{-1}\wedge\delta(S_1^{-1}G)G^{-1}S_1\bigr) &= \operatorname{Tr}\bigl(S_2^{-1}\delta S_2\wedge\delta(S_2^{-1}S_1^{-1}G)G^{-1}S_1S_2\bigr)\\ &= \operatorname{Tr}\bigl(e^{2\pi iK_0}S_2^{-1}\delta S_2e^{-2\pi iK_0}\wedge\delta GG^{-1}\bigr).\end{aligned} \tag{6.28}$$

Hence,

$$\omega = -\frac{1}{4\pi i}\operatorname{Tr}\bigl[(\delta S_1S_1^{-1} + e^{2\pi iK_0}S_2^{-1}\delta S_2e^{-2\pi iK_0})\wedge\delta GG^{-1}\bigr]. \tag{6.29}$$

Formula (6.29) coincides (up to a factor 2) after the change of notations $G = C$, $S_1 = b_+^{-1}$, $S_2e^{-2\pi K_0} = b_-$ with formula (14) in [2], where the symplectic structure on the space of monodromy data for the linear system (6.24) was idenitfied with the symplectic structure of the group G^* dual to $G = \mathrm{GL}_r$.

7. The Whitham equations

It is well-known, that the family of flat $(\varepsilon \neq 0)$-connections on holomorphic vector bundles over an algebraic curve Γ with punctures extends to a smooth family over the whole ε-plane. The central fiber over $\varepsilon = 0$ is identified with the cotangent bundle to the moduli space of holomorphic vector bundles on Γ. The correspondence (2.14) makes these statements transparent.

The space of meromorphic ε-connections with fixed multiplicities $h = \{h_m\}$ of poles is the quotient space $\mathcal{A}_\varepsilon(h)/\mathrm{SL}_r$ of the space of meromorphic differentials $\widetilde{L}_\varepsilon$ such that $\varepsilon^{-1}L_\varepsilon \in \mathcal{A}(h)$. A meromorphic differential $\widetilde{L}_\varepsilon \in \mathcal{A}_\varepsilon(h)$ satisfies the constraints (2.4), (2.5), (2.7) and the condition

$$\operatorname{res}_{\gamma_s} \operatorname{Tr} \widetilde{L} = (\alpha_s^T \beta_s) = \varepsilon. \tag{7.1}$$

The characteristic property of meromorphic ε-connections is that in the neighborhood of the points γ_s they are of the form

$$\widetilde{L} = \varepsilon\, d\Phi_s(z)\Phi_s^{-1}(z) + \Phi_s(z)\widetilde{L}_s(z)\Phi_s^{-1}(z), \tag{7.2}$$

where $\widetilde{L}_s$ and Φ_s are holomorphic at γ_s, and $\det \Psi_s$ has at most simple zero at γ_s. In the earlier work of the author [21] the space $\mathcal{A}_0(h)$ was called the space of Lax matrices, and orbits of the adjoint action of SL_r on subspaces of $\mathcal{A}_0(h)$ with fixed singular parts of the eigenvalues were identified with phase spaces of Hitchin systems.

The space $\mathcal{A}_\varepsilon(h)$ is a bundle over the moduli space $\mathcal{M}_{g,1}(h)$. Let $\widetilde{L}_\varepsilon(\tau)$ be a one-parameter deformation of $\widetilde{L}_\varepsilon$ which preserves the full set of monodromy data (3.15) associated with a holomorphic solution of the equation

$$\varepsilon\, d\Psi = \widetilde{L}\Psi. \tag{7.3}$$

Along the lines of the proof of Lemma 4.1 it can be shown that the singularities of $M = \partial_\tau \Psi\Psi^{-1}$ are of the form $\varepsilon^{-1}\widetilde{L}_\varepsilon/dE$. Therefore, in order to get a smooth at $\varepsilon = 0$ family of isomonodromy equations, it is necessary to make a proper rescaling of coordinates on $\mathcal{M}_{g,1}(h)$. Namely, if we introduce the *fast* coordinates $t_a = e^{-1}T_a$, then the isomonodromy equations are equivalent to the Lax equations

$$\partial_{t_a}\widetilde{L}_\varepsilon - \varepsilon\, dM_a + [\widetilde{L}_\varepsilon,\, M_a] = 0, \tag{7.4}$$

where matrices $M_a = M_a(\widetilde{L}_\varepsilon)$ are defined by the same analytical properties as above in Section 4. Moreover, the corresponding Hamiltonians are given by the same formulae (5.11).

Remark. Here and below we use the coordinates T_a on $\mathcal{M}_{g,1}$ introduced in Section 4, but mainly our arguments do not rely on any specific choice of the coordinates.

As it follows from [21], equations (7.4) for $\varepsilon = 0$

$$\partial_{t_a}\widetilde{L}_0 = [M_a,\, \widetilde{L}_0] \tag{7.5}$$

coincide with the Lax equations for commuting flows of the Hitchin system corresponding to the second order Hamiltonians given by the same formula (5.11). Equations (7.5) describe *isospectral* deformations. If $\widetilde{L}_0 \in \mathcal{A}_0(h)$ is a solution of (7.5), then the spectral curve $\widehat{\Gamma}$ of $\widetilde{L}_0$ defined by the characteristic equation

$$\det(\tilde{k} - \widetilde{L}_0) = \tilde{k}^r + \sum_i u_i \tilde{k}^i = 0 \tag{7.6}$$

is *time-independent*. The spectral transform identifies $\mathcal{A}_0(h)$ with the Jacobian bundle over the moduli space $\mathcal{S}$ of spectral curves. The fiber of this bundle over $\widehat{\Gamma}$ is the Jacobian $J(\widehat{\Gamma})$. The bijective correspondence between $\mathcal{A}(h)$ and the Jacobian bundle over $\mathcal{S}$ can be seen as a parameterization of $\mathcal{A}_0(h)$ in the form $\widetilde{L}_0 = \widetilde{L}_0(\phi \,|\, I)$. Here and below we regard $\widetilde{L}_0(\phi \,|\, I)$ as an abelian function of the variable $\phi \in J(\widehat{\Gamma})$ depending on $I \in \mathcal{S}$. The function $\widetilde{L}_0(\phi \,|\, I)$ takes values in the space of meromorphic matrix differentials on Γ. The motion equations are linearized on the Jacobian of the spectral curve, and therefore, the general solution of (7.5) can be represented in the form $\widetilde{L}_0 = \widetilde{L}_0(Ut \,|\, I)$, where $Ut = \sum_a U_a t_a$, and $U_a = U_a(I)$ are constant vectors depending on I (see details in [21]).

The main goal of this section is to apply ideas of the Whitham averaging method to construct asymptotic solutions of isomonodromy equations (7.4)

$$\widetilde{L}_\varepsilon = \widetilde{L}_0 + \varepsilon \widetilde{L}_1 + \varepsilon^2 \widetilde{L}_2 + \cdots, \quad M_\varepsilon = M_0 + \varepsilon M_1 + \varepsilon^2 M_2 + \cdots, \tag{7.7}$$

where the leading terms are of the form

$$\widetilde{L}_0(\varepsilon^{-1} S(T) \,|\, I(T)), \quad M_0 = (\varepsilon^{-1} S(T) \,|\, I(T)), \tag{7.8}$$

and $T = \varepsilon t$ are *slow* variables. If the vector-function function $S(T)$ satisfies the equation

$$\partial_T S(T) = U(I(T)) = U(T), \quad \text{i.e.,} \quad S(T) = \int^T U(T)\, dT, \tag{7.9}$$

then the leading term of (7.7) satisfies the original equation up to first order in ε. All other terms of the asymptotic series are obtained from non-homogeneous linear equations whose homogeneous part is just the linearization of the original nonlinear equation at the background of the exact solution $\widetilde{L}_0$. In general, the asymptotic series becomes unreliable on scales of the original variables t of order ε^{-1}. In order to have a reliable approximation, one needs to require a special dependence on the parameters $I(T)$. Geometrically, we note that $\varepsilon^{-1} S(T)$ agrees to first order with Ut, and t are the fast variables. Thus $\widetilde{L}_0(\varepsilon^{-1} S(T) \,|\, I(T))$ describes a motion which is to first order the original *fast* periodic motion on the Jacobian, combined with a slow drift on the moduli space of exact solutions. The equations which describe this drift are in general called *Whitham equations*, although there is no systematic scheme to obtain them.

Below we follow the lines of the scheme proposed in [16], where the Whitham equations for general $(2+1)$ integrable soliton systems were derived. First, we introduce sets of Abelian differentials dv_a^r, dv_a^i on spectral curves $\widehat{\Gamma}$. The differentials dv_a^r, dv_a^i are *real normalized*, i. e., their periods are pure imaginary,

$$\operatorname{Re} \oint_c dv_a^r = \operatorname{Re} \oint_c dv_a^i = 0, \quad c \in H^1(\widehat{\Gamma}). \tag{7.10}$$

For indices $a = n$ the corresponding differentials have poles only at the preimages q_n^j of the point q_n on $\widehat{\Gamma}$, where

$$dv_n^r = dk_j(z) + O(1), \quad dv_n^i = i\, dk_j(z) + O(1), \tag{7.11}$$

and $k_j(z)$ is the corresponding branch of the eigenvalue of $\widetilde{L}(z)/dE$, i. e., the corresponding root of the equation

$$\det(k - \widetilde{L}_0/dE) = 0, \quad k = \tilde{k}/dE. \tag{7.12}$$

For indices $a = b_j$ the corresponding differentials are holomorphic on $\widehat{\Gamma}$, with cuts along all preimages $a_j^l \in H^1(\widehat{\Gamma})$ of the cycle $a_j \in H^1(\Gamma)$, and their boundary values on two sides of the cut a_j^l satisfy the relation

$$(dv_{b_j}^r)^+ - (dv_{b_j}^r)^- = dk_l, \quad (dv_{b_j}^i)^+ - (dv_{b_j}^i)^- = i\, dk_l. \tag{7.13}$$

Here, as before, k_l is the corresponding eigenvalue of $\widetilde{L}/dE$. Note that $dv_a^r + idv_a^i$ is a holomorphic differential.

Theorem 7.1. *A necessary condition for the existence of asymptotic solution* (7.7) *of isomonodromy equation* (7.4) *with leading term* (7.8) *and with bounded first correction term* $\widetilde{L}_1$ *are the equations*

$$\partial_{X_a}\tilde{k} = -dv_a^r, \quad \partial_{Y_a}\tilde{k} = -dv_a^i, \tag{7.14}$$

where X_a *and* Y_a *are the real and imaginary parts of the slow variable* $T_a = X_a + iY_a$.

It can be shown along the lines of [16] that equations (7.14) are generating form of the equations on the space $\mathcal{S}$ of spectral curves (see details in [23], [24], [18], [17]).

Equations (7.14) can be written in the form

$$\partial_{T_a}\tilde{k} = -dv_a, \tag{7.15}$$

where

$$\partial_{T_a} = \frac{1}{2}\left(\frac{\partial}{\partial x_a} - i\,\frac{\partial}{\partial y}\right), \quad dv_a = \frac{1}{2}(dv_a^r - i\,dv_a^i). \tag{7.16}$$

Remark. Equation (7.15) is a particular case of exact solutions of the universal Whitham hierarchy. It is connected with the theory of WDVV equations and the Seiberg–Witten theory of $N = 2$ supersymmetric gauge models (see [23], [24], [18], [17]).

Corollary 7.1. *The real parts of periods of the differential* $\tilde{k}$ *over the spectral curve are integrals of Whitham equations. The correspondence*

$$\widetilde{L}_0 \longmapsto \operatorname{Re}\oint_c \tilde{k}, \quad c \in H^1(\widehat{\Gamma}), \tag{7.17}$$

defines a flat connection on the bundle $\mathcal{S}$ *over* $\mathcal{M}_g(h)$

Proof of the theorem. Substitution of series (7.7) into (7.4) gives non-homogeneous linear equation for the first order terms

$$\partial_t\widetilde{L}_1 + [\widetilde{L}_0, M_1] + [\widetilde{L}_1, M_0] = dM_0 - \partial_T\widetilde{L}_0 - t\sum_{i=1}^{g}(\partial_T U_i)\partial_{\phi_i}\widetilde{L}_0, \tag{7.18}$$

where $\partial_T = (\partial_T I)\partial_I$, and ϕ_i are coordinates on fibers of the Jacobian bundle. Here and below, we skip for brevity index a, i. e., $t = t_a$, and $T = T_a$. Let ψ and ψ^* be solutions of the adjoint systems of equations

$$\widetilde{L}_0\psi = \tilde{k}\psi, \qquad \partial_t\psi = M_0\psi, \tag{7.19}$$

$$\psi^*\widetilde{L}_0 = \tilde{k}\psi^*, \qquad \partial_t\psi^* = -\psi^* M_0. \tag{7.20}$$

Here ψ^* is a vector-row, normalized by the condition $\psi^*\psi = 1$. From (7.18), (7.19), (7.20) it follows that

$$\partial_t(\psi^* L_1 \psi) = -\psi^* \left(\partial_T \widetilde{L}_0 - dM_0 + t \sum_{i=1}^{g} (\partial_T U_i) \partial_{\phi_i} \widetilde{L}_0 \right) \psi. \tag{7.21}$$

From the equations

$$\psi^* (\delta \widetilde{L}_0 - \delta \tilde{k}) \psi = -\psi^* (\widetilde{L}_0 - \tilde{k}) \delta \psi = 0, \tag{7.22}$$

and the normalization $\psi^*\psi = 1$ it follows that

$$\psi^* \delta \widetilde{L}_0 \psi = \delta \tilde{k}. \tag{7.23}$$

Variations of $\widetilde{L}_0$ with respect to variables ϕ_i preserve the spectral curve, i. e., for such variations one has $\delta \tilde{k} = 0$. Hence,

$$\psi^* (\partial_{\phi_i} \widetilde{L}_0) \psi = 0. \tag{7.24}$$

Equation (7.23) also implies

$$\partial_T \tilde{k} = \psi^* (\partial_T \widetilde{L}) \psi. \tag{7.25}$$

Equations (7.19), (7.20) imply

$$\psi^* (dM_0) \psi = -\psi^* M_0 \, d\psi + \psi^* (\partial_t \, d\psi) = \partial_t (\psi^* d\psi). \tag{7.26}$$

Hence, (7.21) can be written as

$$\partial_t (\psi^* L_1 \psi) + \partial_T \tilde{k} - \partial_t (\psi^* d\psi) = 0. \tag{7.27}$$

In [21] it was shown that the solution ψ of equations (7.19) is the conventional Baker–Akhiezer function on $\widehat{\Gamma}$. Therefore, it can be written explicitly in terms of Riemann theta-functions of the spectral curve. In [16] the original formulae were adapted for the averaging procedure. In order to complete the proof of the theorem, we do not need these formulae in full. Let us present necessary facts.

The function $\psi = \psi(x, y; P)$, and the dual Baker–Akhiezer function ψ^* considered as functions of real variables x, y can be represented in the form

$$\psi(x, y; P) = \Phi(xU^r + yU^i + \zeta; P) \exp\left(-\int^P x \, dv^r + y \, dv^i \right), \tag{7.28}$$

$$\psi^*(x, y; P) = \Phi^*(xU^i + yU^r + \zeta; P) \exp\left(\int^P x \, dv^r + y \, dv^i \right), \tag{7.29}$$

where U^r, U^i are real $2g$-dimensional vectors, and for each $P \in \widehat{\Gamma}$ the functions $\Phi(\zeta; P)$ and $\Phi^*(\zeta; P)$, as functions of $2g$ real variables $\zeta = (\zeta_1, \ldots, \zeta_{2g})$, have the following monodromy properties:

$$\Phi(\zeta + e_i; P) = w_i \Phi(\zeta), \quad \Phi^*(\zeta + e_i; P) = w_i^{-1} \Phi^*(\zeta; P), \quad |w_i| = 1, \tag{7.30}$$

where e_i are the basis vectors of $\mathbb{R}^{2g}$.

The functions $\widetilde{L}_0$, ψ and ψ^* as functions of the complex variable $t = x + iy$ are meromorphic functions. Therefore, if L_1 is uniformly bounded outside some neighborhood of the singularity locus, then the average value $\langle\partial_t(\psi L_1\psi)\rangle$ in t of the first term in (7.27) equals zero,

$$\langle f(t)\rangle = \lim_{\Lambda_i\to\infty} \Lambda_i^{-1}\int_0^{\Lambda_i} f\,dt. \tag{7.31}$$

It is necessary to make few remarks to clarify the averaging procedure. First of all, we assume that 0 and Λ_i are not in the locus. The integral is taken along the path in the complex plane of the variable t, which does not intersect singularities.

As it follows from (7.28)–(7.30), the average value of the last term in (7.27) does exist but depends on the direction in t-plane. If we consider t as a real variable, then this average equals $-dv^r$. For $t = iy$ it equals $-dv^i$, and therefore the theorem is proved. □

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