1 Baker-Akhiezer Functions and Integrable Systems

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Key ideas of the algebra-geometric methods in the theory of solitons are presented. Unexpected links between various theories in which the same objects emerge repeatedly, albeit under different names, like τ-function in the Whitham theory, partition function in topological filed theories, and prepotential in Seiberg-Witten theory mainly are discussed.

1.1 Introduction

The main goal of this chapter is to present key ideas which unify the algebro-geometric methods in the theory of soliton equations and recent developments in the theory of $N = 2$ supersymmetric gauge models.

Solitons arose originally in the study of shallow water waves. Since then, the notion of soliton equations has widened considerably. It embraces now a wide class of non-linear partial differential equations, which all share the characteristic feature of being expressible as a compatibility condition for an auxiliary pair of linear differential equations. A variety of methods have been developed over the years to construct exact solutions for these equations. Since the middle seventies algebraic geometry has become one of the most powerful tools among them.

In the next section we outline basic elements of the, so-called, finite-gap theory which were originated in (Novikov [1994]; Dubrovin *et al.* [1976], Lax [1975], McKean *et al.* [1975]) in the framework of the Floquet spectral theory of periodic Schrödinger operators combined with a theory of completely integrable Hamiltonian systems. Analytical properties of Bloch solutions of finite-gap Schrödinger operators with respect to an auxiliary spectral parameter, established in this remarkable series of papers, were a starting point in a definition of the Baker-Akhiezer functions which are the core of a general algebro-geometric construction of exact periodic and almost periodic solutions to soliton equations proposed in (Krichever [1976, 1977a, 1977b]).

Section 3 is devoted to brief description of the Whitham method which is a generalization to the case of partial differential equation of the classical Bogolyubov-Krylov averaging method. It turns out that differential equations describing a slow modulation of integrals of finite-gap solutions of soliton equations, called Whitham equations, are deeply connected with a theory of deformations of topological quantum field models. This connection between Whitham theory and the, so-called, Witten-Dijgraaf-Verlinde-Verlinde (WDVV) equations is discussed in Section 4.

In the last section we show that the Seiberg-Witten theory of $N = 2$ supersymmetric gauge theories can be considered on one hand as a part of the Whitham theory and at the same time leads to a new general approach to Hamiltonian theory of soliton equations proposed in (Krichever *et al.* [1977, 1999]).

Our discussion of unexpected links between various theories in which the same objects emerge repeatedly, albeit under different names, like τ-function in the Whitham theory, partition function in topological field theories, and prepotential in Seiberg-Witten theory mainly follows Krichever *et al.* [1999]), where more details can be found.

1.2 Finite-gap Solutions of Integrable Systems

The finite-gap or algebro-geometric integration method is uniformly applicable to all soliton equations. In the case of spatial one-dimensional evolution equations it is instructive enough to consider as a basic example equations that have Lax representation

$$\partial_t L = [A, L], \tag{1}$$

where the unknown functions $\{u_i(x, y, t)\}_{i=0}^{n-2}$, $\{v_j(x, y, t)\}_{j=0}^{m-2}$ are the coefficients of the ordinary differential operators

$$L = \partial_x^n + \sum_{i=0}^{n-2} u_i(x, t)\partial_x^i, \ A = \partial_x^m + \sum_{j=0}^{m-2} v_j(x, t)\partial_x^j. \tag{2}$$

A preliminary classification of equations of the form (1) is by the orders n, m of the operators L and A.

In the case $n = 2$ the operator L is just the usual Schrödinger operator $L = -\partial_x^2 + u(x, t)$, and for $A = \partial_x^3 - 3/2u\partial_x - 3/4u_x$ equation (1) is equivalent to the KdV equation $4u_t - 6uu_x + u_{xxx} = 0$.

From (1) it follows that certain spectral quantities of the operator L are integrals of motion. In the framework of the finite-gap theory these integrals are organized in the form of the so-called *spectral curve*. In all the cases, i.e. finite-dimensional integrable systems, spatial one- or two-dimensional evolution equations, the spectral curve is defined by a characteristic equation

$$R(w, E) = \det(w - T(t, E)) = 0, \tag{3}$$

where $T(t, E)$ is a finite-dimensional matrix depending on a spectral parameter E.

In the case of finite-dimensional (or (0+1)) integrable systems, which have the Lax representation $L_t(t, E) = [A(t, E), L(t, E)]$, where L and A are finite-dimensional matrices depending on the spectral parameter, the matrix $T(t, E)$ defining the spectral curve is the Lax matrix L by itself, i.e. $T(t, E) = L(t, E)$.

In the infinite-dimensional case the spectral curve can be defined for special classes of solutions, only. For spatial one-dimensional systems these classes are singled out by the constraint that there exists an additional operator T which commutes with L and $(\partial_t - A)$. For example, if the coefficients of the operator L of the form (2) are periodic functions of the variable x with period T, then the operator L commutes with the shift operator

$$\hat{T} : y(x) \longmapsto y(x+T). \tag{4}$$

Therefore, the finite-dimensional linear space $\mathcal{L}(E)$ of the solutions of the ordinary differential equation

$$y(x) \in \mathcal{L}(E) : Ly = Ey \tag{5}$$

is invariant with respect to T. Restriction of the shift operator onto $\mathcal{L}$ defines a finite-dimensional linear operator

$$T(E) = \hat{T}|_{\mathcal{L}(E)}. \tag{6}$$

A point Q of the spectral curve Γ is just a pair $Q = (E, w)$ of complex numbers that satisfy (3). They parametrize Bloch eigenfunctions of the operator L, i.e. common eigenfunctions of L and the monodromy operator

$$L\psi(x, Q) = E\psi(x, Q), \quad \psi(x+T, Q) = w\psi(x, Q). \tag{7}$$

In a generic case the corresponding Riemann surface is a smooth surface of infinite genus. If its genus is finite then the corresponding operators are called *finite-gap* or algebro-geometric operators. It should be emphasized that in such a case the Riemann surface defined by the characteristic equation is a singular surface. After resolving the singularities we get a finite genus smooth Riemann surface, i.e. an algebraic curve. For example, let $L = -\partial_x^2 + u(x)$ be the Schrödinger operator with a periodic potential $u(x) = u(x+T)$. Then equation (3) has the form

$$w^2 - 2Q(E)w + 1 = 0, \quad 2Q(E) = Tr\ T(E). \tag{8}$$

The roots ϵ_i of the equation $Q^2(E) = 1$ are points of the periodic or anti-periodic spectrum of the Schrödinger operator. Equation (8) can be rewritten in the form

$$y^2 = \prod_{i=1}^{\infty}(E - \epsilon_i), \quad y = w - Q(E). \tag{9}$$

If all of the edges ϵ_i are distinct then (9) defines smooth infinite genus hyperelliptic Riemann surface. The finite-gap operators correspond to the degenerate case when all but a finite number of eigenvalues of periodic or anti-periodic spectral problem for L are multiple. Let $E_1 < \cdots < E_{2g+1}$ be the simple eigenvalues. Then a finite genus smooth algebraic curve of the Bloch functions is defined by

$$y^2 = \prod_{i=1}^{2g+1}(E - E_i). \tag{10}$$

Note that E_i are the edges of the spectrum bands of L, being considered as an operator in the space of square integrable functions on the whole line.

The finite-gap theory was initiated by the work (Novikov [1974]), where the spectral theory of periodic operators was combined with an approach based on a use of the KdV hierarchy. The KdV equation (as well as any soliton equation) is compatible with an infinite hierarchy of commuting flows. They have the Lax representation

$$\partial_i L = [A_i, L], \quad \partial_i = \frac{\partial}{\partial t_i}, \tag{11}$$

where A_i is an ordinary differential operator of order $2i + 1$. Consider stationary solutions of a linear combination of these flows, i.e. solutions of the ordinary differential equation

$$[L, A] = 0, \quad A = \sum_{i=1}^{g} c_i A_i. \tag{12}$$

As it was shown in (Novikov [1974]), equation (12) is a completely integrable Hamiltonian system. Therefore, its general solution is a quasi-periodic function of x. Periodic solutions are finite-gap potentials.

Let $A(E)$ be the restriction of the operator A commuting with L onto $\mathcal{L}(E)$. The matrix elements of $A(E)$ are polynomial functions of the spectral parameter. Therefore, the characteristic equation $\det(A(E) - w) = 0$ defines an algebraic curve Γ. It turns out that this curve *coincides* with the spectral curve (10).

Note that the operator equation (12) is a particular case of the more general problem of the classification of commuting ordinary differential operators L_n and L_m of orders n and m, respectively. As a purely algebraic problem it was considered and partly solved in the remarkable works of Burchnall and Chaundy (Burchall *et al.* [1922, 1928]) in the 1920s. They proved that for any pair of such operators there exists a polynomial $R(\lambda, \mu)$ in two variables such that $R(L_n, L_m) = 0$. If the orders n and m of these operators are co-prime, $(n, m) = 1$, then for each point $Q = (\lambda, \mu)$ of the curve Γ defined in C^2 by the equation $R(\lambda, \mu) = 0$ there corresponds a unique (up to a constant factor) common eigenfunction $\psi(x, Q)$ of L_n and L_m

$$L_n \psi(x, Q) = \lambda \psi(x, Q); \quad L_m \psi(x, Q) = \mu \psi(x, Q).$$

The logarithmic derivative $\psi_x \psi^{-1}$ is a meromorphic function on Γ. In the general position (when Γ is smooth) it has g poles $\gamma_1(x), \ldots, \gamma_g(x)$ in the affine part of the curve, where g is the genus of Γ. The commuting operators L_n and L_m (in this case of co-prime orders) are uniquely defined by the polynomial R and by a set of g points $\gamma_1(x_0), \ldots, \gamma_g(x_0)$ on Γ.

In such a form, the solution of the problem is one of pure classification: one set is equivalent to the other. Even the attempt to obtain exact formulae for the coefficients of commuting operators had not been made. Baker proposed making the program effective by pointing out that the eigenfunction ψ has analytical properties that were introduced by Clebsch, Gordan and himself as a proper generalization of the notion of exponential functions on Riemann surfaces. The Baker program was rejected by the authors of (Burchnall *et al.* [1922, 1928]) consciously (see the postscript of Baker's paper [1928]) and all these results were forgotten for a long time. This program was realized only in (Krichever [1976, 1977a]) (though at that time the author was not aware of the remarkable results of Burchnal, Chaundy and Baker) where the commuting pairs of ordinary differential operators were considered in connection with the problem of constructing solutions to the KP equation.

Spatial two-dimensional integrable systems of the KP type have an analogue of the Lax representation of the form

$$[\partial_y - L, \partial_t - A] = 0, \tag{13}$$

where, as before, L and A are ordinary differential operators of the form (2) but now with the coefficients depending on the variables x, y, t. In two dimensions in order to single out special classes of solutions for which a spectral curve can be defined one needs to impose *two constraints*. For example, that can be done if we assume that in addition to (13) there exist two ordinary differential operators of orders n and m such that

$$[\partial_y - L, L_n] = 0, \quad [\partial_y - L, L_m] = 0. \tag{14}$$

Such operators commute with each other, and commute with the operator $(\partial_t - A)$. The corresponding spectral curve is a spectral curve of commuting operators L_n, L_m. It does not depend on (x, y, t). (Classification of commuting operators of arbitrary orders was completed in Krichever [1978]).

The common eigenfunction of commuting operators is a particular case of the general definition of the scalar *multi-point multi-variable* Baker-Akhiezer function. Let Γ be a non-singular algebraic curve of genus g with N punctures P_α and fixed local parameters $k_\alpha^{-1}(Q)$ in neighbourhoods of these punctures. For any set of points $\gamma_1, \ldots, \gamma_g$ in general position, there exists a unique (up to constant factor $c(t_{\alpha,i})$) function $\psi(t, Q)$, $t = (t_{\alpha,i})$, $\alpha = 1, \ldots, N$; $i = 1, \ldots,$ such that:

(i) the function ψ (as a function of the variable $Q \in \Gamma$) is meromorphic everywhere except for the points P_α and it has at most simple poles at the points $\gamma_1, \ldots, \gamma_g$ (if all of them are distinct).
(ii) in a neighbourhood of the point P_α the function ψ has the form

$$\psi(t, Q) = \exp\Big(\sum_{i=1}^{\infty} t_{\alpha,i} k_\alpha^i\Big)\Big(\sum_{s=0}^{\infty} \xi_{s,\alpha}(t) k_\alpha^{-s}\Big), \; k_\alpha = k_\alpha(Q). \tag{15}$$

The Baker-Akhiezer function ψ depends on the variables $t = \{t_{1,i}, \ldots, t_{N,i}\}$ as on external parameters.

From the uniqueness of the Baker-Akhiezer function it follows that for each pair (α, n) there exists a unique operator $L_{\alpha,n}$ of the form

$$L_{\alpha,n} = \partial_{\alpha,1}^n + \sum_{j=1}^{n-1} u_j^{(\alpha,n)}(t) \partial_{\alpha,1}^j, \tag{16}$$

where $\partial_{\alpha,i} = \partial/\partial t_{\alpha,i}$, such that

$$\big(\partial_{\alpha,i} - L_{\alpha,n}\big)\psi(t, Q) = 0. \tag{17}$$

The idea of the proof of theorems of this type proposed in (Krichever [1976]) is universal.

For any formal series of the form (15) there exists a unique operator $L_{\alpha,n}$ of the form (16) such that

$$\big(\partial_{\alpha,i} - L_{\alpha,n}\big)\psi(t, Q) = O(k^{-1}) \exp\big(\sum_{i=1}^{\infty} t_{\alpha,i} k_\alpha^i\big). \tag{18}$$

The coefficients of $L_{\alpha,n}$ are differential polynomials with respect to $\xi_{s,\alpha}$. They can be found after substitution of the series (15) into (18).

It turns out that if the series (15) is not formal but is an expansion of the Baker-Akhiezer function in the neighbourhood of P_α, then the congruence (18) becomes an equality. Indeed, let us consider the function

$$\psi_1 = \big(\partial_{\alpha,n} - L_{\alpha,n}\big)\psi(t, Q). \tag{19}$$

It has the same analytical properties as ψ, except for one. The expansion of this function in a neighbourhood of P_α starts from $O(k^{-1})$. From the uniqueness of the Baker-Akhiezer function it follows that $\psi_1 = 0$ and the equality (17) is proved.

A corollary is that the operators $L_{\alpha,n}$ satisfy the compatibility conditions

$$\big[\partial_{\alpha,n} - L_{\alpha,n}, \partial_{\alpha,m} - L_{\alpha,m}\big] = 0. \tag{20}$$

The equations (20) are gauge invariant. For any function $g(t)$ operators $\tilde{L}_{\alpha,n} = gL_{\alpha,n}g^{-1} + (\partial_{\alpha,n}g)g^{(-1)}$ have the same form (16) and satisfy the same operator equations (20). The gauge transformation corresponds to the gauge transformation of the Baker-Akhiezer function $\psi_1(t, Q) = g(t)\psi(t, Q)$.

In the one-point case the Baker-Akhiezer function has an exponential singularity at a single point P_1 and depends on a single set of variables. Let us choose the normalization of the Baker-Akhiezer function with the help of the condition $\xi_{0,1} = 1$, i.e. an expansion of ψ in the neighbourhood of P_1 is

$$\psi(t_1, t_2, \dots, Q) = \exp\big(\sum_{i=1}^{\infty} t_i k^i\big)\big(1 + \sum_{s=1}^{\infty} \xi_s(t) k^{-s}\big). \tag{21}$$

In this case the operator L_n has the form

$$L_n = \partial_1^n + \sum_{i=0}^{n-2} u_i^{(n)} \partial_1^i. \tag{22}$$

If we denote t_1, t_2, t_3 by x, y, t, respectively, then from (20) it follows (for $n = 2, m = 3$) that $u(x, y, t, t_4, \dots)$ satisfies the KP equation $3u_{yy} = (4u_t - 6uu_x + u_{xxx})_X$. The exact formula for these solutions in terms of the Riemann theta-function is based on the exact formula for the Baker-Akhiezer function.

Let us fix the basis of cycles $a_i, b_i,\ i = 1, \dots, g$ on Γ with the canonical matrix of intersections: $a_i \circ a_j = b_i \circ b_j = 0,\ a_i \circ b_j = \delta_{ij}$. The basis of normalized holomorphic differentials $\omega_j(Q),\ j = 1, \dots, g$ is defined by conditions $\oint_{a_i} \omega_j = \delta_{ij}$. The b -periods of these differentials define the so-called Riemann matrix $B_{kj} = \oint_{b_j} \omega_k$. The basic vectors e_k of C^g and the vectors B_k, which are the columns of matrix B, generate a lattice $\mathcal{B}$ in C^g. The g-dimensional complex torus

$$J(\Gamma) = C^g/\mathcal{B},\ \ \mathcal{B} = \sum n_k e_k + m_k B_k,\ \ n_k, m_k \in Z, \tag{23}$$

is called the Jacobian variety of Γ. A vector with coordinates $A_k(Q) = \int_{q_0}^{Q} \omega_k$ defines the Abel map $A : \Gamma \longrightarrow J(\Gamma)$ which depends on the choice of the initial point q_0.

The Riemann matrix has a positive-definite imaginary part. The entire function of g variables

$$\theta(z) = \theta(z|B) = \sum_{m \in Z^g} e^{2\pi i(z,m) + \pi i(Bm,m)}, \tag{24}$$

$$z = (z_1, \dots, z_n),\ \ m = (m_1, \dots, m_n),\ \ (z,m) = z_1 m_1 + \dots + z_n m_n,$$

is called the Riemann theta-function. It has the following monodromy properties

$$\theta(z + e_k) = \theta(z),\ \ \theta(z + B_k) = e^{-2\pi i z_k - \pi i B_{kk}} \theta(z). \tag{25}$$

The function $\theta(A(Q) - Z)$ is a multi-valued function of Q. But according to (25), the zeros of this function are well-defined. For Z in a general position the equation

$$\theta(A(Q) - Z) = 0 \tag{26}$$

has g zeros $\gamma_1, \dots, \gamma_g$. The vector Z and the divisor of these zeros are connected by the relation

$$Z_k = \sum_{i=1}^{g} A(\gamma_s) + \mathcal{K}, \tag{27}$$

where $\mathcal{K}$ is the vector of Riemann constants.

Let us introduce the normalized Abelian differentials $d\Omega^0_{\alpha,i}$ of the second kind. The differential $d\Omega^0_{\alpha,i}$ is holomorphic on Γ except for the puncture P_α. In the neighbourhood of P_α point it has the form

$$d\Omega^0_{\alpha,i} = d(k_\alpha^i + O(1)). \tag{28}$$

"Normalized" means that it has zero a-periods, $\oint_{a_j} d\Omega^0_{\alpha,i} = 0$. Consider the function

$$\psi(t, Q) = \frac{\theta(A(Q) + \sum_{\alpha,j} t_{\alpha,j} U_{\alpha,j} - Z)}{\theta(A(Q) - Z)} \exp\left(\sum_{\alpha,j} t_{\alpha,j} \int_{q_0}^{Q} d\Omega^0_{\alpha,j}\right) \tag{29}$$

where the coordinates of the vector $U_{\alpha,j}$ are equal to

$$U^k_{\alpha,j} = \frac{1}{2\pi i} \oint_{b_k} d\Omega^0_{\alpha,j}. \tag{30}$$

Equations (25–27) imply that ψ is a single valued function on Γ and has all the analytical properties of the Baker-Akhiezer function. That proves the existence of the Baker-Akhiezer function. Let $\tilde{\psi}$ be any function with the same analytical properties. The ratio $\tilde{\psi}/\psi$ is a meromorphic function with at most g poles. The Riemann-Roch theorem implies that such a function is equal to a constant. Hence, the uniqueness of the Baker-Akhiezer function (up to a constant factor) is also proved.

The coefficients of the operators $L_{\alpha,j}$ which are defined by the equations (17) are differential polynomials in the coefficients of the expansions of the second factor in (29) near the punctures. Hence, they can be expressed as differential polynomials in terms of Riemann theta-functions. For example, the algebraic-geometrical solutions of the KP hierarchy have the form

$$u(x, y, t, t_4, \dots) = 2\partial_x^2 \ln \theta(xU_1 + yU_2 + tU_3 + \cdots + Z) + const. \tag{31}$$

The common eigenfunction of commuting operators of co-prime orders is the particular case of a one-point Baker-Akhiezer function corresponding to $t_1 = x, t_2 = 0, t_3 = 0, \dots$. Therefore, the coefficients of such operators (in general position) are differential polynomials in terms of the Riemann theta-functions. This has an important corollary. The coefficients of commuting differential operators of co-prime orders are meromorphic functions of the variable x. Moreover, in general position they are quasi-periodic functions of x. The last statement presents evidence that the theory of commuting operators is connected with the spectral Floquet theory of periodic differential operators. These connections were missing in (Burchnall *et al.* [1922, 1928], Baker [1928]).

Let us introduce *real normalized* Abelian differentials $d\Omega_{\alpha,i}$ of the second kind. The differential $d\Omega_{\alpha,i}$ is holomorphic on Γ except for the puncture P_α. In the neighbourhood of this point it has the same form as $d\Omega^0_{\alpha,i}$, i.e. $d\Omega_{\alpha,i} = d(k_\alpha^i + O(1))$. Real normalization means that for any cycle on Γ the period of the differential is pure imaginary, i.e. $\mathrm{Re}\left(\oint_c d\Omega_{\alpha,i}\right) = 0$.

From (29) it follows that the algebro-geometric solutions corresponding to Γ are periodic functions of the variable $t_{\alpha,j}$ with a period T if and only if the periods of the corresponding differential have the form

$$\oint_c d\Omega_{\alpha,j} = \frac{2\pi i}{T} n_c, \tag{32}$$

where n_c are integers.

The spectral theory of two-dimensional periodic operators was developed in (Krichever [1989]). It was proved that for the operator $(\partial_y - \partial_x^2 + u(x, y))$ with a real analytic periodic (in x and y) potential the spectral curve does exist. Points of this curve $Q = (w_1, w_2) \in \Gamma$ parametrize Bloch solutions of the equation $(\partial_y - \partial_x^2 + u(x, y))\psi(x, y, w_1, w_2) = 0$, i.e. they parametrize pairs of complex numbers (w_1, w_2) such that there exists a solution to the equation with the following monodromy properties:

$$\psi(x + l_1, y, w_1, w_2) = w_1 \psi(x, y, w_1, w_2), \quad \psi(x, y + l_2, w_1, w_2) = w_2 \psi(x, y, w_1, w_2). \tag{33}$$

In a general case the spectral curve has infinite genus. For the algebro-geometric potentials the spectral curve has finite genus and coincides with the spectral curve of a corresponding pair of the commuting differential operators.

The space of algebro-geometric data defining solutions of the full hierarchy of spatially two-dimensional KP type systems is infinite dimensional because it contains a choice of the local coordinates near the punctures. At the same time the space of algebro-geometric solutions of a single equation of the zero-curvature form (13) is finite-dimensional. If L and A are operators of orders n and m with scalar coefficients, then this space can be described as follows (see details in (Krichever *et al.* [1997])).

Let $\mathcal{M}_g(n, m)$ be the space (Γ, E, Q) of pairs of Abelian integrals on a smooth genus g algebraic curve Γ, where E and Q have poles of orders n and m, respectively, at a puncture P_0 . Then we define a local coordinate k^{-1} near the puncture by the equality $k^n = E$. This choice of the local coordinate corresponds to the identification of the variable y with a basic time variable $y = t_n$.

In the presence of a second Abelian integral Q, we can select a second time t, by writing the singular part $Q_+(k)$ of Q as a polynomial in k and setting

$$Q_+(k) = a_1 k + \cdots + a_m k^m, \ t_i = a_i t, \ 1 \le i \le m. \tag{34}$$

This means that we consider the Baker-Akhiezer function $\psi(x, y, t; k)$ with the essential singularity $\exp(kx + k^n y + Q_+(k)t)$, and construct the operators L and A by requiring that $(\partial_y - L)\psi = (\partial_t - A)\psi = 0$. The pair (L, A) provides then a solution of the zero-curvature equation. By rescaling t, we can assume that A is monic.

The proper interpretation of the full geometric data $(\Gamma, E, Q; \gamma_1, \cdots \gamma_g)$ is as a point in the bundle $\mathcal{N}_g^g(n, m)$ over $\mathcal{M}_g(n, m)$, whose fiber is the g-th symmetric power $S^g(\Gamma)$ of the curve:

$$\mathcal{N}_g^g(n, m) \xrightarrow{S^g(\Gamma)} \mathcal{M}_g(n, m) \tag{35}$$

The g-th symmetric power can be identified with the Jacobian of Γ via the Abel map.

More generally, we can construct the bundles $\mathcal{N}_g^k(n, m)$ with fiber $S^k(\Gamma)$ over the bases $\mathcal{M}_g(n, m)$. Thus the bundle $\mathcal{N}_g^{k=1}(n, m) \equiv \mathcal{N}_g(n, m)$ is the analogue in our context of the universal curve.

1.3 Whitham Equations

We have seen that soliton equations exhibit a unique wealth of exact solutions. Nevertheless, it is desirable to enlarge the class of solutions further, to encompass broader data than just rapidly decreasing or quasi-periodic functions. Typical situations arising in practice can involve Heaviside-like boundary conditions in the spatial variable x, or slowly modulated waves which are not exact solutions, but can appear as such over a small scale in both space and time.

The non-linear WKB method (or, as it is now also called, the Whitham method of averaging) is a generalization to the case of partial differential equations of the classical Bogolyubov-Krylov method of averaging. This method is applicable to nonlinear equations which have a moduli space of exact solutions of the form $u_0(Ux + Wt + Z|I)$. Here $u_0(z_1, \ldots, z_g|I)$ is a periodic function of the variables z_i; $U = (U_1, \ldots, U_g)$, $W = (W_1, \ldots, W_g)$ are vectors which like u itself, depend on the parameters $I = (I_1, \ldots, I_N)$, i.e. $U = U(I),\ W = W(I)$. These exact solutions can be used as a leading term for the construction of asymptotic solutions

$$u(x,t) = u_0(\varepsilon^{-1}S(X,T) + Z(X,T)|I(X,T)) + \varepsilon u_1(x,t) + \varepsilon^2 u_2(x,t) + \cdots, \tag{36}$$

where I depend on the *slow* variables $X = \varepsilon x$, $T = \varepsilon t$ and and ε is a small parameter. If the vector-valued function $S(X,T)$ is defined by the equations

$$\partial_X S = U(I(X,T)) = U(X,T), \quad \partial_T S = W(I(X,T)) = W(X,T), \tag{37}$$

then the leading term of (36) satisfies the original equation up to order one in ε. All the other terms of the asymptotic series are obtained from the non-homogeneous linear equations, whose homogeneous part is just the linearization of the original non-linear equation on the background of the exact solution u_0. In general, the asymptotic series becomes unreliable on scales of the original variables x and t of order ε^{-1}. In order to have a reliable approximation, one needs to require a special dependence of the parameters $I(X,T)$. Geometrically, we note that $\varepsilon^{-1}S(X,T)$ agrees to first order with $Ux + Vt$, and x, t are the fast variables. Thus $u(x,t)$ describes a motion which is to first order the original *fast* periodic motion on the Jacobian, combined with a slow drift on the moduli space of exact solutions. The equations which describe this drift are in general called *Whitham equations*, although there is no systematic scheme to obtain them.

One approach for obtaining these equations in the case when the original equation is Hamiltonian is to consider the Whitham equations as also Hamiltonian, with the Hamiltonian function being defined by the average of the original one. In the case when the phase dimension g is greater than one, this approach does not provide a complete set of equations. If the original equation has a number of integrals one may try to get the complete set of equations by averaging all of them. This approach was used in (Flashka *et al.* [1980]) where Whitham equations were *postulated* for the finite-gap solutions of the KdV equation. The Hamiltonian approach for the Whitham equations of (1+1)-dimensional systems was developed in (Dubrovin *et al.* [1983]) where the corresponding bibliography can also be found.

In (Krichever [1988]) a general approach for the construction of Whitham equations for finite-gap solutions of soliton equations was proposed. It is instructive enough to present it in the case of zero-curvature equation (13) with scalar operators.

Recall from the previous section that the coefficients $u_i(x,y,t),\ v_j(x,y,t)$ of the finite-gap operators L_0 and A_0 satisfying (13) are of the form

$$u_i = u_{i,0}(Ux + Vt + Wt + Z|I), \quad v_j = v_{j,0}(Ux + Vt + Wt + Z|I), \tag{38}$$

where $u_{i,0}$ and $v_{j,0}$ are differential polynomials in θ-functions and I is any coordinate system on the moduli space $\mathcal{M}_g(n,m)$. (A helpful example is provided by the solutions (31) of the KP equation, where I is the moduli of a Riemann surface, and U, V, W are the B_k-periods of its normalized differentials $d\Omega_1$, $d\Omega_2$, and $d\Omega_3$.) We would like to construct operator solutions of (13) of the form

$$L = L_0 + \varepsilon L_1 + \cdots, \quad A = A_0 + \varepsilon A_1 + \cdots, \tag{39}$$

where the coefficients of the leading terms have the form

$$u_i = u_{i,0}(\varepsilon^{-1}S(X,Y,T) + Z(X,Y,T)|I(X,Y,T)),$$
$$v_j = v_{j,0}(\varepsilon^{-1}S(X,Y,T) + Z(X,Y,T)|I(X,Y,T)) \tag{40}$$

If I is a system of coordinates on $\mathcal{M}_g(n,m)$, then we may introduce a system of coordinates $(z,\ I)$ on $\mathcal{N}_g^1(n,m)$ by choosing a coordinate along the fiber Γ. The Abelian integrals p, E, Q are multi-valued functions of $(z,\ I)$, i.e. $p = p(z,I)$, $E = E(z,I)$, $Q = Q(z,I)$. If we describe a drift on the moduli space of exact solutions by a map $(X,Y,T) \to I = I(X,Y,T)$, then the Abelian integrals p, E, Q become functions of (z,X,Y,T). The following was established in (Krichever [1988]):

A necessary condition for the existence of the asymptotic solution (4) with leading term (5) and bounded terms L_1 and A_1 is that the equation

$$\frac{\partial p}{\partial z}\left(\frac{\partial E}{\partial T} - \frac{\partial Q}{\partial Y}\right) - \frac{\partial E}{\partial z}\left(\frac{\partial p}{\partial T} - \frac{\partial Q}{\partial X}\right) + \frac{\partial Q}{\partial z}\left(\frac{\partial p}{\partial Y} - \frac{\partial E}{\partial X}\right) = 0 \tag{41}$$

is satisfied.

The equation (41) is called the Whitham equation for (13). It can be viewed as a generalized dynamical system on $M_g(n,m)$, i.e., a map $(X,Y,T) \to M_g(n,m)$. Some of its important features are:

- Even though the original two-dimensional system may depend on y, Whitham solutions which are Y-independent are still useful. As we shall see later, this particular case has deep connections with topological field theories. If we choose the local coordinate z along the fiber as $z = E$, then the equation simplifies to

$$\partial_T p = \partial_X Q. \tag{42}$$

- Naively, the Whitham equation seems to impose an infinite set of conditions, since it is required to hold at every point of the fiber Γ. However, the functions involved are all Abelian integrals, and their equality over the whole of Γ can actually be reduced to a finite set of conditions.
- The equation (41) can be represented in a manifestly invariant form without explicit reference to any local coordinate system z. Given a map $(X,Y,T) \to M_g(n,m)$, the pull-back of the bundle $\mathcal{N}_g^1(n,m)$ defines a bundle over a space with coordinates X, Y, T. The total space $\mathcal{N}^4$ of this bundle is 4-dimensional. Let us introduce on it the one-form

$$\alpha = pdX + EdY + QdT, \tag{43}$$

Then (41) is equivalent to the condition that the wedge product of $d\alpha$ with itself be zero (as a 4-form on $\mathcal{N}^4$)

$$d\alpha \wedge d\alpha = 0. \tag{44}$$

- It is instructive to present the Whitham equation (41) in yet another form. Because (41) is invariant with respect to a change of local coordinate we may use $p = p(z, I)$ by itself as a local coordinate. Then we may view E and Q as functions of p, X, Y and T, i.e. $E = E(p, X, Y, T)$, $Q = Q(p, X, Y, T)$. With this choice of local coordinate (41) takes the form
$$\partial_T E - \partial_Y Q + \{E, Q\} = 0, \tag{45}$$
where $\{\cdot, \cdot\}$ stands for the usual Poisson bracket of two functions of the variables p and X, i.e. $\{f, g\} = f_p g_X - g_p f_X$.
- Above we had focused on constructing an asymptotic solution for a single equation. This corresponds to a choice of A, and thus of an Abelian differential Q, and the Whitham equation is an equation for maps from (X, Y, T) to $M_g(n, m)$. As in the case of the KP and other hierarchies, we can also consider a whole hierarchy of Whitham equations. This means that the Abelian integral Q is replaced by the real normalized Abelian integral Ω_i which has the following form
$$\Omega_i = k^i + O(k^{-1}), \quad k^n = E,$$
in a neighbourhood of the puncture P. The whole hierarchy may be written in the form (44) where we set now
$$\alpha = \sum_i \Omega_i dT_i.$$
In (Krichever [1988]) a construction of exact solutions to the Whitham equations (41) was proposed. We present the most important special case of this construction, which is also of interest to topological field theories and supersymmetric gauge theories. It should be emphasized that for these applications, the definition of the hierarchy should be slightly changed. Namely, the Whitham equations describing modulated waves in soliton theory are equations for Abelian differentials with a real normalization. In what follows we shall consider the same equations, but where the real-normalized differentials are replaced by differentials with the complex normalization $\oint_{A_k} d\Omega = 0$. The two types of normalization coincide on the subspace corresponding to M-curves, which is essentially the space where all solutions are regular and where the averaging procedure is easily implemented. Thus, the two forms of the Whitham hierarchy can be considered as different extensions of the same hierarchy. The second one is an analytic theory, and we shall henceforth concentrate on it.

In the rest of this chapter we shall restrict ourselves to the hierarchy of "algebraic geometric solutions" of Whitham equations, that is, solutions of the following stronger version of the equations (45)
$$\partial_{T_i} E = \{\Omega_i, E\}. \tag{46}$$
We note that the original Whitham equations can actually be interpreted as consistency conditions for the existence of an E satisfying (46). Furthermore, the solutions of (46) can be viewed in a sense as "Y-independent" solutions of Whitham equations. They play the same role as Lax equations in the theory of (2+1)-dimensional soliton equations. As stressed earlier, Y-independent solutions of the Whitham hierarchy can be considered even for two-dimensional systems where the y-dependence is non-trivial in general.

Equations (46) define a system of commuting flows on the moduli space of Abelian integrals. For the one puncture case this space is a union of the spaces $\mathcal{M}_g(n)$ of Abelian integrals with the pole of order n at the puncture. The complex dimension of $\mathcal{M}_g(n)$ is equal to $dim\,\mathcal{M}_g(n) = 4g + n - 1$. Let us describe a special system of coordinates for it.

The first $2g$ coordinates are still the periods of dE,

$$T_{A_i,E} = \oint_{A_i} dE, \; T_{B_i,E} = \oint_{B_i} dE. \tag{47}$$

The differential dE has $2g + n - 1$ zeros (counting multiplicities). When all the zeroes are simple, we can supplement (47) by the $2g + n - 1$ critical values E_s of the Abelian differential E, i.e.

$$E_s = E(q_s), \;\; dE(q_s) = 0, \;\; s = 1, \dots, 2g + n - 1. \tag{48}$$

Let $\mathcal{D}'$ be the open set in $\mathcal{M}_g(n)$ where the zero divisors of dE and dp, namely the sets $\{z|dE(z) = 0\}$ and $\{z|dp(z) = 0\}$, do not intersect and where all zeros of dE are simple. As shown in Krichever *et al.* [1997], the set $(T_{A_i,E}, T_{B_i,E}, E_s)$ define a local coordinate system on $\mathcal{D}^0$.

The Whitham equations (46) define a system of commuting meromorphic vector fields (flows) on $\mathcal{M}_g(n)$ which are holomorphic on $\mathcal{D}' \subset \mathcal{M}_g(n)$ and have the form

$$\frac{\partial}{\partial T_j} T_{A_i,E} = 0, \;\; \frac{\partial}{\partial T_j} T_{B_i,E} = 0, \;\; \partial_{T_j} E_s = \frac{d\Omega_j}{dp}(q_s)\partial_X E_s. \tag{49}$$

An important consequence of (49) is that the space $\mathcal{M}_g(n)$ admits a natural foliation by the joint level sets of the functions $T_{A_i,E}, \; T_{B_i,E}$. The leaves of the foliation are smooth $(2g + n - 1)$-dimensional submanifolds, and are invariant under the flows of the Whitham hierarchy (46).

A special case of the construction of exact solutions to (46) in [Krichever [1988]) may now be described as follows: the moduli space $\mathcal{M}_g(n, m)$ provides the solutions of the first $n + m$-flows of (46) parametrized by $3g$ constants, which are the set

$$T_{A_i,Q} = \oint_{A_i} dQ, \; T_{B_i,Q} = \oint_{B_i} dQ, \; a_i = \oint_{A_i} QdE. \tag{50}$$

Let us consider the joint level set of functions (47, 50). Then the functions

$$T_i = \frac{1}{i} Res_{P_0}(E^{-i/n} QdE) \tag{51}$$

define coordinates on its open set $\mathcal{D}'$ where the zero divisors of dE and dQ do not intersect. The projection

$$\mathcal{M}_g(n, m) \longrightarrow \mathcal{M}_g(n) : \;\; (\Gamma, E, Q) \longmapsto (\Gamma, E) \tag{52}$$

defines (Γ, E) as a function of the coordinates on $\mathcal{M}_g(n, m)$. For each fixed set of parameters $T_{A_i,E}, \; T_{B_i,E}, T_{A_i,Q}, T_{A_i,Q}, \; a_i$, the map $(T_i)_{i=1}^{i=n+m} \rightarrow \mathcal{M}_g(n)$ satisfies the Whitham equations (46).

For the proof of this statement it is enough to note that if we use $E(z)$ as a local coordinate on Γ, then as we saw earlier, the equations (46) are equivalent to the equations $\partial_{T_i} p(E, T) = \partial_X \Omega_i(E, T)$. These are the compatibility conditions for the existence of a generating function for all the Abelian differentials $d\Omega_i$. In fact, if we set

$$dS = QdE, \tag{53}$$

then it turns out that

$$\partial_{T_i} dS = d\Omega_i, \ \partial_X dS = dQ, \tag{54}$$

(For the proof of (54), it is enough to check that the right and the left hand sides of it have the same analytical properties.)

Consider now the second Abelian integral Q as a function of the same parameters T_i, $1 \le i \le n + m$. Then $Q(p, T)$ satisfies the same equations as E, i.e.

$$\partial_{T_i} Q = \{\Omega_i, Q\}. \tag{55}$$

Furthermore,

$$\{E, Q\} = 1. \tag{56}$$

We note that (56) can be viewed as a Whitham version of the so-called string equation (or Virasoro constraints) in a non-perturbative theory of 2-d gravity (Douglas [1990], Witten [1991]).

The solution of the Whitham hierarchy can be summarized in a single τ-function defined as follows. The key underlying idea is that suitable submanifolds of $\mathcal{M}_g(n, m)$ can be parametrized by Whitham times T_A, to each of which is associated a "dual" time T_{DA}, and an Abelian differential $d\Omega_A$, which generates with the help of equation (46) the T_A-flow.

Recall that the coefficients of the pole of dS determine $n + m$ Whitham times (51). Their dual variables are

$$T_{Dj} = Res_P(z^{-j} dS), \tag{57}$$

and the associated Abelian differentials are the familiar $d\Omega_i$ of (28) (complex normalized). When $g > 0$, the moduli space $\mathcal{M}_g(n, m)$ has in addition $5g$ more parameters. We consider only the foliations for which $3g$ parameters $T_{A_i,E}$, $T_{B_i,E}$, and $T_{A_i,Q}$ (defined by (47, 50) are fixed.

Thus the case $g > 0$ leads to two more sets of g Whitham times, namely each a_k and $T_{B_k,Q}$. Their dual variables are

$$a_{Dk} = -\frac{1}{2\pi i} \oint_{B_k} dS, \quad T^Q_{Dk} = \frac{1}{2\pi i} \oint_{A_k^-} E dS. \tag{58}$$

(Because EdS has a jump on the cycle, one has to be careful in choosing a side of integration. The superscript A_k^- here means the left hand side of the cut with respect to the natural orientation.) The corresponding Abelian differentials are respectively the holomorphic differentials $d\omega_k$ and the differentials $d\Omega^E_k$, defined to be holomorphic everywhere on Γ except along the A_j cycles, where they have discontinuities

$$d\Omega_k^{E+} - d\Omega_k^{E-} = \delta_{jk} dE. \tag{59}$$

We denote the collection of all $2g+n+m$ times by $T_A = (T_j, a_k, T_k^E = T_{B_k,Q})$.

We can now define the τ-function of the Whitham hierarchy by

$$\ln \tau(T) = \mathcal{F}(T) = \frac{1}{2}\sum_A T_A T_{DA} + \frac{1}{4\pi i}\sum_{k=1}^{g} a_k T_k^E E(A_k \cap B_k). \tag{60}$$

where $A_k \cap B_k$ is the point of intersection of the A_k and B_k cycles. Note that the definition of the τ-function for a general case of the universal Whitham hierarchy (for which a corresponding moduli space is the space of curves with fixed pair of Abelian integrals with several poles) is given by the same formula. The only difference is that there are more times and more corresponding differentials (see (Krichever *et al.* [1997, 1999]).

As shown in Krichever [1994] the derivatives of $\mathcal{F}$ with respect to the $2g+n+m$ Whitham times T_A are given by

$$\begin{aligned}
\partial_{T_A}\mathcal{F} &= T_{DA} + \frac{1}{2\pi i}\sum_{k=1}^{g} \delta_{a_k,A} T_k^E E(A_k \cap B_k),\\
\partial^2_{T_i,T_j}\mathcal{F} &= Res_P(z^i d\Omega_j),\\
\partial^2_{a_k,A}\mathcal{F} &= \frac{1}{2\pi i}\left(E(A_k \cap B_k)\delta_{(E,k),A} - \oint_{B_k} d\Omega_A\right),\\
\partial^2_{(E,k)A}\mathcal{F} &= \frac{1}{2\pi i}\oint_{A_k} E\, d\Omega_A,\\
\partial^3_{ABC}\mathcal{F} &= \sum_{q_s} Res_{q_s}\left(\frac{d\Omega_A d\Omega_B d\Omega_C}{dE dQ}\right).
\end{aligned} \tag{61}$$

These formulae show that the τ-functions encodes the whole hierarchy, because the coefficients of expansions of the differentials at the puncture, as well as their periods are given by derivatives of τ. (Note that formulae (61) require some modifications in the multi-puncture case for differentials with nonzero residues (see D'Hoker *et al.* [1997]).)

1.4 Topological Landau-Ginzburg Models on Riemann Surfaces

In general, a two-dimensional quantum field theory is specified by the correlation functions $< \phi(z_1)\cdots\phi(z_N) >_g$ of its local physical observables $\phi_i(z)$ on any surface Γ of genus g. Here $\phi_i(z)$ are operator-valued tensors on Γ. The operators act on a Hilbert space of states with a designated vacuum state $|\Omega>$. Topological field theories are theories where the correlation functions are actually independent of the insertion points z_i. Thus they depend only on the labels of the fields ϕ_i and the genus g of Γ. This independence implies that for all practical purposes, the operator product $\phi_i(z_i)\phi_j(z_j)$ can be replaced by the formal operator algebra

$$\phi_i\phi_j = \sum_k c_{ij}^k \phi_k. \tag{62}$$

The associativity of operator compositions translates into the associativity of the operator algebra (62). Furthermore, the operator algebra is commutative.

As shown in (Dijkgraaf *et al.* [1990, 1991a, 1991b], the partition function $\mathcal{F}(x_1, \dots, x_n)$ for the marginal deformations of a topological field theory with n primary fields $\phi_1, \dots, \phi_n$ satisfies an overdetermined system of equations which are equivalent to the condition that the commutative algebra with generators ϕ_k and the structure constants defined by the third derivatives of $\mathcal{F}$:

$$c_{klm}(x) = \frac{\partial^3 \mathcal{F}(x)}{\partial x^k \partial x^l \partial x^m}, \tag{63}$$

$$\phi_k \phi_l = c_{kl}^m(x)\, \phi_m;\ c_{kl}^m = c_{kli}\, \eta^{im};\ \eta_{ki}\eta^{im} = \delta_k^m, \tag{64}$$

is an associative algebra, i.e.

$$c_{ij}^k(x) c_{km}^l(x) = c_{jm}^k(x) c_{ik}^l(x) \tag{65}$$

In addition, it is required that there exist constants r^m such that the constant metric η in (64) is equal to

$$\eta_{kl} = r^m c_{klm}(x). \tag{66}$$

In terms of $\mathcal{F}$ the conditions (65) become a system of non-linear equations called the Witten-Dijkgraaf-Verlinde-Verlinde (WDVV) equations. In recent years these equations have become a key element of the theory of Gromov-Witten invariants and have been applied for solving various problems of enumerative geometry.

In the original work (Dijkgraaf *et al.* [1991a,b]) a solution to WDVV equations for some topological Landau-Ginzburg theories was found. In Krichever [1992], it was noted that the calculation of Dijkgraaf [1991], are similar to the construction of solutions to the dispersionless Lax equations which are the zero genus case of the Whitham hierarchy. The results of Krichever [1992] were generalized for higher genus Whitham hierarchies for Lax equations in Dubrovin [1992]. The general case of the universal Whitham hierarchy was considered in Krichever [1994].

Let us consider the space $\mathcal{M}_g(n) = \{\Gamma, E\}$ of normalized Abelian integrals on genus g curves with a single pole of order n at the puncture (for simplicity, we consider only the one-puncture case). As before, we identify $\mathcal{M}_g(n)$ with $\mathcal{M}_g(n, 1)$ by the choice $dQ = d\Omega_1$. The relevant leaf within $\mathcal{M}_g(n, 1)$ is of dimension $n - 1 + 2g$ and is given by the constraints

$$T_n = 0,\ T_{n+1} = \frac{n}{n+1},$$

$$\oint_{A_k} dE = 0,\ \oint_{B_k} dE = fixed,\ \oint_{A_k} dQ = 0. \tag{67}$$

The leaf is parametrized by the $(n-1)$ Whitham times T_A, $A = 1, \cdots, n-1$, and by the periods a_j and $T_j^E = T_{B_j,Q}$ defined by (50). The fields ϕ_A of the theory can be identified with $d\Omega_j/dQ$. We take the $2g$ additional fields to be given by $d\omega_j/dQ$ and $d\Omega_j^E/dQ$, where the differentials $d\omega_j$ and $d\Omega_j^E$ are the ones associated to a_j and T_j^E, as described earlier.

Let η_{AB} and c_{ABC} be defined by

$$\eta_{A,B} = \sum_{q_s} Res_{q_s} \frac{d\Omega_A d\Omega_B}{dE}, \quad c_{ABC} = \sum_{q_s} Res_{q_s} \frac{d\Omega_A d\Omega_B d\Omega_C}{dEdQ}, \tag{68}$$

where q_s are the zeroes of dE, and the indices A,B,C are running this time through the augmented set of $n-1+2g$ indices given by $T_A = (T_i, a_j, T_j^E)$. Then $\eta_{i,j} = \delta_{i+j,n}$, $\eta_{a_j,(E,k)} = \delta_{j,k}$. All other pairings vanish.

The algebra $\phi_A\phi_B = c_{AB}^C\phi_C$ can be identified with the algebra of functions at zeros q_s of the differential dE which is obviously associative. From (61) it follows that $\partial^3_{ABC}\mathcal{F}(T) = c_{ABC}$. We have $\eta_{AB} = c_{1AB}$, also. Therefore, the τ-function of the Whitham hierarchy $\mathcal{F}(T_i, a_j, T_j^E)$ restricted to the leaf (67) is a solution of the WDVV equations.

Remarkably, the larger spaces $\mathcal{M}_g(n,m)$ can accommodate the gravitational descendants of the fields ϕ_A. More precisely, consider for $g=0$ the leaf of the space $\mathcal{M}_0(n, mn+1)$ given by the following normalization

$$T_{in} = 0, \ i = 1, \dots, m, \quad T_{nm+1} = \frac{nm}{nm+1}.$$

The space of Whitham times is automatically increased to the correct number by taking all the coefficients of QdE. The additional $m(n-1)$ fields may be identified with the first m gravitational descendants of the primary fields. Namely, the p-th descendant $\sigma_p(\phi_i)$ of the primary field ϕ_i is just $d\Omega_{pn+i}/dQ$. This statement is a direct corollary of the following result proved in Krichever [1994].

The correlation functions given by $< \phi_A\phi_B\phi_C >= \partial^3_{ABC}\mathcal{F}$ *with* $\sigma_p(\phi_i) = d\Omega_{i+pn}/dQ$ *satisfy the factorization properties for descendant fields*

$$< \sigma_p(\phi_i)\phi_B\phi_C > = < \sigma_{p-1}(\phi_i)\phi_j > \eta^{jk} < \phi_k\phi_B\phi_C >,$$

where ϕ_i, $i = 1, \dots, n-1$ *are primary fields, and* ϕ_A *are all fields (including descendants).* Factorization properties for descendant fields were derived by Witten [1988a, 1988b, 1991, 1992].

1.5 Seiberg-Witten Solutions of N=2 SUSY Gauge Theories

Moduli spaces of geometric structures are appearing increasingly frequently as the key to the physics of certain supersymmetric gauge or string theories. One recurring feature is a moduli space of degenerate vacua in the physical theory. The physics of the theory is then encoded in a Kähler geometry on the space of vacua, or, in presence of powerful constraints such as N=2 supersymmetry, in an even more restrictive *special geometry*, where the Kähler potential is dictated by a single holomorphic function $\mathcal{F}$, called the prepotential.

In Seiberg [1994a,b] Seiberg and Witten introduced the following fundamental ansatz that for $N=2$ SUSY gauge theories:

(i) the quantum moduli space should be parametrized a family of Riemann surfaces $\Gamma(a)$, now known as the spectral curves of the theory;

(ii) on each $\Gamma(a)$, there is a meromorphic one-form $d\lambda$, such that its derivatives along the moduli space are meromorphic differentials;

(iii) $\mathcal{F}$ is determined by the periods of $d\lambda$

$$a_k = \oint_{A_k} d\lambda, \quad a_{D,k} = \frac{1}{2\pi i}\oint_{B_k} d\lambda, \quad \frac{\partial \mathcal{F}}{\partial a_k} = a_{D,k}. \tag{69}$$

In Gorsky *et al.* [1995], it was noticed that the moduli space of curves for $SU(N)$ theories can be identified with spectral curves of the N-periodic Toda lattice. It was also noted that the generating differential $d\lambda$ coincides with the generating differential dS (c.f. 53) of the Whitham hierarchy. A general approach for solution of the Seiberg-Witten ansatz was developed in Krichever *et al.* [1997]. We present here key elements of this approach.

Let now $n = (n_\alpha),\ m = (m_\alpha),\ \alpha = 1, \ldots, N$, be multi-indices, and $\mathcal{M}_g(n, m)$ be the moduli space (Γ, E, Q) of pairs of Abelian differentials on Γ with poles of orders n_α and m_α at punctures P_α. The dimension of this space is equal to

$$\dim \mathcal{M}_g(n, m) = 5g - 3 + 3N + \sum_{\alpha=1}^{N}(n_\alpha + m_\alpha). \tag{70}$$

The Whitham coordinates on this space can be introduced in a similar way to the one puncture case. The Abelian integral E defines a coordinate system z_α near each P_α by

$$E = z_\alpha^{-n_\alpha} + R_\alpha^E \log z_\alpha,$$

(for simplicity we assume that n_α is strictly positive). Then the formulae

$$T_{\alpha,i} = -\frac{1}{i} Res_{P_\alpha}(z_\alpha^i Q dE), \quad T_{\alpha,0} = Res_{P_\alpha}(Q dE), \tag{71}$$

define $\sum_{\alpha=1}^N (n_\alpha + m_a) + N - 1$ parameters ($\sum_\alpha T_{\alpha,0} = 0$).

The remaining parameters needed to parametrize $\mathcal{M}_g(n, m)$ consist of the $2N-2$ residues of dE and dQ

$$R_\alpha^E = Res_{P_\alpha} dE, \quad R_\alpha^Q = Res_{P_\alpha} dQ, \quad \alpha = 2, \cdots, N, \tag{72}$$

and $5g$ parameters which are the periods of dE, dQ and a-periods of $dS = q dE$ given by (47, 50).

In Krichever *et al.* [1997], it was shown that the joint level sets of all parameters except $a_k = \oint_{A_k} dS$ define a smooth foliation of the open set $\mathcal{D}'$ of $\mathcal{M}_g(n, m)$, which is independent of the choices made to define the coordinates themselves. This intrinsic foliation is central to the Seiberg-Witten theory and the Hamiltonian theory of soliton equations. We shall refer to it as the *canonical foliation*.

Our goal is to construct now a symplectic form ω on the complex $2g$-dimensional space obtained by restricting the fibration $\mathcal{N}_g^g(n, m)$ to a g-dimensional leaf $\mathcal{M}$ of the canonical foliation of $\mathcal{M}_g(n, m)$.

Let us consider the Abelian integrals E and Q as multi-valued functions on the fibration. Despite their multivaluedness, their differentials along any leaf of the canonical fibration are well-defined. In fact, E and Q are well-defined in a small neighbourhood of the puncture P_1. The ambiguities in their values anywhere on each Riemann surface consist only of integer combinations of their residues or periods along closed cycles. Thus these ambiguities are constant along any leaf of the canonical foliation, and disappear upon differentiation. The differentials along the fibrations obtained this way will be denoted by δE and δQ. Restricted to vectors tangent to the fiber, they reduce to the differentials dE and dQ. These arguments show that (Krichever *et al.* [1997])
the following two-form on the fibration $\mathcal{N}^g(n,m)$ restricted to a leaf $\mathcal{M}$ of the canonical foliation of $\mathcal{M}_g(n,m)$

$$\omega_{\mathcal{M}} = \delta\Big(\sum_{i=1}^{g} Q(\gamma_i)dE(\gamma_i)\Big) = \sum_{i=1}^{g} \delta Q(\gamma_i) \wedge dE(\gamma_i) \tag{73}$$

defines a holomorphic symplectic form which is equal to

$$\omega_{\mathcal{M}} = \sum_{i=1}^{g} da_i \wedge d\phi_i, \tag{74}$$

where ϕ_k are canonical coordinates on the Jacobian of the curve.

Note that the first set of formulae (61) implies that the restriction of the logarithm of the τ-function of the Whitham hierarchy on a leaf of the canonical foliation satisfies relations (69) for the prepotential, and therefore, the function $\mathcal{F}(T)$ given by (60) is a solution of the Seiberg-Witten ansatz. Although the results presented above suggest deep relations between N=2 gauge theories, soliton equations, their Whitham theory, and Landau-Ginzburg type models, such relations are still not fully understood at the present time. Nevertheless, the parallels between these fields allows us to apply to the study of the prepotential $\mathcal{F}$ of gauge theories the methods developed in the theory of solitons. In D'Hoker *et al.* [1997], with the help of these methods, the renormalization group equation for the prepotential $\mathcal{F}$ for SU(N_c) gauge theories with $N_f < 2N_c$ hypermultiplets of masses m_j in the fundamental representation was derived. It was shown that this equation is powerful enough to generate explicit expressions for the contributions of instanton processes to any order.

We conclude this chapter by a discussion of connections of the symplectic form (73) with the Hamiltonian theory of soliton equations. The Hamiltonian theory of finite-dimensional and spatial one-dimensional soliton equations is a rich subject which has been developed extensively over the years (see Faddeev *et al.* [1987], Dickey [1991]). However, until recently much less was known about the 2D case. In Krichever *et al.* [1997, 1999], a new algebro-geometric approach to the Hamiltonian theory of soliton equations was developed. This approach is uniformly applicable for all integrable systems: finite-dimensional, spatial one- or two-dimensional evolution equations. Its universality is based on a universal symplectic form which can be defined on a space of operators in terms of operators and their eigenfunctions, only. For simplicity we consider here the Lax equations (1) for the operators (2) with scalar coefficients.

Let $\psi(x, k)$ be a formal solution of the form

$$\psi(x, k) = e^{kx}(1 + \sum_{s=1}^{\infty} \xi_s(x)k^{-s}) \tag{75}$$

to the equation $L\psi = k^n\psi$, normalized by the condition $\psi(0, k) = 1$. The coefficients h_i of the expansion

$$\partial_x \ln \psi = k + \sum_{s=1}^{\infty} h_s k^{-s} \tag{76}$$

are differential polynomials in the coefficients u_i of L, i.e $h_i = h_i(u)$. They are densities of integrals of motions $H_s = \int h_i(u)dx$ of the Lax equation (1). Let us introduce the dual formal solution

$$\psi^* = e^{-kx}(1 + \sum_{s=1}^{\infty} \xi_s^*(x)k^{-s}) \tag{77}$$

of the formal adjoint equation $\psi^* L = k^n\psi^*$, normalized by the condition $\int \psi^*\psi dx = 1$.

The main ingredients are the one-forms δL and $\delta\Psi_0$. The one-form δL is given by

$$\delta L = \sum_{i=0}^{n-2} \delta u_i \partial_x^i,$$

and can be viewed as an operator-valued one-form on the space of operators L. Similarly, the coefficients of the series ψ are explicit integro-differential polynomials in u_i. Thus $\delta\psi$ can be viewed as a one-form on the space of operators with values in the space of formal series.

Consider the following two-form on the space of operators L

$$\omega = \text{Res}_\infty \left(\int (\psi^* \delta L \wedge \delta\psi) dx \right) dp, \tag{78}$$

where $p = k + \sum_s H_s k^{-s}$. In Krichever *et al.* [1999] it was shown that on the subspaces of the operators L defined by the constrains $\{H_i = const,\ i = 1, \ldots, n-1\}$ the form ω

(i) *defines a symplectic structure, i.e. a closed non-degenerate two-form;*
(ii) *the form ω is actually independent of the normalization point ($x = 0$) for the formal Bloch solution $\psi(x, k)$;*
(iii) *the flows (1) are Hamiltonian with respect to this form, with the Hamiltonians $2nH_{m+n}(u)$.*

Consider now the leaves $\mathcal{M}^0$ of the canonical foliation on $\mathcal{M}_g(n, 1)$ corresponding to zero values of variables $\oint dE = 0$. These leaves correspond to spectral curves of one-dimensional finite-gap Lax operators, i.e. we have a geometric map of the Jacobian bundle $\mathcal{N}^0$ over $\mathcal{M}^0$ to the space of operators

$$\mathcal{G} : \mathcal{N}^0 \longrightarrow (L).$$

A connection of the Hamiltonian theory of soliton equations with the previous construction of algebro-symplectic structures associated with the Seiberg-Witten theory was established in Krichever [1997]. Namely, it was proved that

(iv) *the restriction of the symplectic form ω given by (78) via the geometric map to the leaf $\mathcal{N}^0$ of the canonical foliation is holomorphic symplectic form equals*

$$\omega_{\mathcal{M}^0} = \mathcal{G}^*(\omega) = \sum_{k=1}^{g} da_k \wedge d\phi_k.$$

The results presented above are a particular case of more general settings. It turns out the the algebro-geometric symplectic structure (73) on the general leaves of the canonical foliation on $\mathcal{M}_g(n, 1)$ is a restriction of the basic symplectic structure for 2D soliton equations. The symplectic structure on leaves of the foliation for $\mathcal{M}_g(n, m)$ for $m > 1$ is the restriction of higher symplectic forms for soliton equations. It is necessary to emphasize that the same soliton equations are Hamiltonian flows with respect to all these structures but generated by different Hamiltonians. A variety of examples which show the universality of our approach can be found in Krichever *et al.* [1997, 1999].

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