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Spin Chain Models with Spectral Curves from M Theory

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Abstract: We construct the integrable model corresponding to the $\mathcal{N} = 2$ supersymmetric $SU(N)$ gauge theory with matter in the antisymmetric representation, using the spectral curve found by Landsteiner and Lopez through M Theory. The model turns out to be the Hamiltonian reduction of a $N + 2$ periodic spin chain model, which is Hamiltonian with respect to the universal symplectic form we had constructed earlier for general soliton equations in the Lax or Zakharov–Shabat representation.

1. Introduction

The main goal of this paper is to construct the integrable model which corresponds to the $\mathcal{N} = 2$ SUSY $SU(N)$ Yang–Mills theory with a hypermultiplet in the antisymmetric representation. The 1994 work of Seiberg and Witten [1] had shown that the Wilson effective action of $\mathcal{N} = 2$ SUSY Yang–Mills theory is determined by a fibration of spectral curves Γ equipped with a meromorphic one-form $d\lambda$, now known as the Seiberg–Witten differential. It was soon recognized afterwards [2–4] that this set-up is indicative of an underlying integrable model, with the vacuum moduli of the Yang–Mills theory corresponding to the action variables of the integrable model. In fact, in the special case of hyperelliptic curves, a similar set-up for the construction of action variables as periods of a meromorphic differential had been introduced in [5]. This unexpected relation between $\mathcal{N} = 2$ Yang–Mills theories on one hand and integrable models has proven to be very beneficial for both sides. The Seiberg–Witten differential has led to a universal symplectic form for soliton equations in the Lax or Zakharov–Shabat representation [6,7]. The connection with integrable models has helped solve the $SU(N)$ Yang–Mills theory with a hypermultiplet in the adjoint representation [4,8], as well as pure Yang–Mills theories with arbitrary simple gauge groups G [3]. Conversely, the connection with

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Yang–Mills theories has led to new integrable models, such as the twisted Calogero–Moser systems associated with Yang–Mills theories with non-simply laced gauge group and matter in the adjoint representation [9], and the elliptic analog of the Toda lattice [10]. [1]

Despite all these successes, we still do not know at this moment how to identify or construct the correct integrable model corresponding to a given Yang–Mills theory. This is a serious drawback, since the integrable model can be instrumental in investigating key physical issues such as duality, the renormalization group, or instanton corrections [13–15]. At the same time, the list of spectral curves continues to grow, thanks in particular to methods from M theory [16,17] and geometric engineering [18]. It seems urgent to develop methods which can identify the correct integrable model from a given spectral curve and Seiberg–Witten differential.

In the case of interest in this paper, namely the $SU(N)$ gauge theory with antisymmetric matter, the Seiberg–Witten differential and spectral curve had been found by Landsteiner and Lopez [17] using branes and M theory. The Seiberg–Witten differential $d\lambda$ is given by

$$d\lambda = x\frac{dy}{y}. \tag{1.1}$$

The spectral curve is of the form

$$y^3-(3\Lambda^{N+2}+x^2\sum_{i=0}^{N}u_ix^i)y^2+(3\Lambda^{N+2}+x^2\sum_{i=0}^{N}(-)^iu_ix^i)\Lambda^{N+2}y-\Lambda^{3(N+2)}=0, \tag{1.2}$$

where Λ is a renormalization scale. For the $SU(N)$ gauge theories, one restricts to $u_N = 1$, $u_{N-1} = 0$, so that the moduli dimension is $N-1$, which is the rank of the gauge group $SU(N)$. The Landsteiner–Lopez curve (1.2) and differential (1.1) have been studied extensively by Ennes, Naculich, Rhedin, and Schnitzer [20]. In particular, they have verified that the curve and differential do reproduce the correct perturbative behavior of the prepotential predicted by asymptotic freedom. The problem which we wish to address here is the one of finding a dynamical system which is integrable in the sense that it admits a Lax pair, and which corresponds to the Landsteiner–Lopez curve and Seiberg–Witten differential (1.1) in the sense that its spectral curve is of the form (1.2), and its action variables are the periods of $d\lambda$ along $N-1$ suitable cycles on Γ.

We have succeeded in constructing two integrable spin chain models, whose spectral curves are given exactly by the Landsteiner–Lopez curves. However, the action variables of the desired integrable model must be given by $d\lambda = x\frac{dy}{y}$, and here the two models differ significantly. For one model, referred to as *the odd divisor spin model*, the 2-form resulting from $d\lambda$ vanishes identically. For the other, referred to as *the even divisor spin model*, the Hamiltonian reduction of the 2-form resulting from $d\lambda$ to the moduli space of vacua $\{u_N = 1, \quad u_{N-1} = 0\}$ is non-degenerate, and the reduced system is indeed Hamiltonian with respect to this symplectic form, with Hamiltonian $H = u_{N-2}$. Thus the latter model is the integrable system we are looking for.

[1] We refer to [11,12] for more complete lists of references.

Our main result is as follows[2]. Let q_n, p_n be 3-dimensional vectors which are $N+2$ periodic, i.e. $p_{n+N+2} = p_n$, $q_{n+N+2} = q_n$, and satisfy the constraints

$$p_n^T q_n = 0, \tag{1.3}$$

$$p_n = g_0 p_{-n-1}, \quad q_n = g_0 q_{-n-1}, \tag{1.4}$$

where g_0 is the diagonal matrix

$$g_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{1.5}$$

Consider the dynamical system

$$\dot{p}_n = \frac{p_{n+1}}{p_{n+1}^T q_n} + \frac{p_{n-1}}{p_{n-1}^T q_n} + \mu_n p_n, \quad \dot{q}_n = -\frac{q_{n+1}}{p_n^T q_{n+1}} - \frac{q_{n-1}}{p_n^T q_{n-1}} - \mu_n p_n \tag{1.6}$$

for some scalar functions $\mu_n(t)$. The system is invariant under the gauge group G generated by the following gauge transformations:

$$p_n \to \lambda_n p_n, \quad q_n \to \lambda_n^{-1} q_n, \tag{1.7}$$

$$p_n \to W^T p_n, \quad q_n \to W^{-1} q_n. \tag{1.8}$$

Here W is a 3×3 matrix which commutes with g_0, $W g_0 = g_0 W$. Define the 3×3 matrices $L(x)$ and $M(x)$ by

$$L(x) = \prod_{n=0}^{N+1} (1 + x q_n p_n^T), \quad M(x) = x \left(\frac{q_{N+1} p_0^T}{p_0^T q_{N+1}} - \frac{q_0 p_{N+1}^T}{p_{N+1}^T q_0} \right). \tag{1.9}$$

Main Theorem. • *The dynamical system* (1.6) *is equivalent to the Lax equation*

$$\dot{L}(x) = [M(x), L(x)]; \tag{1.10}$$

- *The spectral curves* $\Gamma = \{R(x, y) \equiv \det\big(yI - L(x)\big) = 0\}$ *are invariant under the flow* (1.6)*, and are exactly the curves of the Landsteiner–Lopez form (1.2) (with* Λ^{N+2} *normalized to* 1*);*
- *There is a natural map* $(q_n, p_n) \to (\Gamma, D)$ *from the space of all spin chains satisfying the constraints* (1.3,1.4) *to the space of pairs* (Γ, D)*, where* Γ *is a Landsteiner–Lopez curve, and* $D = \{z_1, \cdots, z_{2N+1}\}$ *is a divisor whose class* $[D] = [D^\sigma]$ *is symmetric under the involution*

$$\sigma: \quad (x, y) = z \to z^\sigma = (-x, y^{-1}). \tag{1.11}$$

For a given (q_n, p_n)*,* D *is the set of poles of the Bloch function* ψ_0*,* $L(x)\psi_0 = y\psi_0(x)$*;*

[2] The notation is explained in greater detail in Sects. 3 and 5.

- *Let $\mathcal{M}_0$ be the space of pairs $\{\Gamma, [D]\}$, where Γ is a Landsteiner–Lopez curve with $u_N = 1, u_{N-1} = 0$, and $[D]$ is a divisor class which is symmetric under the involution σ. Then the space $\mathcal{M}_0$ has dimension $2(N-1)$. The map $(q_n, p_n) \to (\Gamma, D)$ descends to a map between the two spaces*

$$\{(q_n, p_n)\}/G \leftrightarrow \mathcal{M}_0, \tag{1.12}$$

where on the left-hand side, we have factored out the gauge group G from the space of periodic spin chains satisfying the constraints (1.3,1.4)*. At a generic curve Γ and a divisor $[D]$ in general position, the map* (1.12) *is a local isomorphism.*
- *Let the action variables a_i and the angle variables ϕ_i be defined on the space $\mathcal{M}_0$ by*

$$a_i = \oint_{A_i} d\lambda, \qquad \phi_i = \sum_{i=1}^{2N+1} \int^{z_i} d\omega_i, \tag{1.13}$$

where $\{A_i\}_{1 \le i \le N-1}$ and $\{d\omega_i\}_{1 \le i \le N-1}$, are respectively a basis for the even cycles and a basis for the even holomorphic differentials on Γ. Then

$$\omega = \sum_{i=1}^{N-1} \delta a_i \wedge \delta\phi_i \tag{1.14}$$

defines a symplectic form on the $2(N-1)$-dimensional space $\mathcal{M}_0$;
- *The dynamical system* (1.6) *is Hamiltonian with respect to the symplectic form* (1.14)*. The Hamiltonian is $H = u_{N-2}$.*

In terms of the (q_n, p_n) dynamical variables, the Hamiltonian can be expressed under the form

$$\begin{aligned} H &= \frac{u_{N-2}}{u_N} - \frac{u_{N-1}^2}{2u_N^2} \\ &= \sum_{n=0}^{N+1} \frac{(p_n^T q_{n-3})}{(p_n^T q_{n-1})(p_{n-1}^T q_{n-2})(p_{n-2}^T q_{n-3})} - \frac{(p_n^T q_{n-2})^2}{2(p_n^T q_{n-1})^2(p_{n-1}^T q_{n-2})^2}, \end{aligned} \tag{1.15}$$

where we have used the constraint $u_N = 1, u_{N-1} = 0$ to write H as $H = \frac{u_{N-2}}{u_N} - \frac{u_{N-1}^2}{2u_N^2}$.

We would like to note the similarity of the Lax matrix L in (1.9) to the 2×2 Lax matrix used in [21] for the integration of a quasi-classical approximation to a system of reggeons in QCD.

A key tool in our analysis is the construction of [6,7], which shows that symplectic forms constructed in terms of Seiberg–Witten differentials can also be constructed directly in terms of the Lax representation of integrable models. The latter are given by the following universal formula [6,7]:

$$\omega = \frac{1}{2} \sum_\alpha \mathrm{Res}_{P_\alpha} \left\langle \psi_{n+1}^* \delta L_n(x) \wedge \delta\psi_n \right\rangle dx, \tag{1.16}$$

where ψ_n and ψ_{n+1}^* are the Bloch and dual Bloch functions of the system, and P_α are marked punctures on the spectral curve Γ. In the present case, P_α are the 3 points on Γ above $x = \infty$.

Finally, we note that the odd divisor spin model (which we describe in Sects. 3.1 and 6) may be of independent interest. Although the symplectic form associated to the Seiberg-differential $x\frac{dy}{y}$ is degenerate in this case, the model does admit a Hamiltonian structure with non-degenerate symplectic form, but one which is associated rather with the form $d\lambda_{(1)} = \ln y \frac{dx}{x}$. As suggested in [19], the form $\ln y \frac{dx}{x}$ is also indicative of supersymmetric Yang–Mills theories, but in 5 or 6 dimensions with $\mathcal{N} = 1$ supersymmetry.

2. Geometry of the Landsteiner–Lopez Curve

We begin by identifying the geometric features of the generic Landsteiner–Lopez curve which will play an important role in the sequel. Fixing the normalization $\Lambda^{N+2} = 1$, we can write

$$\Gamma : \quad R(x, y) \equiv y^3 - f(x)y^2 + f(-x)y - 1 = 0, \tag{2.1}$$

where $f(x)$ is a polynomial of the form

$$f(x) = 3 + x^2 P_N(x), \quad P_N(x) = \sum_{i=0}^{N} u_i x^i. \tag{2.2}$$

The parameters $u_0, \cdots, u_N$ are the moduli of the Landsteiner–Lopez curve.

• The Landsteiner–Lopez curve Γ is a three-fold covering of the complex plane in the x variable. It is invariant under the involution σ defined in (1.11). The important points on Γ are the singular points, the points above $x = \infty$, and the branch points. We discuss now all these points in turn. • The singular points are the points where

$$\partial_x R(x, y) = \partial_y R(x, y) = 0. \tag{2.3}$$

The generic Landsteiner–Lopez curve has exactly one singular point, namely $(x, y) = (0, 1)$. At this point, Eq. (2.1) has a triple root, and all three sheets of the curve intersect. For generic values of the moduli u_i, all three solutions y of $R(x, y) = 0$ can be expressed as power series in x in a neighborhood of $x = 0$,

$$y(x) = 1 + \sum_{i=1}^{\infty} y_i x^i. \tag{2.4}$$

In fact, we can substitute (2.4) into (2.1) to find recursively all coefficients y_i, with the first coefficient y_1 a solution of

$$y_1^3 - u_0 y_1 + 2u_1 = 0. \tag{2.5}$$

For generic u_0, u_1, this equation does admit three distinct solutions for y_1, which lead in turn to the three distinct solutions. These three distinct solutions provide effectively a smooth resolution of the curve Γ, where the crossing point $y = 1$ above $x = 0$ has been separated into 3 distinct points Q_α, $1 \le \alpha \le 3$. Under the involution σ, the leading terms in the three solutions (2.4) transform as

$$(x, 1 + y_1 x + \cdots) \to (-x, (1 - y_1 x + \cdots)^{-1}) = (-x, 1 + y_1 x + \cdots). \tag{2.6}$$

Since the three solutions y_1 of Eq. (2.5) are distinct for generic values of the moduli u_i, we see that each of the three points Q_α above $x = 0$ are fixed under the involution σ.

• For generic values of the moduli u_i, there are also three distinct branches of $y(x)$ near $x = \infty$. A first branch $y(x) = O(x^{N+2})$ with a pole of order $N+2$ can be readily found

$$y(x) = x^{N+2}(u_N + u_{N-1}x^{-1} + u_{N-2}x^{-2} + \cdots). \tag{2.7}$$

(The first three coefficients in $y(x)$ turn out to be exactly the first three coefficients u_N, u_{N-1} and u_{N-2} in the polynomial $P_N(x)$ of (2.2).) We denote by P_1 the corresponding point above $x = \infty$. In view of the involution σ, a second branch $y(x) = O(x^{-(N+2)})$ with a zero of order $N+2$ exists which is the image of the first branch under σ

$$y(x) = (-x)^{-(N+2)} \frac{1}{u_N} \left(1 + \frac{u_{N-1}}{u_N} x^{-1} + \frac{u_{N-1}^2 - u_N u_{N-2}}{u_N^2} x^{-2} + \cdots \right). \tag{2.8}$$

The corresponding point above $x = \infty$ is denoted P_3. Finally, the involution σ implies that the third branch $y(x)$ is regular and fixed under σ

$$y(x) = (-)^{N+2}\left[1 + O\left(\frac{1}{x}\right)\right]. \tag{2.9}$$

Denoting the corresponding point above $x = \infty$ by P_2, we have

$$\sigma : P_1 \leftrightarrow P_3, \quad \sigma : P_2 \leftrightarrow P_2. \tag{2.10}$$

• The branching points of Γ over the x-plane are just the zeroes on Γ of the function $\partial_y R(x, y)$ which are different from the singular points Q_α. This function has a pole of order $2(N+2)$ at P_1 and a pole of order $(N+2)$ at each of the points P_2 and P_3. Therefore, it has $4N+8$ zeros. At each of the points Q_α the function $\partial_y R(x, y)$ has zeros of order 2. Hence

$$\#\{\text{Branch Points}\} = 4N + 2. \tag{2.11}$$

Note that for generic moduli u_i, neither 0 nor ∞ is a branch point, in view of our previous discussion. Also for generic u_i, we can assume that the ramification index at all branch points is 2. Thus the total branching number is just the number of branch points. Since the number of sheets is 3, the Riemann-Hurwitz formula can be written as $g(\Gamma) = -3 + \frac{1}{2}(4N+2) + 1$ in this case. Thus the genus $g(\Gamma)$ of the curve Γ is

$$g(\Gamma) = 2N - 1. \tag{2.12}$$

• For generic moduli u_i, the involution $\sigma : \Gamma \to \Gamma$ has exactly four fixed points, namely the three points Q_α above $x = 0$ and the point P_2 above $x = \infty$. That implies that the factor-curve Γ/σ has genus

$$g(\Gamma/\sigma) = N - 1. \tag{2.13}$$

The involution σ induces an involution of the Jacobian variety $J(\Gamma)$ of Γ. The odd part $J^{Pr}(\Gamma)$ of $J(\Gamma)$ is the Prym variety and the even part is isogenic to the Jacobian $J(\Gamma/\sigma)$ of the factor-curve Γ/σ. The dimension of the space of divisors $[D]$ which are even under σ is equal to $\dim J(\Gamma/\sigma) = N - 1$.

3. The Spin Models

We introduce two systems with the same family of spectral curves (2.1). One system has non-trivial dynamics along the even while the other system has non-trivial dynamics along the odd (Prym) directions of the Jacobian. The system corresponding to the $SU(N)$ Yang–Mills theory with a hypermultiplet in the anti-symmetric representation is the even system. We sketch here the outline of the construction of both models, leaving the full discussion to Sects. 4-5.

Both models are periodic spin chain models, with a 3-dimensional complex vector at each site. We view three-dimensional vectors s as column vectors, with components s_α, $1 \leq \alpha \leq 3$. We denote by s^T the transpose of s, which is then a three-dimensional row vector, with components s^α. In particular, $s^T s$ is a scalar, while ss^T is a 3×3 matrix. Since the odd divisor spin model is simpler, we begin with it.

3.1. The odd divisor spin model. The odd divisor spin model is a $(N+2)$-periodic chain of complex three-dimensional vectors $s_n = s_{N+n+2},\ s_n = (s_{n,\alpha}),\ \alpha = 1, 2, 3,$ subject to the constraint

$$s_n^T s_n = \sum_{\alpha=1}^{3} s_n^\alpha s_{n,\alpha} = 0, \tag{3.1}$$

and the following equations of motion:

$$\dot{s}_n = \frac{s_{n+1}}{s_{n+1}^T s_n} - \frac{s_{n-1}}{s_{n-1}^T s_n}. \tag{3.2}$$

The constraint (3.1) and the equations of motion are invariant under transformation of the spin chain by a matrix V satisfying the condition $V^T V = I$,

$$s_n \to V s_n. \tag{3.3}$$

The odd divisor spin model is integrable in the sense that the equations of motion are equivalent to a Lax pair. To see this, we define the 3×3 matrices $L_n(x)$ and $M_n(x)$ by

$$L_n(x) = 1 + x\, s_n s_n^T, \tag{3.4}$$

$$M_n(x) = x \frac{1}{s_n^T s_{n-1}} (s_{n-1} s_n^T + s_n s_{n-1}^T). \tag{3.5}$$

Then the compatibility condition for the system of equations

$$\psi_{n+1} = L_n(x)\psi_n, \tag{3.6}$$

$$\dot{\psi}_n = M_n(x)\psi_n \tag{3.7}$$

is given by

$$\dot{L}_n(x) = M_{n+1}(x)L_n(x) - L_n(x)M_n(x). \tag{3.8}$$

A direct calculation shows that for $L_n(x)$ and $M_n(x)$ defined as in (3.5), this equation is equivalent to the equations of motion (3.2) for the spin model. Define now the monodromy matrix $L(x)$ by

$$L(x) = L_{N+1}(x) \cdots L_0(x) = \prod_{n=0}^{N+1} L_n(x), \tag{3.9}$$

where the ordering in the product on the right-hand side starts by convention with the lowest indices on the right. Then $L(x)$ and $M(x) = M_0(x)$ form themselves a Lax pair

$$\dot{L}(x) = [M(x), L(x)]. \tag{3.10}$$

This is easily verified using (3.8), since

$$\begin{aligned}
\dot{L}(x) &= \sum_{k=0}^{N+1} \prod_{n=k+1}^{N+1} L_n(x)\, \dot{L}_k \times \prod_{n=0}^{k-1} L_n(x) && (3.11)\\
&= \sum_{k=0}^{N+1} \prod_{n=k+1}^{N+1} L_n(x)(M_{k+1}L_k - L_k M_k) \prod_{n=0}^{k-1} L_n(x) && (3.12)\\
&= \sum_{k=0}^{N+1} \prod_{n=k+1}^{N+1} L_n(x)\, M_{k+1} \prod_{n=0}^{k} L_n(x) - \sum_{k=0}^{N+1} \prod_{n=k}^{N+1} L_n(x)\, M_k \prod_{n=0}^{k-1} L_n(x) && (3.13)\\
&= M_{N+2}L(x) - L(x)M_0(x). && (3.14)
\end{aligned}$$

In particular, the characteristic equation of $L(x)$ is time-independent and defines a time-independent spectral curve

$$\Gamma = \{(x, y); 0 = R(x, y) \equiv \det (yI - L(x))\}. \tag{3.15}$$

We assert that these spectral curves are Landsteiner–Lopez curves (2.1). In fact, it follows immediately from the expression (3.5) that $\det L_n(x) = 1$, $L_n(x) = L_n(x)^T$, and $L_n(x)^{-1} = L(-x)$. Thus

$$\det L(x) = 1, \quad L(x)^{-1} = L(-x). \tag{3.16}$$

These two equations imply that $\det(yI - L(x))$ is of the form (2.1) for some polynomial $f(x)$. To obtain the expression (2.2) for $f(x)$, it suffices to observe that

$$f(x) = \mathrm{Tr}\, L(x) = \mathrm{Tr}\, (1 + x \sum_{n=0}^{N+1} s_n s_n^T) + O(x^2) = 3 + O(x^2). \tag{3.17}$$

Define the moduli u_i of the curve $R(x, y) = 0$ as in (2.1) by $f(x) = 3 + x^2 \sum_{i=0}^{N} u_i x^i$. Then the correspondence between the dynamical variables s_n, $0 \le n \le N + 1$, and the moduli u_i is given by

$$u_i = \sum_{I_i} s_{n_1}^T s_{n_2} s_{n_2}^T s_{n_3} \cdots s_{n_{i-1}}^T s_{n_i}, \tag{3.18}$$

where the summation runs over the set I_i of all ordered i-th multi-indices $n_1 < n_2 < \cdots < n_i$.

To obtain the phase space of the model, we consider the space of all $(N+2)$-periodic spin chains s_n, subject to the constraint (3.1), and modulo the equivalence $s_n \sim V s_n$, where V is a matrix satisfying $V^T V = I$. The dimension of this space is

$$\dim \{s_n\}/\{s_n \sim V s_n\} \ = \ 2N + 1. \tag{3.19}$$

Indeed, the $(N+2)$-periodic spin chains s_n have $3(N+2)$ degrees of freedom. The constraint (3.1) removes $N+2$ degrees of freedom, and the equivalence $s_n \sim V s_n$ removes 3 others, since the dimension of the matrices V with $V^T V$ is 3. A $2N$-dimensional symplectic manifold $\mathcal{L}^{\mathrm{odd}}$ is obtained by setting

$$\mathcal{L}^{\mathrm{odd}} \ = \ \{s_n;\ u_N = constant\ \}/\{s_n \sim V s_n\}. \tag{3.20}$$

On the space $\mathcal{L}^{\mathrm{odd}}$, the system is Hamiltonian with respect to the symplectic form defined by the differential $d\lambda_{(1)} = (\ln x)\frac{dy}{y}$, with Hamiltonian

$$H_{(1)} = \frac{u_{N-1}}{u_N} = \sum_{n=0}^{N+1} \frac{(s_{n+1}^T s_{n-1})}{(s_{n+1}^T s_n)(s_n^T s_{n-1})}. \tag{3.21}$$

The action-variables are the periods of the differential $d\lambda_{(1)} = -(\ln x)\frac{dy}{y}$ over a basis of N cycles for the curve Γ, which are odd under the involution σ. If the curve Γ is viewed as a two–sheeted cover of Γ/σ, these N odd curves can be realized as the N cuts along which the sheets are to be glued.

3.2. The even divisor spin model. The even divisor spin model is the Hamiltonian reduction of a periodic spin chain model which incorporates a natural gauge invariance.

The starting point is a $(N+2)$-periodic chain of pairs of three-dimensional complex vectors $p_n = (p_{n,\alpha}),\ \ q_n = (q_{n,\alpha}),\ 1 \leq \alpha \leq 3$, satisfying the constraints (1.3). We impose the equations of motion (1.6). As noted before, the constraints and the equations of motion are invariant under the gauge transformations (1.7,1.8). In particular, a gauge fixed version of the equations of motion (1.6) is

$$\dot{p}_n = \frac{p_{n+1}}{p_{n+1}^T q_n} + \frac{p_{n-1}}{p_{n-1}^T q_n}, \quad \dot{q}_n = -\frac{q_{n+1}}{p_n^T q_{n+1}} - \frac{q_{n-1}}{p_n^T q_{n-1}}. \tag{3.22}$$

This version follows from the other one by the gauge transformation

$$p_n \to \lambda_n(t) p_n,\ \ q_n \to \lambda_n^{-1}(t) q_n,\ \ \lambda(t) = \exp\left(-\int^t \mu_n(t')dt'\right). \tag{3.23}$$

We shall see in the next section that the system (1.6) admits a Lax representation.

A reduced system is defined as follows. We impose the additional constraints (1.4). With these constraints, the spectral curves of the system are the Landsteiner–Lopez curves (2.1). The dimension of the phase space $\mathcal{M}$ of all (q_n, p_n) subjected to the previous constraints and divided by the gauge group G of (1.7,1.8), is

$$\dim \mathcal{M} \equiv \dim \{(q_n, p_n)\}/G = 2N. \tag{3.24}$$

To see this, assume that N is even (the counting for N odd is similar). Then the constraint (1.4) reduces the number of degrees of the $(N+2)$-periodic spin chain (q_n, p_n) to the number $3(N+2)$ of a $(N+2)$-periodic spin chain. The constraint (1.3) and the gauge transformation (1.7) each eliminates $\frac{N}{2}+1$ degrees of freedom. Now the dimension of the space of matrices W satisfying $Wg_0 = g_0 W$ is 5. However, in the gauge transformation (1.8), the matrices W which are diagonal have already been accounted for in the gauge transformation (1.7). Altogether, we arrive at the count which we announced earlier.

The phase space $\{(q_n, p_n)\}/G$ itself can be reduced further, to a lower-dimensional phase space defined by suitable constraints on the moduli space $(u_0, \cdots, u_N)$. It turns out that there are 2 possible natural further reductions, each related to its own choice of differential $d\lambda$ and corresponding Hamiltonian structure:

- On the $(2N-2)$-dimensional phase space defined by the constraints

$$\mathcal{M}_0 = \{(q_n, p_n);\ u_N = 1,\ u_{N-1} = 0\}/G \tag{3.25}$$

the system is Hamiltonian with respect to the symplectic form defined by the differential $d\lambda = x\frac{dy}{y}$. Here we have used the same notation for the space just introduced and the space $\mathcal{M}_0$ described in the Main Theorem, in anticipation of their isomorphism which will be established later in Sect. 4. The Hamiltonian is given by $H = u_{N-2}$ or equivalently by (1.15).

The action-variables are periods of $d\lambda$ along a basis of $N-1$ cycles A_i of Γ which are even under the involution σ. (Equivalently, the A_i correspond to a basis of cycles for the factor curve Γ/σ.) This is the desired integrable Hamiltonian system, corresponding to the $\mathcal{N}=2$ supersymmetric $SU(N)$ Yang–Mills theory with a hypermultiplet in the anti-symmetric representation.

- On the $(2N-2)$-dimensional phase space $\mathcal{M}_2$ defined by the constraints

$$\mathcal{M}_2 = \{(q_n, p_n);\ u_0 = \text{constant},\ u_1 = \text{constant}\}/G \tag{3.26}$$

the system is Hamiltonian with respect to the symplectic form defined by the differential $d\lambda_{(2)} = -\frac{1}{x}\frac{dy}{y}$. This symplectic form coincides with the natural form

$$\omega = \sum_n dp_n^T \wedge dq_n \tag{3.27}$$

with respect to which the system (1.6) is manifestly Hamiltonian, with Hamiltonian

$$H(p,q) = \ln u_N = \frac{1}{2}\sum_{n=0}^{N+1} \ln\left[(p_n^+ q_{n-1})(p_{n-1}^+ q_n)\right]. \tag{3.28}$$

The action-variables are the periods of the differential $d\lambda_{(2)} = -\frac{dy}{xy}$ over again the even cycles A_i of the earlier case.

4. The Direct and Inverse Spectral Transforms

We concentrate now on the even divisor spin model. The main goal of this section is to describe the map stated in the main theorem, which associates to the spin chain (q_n, p_n) a geometric data $(\Gamma, [D])$,

$$(q_n, p_n) \to (\Gamma, [D]). \tag{4.1}$$

The curve Γ is obtained by showing that the dynamical system (1.6) for (p_n, q_n) admits a Lax representation $\dot{L}(x) = [M(x), L(x)]$, in which case Γ is the spectral curve $\{\det(yI - L(x)) = 0\}$. The Lax operator $L(x)$ also gives rise to the Bloch function, which is essentially its eigenvector. The divisor D is obtained by taking the divisor of poles of the Bloch function. A characteristic feature of the even divisor spin model is that the equivalence class of this divisor $[D]$ is *even* under the involution σ. The map (4.1) descends to a map from the space of equivalence classes of (q_n, p_n) under the gauge group G to the space of geometric data $(\Gamma, [D])$. These two spaces are of the same dimension $2N$: we saw this in (3.24) for the first space, while for the second, the number $2N$ of parameters is due to $N+1$ parameters for the Landsteiner–Lopez curves (including u_N and u_{N-1}), and $N-1$ parameters for the even divisors $[D]$. It is a fundamental fact in the theory that the map (4.1) becomes then a bijective correspondence of generic points

$$\{q_n, p_n)\}/G \leftrightarrow \{(\Gamma, [D])\}. \tag{4.2}$$

We shall refer to the construction $\rightarrow$ described above as *the direct problem*. The reverse construction $\leftarrow$, which recaptures the dynamical variables (p_n, q_n) from the geometric data $(\Gamma, [D])$ will be referred to as *the inverse problem*. As usual in the geometric theory of solitons [22], it will be based on the construction of a Baker–Akhiezer function. We now provide the details.

4.1. The Lax representation. We exhibit first the Lax representation for the system (1.6). The desired formulas can be obtained from a slight modification of the easier odd spin model treated in Sect. 3.1. Let p_n, q_n be $(N+2)$-periodic, three-dimensional vectors satisfying $p_n^T q_n = 0$, and define matrix-valued functions $L_n(x)$ and $M_n(x)$ by

$$L_n(x) = 1 + x\, q_n p_n^T, \quad M_n(x) = x \left(\frac{q_{n-1} p_n^T}{p_n^T q_{n-1}} - \frac{q_n p_{n-1}^T}{p_{n-1}^T q_n} \right). \tag{4.3}$$

Then a direct calculation shows that the matrix functions $L_n(x)$ and $M_n(x)$ satisfy the Lax equation

$$\partial_t L_n = M_{n+1} L_n - L_n M_n \tag{4.4}$$

if and only if the vectors p_n and q_n satisfy the equations of motion (1.6).

As before, Eq. (4.4) is a compatibility condition for the linear system $\psi_{n+1} = L_n(x)\psi_n$, $\dot{\psi}_n = M_n(x)\psi_n$. To obtain the spectral curve Γ, we observe that the same arguments as in the case of the odd spin model show that the matrix $M(x) = M_0(x)$ and the monodromy matrix $L(x)$ defined by $L(x) = \prod_{n=0}^{N+1} L_n(x)$ form again a Lax pair

$$\dot{L}(x) = [M(x), L(x)]. \tag{4.5}$$

Thus the spectral curve $\Gamma = \{(x, y);\ R(x, y) \equiv \det(yI - L(x)) = 0\}$ is time-independent and well-defined. We have used here the same notation $R(x, y)$ as for (2.1), since the equation $\det(yI - L(x))$ is indeed of the Landsteiner–Lopez form. To see this, we note that $\det L_n(x) = 1$ and $L_n(-x) = L_n(x)^{-1}$. Together with the constraint (1.4), this implies

$$\det L(x) = 1, \quad L(-x) = g_0 L^{-1}(x) g_0. \tag{4.6}$$

But we also have near $x=0$

$$\mathrm{Tr}\, L(x) = \mathrm{Tr}\,(1 + x\sum_{n=0}^{N} q_n p_n^T) + O(x^2) = 3 + 0(x^2), \tag{4.7}$$

so that $\det(yI - L(x))$ is of the form (2.1).

We observe that the expression $R(x,y) = \det(yI - L(x))$ is invariant with respect to the gauge transformations (1.7) and (1.8). Therefore, if we write $R(x,y)$ in the Landsteiner–Lopez form (2.1) with moduli u_i, the moduli u_i are well-defined functions on the factor-space $\mathcal{M}$. In analogy with the odd spin case, u_i can be written in terms of the dynamical variables (p_n, q_n) as

$$u_k = \sum_{I_k} (p_{i_1}^+ q_{i_2})(p_{i_2}^+ q_{i_3}) \cdots (p_{i_k}^+ q_{i_1}). \tag{4.8}$$

Here the summation is again over sets I_k of multi-indices $I = (i_1 < i_2 < \ldots < i_k)$.

4.2. General properties of Bloch functions. The points $Q=(x,y)$ of the spectral curve $\Gamma = \{(x,y); \det(yI - L(x)) = 0\}$ parametrize the Bloch functions $\{\psi_n(Q)\}_{0\le n\le N+1}$ of the spin model. We begin by recalling the definition of Bloch functions, and by describing their main properties in the case of our model.

• We fix a generic choice of moduli parameters u_i. Then the matrix $L(x)$ has 3 distinct eigenvalues y, except possibly at a finite number of points x. Let $Q=(x,y)$. The Bloch solution $\psi_n(Q)$ for the spin model $\{L_n(x)\}_{0\le n\le N+1}$ is the function $\psi_n(Q)$ with the following properties:

$$\Psi_{n+1}(Q) = L_n(x)\Psi_n(Q), \quad \Psi_{N+n+2}(Q) = y\Psi_n(Q). \tag{4.9}$$

These equations determine $\psi_n(Q)$ only up to a multiplicative constant. To normalize $\psi_n(Q)$, we observe that for generic moduli parameters u_i, there are only finitely many points Q, where the eigenvector $\psi_0(Q)$ of the matrix $L(x)$ satisfies the linear constraint $\sum_{\alpha=1}^3 \psi_0^\alpha(Q) = 0$. Outside of these points, we can fix $\psi_n(Q)$ by the following normalization condition:

$$\sum_{\alpha=1}^{3} \psi_0^\alpha = 1. \tag{4.10}$$

The Bloch function $\psi_n(Q)$ is then determined on the spectral curve Γ outside of a finite number of points, and hence uniquely on Γ. Furthermore, the components of $\psi_n(Q)$ are meromorphic functions on Γ. This follows from the constraint (4.10) and the equation $L(x)\psi_0(Q) = y\psi_0(Q)$. They imply that $\psi_0(Q)$ is a rational expression in y and in the entries of the matrix $(L_{\alpha\beta}(x) - L_{\alpha 3}(x))_{\substack{1\le\alpha\le 3\\ 1\le\beta\le 2}}$, in view of Cramer's rule for solving inhomogeneous systems of linear equations. Since x, y and $L_{\alpha\beta}(x)$ are all meromorphic functions on Γ, our assertion follows.

• The exceptional points excluded in the preceding construction of Bloch functions are the points where $L(x)$ has multiple eigenvalues, and the points where the eigenvector $\psi_0(Q)$ lies in the linear subspace of equation $\sum_{\alpha=1}^3 \psi_0^\alpha(Q) = 0$. By restricting ourselves to generic values of the moduli u_i, we can make the convenient assumption that these two sets of points are disjoint. In this case, it is evident that at points where $\sum_{\alpha=1}^{3}\psi_0^\alpha(Q)=0$, the function $\psi_0(Q)$ develops a pole.

Consider now a point $x_0\neq 0$, where the matrix $L(x)$ has a multiple eigenvalue. Let $(x-x_0)^{1/b}$ be the local holomorphic coordinate centered at the points Q lying above x_0, where the branching index b can be either 1 or 2. (We can exclude the possibility $b=3$ by a genericity assumption on the moduli u_i.) The holomorphic function y on the surface Γ can be expanded as

$$y=y_0+\epsilon y_1(x-x_0)^{1/b}+O(x-x_0), \tag{4.11}$$

where $\epsilon^b=1$ is a root of unity. If $b=1$, it follows that

$$\partial_x R(x_0,y_0)=\partial_y R(x_0,y_0)=0, \tag{4.12}$$

which means that the curve is singular at (x_0,y_0). By a genericity assumption on the moduli u_i, the only singular point on Γ is at $x_0=0$, and this possibility has been excluded. Thus $b=2$, and the curve Γ has a branch point at x_0 if and only if $L(x_0)$ has multiple eigenvalues. The matrix $L(x_0)$ can now be shown to be a Jordan cell, i.e., $L(x_0)$ is of the form

$$L(x_0)=\begin{pmatrix}\lambda_1 & \mu & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{pmatrix} \tag{4.13}$$

in a suitable basis, for some $\mu\neq 0$ and $\lambda_1=\lambda_2\neq\lambda_3$. In fact, $L(x_0)$ has only one double eigenvalue by genericity assumptions on u_i. The three branches of the function y consist then of one branch which is of the form $\lambda_3+y_1(x-x_0)+\cdots$ and is holomorphic in the variable $x-x_0$. The other two branches are of the form

$$y=\lambda_1\pm y_1(x-x_0)^{1/2}+\cdots. \tag{4.14}$$

We must have $y_1\neq 0$, for otherwise $y=\lambda_1+O(x-x_0)$, and the same argument which ruled out the branching index $b=1$ would imply that Γ is singular at x_0. Now for x near but distinct from x_0, the Bloch function $\psi_0(Q)$ also has 3 distinct branches. Let $\psi_\pm$ be the branches corresponding to the eigenvalues in (4.14), and expand them as

$$\psi_\pm=\psi_\pm^{(0)}+(x-x_0)^{1/2}\psi_\pm^{(1)}+O(x-x_0). \tag{4.15}$$

Up to $O(x-x_0)$, the eigenvector condition can be expressed as

$$L(x)(\psi_\pm^{(0)}+(x-x_0)^{1/2}\psi_\pm^{(1)})=(\lambda_1\pm y_1(x-x_0)^{1/2})(\psi_\pm^{(0)}+(x-x_0)^{1/2}\psi_\pm^{(1)}). \tag{4.16}$$

This is equivalent to

$$L(x_0)\psi_\pm^{(0)}=\lambda_1\psi_\pm^{(0)},\quad (L(x_0)-\lambda_1)\psi_\pm^{(1)}=\pm y_1\psi_\pm^{(0)}. \tag{4.17}$$

Clearly, this equation admits no solution if $L(x_0)$ is diagonal. Thus $L(x_0)$ is of the form (4.13) with $\mu\neq 0$. We can now identify the coefficients $\psi_\pm^{(0)}$ and $\psi_\pm^{(1)}$ in the Puiseux expansion (4.15). The eigenspace of $L(x_0)$ corresponding to the eigenvalue λ_1 is one-dimensional and generated by a single vector ϕ_1, which we can take to satisfy the normalization condition (4.10). Evidently, $\psi_\pm^{(0)}=\phi_1$. Let ϕ_2 be the second basis vector in the basis with respect to which $L(x_0)$ takes the Jordan form (4.13), i.e., $L(x_0)\phi_2 = y_1\phi_2 + \mu\phi_1$. Then the second equation above is solved by

$$\psi_{\pm}^{(1)} = \pm(\frac{y_1}{\mu}\phi_2 + \nu\phi_1), \tag{4.18}$$

where the constant ν is chosen so that $\sum_{\alpha=1}^{3} \psi_{\pm}^{(1),\alpha} = 0$.

• Outside a finite number of points x, the matrix $L(x)$ has 3 distinct eigenvalues $y(a)$ and three distinct eigenfunctions $\psi_0(a)$, $1 \leq a \leq 3$, normalized uniquely by the condition (4.10). The function

$$\det{}^2\big\{\psi_0(1)\ \psi_0(2)\ \psi_0(3)\big\} \tag{4.19}$$

is independent of the ordering of both $\psi_0(a)$ and the corresponding eigenvalues $y(a)$. By the preceding observations, it can be expressed as a rational function of x and $y(a)$, which is also symmetric under permutations of $y(a)$. Thus it is actually an unambiguous and rational function of x. We observe that the function $\det^2\big\{\psi_0(1)\ \psi_0(2)\ \psi_0(3)\big\}$ vanishes at exactly those values of x which are branch points for the spectral curve $\det(yI - L(x)) = 0$. Indeed, we saw earlier that the branch points x_0 are exactly the points where $L(x_0)$ has multiple eigenvalues. Outside points x_0 where $L(x_0)$ has multiple eigenvalues, the determinant (4.19) is readily seen to be $\neq 0$ (it may be infinite, because of the normalization (4.10)). Conversely, assume that x_0 is a branch point. Then our preceding discussion shows that for x near x_0

$$\det{}^2\big\{\psi_0(1)\ \psi_0(2)\ \psi_0(3)\big\}(x) = (x - x_0)\det{}^2\big\{\phi_1\ \phi_2\ \psi_0(3)\big\} + O(x - x_0)^{3/2}. \tag{4.20}$$

This shows that $\det^2\big\{\psi_0(1)\ \psi_0(2)\ \psi_0(3)\big\}(x_0) = \lim_{x\to x_0}\det^2\big\{\psi_0(1)\ \psi_0(2)\ \psi_0(3)\big\}(x) = 0$, establishing the observation. Furthermore, since the vectors ϕ_1, ϕ_2, and $\psi_0(3)$ are linearly independent by construction, we obtain the important fact that the order of vanishing of the square of the determinant in (4.19) at a branch point is exactly 1. (More generally, for an arbitrary branching index b, the order of vanishing of the square of the determinant is equal to $b-1$, although we do not need this more general version here, thanks to our genericity assumption on the moduli u_i.)

• We can now determine the number of poles of the Bloch function $\psi_0(Q)$ outside of the points P_a above $x = \infty$. Clearly, this number is half of the number of poles of the expression (4.19) outside of $x = \infty$. Now at $x = \infty$, we saw that the operator $L(x)$ has 3 eigenvalues, so that (4.19) does not vanish there. Furthermore, we shall show later that $\psi_0(Q)$ is finite at all three points above $x = \infty$. Thus (4.19) has neither a zero nor a pole at $x = \infty$. In view of the preceding discussion, the number of zeroes of (4.19) is equal to the number of branch points of Γ. We showed earlier, using the Riemann-Hurwitz formula, that the number of branch points of Γ is $4N + 2$. It follows that the number of poles, and hence of zeroes of $\psi_0(Q)$ on Γ is $2N + 1$.

• The poles of $\psi_n(Q)$ outside of the points P_a lying above $x = \infty$ are independent of n. To see this, we let $S_0(x)$ be the 3×3 identity matrix I, and set $S_n(x) = L_{n-1}(x)S_{n-1}(x) = \prod_{0\leq k\leq n-1} L_k(x)$. Then $\psi_n(Q)$ can be expressed as

$$\psi_n(Q) = L_{n-1}(x)\psi_{n-1}(Q) = \prod_{0\leq k\leq n-1} L_k(x)\psi_0(Q) = S_n(x)\psi_0(x). \tag{4.21}$$

This shows that the poles of $\psi_n(Q)$ outside of P_a can only occur at the poles of $\psi_0(Q)$. For generic values of the moduli u_i, we can assume that all the poles of $\psi_n(Q)$, $0 \leq n \leq$

$N+1$, are exactly of the same order 1 when they occur outside of the points P_a above $x=\infty$.

• Let $D=\{z_1,\cdots,z_{2N+1}\}$ be the divisor of poles of the Bloch function $\psi_n(Q)$. Then a fundamental property of the even divisor spin chain model is the invariance of the equivalence divisor class $[D]$ of D under the involution σ

$$[D]=[D^\sigma]. \tag{4.22}$$

In other words, there exists a meromorphic function on Γ with poles at z_n and z_n^σ. This is a consequence of how $L(x)$ transforms under the involution $x\to -x$, $y\to y^{-1}$,

$$L(-x)=g_0L(x)^{-1}g_0. \tag{4.23}$$

This transformation rule implies that $g_0\psi_0(Q)$ is a Bloch function at $(-x,y^{-1})$ if $\psi_0(Q)$ is a Bloch function at (x,y). Thus $g_0\psi_0(Q)$ must coincide with $\psi_0(Q^\sigma)$ up to normalization

$$g_0\psi_0(Q)=f(Q)\psi_0(Q^\sigma). \tag{4.24}$$

Since both $\psi_0(Q)$ and $\psi_0(Q^\sigma)$ are meromorphic functions, the function $f(Q)$ is meromorphic. This proves (4.22).

We summarize the discussion in the following lemma:

Lemma 4.1. *The vector-function $\psi_n(Q)$ is a meromorphic vector-function on Γ. Outside the punctures P_a (which are the points of Γ situated over $x=\infty$) it has $g+2=2N+1$ poles $\{z_1,\ldots,z_{2N+1}\}$, which are n-independent. The divisor class $[D]$ of D is invariant with respect to the involution σ, i.e. there exists a function $f(Q)$ on Γ with poles at z_j and zeroes at z_j^σ.*

4.3. The direct problem. In the previous discussion, we made use only of the fact that the curve $R(x,y)=0$ is the spectral curve of a matrix $L(x)$ which satisfies the involution condition $L(-x)=L(x)^{-1}$. In particular, the discussion applies for generic values of the moduli u_i parametrizing the curves.

We consider now the direct problem for the system (1.6), where the matrix $L(x)$ arises more specifically in terms of the dynamical variables (q_n,p_n) as $L(x)=\prod_{n=0}^{N+1}L_n(x)=\prod_{n=0}^{N+1}(1+xq_np_n^T)$. The discussion in the previous section has provided a precise description of the right-hand side of the map (4.1). It is also evident that the map descends to the equivalence classes of (q_n,p_n) under the gauge group G.

It is convenient to exploit the gauge transformation (1.8) to normalize the Bloch functions at $x=0$. First, we observe that $L_n(0)=I$ for all n, so that $\psi_n(Q_a)$ is independent of n. Furthermore, the Lax operator $L(x)$ can be written near $x=0$ as

$$L(x)=I+xT+O(x^2), \tag{4.25}$$

where the matrix T is given by

$$T=\sum_{n=0}^{N+1}q_np_n^T. \tag{4.26}$$

In particular, T satisfies the condition

$$T = g_0 T g_0. \tag{4.27}$$

in view of the constraint $p_n = g_0 p_{-n-1}, q_n = g_0 q_{-n-1}$. Next, recall from our discussion of the Landsteiner–Lopez curve in Sect. 2 that T has 3 distinct eigenvalues $y_1(Q_a)$, and that y can be expanded as $y = 1 + y_1(Q_\alpha)x + O(x^2)$ near Q_a. Expanding $\psi_0(Q)$ near Q_a as $\psi_0(Q) = \psi_0(Q_a) + O(x)$, and using the preceding expansion for $L(x)$, the condition $L(x)\psi_0(Q) = y\psi_0(Q)$ for Bloch functions can be rewritten as

$$(I + xT)(\psi_0(Q_a) + x\psi_0'(Q_a)) = (1 + y_a x)(\psi_0(Q_a) + x\psi_0'(Q_a)) + O(x^2). \tag{4.28}$$

This implies

$$T\psi_0(Q_a) = y_a\psi_0(Q_a) \tag{4.29}$$

i.e., $\psi_0(Q_a)$ are precisely the three eigenvectors of T, corresponding to the eigenvalues y_a. If we let $\Psi_0(0)$ be the 3×3 matrix whose columns are the vectors $\psi_0(Q_a)$, then the transformation law (4.27) implies that $\Psi_0(0)$ satisfies the condition

$$\Psi_0(0) = g_0\Psi_0(0)g_0. \tag{4.30}$$

Now the transformation (1.8) on (q_n, p_n) does not change the curve Γ and the divisor D, but changes the matrix $\Psi_0(0)$ into $W\Psi_0(0)$. But $\Psi_0(0)$ commutes with the matrix g_0, and hence so does its inverse. This means that the inverse qualifies as one of the gauge transformations W allowed in (1.8). Under such a gauge transformation W, the Bloch function $\Psi_0(0)$ gets transformed to the identity

$$\Psi_0(0) = I. \tag{4.31}$$

Henceforth we can assume then this normalization, and p_n, q_n satisfies the condition

$$T_\alpha^\beta = \sum_{n=0}^{N+1} q_{n,\alpha} p_n^\beta = y_1^\alpha \delta_\alpha^\beta. \tag{4.32}$$

Our main task is to establish that the map (4.2) is generically locally invertible. This is the goal of the next section on the inverse spectral problem, but in order to motivate the constructions given there, we identify here the basic behavior of the Bloch function $\psi_n(x, y)$ near the points P_α above $x = \infty$. For (x, y) near P_α, set

$$\psi_n(x, y) = x^{p_{n\alpha}} \sum_{k=0}^{\infty} \psi_{n,k}(P_\alpha)x^{-k}. \tag{4.33}$$

Here $p_{n\alpha}$ is the order of the pole (or zero when $p_{n\alpha} < 0$) of $\psi_n(x, y)$ near P_α, which may vary with both n and α. The following lemma identifies the coefficients $\psi_{n,k}(P_\alpha)$ up to normalization:

Lemma 4.2. • *In the neighborhood of the puncture P_1 (where $y = O(x^{N+2})$), the vector-function ψ_n has a pole of order n and the leading coefficient $\psi_{n,0}(P_1)$ of its expansion is equal to*

$$\psi_{n,0}(P_1) = \alpha_n q_{n-1}, \tag{4.34}$$

where the scalar α_n satisfy the recurrence relation

$$\alpha_{n+1} = (p_n^T q_{n-1})\alpha_n. \tag{4.35}$$

The next coefficient $\psi_{n,1}(P_1)$ satisfies

$$\psi_{n+1,1} = \psi_{n,0} + q_n(p_n^T \psi_{n,1}). \tag{4.36}$$

- *In the neighborhood of the puncture P_3 (where $y = O(x^{-N-2})$) the vector-function ψ_n has a zero of order n and the leading coefficient $\psi_{n,0}(P_3)$ of its expansion is equal to*

$$\psi_{n,0}(P_3) = \beta_n q_n, \tag{4.37}$$

where the scalar β_n satisfies the recurrence relation

$$\beta_{n+1} = -\frac{1}{(p_n^T q_{n+1})}\beta_n. \tag{4.38}$$

- *In the neighborhood of the puncture P_2 (where $y = 1$) the vector-function ψ_n is regular and its evaluation $\psi_{n,0}(P_2)$ at P_2 is orthogonal to both p_n and p_{n-1}, i.e.,*

$$p_n^T \psi_{n,0}(P_2) = p_{n-1}^T \psi_{n,0}(P_2) = 0. \tag{4.39}$$

Proof. First, we show that for generic moduli, the Bloch function $\psi_0(x, y)$ is regular near each P_α. Observe that $\psi_{N+2}(x, y) = L(x)\psi_0(x, y) = y\psi_0(x, y)$. Now the relation $\psi_{n+1} = L_n(x)\psi_n$ can be inverted to produce

$$\psi_n(x, y) = L_n(x)^{-1}\psi_{n+1}(x, y) = (1 - xq_n p_n^T)\psi_{n+1}(x, y). \tag{4.40}$$

Applying this relation $N + 2$ times, we may write

$$\psi_0(x, y) = y(1 - xq_0 p_0^T) \cdots (1 - xq_{N+1} p_{N+1}^T)\psi_0(x, y). \tag{4.41}$$

Consider first the neighborhood of the point P_3, where y is of order $x^{-(N+2)}$. If $\psi_0(x, y)$ admits the expansion (4.33) near P_3 with $\psi_{0,0}(P_3) \neq 0$, then we must have

$$\psi_{0,0} = (-)^{N+2} q_0(p_0^T q_1) \cdots (p_N^T q_{N+1})(p_{N+1}^T \psi_{0,0}). \tag{4.42}$$

This shows that $\psi_{0,0}(P_3)$ is proportional to the vector q_0, say $\psi_{0,0} = \beta q_0$. Now recall that the Bloch function $\psi_0(x, y)$ satisfies the normalization condition (4.10) throughout. This implies that $\sum_{\alpha=1}^3 \psi_{0,0}(P_3) = 0$ if the order $n_0(P_3)$ of the pole of $\psi_0(x, y)$ at P_3 is positive. For generic values of the moduli of the curve Γ, we may assume that $\sum_{\alpha=1}^3 q_{0\alpha} \neq 0$. It follows that $\beta_0 = 0$ and hence $\psi_{0,0}(P_3) = 0$, which contradicts the definition of $\psi_{0,0}(P_3)$. This shows that $n_0(P_3) = 0$, and the Bloch function $\psi_0(x, y)$ is regular at P_3. The argument near P_1 is similar and even more direct, just using the equation $y\psi_0(x,y) = \psi_{N+2}(x,y) = \prod_{n=0}^{N+1}(1 + xq_n p_n^T)\psi_0(x,y)$. It shows, incidentally, that the leading coefficient $\psi_{0,0}(P_1)$ is proportional to q_{N+1}. At P_2, the regularity of $\psi_0(x,y)$ follows from the regularity of $\psi_0(x,y)$ at the other two points P_1 and P_3, and from the fact that for generic moduli, the determinant (4.19) is regular.

It is now easy to see that the functions $\psi_n(x,y)$ have the zeroes and poles spelled out in Lemma 4.2. The recurrence relations stated there can also be read off the defining relations $\psi_{n+1} = L_n(x)\psi_n(x)$. For example, near P_1, we find

$$x^{n+1}\left(\psi_{n+1,0} + \frac{1}{x}\psi_{n+1,1} + \cdots\right) = x^n(1 + xq_n p_n^T)\left(\psi_{n,0} + \frac{1}{x}\psi_{n,1} + \cdots\right). \tag{4.43}$$

This implies

$$\psi_{n+1,0} = q_n(p_n^T\psi_{n,0}), \tag{4.44}$$

$$\psi_{n+1,1} = \psi_{n,0} + q_n(p_n^T\psi_{n,1}). \tag{4.45}$$

The relations (4.35, 4.36) follow. Near P_2, we write instead

$$x^{-n}\left(\psi_{n,0} + \frac{1}{x}\psi_{n,1} + \cdots\right) = x^{-n-1}(1 - xq_n p_n^T)\left(\psi_{n+1,0} + \frac{1}{x}\psi_{n+1,1} + \cdots\right). \tag{4.46}$$

This implies

$$\psi_{n,0} = -q_n(p_n^T\psi_{n+1,0}), \tag{4.47}$$

$$\psi_{n,1} = \psi_{n+1,0} - q_n(p_n^T\psi_{n+1,1}). \tag{4.48}$$

which gives (4.37, 4.38). Finally near P_2, we get

$$\psi_{n+1,0} + \frac{1}{x}\psi_{n+1,1} + \cdots = \left(1 + xq_n p_n^T)(\psi_{n,0} + \frac{1}{x}\psi_{n,1} + \cdots\right). \tag{4.49}$$

This implies that $p_n^T\psi_{n,0} = 0$. Furthermore, $\psi_{n+1,0} = \psi_{n,0} + q_n(p_n^T\psi_{n,1})$. Multiplying on the left by p_n^T, we conclude that $p_n^T\psi_{n+1,0} = 0$. This establishes (4.39), and Lemma 4.2 is proved. □

4.4. The inverse spectral problem. It is now a standard procedure in the geometric theory of soliton equations to solve the inverse problem using the concept of the Baker–Akhiezer function originally proposed in [22]. The main properties of the Baker–Akhiezer function in our model are the following.

• Let Γ be a Landsteiner–Lopez curve defined by Eq. (2.1). Then for a divisor D of degree $g+2 = 2N+1$ in general position, there exists a unique vector-function $\phi_n(t,Q)$ such that:

(a) $\phi_n(t,Q)$ is meromorphic on Γ outside the punctures P_1, P_3. It has at most simple poles at the points z_i of the divisor D (if all of them are distinct);

(b) In the neighborhood of the punctures P_1 and P_3, it has respectively the form

$$\phi_n = x^n e^{xt} \left(\sum_{k=0}^{\infty} \phi_{n,k}(P_1) x^{-k} \right), \quad Q \to P_1, \tag{4.50}$$

$$\phi_n = x^{-n} e^{-xt} \left(\sum_{k=0}^{\infty} \phi_{n,k}(P_3) x^{-k} \right), \quad Q \to P_3. \tag{4.51}$$

(c) At the points Q_a, $\phi_n(Q)$ is regular, and $\phi_n(Q_a)$ is equal to

$$\phi_{n,\alpha}(t, Q_\beta) = \delta_{\alpha,\beta}. \tag{4.52}$$

The arguments establishing the existence of the Baker–Akhiezer function ϕ_n are well-known, so we shall be brief. First, we recall that as shown in [22] for any algebraic curve with two punctures, any fixed local coordinate in the respective neighborhoods of the punctures, and for any divisor D of degree g there exists a unique (up to a constant factor) function with the analytic properties stated above. Now let (P_1, P_3) be the punctures, and let x^{-1} be the local coordinate near either one of the punctures. We can easily show that if D has degree $g+2$, the dimension of the space of such functions is equal to 3. We form the 3-dimensional vector whose components are just the three independent functions from this space. This 3-dimensional vector is unique up to multiplication by a constant matrix. We fix this matrix by the normalization condition (4.52). This establishes our claim.

The function $\phi_n(t, Q)$ can be written explicitly in terms of the Riemann θ-function associated with Γ. The θ-function is an entire function of $g = 2N-1$ complex variables $z = (z_1, \ldots, z_g)$, and is defined by its Fourier expansion

$$\theta(z_1, \ldots, z_g) = \sum\nolimits_{m \in \mathbf{Z}^g} e^{2\pi i <m,z> + \pi i <\tau m, m>},$$

where $\tau = \tau_{ij}$ is the period matrix of Γ. The θ-function has the following monodromy properties with respect to the lattice $\mathbf{Z}^g + \tau \mathbf{Z}^g$:

$$\theta(z+l) = \theta(z), \qquad \theta(z+\tau l) = \exp[-i\pi < \tau l, l > -2i\pi < l, z >]\, \theta(z),$$

where l is an integer vector, $l \in \mathbf{Z}^g$. The complex torus $J(\Gamma) = \mathbf{C}^g / \mathbf{Z}^g + \tau \mathbf{Z}^g$ is the Jacobian variety of the curve Γ. The Abel transform

$$\Gamma \ni Q \to A_k(Q) = \int_{Q_0}^{Q} d\omega_k$$

imbeds the curve Γ into its Jacobian variety. Here $d\omega_k$ is a basis of g holomorphic differentials, normalized as dual to the A-cycles of a symplectic homology basis for Γ.

According to the Riemann–Roch theorem, for each divisor $D = z_1 + \ldots + z_{g+2}$ in the general position, there exists a unique meromorphic function $r_\alpha(Q)$ with $r_\alpha(Q_\beta) = \delta_{\alpha\beta}$ and D as the divisor of its poles. It can be written explicitly as (see details in [23]):

$$r_\alpha(Q) = \frac{f_\alpha(Q)}{f_\alpha(Q_\alpha)}, \quad f_\alpha(Q) = \theta(A(Q) + Z_\alpha) \frac{\prod_{\beta \neq \alpha} \theta(A(Q) + F_\beta)}{\prod_{m=1}^{l} \theta(A(Q) + S_m)},$$

where

$$F_\beta = -\mathcal{K} - A(Q_\beta) - \sum_{j=1}^{g-1} A(z_j), \quad S_m = -\mathcal{K} - A(z_{g-1+m}) - \sum_{j=1}^{g-1} A(z_j),$$

$$Z_\alpha = Z_0 - A(R_\alpha), \qquad Z_0 = -\mathcal{K} - \sum_{j=1}^{g+2} A(z_j) + \sum_{\alpha=1}^{3} A(Q_\alpha),$$

where $\mathcal{K}$ is the vector of Riemann constants.

Let $d\Omega_0$ and $d\Omega_1$ be the unique normalized meromorphic differentials on Γ, which are holomorphic outside P_1 and P_3, and with the property that $d\Omega_0$ has simple poles at the punctures with residues ∓ 1, $d\Omega_1$ is regular at P_3, and has the form $d\Omega_1 = dx(1 + O(x^{-1}))$ at P_1. The normalization means that the differentials have zero periods around A-cycles

$$\oint_A d\Omega_0 = \oint_A d\Omega_1 = 0.$$

We observe that the differential $d\Omega_1^\sigma(Q) = d\Omega(Q^\sigma)$ has a pole only at P_3, and is there of the form $-dx(1 + O(x^{-1}))$.

Let V and U be the vectors whose components are the B-periods of the differentials $d\Omega_0$ and $d\Omega_1$ respectively

$$V = \frac{1}{2\pi i} \oint_B d\Omega_1, \; U = \frac{1}{2\pi i} \oint_B d\Omega_0.$$

The Baker–Akhiezer function $\phi_n(t, Q)$ is given by

$$\phi_{n,\alpha}(t, Q) = r_\alpha(Q) \frac{\theta(A(Q) + tU^+ + nV + Z_\alpha)\, \theta(Z_0)}{\theta(A(Q) + Z_\alpha)\, \theta(tU^+ + nV + Z_0)} \exp\left(\int_{Q_\alpha}^{Q} n d\Omega_0 + t d\Omega^+ \right), \tag{4.53}$$

where $d\Omega^+ = d\Omega_1 + d\Omega_1^\sigma$ and $U^+ = U + U^\sigma$.

• The Baker–Akhiezer function ϕ_n is a Bloch function, in the sense that

$$\phi_{N+2+n}(t, Q) = y\phi_n(t, Q). \tag{4.54}$$

This is just a consequence of the fact that both sides of the equation satisfy the criteria for the Baker–Akhiezer function, and that the Baker–Akhiezer function is unique. Similarly, the uniqueness of the Baker–Akhiezer function implies that, if the divisor D is equivalent to D^σ, then the function ϕ_n satisfies

$$\phi_n(t, Q) = g_0 \phi_{-n}(t, Q^\sigma) f(Q), \tag{4.55}$$

where $f(Q)$ is a function with poles at γ_s and zeros at γ_s^σ. Without loss of generality, we may assume that $f(Q_1) = f(Q_3) = -f(Q_2) = 1$.

• Let $\phi_n(t, Q)$ be the Baker–Akhiezer function corresponding to Γ and the divisor D of degree $2g + 2$. Let $p_n(t)$ be a vector orthogonal to $\phi_{n,0}(P_3, t)$ (the leading term in the expansion (4.50)), and to $\phi_n(t, P_2)$, i.e.

$$p_n^T \phi_{n,0}(P_3) = p_n^T \phi_n(t, P_2) = 0, \tag{4.56}$$

and q_n be the vector

$$q_n = \frac{\phi_{n,0}(P_1)}{p_n^T \phi_{n-1,0}(P_1)}. \tag{4.57}$$

The vector functions p_n, q_n are then $(N+2)$-periodic and mutually orthogonal, and they satisfy the contraint (4.31). As can be expected from the gauge invariance (1.7) in the direct problem, the functions $p_n(t)$ and $q_n(t)$ which we obtain this way are defined only up to a multiplier $\mu_n(t)$. However, the operators L_n and $M_n(x)$ are uniquely defined by the expression (4.3). Furthermore, again by uniqueness of the Baker–Akhiezer function, the Baker–Akhiezer function $\phi_n(Q)$ satisfies

$$\psi_{n+1}(t, Q) = L_n(x)\psi_n(t, Q), \ \ (\partial_t - M_n(x))\psi_n(t, Q) = 0. \tag{4.58}$$

Thus the vector function $(q_n(t), p_n(t))$ is a solution of the dynamical system (1.6). If the equivalence class of the divisor D is invariant with respect to σ, then (p_n, q_n) satisfies in addition the relation (1.4).

• The Baker–Akhiezer function $\phi_n(t, Q)$ satisfies the same defining Bloch property (4.54) as the Bloch function $\psi_n(Q)$, except for the different normalizations, which is (4.52) in the case of $\phi_n(t, Q)$ and (4.10) in the case of $\psi_n(Q)$. It follows that

$$\psi_n(t, Q) = r^{-1}(t, Q)\phi_n(t, Q), \ \ r(t, Q) = \sum_{\alpha=1}^{3} \phi_{0,\alpha}(Q) \tag{4.59}$$

is a Bloch solution of (4.9) normalized by the condition (4.10). This leads to the following description of the dynamical system (1.6).

Let $p_n(t), q_n(t)$ be vector functions (subject to constraints (1.3, 1.4, 4.31)) which satisfy Eqs. (1.6). Then the t-dependence of the divisor D under the map (4.1)

$$(p_n(t), q_n(t)) \longmapsto \{\Gamma, D(t) = \sum_{j=1}^{2N+1} z_j(t)\} \tag{4.60}$$

coincides with the dynamics of the zeroes of the function $r(t, Q)$ given by (4.59). The dynamics of the Bloch eigenfunction of (4.9) (i.e. normalized by (4.10)) are described by

$$(\partial_t - M_n(t, x))\psi_n(t, Q) = \mu(t, Q)\psi_n(t, Q), \quad \mu = -\partial_t \ln r(t, Q). \tag{4.61}$$

We observe that the linearization of the equations of motion on the Jacobian of the curve is a direct corollary of the linear dependence on t of the exponential factor in the expansion of $\psi_n(t, Q)$ near the punctures.

• As we saw earlier, the normalization (4.31) can be achieved by the action (1.8) of a subgroup of matrices W which commutes with g_0. In order to get $\mathcal{M}$, we have to consider in addition the action of diagonal matrices W. The basic observation is the following:

Let the Baker–Akhiezer functions $\psi_n(t, Q)$ and $\psi'(t, Q)$ correspond to equivalent divisors D and D', respectively. Then

$$\psi_n'(t, Q) = W\psi_n(t, Q)h(Q), \tag{4.62}$$

where $h(Q)$ is a function with poles at D and zeros at D', and W is a diagonal matrix $W = h^{-1}(Q_\alpha)\delta_{\alpha,\beta}$. To establish (4.62), it suffices to check that both sides of the equation have the same analytical properties. Equation (4.62) implies that the vectors p_n, q_n defined by equivalent divisors are related by a transformation (1.8) with a diagonal matrix W. Altogether, we have established the following part of the Main Theorem of Sect. 1:

Theorem 1. *The map (4.1) identifies the reduced phase space $\mathcal{M}$ with a bundle over the space of algebraic curves Γ defined by (2.1) with $f_N(x)$ of the form (2.2). At generic data, the map has bijective differential. The fiber of the bundle is the Jacobian $J(\Gamma_0)$ of the factor-curve $\Gamma_0 = \Gamma/\sigma$,*

$$\mathcal{M} = \{\Gamma, [D] \in J(\Gamma_0)\}. \tag{4.63}$$

5. Hamiltonian Theory and Seiberg–Witten Differential: The Even Divisor Model

We come now to the crucial issue of how to determine the symplectic forms with respect to which the system (1.6) is Hamiltonian. For this, we rely on the Hamiltonian approach proposed in [6] and [7] for general soliton equations expressible in terms of Lax or Zakharov–Shabat equations. This approach was effective in the study of gauge theories with matter in the fundamental representation. Further applications were given in [10] and [24]. We review its main features.

5.1. The symplectic forms in terms of the Lax operator. In order to find the Hamiltonian structure of the equations starting with the Lax operator, we need to identify a two-form on the phase space $\mathcal{M}$ of vectors (q_n, p_n), written in term of the Lax operator $L(x)$. Candidates for such two-forms are

$$\omega_{(m)} = \frac{1}{2}\sum_{\alpha=1}^{3} \mathrm{Res}_{P_\alpha} \left\langle \Psi_{n+1}^*(Q)\delta L_n(x) \wedge \delta\Psi_n(Q)\right\rangle \frac{dx}{x^m}. \tag{5.1}$$

The various expressions in this equation are defined as follows. The notation $\langle f_n\rangle$ stands for a sum over one period of the periodic function f_n:

$$\langle f_n\rangle = \sum_{n=0}^{N+1} f_n. \tag{5.2}$$

The expression $\psi_n^*(Q)$ is the dual Baker–Akhiezer function, which is the row-vector solution of the equation

$$\psi_{n+1}^*(Q)L_n(z) = \psi_n^*(Q), \quad \psi_{N+2}^*(Q) = y^{-1}\psi_0^*(Q), \tag{5.3}$$

normalized by the condition

$$\psi_0^*(Q)\psi_0(Q) = 1. \tag{5.4}$$

Note that (4.9) and (5.3) imply that $\psi_{n+1}^*\psi_{n+1} = \psi_{n+1}^*(L_n(x)\psi_n) = (\psi_{n+1}^*L_n(x))\psi_n = \psi_n^*\psi_n$ does not depend on n. We would also like to emphasize that, unlike the Bloch function $\psi_n(Q)$ which does not have n-independent zeroes, the normalization (5.4) allows the dual Bloch function $\psi_n^*(Q)$ to have such zeroes. In fact, they occur at the poles of $\psi_n(Q)$.

In (5.1), the differential δ denotes the exterior differential with respect to the moduli parameters of $\mathcal{M}$. (This is in order to distinguish δ from the differential d, which is the exterior differential on the surface Γ.) Thus the external differential $\delta L_n(z)$ can be viewed as a one-form on $\mathcal{M}$, valued in the space of operator-valued meromorphic functions on Γ. Similarly the Bloch function $\psi_n(Q)$ and dual Bloch functions $\psi_n^*(Q)$ are functions on $\mathcal{M}$, valued respectively in the space of column-vector-valued and the space of row-vector-valued meromorphic functions on Γ. It follows that $\delta\psi_n(Q)$ is a one-form on $\mathcal{M}$, valued in the space of column-vector-valued meromorphic functions on Γ. The expression $\psi_{n+1}^*\delta L_n(x)\wedge\delta\psi_n(x)$ is then a two-form on $\mathcal{M}$, valued in the space of meromorphic functions on Γ, and for each m integer, the expression

$$\Omega_{(m)} = \left\langle \psi_{n+1}^*\delta L_n(x)\wedge\delta\psi_n(x)\right\rangle \frac{dx}{x^m} \tag{5.5}$$

is a meromorphic 1-form on Γ. This justifies (5.1) as a two-form on $\mathcal{M}$.

In (5.1), we have allowed for a later choice of an integer m. We shall see shortly that holomorphicity requirements restrict to $0\le m\le 2$, and that the symplectic form of the $\mathcal{N}=2$ SUSY with a hypermultiplet in the anti-symmetric representation is obtained by setting $m=0$.

Sometimes it is useful to think of the symplectic form ω as

$$\omega_{(m)} = \frac{1}{2}\mathrm{Res}_{x=\infty}\,\mathrm{Tr}\left\langle\left(\Psi_{n+1}^{-1}(x)\delta L_n(x)\wedge\delta\Psi_n(x)\right)\right\rangle\frac{dx}{x^m}, \tag{5.6}$$

where $\Psi_n(x)$ is a matrix with the columns $\psi_n(Q_j(x)),\ Q_j(x)=(x,y_j)$ corresponding to different sheets of Γ. The matrix $\Psi_n(x)$ is of course not defined globally. Note that $\psi_n^*(Q)$ are the rows of the matrix $\Psi_n^{-1}(x)$. That implies that $\Psi_n^*(Q)$ as a function on the spectral curve is meromorphic outside the punctures, has poles at the branching points of the spectral curve, and zeroes at the poles z_j of $\Psi_n(Q)$. These analytical properties will be crucial in the sequel.

5.2. The symplectic forms in terms of x and y. A remarkable property of the symplectic form defined by (5.1) in terms of the Lax operator $L(x)$ is that it can, under quite general circumstances, be rewritten in terms of the meromorphic functions x and y on the spectral curve Γ. More precisely, we have

$$\omega_{(m)} = -\sum_{i=1}^{2N+1}\delta\ln y(z_i)\wedge\frac{\delta x}{x^m}(z_i). \tag{5.7}$$

The meaning of the right-hand side of this formula is as follows. The spectral curve is equipped by definition with the meromorphic functions $y(Q)$ and $x(Q)$. Their evaluations $x(z_i),\ y(z_i)$ at the points z_i define functions on the space $\mathcal{M}$, and the wedge product of their external differentials is a two-form on $\mathcal{M}$.

The proof of the formula (5.7) is very general and does not rely on any specific form of L_n. For the sake of completeness we present it here in detail, although it is very close to the proof of Lemma 5.1 in [24].

Recall that the expression $\Omega_{(m)}$ defined in (5.5) is a meromorphic differential on the spectral curve Γ. Therefore, the sum of its residues at the punctures P_α is equal to the opposite of the sum of the other residues on Γ. For $m \leq 2$, the differential $\Omega_{(m)}$ is regular at the points situated over $x=0$, thanks to the normalization (4.31), which insures that $\delta\psi_n(Q) = O(x)$. Otherwise, it has poles at the poles z_i of $\psi_n(Q)$ and at the branch points s_i, where we have seen that $\psi^*_{n+1}(Q)$ has poles. We analyze in turn the residues at each of these two types of poles.

First, we consider the poles z_i of $\psi_n(Q)$. By genericity, these poles are all distinct and of first order, and we may write

$$\psi_n \equiv \psi_{n,0}(z_i)\frac{1}{x-x(z_i)} + \cdots . \tag{5.8}$$

It follows that $\delta\psi_n$ has a pole of second order at z_i

$$\delta\psi_n = \psi_{n,0}(z_i)\frac{\delta x(z_i)}{(x-x(z_i))^2} + \cdots . \tag{5.9}$$

In view of the fact that ψ^*_{n+1} has a simple zero at z_i and hence can be expressed as

$$\psi^*_{n+1} \equiv \psi^*_{n+1,0}(x-x(z_i)) + \cdots , \tag{5.10}$$

we obtain

$$\mathrm{Res}_{z_i}\,\Omega_{(m)} = \left\langle \psi^*_{n+1,0}\delta L_n\psi_n\right\rangle \wedge \frac{\delta x}{x^m}(z_i) = \left\langle \psi^*_{n+1}\delta L_n\psi_n\right\rangle \wedge \frac{\delta x}{x^m}(z_i). \tag{5.11}$$

The key observation now is that the right-hand side can be rewritten in terms of the monodromy matrix $L(x)$. In fact, the recursive relations (4.9) and (5.3) imply that

$$\left\langle \psi^*_{n+1}\delta L_n\psi_n\right\rangle = \left\langle \psi^*_{N+2}\left(\prod_{m=n+1}^{N+1} L_m\right)\delta L_n\left(\prod_{m=0}^{n-1} L_m\right)\psi_0\right\rangle \tag{5.12}$$

$$= \sum_{n=0}^{N+1}\psi^*_{N+2}\left(\prod_{m=n+1}^{N+1} L_m\right)\delta L_n\left(\prod_{m=0}^{n-1} L_m\right)\psi_0 \tag{5.13}$$

$$= \psi^*_{N+2}\delta L\psi_0 = \psi_0\delta\ln y\psi_0. \tag{5.14}$$

In the last equality, we have used the standard formula for the variation of the eigenvalue of an operator, $\psi^*_0\delta L\psi_0 = \psi^*_0\delta y\psi_0$. Altogether, we have found that

$$\mathrm{Res}_{z_i}\,\Omega_{(m)} = \delta\ln y(z_i)\wedge\frac{\delta x}{x^m}(z_i). \tag{5.15}$$

The second set of poles of $\Omega_{(m)}$ is the set of branching points s_i of the cover. The pole of ψ^*_n at s_i cancels with the zero of the differential dx, $dx(s_i)=0$, considered as a differential on Γ. The vector-function ψ_n is holomorphic at s_i. However, $\delta\psi_n$ can develop a pole as we see below. If we take an expansion of ψ_n in the local coordinate $(x-x(s_i))^{1/2}$ (in general position when the branching point is simple) and consider its variation we get

$$\psi_n = \psi_{n,0} + \psi_{n,\pm}(x-x(s_i))^{1/2} + \cdots , \tag{5.16}$$

$$\delta\psi_n = -\frac{1}{2}\psi_{n,\pm}\frac{\delta x(s_i)}{(x-x(s_i))^{1/2}} + \cdots . \tag{5.17}$$

Comparing with $\frac{d\psi_n}{dx} = \frac{1}{2}\psi_{n,\pm}\frac{1}{(x-x(s_i))^{1/2}} + \cdots$, we may write

$$\delta\psi_n = -\frac{d\psi_n}{dx}\delta x(s_i) + O(1). \tag{5.18}$$

This shows that $\delta\Psi_n$ has a simple pole at s_i. Similarly, we may write

$$\delta y = -\frac{dy}{dx}\delta x(s_i) + O(1). \tag{5.19}$$

The identities (5.18) and (5.19) imply that

$$\mathrm{Res}_{s_i}\,\Omega_{(m)} = \mathrm{Res}_{s_i}\left[\left\langle\psi^*_{n+1}\delta L_n d\psi_n\right\rangle \wedge \frac{\delta y\,dx}{x^m dy}\right]. \tag{5.20}$$

Arguing as for (5.12), this can be rewritten as

$$\mathrm{Res}_{s_i}\,\Omega_{(m)} = \mathrm{Res}_{s_i}\left[\left(\psi^*_{N+2}\delta L d\psi_0\right) \wedge \frac{\delta y dx}{x^m dy}\right]. \tag{5.21}$$

Due to the antisymmetry of the wedge product, we may replace δL in (5.21) by $(\delta L - \delta y)$. Then using the identities

$$\psi^*_{N+2}(\delta L - \delta y) = \delta\psi^*_{N+2}(y - L), \tag{5.22}$$

$$(y - L)d\psi_0 = (dL - dy)\psi_0, \tag{5.23}$$

which result from $\psi^*_{N+2}(L - y) = (L - y)\psi_0 = 0$, we obtain

$$\mathrm{Res}_{s_i}\,\Omega = \mathrm{Res}_{s_i}\left(\delta\psi^*_{N+2}(dL - dy)\psi_0\right) \wedge \frac{\delta y dx}{x^m dy}. \tag{5.24}$$

Now the differential dL does not contribute to the residue, since $dL(s_i) = 0$. Furthermore, $\psi^*_{N+2}\psi_0 = y^{-1}\psi^*_0\psi_0 = y^{-1}$. Thus $\delta\psi^*_{N+2}\psi_0 = -\psi^*_{N+2}\delta\psi_0 - y^{-2}\delta y$. Exploiting again the antisymmetry of the wedge product, we arrive at

$$\mathrm{Res}_{s_i}\,\Omega = \mathrm{Res}_{s_i}\left(\psi^*_{N+2}\delta\psi_0\right) \wedge \delta y\frac{dx}{x^m}. \tag{5.25}$$

Recall that we have normalized the Bloch function $\psi_0(Q)$ at $x = 0$ by (4.31), and that near $x = 0$, the function y is of the form (2.4). Thus $\delta\psi_0 = O(x)$ and $\delta y = O(x)$ near $x = 0$, and the differential form

$$\left(\psi^*_{N+2}\delta\psi_0\right) \wedge \delta y\frac{dx}{x^m} \tag{5.26}$$

is holomorphic at $x = 0$ for $0 \le m \le 2$. It is manifestly holomorphic at all the other points of Γ, except at the branching points s_i and the poles $z_1, \cdots, z_{2N+1}$. Therefore

$$\sum_{s_i}\mathrm{Res}_{s_i}\left(\psi^*_{N+2}\delta\psi_0\right) \wedge \delta y\frac{dx}{x^m} = -\sum_{i=1}^{2N+1}\mathrm{Res}_{z_i}\left(\psi^*_{N+2}\delta\psi_0\right) \wedge \delta y\frac{dx}{x^m}. \tag{5.27}$$

Using again the expressions (5.16, 5.18) for ψ_0 and $\delta\psi_0$, and the fact that $\psi^*_{N+2} = y^{-1}\psi^*_0$, the right-hand side of (5.27) can be recognized as

$$\sum_{i=1}^{2N+1} \delta \ln y(z_i) \wedge \frac{\delta x(z_i)}{x^m(z_i)}. \tag{5.28}$$

The sum of (5.15) and (5.28) gives (5.7), since

$$2\omega_{(m)} = -\sum_{i=1}^{2N} \mathrm{Res}_{z_i}\Omega_{(m)} - \sum_{s_i} \mathrm{Res}_{s_i}\Omega_{(m)}. \tag{5.29}$$

The identity (5.7) is proved.

5.3. Action-angle variables and Seiberg–Witten differential. The expression (5.7) for the symplectic form $\omega_{(m)}$ suggests its close relation with the following one-form on Γ:

$$d\lambda_{(m)} = \ln y\, \frac{dx}{x^m}. \tag{5.30}$$

Strictly speaking, the form $d\lambda_{(m)}$ is not a meromorphic differential in the usual sense, because of the multiple-valuedness of $\ln\, y$. However, the ambiguities in $\ln\, y$ are fixed multiples of $2\pi i$, which disappear upon differentiation. Thus, the form $d\lambda_{(m)}$ is no different from the usual meromorphic differentials, as far as the construction of symplectic forms is concerned. Also, the form $d\lambda_{(m)}$ and the form $\frac{1}{m-1}x^{-m+1}\frac{dy}{y}$ (for $m \neq 1$; for $m = 1$, $-(\ln\, x)\frac{dy}{y}$) differ by an exact differential, and we shall not distinguish between them. From this point of view, the Seiberg–Witten form (1.1) can be identified with the form $-d\lambda_{(0)}$.

Our spin chain model has led so far to a $2N$-dimensional phase space $\mathcal{M}$, equipped with several candidate symplectic forms $\omega_{(m)}$, $1 \le m \le 2$. We still have to reduce $\mathcal{M}$ to a $(2N-2)$-dimensional phase space, and to identify the correct symplectic form. Remarkably, both selections are tied to a key physical requirement for the one-form which corresponds to the Seiberg–Witten of a $\mathcal{N} = 2$ SUSY gauge theory, namely the holomorphicity of its variations under moduli deformations.

It is an important feature of $\mathcal{N} = 2$ Yang–Mills theories that the masses of the theory are not renormalized. Since the masses of the theory correspond to the poles of the Seiberg–Witten differential $d\lambda$, it follows that $\delta d\lambda$ must be holomorphic. Thus we need to examine the poles of $\delta\, d\lambda = \delta \ln y \frac{dx}{x^m}$, and identify the subvarieties of $\mathcal{M}$ along which $\delta d\lambda$ is holomorphic. There are 3 such subvarieties, corresponding to the choices of m:

- On the variety $\mathcal{M}_2 = \mathcal{M} \cap \{u_0 = c_0,\ u_1 = c_1\}$, the differential $\delta\, d\lambda_{(2)} = (\delta \ln y)\frac{dx}{x^2}$ has no pole at Q_α, since $y = 1 + O(x^2)$ near $x = 0$. On the other hand, the differential $\frac{dx}{x^2}$ vanishes at $x = \infty$, so $\delta \ln y \frac{dx}{x^2}$ is also holomorphic there, and $\delta d\lambda_{(2)}$ is holomorphic.
- On the variety $\mathcal{M}_0 = \mathcal{M} \cap \{u_N = 1,\ u_{N-1} = 0\}$, the differential $\delta\, d\lambda_{(0)} = (\delta \ln y)dx$ is automatically holomorphic at $x = 0$. Near ∞, in view of the expansion () for y, we have $\delta \ln y = O(x^2)$ if we vary only the moduli within $\mathcal{M}_2$. Thus $\delta d\lambda_{(0)}$ is holomorphic.
- On the variety $\mathcal{M}_1 = \mathcal{M} \cap \{u_N = 1\}$, the differential $\delta\, d\lambda_{(1)} = (\delta \ln y)\frac{dx}{x}$ is still holomorphic, because $\delta\, \ln y = O(x)$. Near $x = \infty$, the sole constraint $\{u_{N-1} = 1\}$ suffices to guarantee that $\delta \ln y = O(\frac{1}{x})$. Thus $\delta d\lambda_{(1)}$ is holomorphic.

When m and hence $d\lambda_{(m)}$ is even under the involution σ, action-angle variables can be introduced as follows. Restricted to $\mathcal{M}_{(m)}$, $\delta d\lambda_{(m)}$ is holomorphic, and hence can be expressed for suitable coefficients δa_i as

$$\delta d\lambda_{(m)} = \sum_{i=1}^{2N-1} (\delta a_i) d\omega_i, \tag{5.31}$$

where $d\omega_i$ is a basis of $2N-1$ holomorphic one-forms on Γ. Since $d\lambda_{(m)}$ is even, only holomorphic one-forms $d\omega_i$ which are even can occur on the right-hand side. We identify such forms with forms on Γ/σ. We choose a symplectic homology basis A_i, B_i and a dual basis of holomorphic forms $d\omega_i$, $1 \leq i \leq N-1$, for the factor curve Γ/σ. The variables a_i and a_{Di} can then be defined by

$$a_i = \oint_{A_i} d\lambda_{(m)}, \quad a_{Di} = \oint_{B_i} d\lambda_{(m)}. \tag{5.32}$$

The interpretation of the variables a_i is as action variables from the viewpoint of the spin model and as vacuum moduli from the viewpoint of the $\mathcal{N}=2$ SUSY gauge theory. Evidently, their variations coincide with the δa_i of Eq. (5.31).

Next, the angle variables ϕ_i, $1 \leq i \leq N-1$, are defined by

$$D = \{z_1, \cdots, z_{2N+1}\} \longmapsto \phi_i = \sum_{j=1}^{2N+1} \int^{z_j} d\omega_i. \tag{5.33}$$

We claim now that, for m even, the symplectic form $\omega_{(m)}$ is a genuine symplectic form when restricted to $\mathcal{M}_{(m)}$, and that a_i and ϕ_i as defined above are action-angle coordinates for $\omega_{(m)}$

$$\omega_{(m)} = \sum_{i=1}^{N-1} \delta a_i \wedge \delta\phi_i \quad \text{on } \mathcal{M}_{(m)}. \tag{5.34}$$

To see this, we evaluate the two-form $\delta\big(\sum_{j=1}^{2N+1} \int_{Q_0}^{z_j} \delta\, d\lambda\big)$ in two different ways. Substituting in (5.31), we find that it is equal to

$$\delta(\sum_{i=1}^{N-1} \delta a_i\, \phi_i) = \sum_{i=1}^{N-1} \delta\phi_i \wedge \delta a_i. \tag{5.35}$$

On the other hand, we can also write

$$\delta\left(\sum_{j=1}^{2N+1} \int_{Q_0}^{z_j} \delta\, d\lambda\right) = \delta\left(\sum_{j=1}^{2N+1} \int_{Q_0}^{z_j} (\delta \ln\, y)\frac{dx}{x^m}\right) = \sum_{j=1}^{2N+1} \frac{\delta x(z_j)}{x^m(z_j)} \wedge (\delta \ln\, y)(z_j). \tag{5.36}$$

Comparing the two formulas, and making use of (5.7), we obtain the desired equation (5.34).

We observe that for the present even divisor spin model, the space $\mathcal{M}_1$ and the form $d\lambda_{(1)}$ are not applicable. In fact, there are difficulties with both the dimension of $\mathcal{M}_1$

which is odd, and the angle variables ϕ_i defined by (5.33), which would vanish identically because the class of the divisor D is even.

For the $\mathcal{N} = 2$ SUSY Yang–Mills theory with a hypermultiplet in the antisymmetric representation, the spectral curves are given by $\mathcal{M}_0$. The symplectic form is then $\omega_{(0)}$, which provides an independent check of the choice of Seiberg–Witten form found by Landsteiner and Lopez.

5.4. The Hamiltonian of the Flow. We show now that the even divisor spin model is a Hamiltonian system. More precisely, restricted to each of the phase spaces $\mathcal{M}_{(0)}$ or $\mathcal{M}_{(2)}$, the system is Hamiltonian with the corresponding symplectic form, with a corresponding Hamiltonian. We would like to stress that, once again, the arguments to these ends are quite general, and use only the expression for $\omega_{(m)}$ in terms of the Lax operator.

Lemma 5.1. *Let m be either 0 or 2. Then Eqs. (1.6) restricted on $\mathcal{M}_{(m)}$ are Hamiltonian with respect to the symplectic form $\omega_{(m)}$ given by (5.1). The Hamiltonians $H_{(m)}$ are given by*

$$H_{(0)} = u_{N-2}, \tag{5.37}$$

$$H_{(2)} = \ln u_N = \sum_{n=0}^{N+1} \ln(p_n^+ q_{n-1}) = \frac{1}{2}\sum_{n=0}^{N+1} \ln[(p_n^+ q_{n-1})(p_{n-1}^+ q_n)]. \tag{5.38}$$

Proof. By definition, a vector field ∂_t on a symplectic manifold is Hamiltonian, if its contraction $i_{\partial_t}\omega(X) = \omega(X, \partial_t)$ with the symplectic form is an exact one-form $\delta H(X)$. The function H is the Hamiltonian corresponding to the vector field ∂_t. Thus

$$i_{\partial_t}\omega_{(m)} = \frac{1}{2}\sum_\alpha \mathrm{Res}_{P_\alpha}\left(\left\langle \psi_{n+1}^* \delta L_n \dot{\psi}_n\right\rangle - \left\langle \psi_{n+1}^* \dot{L}_n \delta\psi_n\right\rangle\right)\frac{dx}{x^m}. \tag{5.39}$$

Now under the flow (1.6), the Lax operators $L_n(x)$ flow according to the Lax equation (4.4), while the Bloch function ψ_n flow according to (4.61). Consequently,

$$i_{\partial_t}\omega_{(m)} = \frac{1}{2}\sum_\alpha \mathrm{Res}_{P_\alpha}$$
$$\left(\left\langle \psi_{n+1}^* \delta L_n (M_n + \mu)\psi_n\right\rangle - \left\langle \psi_{n+1}^* (M_{n+1}L_n - L_n M_n)\delta\psi_n\right\rangle\right)\frac{dx}{x^m}. \tag{5.40}$$

Since $L_n\psi_n = \psi_{n+1}$, it follows that

$$\psi_{n+1}^* M_{n+1} L_n \delta\psi_n = \psi_{n+1}^* M_{n+1}\psi_{n+1} - \psi_{n+1}^* M_{n+1}\delta L_n \psi_n.$$

Upon averaging in n, we obtain

$$\left\langle \psi_{n+1}^*(M_{n+1}L_n - L_n M_n)\delta\psi_n\right\rangle = -\left\langle \psi_{n+1}^* M_{n+1}\delta L_n \psi_n\right\rangle. \tag{5.41}$$

For all n, both $\delta L_n(x)$ and $M_n(x)$ vanish at $x = 0$. The differential form

$$\left\langle \psi_{n+1}^* \left(\delta L_n M_n + M_{n+1}\delta L_n\right)\psi_n\right\rangle \frac{dx}{x^m} \tag{5.42}$$

is thus holomorphic at $x = 0$, in both cases $m = 0$ and $m = 2$. As we have seen, outside of $x = \infty$, the poles of ψ_{n+1}^* are at the branch ponits and are cancelled by the zeroes of dx there, while the poles of ψ_n are cancelled by the zeroes of ψ^*_{n+1}. Thus the above differential form is holomorphic outside of $x = 0$. The sum of its residues at P_α must be zero

$$\sum_\alpha \mathrm{Res}_{P_\alpha} \left\langle \psi^*_{n+1} \left(\delta L_n M_n + M_{n+1} \delta L_n\right) \psi_n \right\rangle \frac{dx}{x^m} = 0. \tag{5.43}$$

The expression (5.40) for $i_{\partial_t}\omega_{(m)}$ reduces to

$$i_{\partial_t}\omega_{(m)} = \frac{1}{2} \sum_\alpha \mathrm{Res}_{P_\alpha} \left(\langle \psi^*_{n+1} \delta L_n \psi_n \rangle \mu(Q, t) \right) \frac{dx}{x^m}. \tag{5.44}$$

Applying the arguments leading to (5.12), we obtain

$$i_{\partial_t}\omega_{(m)} = \frac{1}{2} \sum_\alpha \mathrm{res}_{P_\alpha} \delta(\ln y) \mu(t, Q) \frac{dx}{x^m}\ . \tag{5.45}$$

As follows from (4.50,4.51), and (4.61) the function $\mu(t, Q)$ is holomorphic at P_2, while it has the following expansion at the punctures P_1, P_3:

$$\mu(t, Q) = -x + O(1), \quad Q \to P_1; \quad \mu(t, Q) = x + O(1), \quad Q \to P_3. \tag{5.46}$$

We consider now the cases $m = 2$ and $m = 0$ separately. When $m = 2$, the form $\mu \frac{dx}{x^2}$ is regular at P_2, and has simple poles with opposite residues at P_1 and P_3. Since $\delta \ln y = \delta u_N + O(\frac{1}{x})$ near P_1, it follows immediately that

$$i_{\partial_t}\omega_{(2)} = \delta(\ln u_N). \tag{5.47}$$

When $m = 0$, we observe that the form $(\delta \ln y)dx$ is regular at $x = \infty$. Indeed, the constraints $u_N = 1$, $u_{N-1} = 0$ defining the phase space $\mathcal{M}_0$ in this case imply that $\delta \ln y = O(\frac{1}{x^2})$ near all three points P_1, P_2, and P_3. For P_1 and P_3, this statement is a direct consequence of (2.7) and (2.8). For P_2, this follows from the fact that three roots y_α of the Landsteiner–Lopez curve (2.1) must satisfy $\prod_{\alpha=1}^3 y_\alpha = 1$. Returning to the residues in (5.45), we see that the point P_2 does not contribute. As for the points P_1 and P_3, they contribute exactly the coefficient u_{N-2} in the expansions (2.7) and (2.8) for y,

$$i_{\partial_t}\omega_{(0)} = \delta u_{N-2}. \tag{5.48}$$

The lemma is proved. $\square$

5.5. The symplectic form in terms of (p_n, q_n). The expression (5.1) for the symplectic forms $\omega_{(m)}$ in terms of the Lax operator also provides a straightforward way of writing $\omega_{(m)}$ in terms of the dynamical variables (q_n, p_n). Such an expression for the form $\omega_{(0)}$ appears complicated. But it is quite simple for the form $\omega_{(2)}$, and we derive it here.

We have $\delta L_n = x\, \delta(q_n p_n^T)$, and the contributions of the three points P_a above $x = \infty$ can be evaluated as follows.

At the point P_1, $y = O(x^{N+2})$, $\psi_n = O(x^n)$, $\psi_{n+1} = O(x^{-(n+1)})$, and thus the differential $\langle \psi^*_{n+1} \delta L_n \wedge \delta \psi_n \rangle \frac{dx}{x^2}$ is regular. The residue at P_1 vanishes.

At the point P_2, ψ_n and ψ^*_{n+1} are regular. Using the same notation as in (4.33), we write

$$\psi_n = \psi_{n,0} + \psi_{n,1} x^{-1} + \cdots , \tag{5.49}$$

$$\psi^*_{n+1} = \psi^*_{n+1,0} + \psi^*_{n+1,1} x^{-1} + \cdots . \tag{5.50}$$

In analogy with (ref), from the equation

$$\psi^*_{n+1} = \psi^*_n L_n(x)^{-1} = \psi^*_n (1 - x q_n p_n^T), \tag{5.51}$$

it follows that

$$\psi^*_{n,0} q_n = \psi^*_{n+1,0} q_n = 0. \tag{5.52}$$

The residue at P_2 is then readily identified

$$\mathrm{Res}_{P_2} \left\langle \psi^*_{n+1} \delta L_n \wedge \delta \psi_n \right\rangle \frac{dx}{x^2} = \mathrm{Res}_{P_2} \langle \psi^*_{n+1,0} \delta (q_n p_n^T) \wedge \delta \psi_{n,0} \rangle \frac{dx}{x} \tag{5.53}$$

$$= - \langle \psi^*_{n+1,0} \delta q_n \wedge (\delta p_n^T) \psi_{n,1} \rangle \tag{5.54}$$

$$\equiv I. \tag{5.55}$$

At the point P_3, $y = O(x^{-N-2})$, and

$$\psi_n = \psi_{n,0} x^{-n} + \psi_{n,1} x^{-n-1} + \cdots , \tag{5.56}$$

$$\psi^*_{n+1} = \psi^*_{n+1,0} x^{n+1} + \psi^*_{n+1,1} x^n + \cdots . \tag{5.57}$$

It follows that the residue is given by

$$\mathrm{Res}_{P_3} \left\langle \psi^*_{n+1} \delta L_n \wedge \delta \psi_n \right\rangle \frac{dx}{x^2}$$
$$= \left[\psi^*_{n+1,0} \delta (q_n p_n^T) \wedge \delta \psi_{n,1} + \psi^*_{n+1,1} \delta (q_n p_n^T) \wedge \delta \psi_{n,0} \right] \tag{5.58}$$

We now make use of Eq. (5.51) to derive recursion relations between the coefficients of ψ^*_n,

$$\psi^*_{n+1,0} = - \psi^*_{n,0} q_n p_n^T , \quad \psi^*_{n+1,1} = \psi^*_{n,0} - \psi^*_{n,1} q_n p_n^T . \tag{5.59}$$

They imply that

$$\psi^*_{n+1,0} q_n = 0, \quad \psi^*_{n+1,1} q_n = \psi^*_{n,0} q_n . \tag{5.60}$$

As a consequence, the first term on the right-hand side of () simplifies to

$$\psi^*_{n+1,0} \delta (q_n p_n^T) \wedge \delta \psi_{n,1} = \psi^*_{n+1,0} \delta q_n \wedge p_n^T \delta \psi_{n,1} . \tag{5.61}$$

Now recall that we introduced the coefficient β_n by $\psi_n = \beta_n q_n$. Comparing with the equation (), we obtain

$$\beta_n = - p_n^T \psi_{n,1} \tag{5.62}$$

and the preceding term becomes

$$\psi^*_{n+1,0} \delta (q_n p_n^T) \wedge \delta \psi_{n,1} = - \psi^*_{n+1,0} \delta q_n \wedge \delta \beta_n - \psi^*_{n+1,0} (\delta q_n \wedge \delta p_n^T) \psi_{n,1} . \tag{5.63}$$

On the other hand, $p_n^T\psi_{n,0}=0$, and the second term on the right-hand side of () can be rewritten as

$$\psi^*_{n+1,1}\delta(q_np_n^T)\wedge\delta\psi_{n,0}=\psi^*_{n+1,1}q_n\delta p_n^T\wedge\delta\psi_{n,0}-\psi^*_{n+1,1}\delta q_n\wedge(\delta p_n^T)\psi_{n,0}. \quad (5.64)$$

Altogether, we obtain the following expression for the residue at P_3:

$$\mathrm{Res}_{P_3}\left\langle\psi^*_{n+1}\delta L_n\wedge\delta\psi_n\right\rangle\frac{dx}{x^2}=\mathrm{II}+\mathrm{III}, \quad (5.65)$$

where the terms II and III are defined by

$$\mathrm{II}=-[\psi^*_{n+1,0}(\delta q_n\wedge\delta p_n^T)\psi_{n,1}+\psi^*_{n+1,1}(\delta q_n\wedge\delta p_n^T)\psi_{n,0}], \quad (5.66)$$

$$\mathrm{III}=-(\psi^*_{n+1,0}\delta q_n\wedge\delta\beta_n-\psi^*_{n+1,1}q_n\delta p_n^T\wedge\delta\psi_{n,0}). \quad (5.67)$$

We claim that the term III can be simplified to

$$\mathrm{III}=-\delta p_n^T\wedge\delta q_n. \quad (5.68)$$

In fact, in view of the recursion relations (5.59) and the fact that $\psi_n=\beta_nq_n$, it can be rewritten as

$$\mathrm{III}=-\psi^*_{n,0}q_n)\big(p_n^T\delta q_n\wedge\delta\beta_n+\delta p_n^T\wedge(\delta\beta_n)q_n+\delta p_n^T\wedge\beta_n\delta q_n\big). \quad (5.69)$$

The first two terms on the right-hand side cancel, since $p_n^Tq_n=0$. As for the remaining term, we note that the normalization $\psi_n^*\psi_n=1$ implies near P_3

$$1=(\psi^*_{n,0}x^{-n}+O(x^{-n-1}))(\beta_nq_nx^n+O(x^{n-1}))=\psi^*_{n,0}\beta_nq_n+O(x^{-1}) \quad (5.70)$$

from which it follows that $\psi^*_{n,0}\beta_nq_n=1$. The identity (5.68) is established.

Finally, it is readily seen that the remaining terms I and II combine into

$$\mathrm{I}+\mathrm{II}=-\sum_{a=1}^{3}\mathrm{Res}_{P_a}\left\langle\psi^*_{n+1}\delta q_n\wedge\delta p_n^T\psi_n\right\rangle\frac{dx}{x}. \quad (5.71)$$

But the 1-form $\left\langle\psi^*_{n+1}\delta q_n\wedge\delta p_n^T\psi_n\right\rangle\frac{dx}{x}$ is meromorphic on the space Γ, with poles only at the points P_a above $x=\infty$ and Q_a above $x=0$. We can deform then contours and rewrite II+III as residues at Q_a,

$$\mathrm{I}+\mathrm{II}=\sum_{a=1}^{3}\mathrm{Res}_{Q_a}\left\langle\psi^*_{n+1}\delta q_n\wedge\delta p_n^T\psi_n\right\rangle\frac{dx}{x}. \quad (5.72)$$

At $x=0$, we have $\psi^*_{n+1}=\psi^*_n$, and this expression is determined by the normalization condition (4.31) on the matrix W. In terms of ψ_n, the normalization (4.31) can be restated as the normalization condition $\psi^*_n(0)\psi_n^T=I$ as an identity between 3×3 matrices. Thus $\mathrm{I}+\mathrm{II}=3\,\langle\delta q_n\wedge\delta p_n\rangle$, and we obtain the final formula for the symplectic form ω in terms of p_n and q_n,

$$\omega=2\sum_{n=0}^{N+1}\delta q_n^T\wedge\delta p_n. \quad (5.73)$$

6. Hamiltonian Theory and Seiberg–Witten Differential: The Odd Divisor Model

The main difference between the even and the odd divisor spin models is in the parity of the divisor D of poles of the Bloch function $\psi_n(Q)$. For the odd divisor spin model, D is essentially odd under the involution $\sigma : (x, y) \to (-x, y^{-1})$ in the following sense:

$$[D] + [D^\sigma] = K + 2\sum_{\alpha=1}^{3} P_\alpha. \tag{6.1}$$

Here K is the canonical class, which is the divisor class of any meromorphic 1-forms on Γ. As in the case of the even divisor spin model, the relation (6.1) is a consequence of the transformation of $L(x)$ under σ, which is in this case $L(-x) = (L(x)^{-1})^T$. This implies that $\psi_0(Q^\sigma)$ and $\psi_0(Q)^*$ are both dual Bloch functions for $L(x)$, and thus

$$\psi_0^*(Q) = \psi_0(Q^\sigma) f(Q), \tag{6.2}$$

where $f(Q)$ is a meromorphic function on Γ. But the zeroes of the dual Bloch function ψ_0^* are exactly the poles of $\psi_0(Q)$, while its poles are exactly the branch points of the surface Γ. Thus the preceding equation implies the following equation for divisor classes

$$[\text{branch points}] - [D] = [D^\sigma]. \tag{6.3}$$

To determine the divisor of the branch points of Γ, we consider the differential dx, viewed as a meromorphic form on Γ. Since dx has a pole of order 2 at each P_a, and a zero at each branch point, we have [branch points] $- 2\sum_{a=1}^{3} P_a = K$, and the desired relation (6.1) follows.

• We discuss briefly the direct and the inverse problems for the odd divisor spin system. Once the difference in parity of the divisor of poles of the Bloch functions is taken into account, the direct problem is treated in exactly the same way as before. As for the inverse problem, we need only a few minor modifications in expansions near the punctures P_1, P_3, which we give (cf. (4.50, 4.51) now

$$\phi_n = x^n e^{xt}\left(\sum_{k=0}^{\infty} \phi_{n,k}(P_1)x^{-k}\right), \quad Q \to P_1, \tag{6.4}$$

$$\phi_n = x^{-n} e^{xt}\left(\sum_{k=0}^{\infty} \phi_{n,k}(P_3)x^{-k}\right), \quad Q \to P_3, . \tag{6.5}$$

They lead to minor modifications in the exact formulas for the Baker–Akhiezer function $\phi_n(t, Q)$ (cf. (4.53)):

$$\phi_{n,\alpha}(t, Q) = r_\alpha(Q)\frac{\theta(A(Q) + tU^- + nV + Z_\alpha)\,\theta(Z_0)}{\theta(A(Q) + Z_\alpha)\,\theta(tU^- + nV + Z_0)} \exp\left(\int_{Q_\alpha}^{Q} nd\Omega_0 + td\Omega^-\right), \tag{6.6}$$

where $d\Omega^- = d\Omega_1 - d\Omega_1^\sigma$ and $U^- = U - U^\sigma$.

We show that if the divisor D satisfies (6.1), then the corresponding Baker–Akhiezer function satisfies the relation

$$\phi_n^*(t, Q) = \phi_n^T(t, Q^\sigma) f(Q), \tag{6.7}$$

where as before ϕ_n^* are the rows of the matrix inverse to the matrix

$$\Phi_{n,\alpha}^{\beta}(x) = \phi_{n,\alpha}(P_\beta). \tag{6.8}$$

Here the points $P_\alpha(x)$ are the three preimages of x on Γ on different sheets. Of course, the matrix $\Phi_n(x)$ does depend on the ordereing of sheets, but one can check that if for $P_\alpha(x)$ we define $\phi_n^*(P_\alpha)$ as the corresponding row of the inverse matrix, then ϕ_n^* is well-defined. As before ϕ_n^* has poles at all the branching points and zeroes at the points of the divisor D.

To establish (6.7), we show that

$$\sum_\alpha \phi_{n,\alpha}(t, P_\gamma(x))\phi_{n,\beta}^\sigma(t, P_\gamma)f(P_\gamma) = \delta_{\alpha,\beta}. \tag{6.9}$$

Indeed, from (6.4) and (6.5), it follows that the function $\phi_{n,\alpha}(t, Q)\phi_{n,\beta}(Q^\sigma)f(Q)$ is holomorphic everywhere except at the branching points (the poles and the essential singularities at the punctures P_α over $x = \infty$ cancel each other; there are no poles at D and D^σ because $f(Q)$ has zeros at these points). Therefore, the left-hand side of the above equation is a holomorphic function of x (the poles at the branching points cancel upon the summation). Hence it is a constant, which can be found by taking $x = 0$.

The uniqueness of ϕ_n and the relation (6.7) implies as before that it satisfies the equation

$$\phi_{n+1} = L_n(x)\phi_n, \partial_t\phi_n = M_n(x)\phi_n, \tag{6.10}$$

where L_n and M_n have the form (3.4,3.5).

• We come now to the Hamiltonian structure of the odd divisor spin model. Recall that we had introduced the space $\mathcal{M}^{\text{odd}}$ of spin chains. Solving the direct and inverse spectral problem as in the case of the even divisor spin model, we can identity $\mathcal{M}^{\text{odd}}$ with the space of geometric data

$$\mathcal{M}_1^{\text{odd}} \leftrightarrow \{\Gamma, D;\ [D] + [D^\sigma] = K + 2\sum_{\alpha=1}^{3} P_\alpha\}. \tag{6.11}$$

We can verify that the space on the right-hand side is $2N + 1$ dimensional, as it should be: there are $N + 1$ moduli parameters for the curve Γ, and N parameters for the antisymmetric divisor $[D]$. The same discussion as in Sects 5.3 and 5.4 for the even divisor spin model shows that, in the present case, the only candidate for the symplectic form is the form $\omega_{(1)}$, restricted to the $2N$-dimensional phase space $\mathcal{M}_1^{\text{odd}}$ defined by

$$\mathcal{M}_1^{\text{odd}} = \mathcal{M}^{\text{odd}} \cap \{u_N = 1\}. \tag{6.12}$$

The corresponding action and angle variables are now given by

$$a_i = \oint_{A_i^{\text{odd}}} d\lambda_{(1)}, \quad \phi_i = \sum_{j=1}^{2N+1} \int^{z_j} d\omega_i^{\text{odd}}, \quad 1 \le i \le N, \tag{6.13}$$

where $d\omega_i^{\text{odd}}$ and A_i^{odd} are respectively a basis of odd holomorphic differentials and a basis of odd A-cycles. We have then as before

$$\omega_{(1)} = \sum_{j=1}^{N} \delta a_j \wedge \delta\phi_j. \tag{6.14}$$

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