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Vacuum curves of elliptic L-operators and representations of Sklyanin algebra

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Abstract

An algebro-geometric approach to representations of Sklyanin algebra is proposed. To each 2×2 quantum L-operator an algebraic curve parametrizing its possible vacuum states is associated. This curve is called the vacuum curve of the L-operator. An explicit description of the vacuum curve for quantum L-operators of the integrable spin chain of XYZ type with arbitrary spin ℓ is given. The curve is highly reducible. For half-integer ℓ it splits into $\ell+\frac{1}{2}$ components isomorphic to an elliptic curve. For integer ℓ it splits into ℓ elliptic components and one rational component. The action of elements of the L-operator to functions on the vacuum curve leads to a new realization of the Sklyanin algebra by difference operators in two variables restricted to an invariant functional subspace.

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1 Introduction

The Yang-Baxter equation

$$R^{23}(u-v)R^{13}(u)R^{12}(v) = R^{12}(v)R^{13}(u)R^{23}(u-v) \tag{1.1}$$

is a key relation of the theory of quantum integrable models[1]. Each solution of eq. (1.1) generates a hierarchy of integrable models. The commutation relations for elements of quantum L-operators of this hierarchy are given by the "intertwining" equation

$$R^{23}(u-v)L^{13}(u)L^{12}(v) = L^{12}(v)L^{13}(u)R^{23}(u-v) \tag{1.2}$$

Here L is an operator in the tensor product $\mathbf{C}^N \otimes \mathbf{C}^n$ and all the factors in (1.2) are operators in the tensor product $\mathbf{C}^N \otimes \mathbf{C}^n \otimes \mathbf{C}^n$.

Let $n = 2$ in (1.1); then the most general R-matrix with elliptic dependence of the spectral parameter u corresponds to the famous 8-vertex model (or, equivalently, to the XYZ magnet):

$$R(u) = \sum_{a=0}^{3} W_a(u+\eta)\sigma_a \otimes \sigma_a \tag{1.3}$$

Here σ_a are Pauli matrices (σ_0 is the unit matrix), $W_a(u)$ are functions of u with parameters η and τ:

$$W_a(u) = \frac{\theta_{a+1}(u|\tau)}{\theta_{a+1}(\eta|\tau)} \tag{1.4}$$

(the Jacobi θ-functions are listed in the Appendix).

In [1], [2] Sklyanin reformulated the problem of solving equation (1.2) in terms of representations of an algebra with four generators $S_0, S_\alpha,\ \alpha = 1,2,3$, subject to the homogeneous quadratic relations

$$\begin{aligned} [S_0, S_\alpha]_- &= iJ_{\beta\gamma}[S_\beta, S_\gamma]_+ \\ [S_\alpha, S_\beta]_- &= i[S_0, S_\gamma]_+ \end{aligned} \tag{1.5}$$

Here and below $\{\alpha,\beta,\gamma\}$ is any *cyclic* permutation of $\{1,2,3\}$, $[A,B]_\pm = AB \pm BA$. The structure constants have the form

$$J_{\alpha\beta} = \frac{J_\beta - J_\alpha}{J_\gamma} \tag{1.6}$$

where J_α are arbitrary parameters. The algebra generated by the S_a with relations (1.5) and structure constants (1.6) is called *Sklyanin algebra*. There is a two-parametric family of such algebras. The relations of the Sklyanin algebra imposed on S_a are equivalent to the condition that the L-operator of the form

$$L(u) = \sum_{a=0}^{3} W_a(u)\, S_a \otimes \sigma_a \tag{1.7}$$

[1]We use the following standard notation: let R be a linear operator acting in the tensor product $\mathbf{C}^n \otimes \mathbf{C}^n$, then $R^{ij},\ i,j = 1,2,3$ is an operator in the tensor product $\mathbf{C}^n \otimes \mathbf{C}^n \otimes C^n$ which acts as R in the tensor product of i-th and j-th factors of the triple tensor product and as identity operator on the factor left.

(considered as an operator in $\mathcal{H} \otimes \mathbf{C}^2$, where $\mathcal{H}$ is a module over the algebra) satisfies eq. (1.2). Hence, any finite-dimensional representation of the Sklyanin algebra provides a solution to eq. (1.2).

As it was shown in [2], the operators S_a, $a = 0, \ldots, 3$ admit a realization as second order difference operators in the space of meromorphic functions $F(z)$ of a complex variable z. One of the series of such representations (called the principal analytic series or series a) in [2]) is

$$(S_a F)(z) = \frac{(i)^{\delta_{a,2}} \theta_{a+1}(\eta)}{\theta_1(2z)} \Big(\theta_{a+1}(2z - 2\ell\eta) F(z+\eta) - \theta_{a+1}(-2z - 2\ell\eta) F(z-\eta) \Big) \tag{1.8}$$

(hereafter $\theta(z) \equiv \theta(z|\tau)$). A straightforward but tedious computation shows that the operators (1.8) for any τ, η, ℓ satisfy the commutation relations (1.5) with the following values of the structure constants:

$$J_\alpha = \frac{\theta_{\alpha+1}(0)\theta_{\alpha+1}(2\eta)}{\theta^2_{\alpha+1}(\eta)} \tag{1.9}$$

Therefore, τ and η parametrize the structure constants, while ℓ characterizes a representation.

In [3], a connection of the representation theory of the Sklyanin algebra with the finite-gap theory of soliton equations was found. It has been proved that for integer ℓ the operator S_0 is algebraically integrable and, therefore, is a difference analogue of the classical Lame operator

$$\mathcal{L} = -\frac{d^2}{dx^2} + \ell(\ell+1)\wp(x)$$

which can be obtained from S_0 in the limit $\eta \to 0$. Finite gap properties of higher Lame operators (for arbitrary integer values of ℓ) were established in [4]. Algebraic integrability of S_0 implies, in particular, extremely unusual spectral properties of this operator. Putting $F_n = F(n\eta + z_0)$, we assign to (1.8) the difference Schrödinger operators

$$S_a F_n = A^a_n F_{n+1} + B^a_n F_{n-1} \tag{1.10}$$

with quasiperiodic coefficients. The spectrum of a generic operator of this form in the space $l^2(\mathbf{Z})$ (square integrable sequences F_n) has a Cantor set type structure. If η is a rational number, $\eta = P/Q$, the operators (1.10) have Q-periodic coefficients. In general, Q-periodic difference Schrödinger operators have Q unstable bands in the spectrum.

It was shown [3] that the operator S_0 given by eq. (1.8) for positive integer values of ℓ and *arbitrary* η has 2ℓ unstable bands in the spectrum. Its Bloch functions $\psi(z)$ are parametrized by points of a hyperelliptic curve of genus 2ℓ defined by the equation

$$y^2 = P(\varepsilon) = \prod_{i=1}^{2\ell+1} (\varepsilon^2 - \varepsilon_i^2) \tag{1.11}$$

Considered as a function on the curve, ψ is the Baker-Akhiezer function. Moreover, Bloch eighenfunctions $\psi(z, \pm\varepsilon_i)$ of the operator S_0 at the edges of bands span an invariant functional subspace for all the operators S_a. The corresponding $4\ell+2$-dimensional representation of the Sklyanin algebra is a direct sum of two equivalent $2\ell+1$-dimensional irreducible representations of the Sklyanin algebra.

As it is well-known from the early days of the finite-gap theory, the ring of operators commuting with a finite-gap operator is isomorphic to a ring of meromorphic functions on the corresponding spectral curve with poles at "infinite points". For difference operators it was proved in [5], [6]. Therefore, the ring of the operators commuting with S_0 is generated by S_0 and an operator D such that

$$D^2 = P(S_0) = \prod_{i=1}^{2\ell+1} (S_0^2 - \varepsilon_i^2)$$

In [7], for any algebraic curve Γ of genus g with two punctures, a special basis in the ring $\mathcal{M}$ of meromorphic functions A_i with poles at the punctures was introduced. These functions define *the almost graded structure* in $\mathcal{M}$, i.e. the product of two basis functions has the form

$$A_i A_j = \sum_{k=-g/2}^{g/2} c_{i,j}^k A_{i+j+k}$$

Therefore, for any finite-gap difference operator S there exist commuting with S operators M_i such that $M_i \psi = A_i \psi$, where ψ is the Baker-Akhiezer eigenfunction of S. The ring generated by the operators M_i has the same almost graded structure as the ring $\mathcal{M}$. A particular case of the last result corresponding to the ring of operators commuting with the Sklyanin operator S_0 was recently rediscovered in [8].

The established connection of the Sklyanin operator S_0 with the algebro-geometric theory of soliton equations is a part of the theory of integrable multi-dimensional differential or difference linear operators with elliptic coefficients. It turns out that the spectral theory of such operators is isomorphic to the theory of finite-dimensional integrable systems. Among them are spin generalizations of the Calogero-Moser system, the Ruijesenaars-Schneider systems and nested Bethe ansatz equations (see [9], [3] and [10], respectively).

In [3], we have suggested a relatively simple way to derive the realization (1.8) of the Sklyanin algebra by difference operators. Our approach clarifies their origin as diference operators acting on the *vacuum curve* of the L-operator (1.7). This notion, introduced by one of the authors [11], proved to be useful in analysis of the Yang-Baxter equation by methods of algebraic geometry. The construction of vacuum curve and vacuum vectors is a suitable generalization of a key property of the elementary R-matrix (1.3), which was used by Baxter in his famous solution of the 8-vertex model and called by him "pair-propagation through a vertex" [12]. In that particular case the vacuum curve is an elliptic curve.

In this paper we present a more detailed analysis of the vacuum curve of the higher spin L-operator (1.7). The main result of this work is a new realization of the Sklyanin algebra by difference operators acting in *two* variables rather than one (Sects. 5, 6). Remarkably, the "finite-gap" operator S_0 in this new realization preserves its form, while the form of the other three generators changes drastically, with both variables entering them in a non-trivial way. This realization naturally follows from the explicit construction of the vacuum curve of the elliptic L-operator and corresponding vacuum vectors.

Let us recall the main definitions. Consider an *arbitrary* L-operator L with two-dimensional auxiliary space $\mathbf{C}^2$, i.e., an arbitrary $2N \times 2N$ matrix represented as 2×2

matrix whose matrix elements are $N \times N$ matrices L_{11}, L_{12}, L_{21}, L_{22}:

$$L = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}. \tag{1.12}$$

They act in a linear space $\mathcal{H} \cong \mathbf{C}^N$ which is called the *quantum space* of the L-operator.

Let $X \in \mathcal{H}$, $|U\rangle = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} \in \mathbf{C}^2$ be two vectors such that

$$L\Big(X \otimes |U\rangle\Big) = Y \otimes |V\rangle \tag{1.13}$$

where $Y \in \mathcal{H}$, $|V\rangle \in \mathbf{C}^2$ are some vectors. Suppose (1.13) holds; then the vector X is called a *vacuum vector* of the L-operator. Multiplying (1.13) from the left by the covector ${}^{\perp}\langle V| = (V_2, -V_1)$, orthogonal to $|V\rangle$, we get the necessary and sufficient condition for existence of the vacuum vectors:

$${}^{\perp}\langle V|L|U\rangle X = 0 \tag{1.14}$$

Here ${}^{\perp}\langle V|L|U\rangle$ is an operator in $\mathcal{H}$. The *vacuum curve* is defined by the equation

$$\det\Big({}^{\perp}\langle V|L|U\rangle\Big) = 0 \tag{1.15}$$

By construction, it is embedded into $\mathbf{C}P^1 \times \mathbf{C}P^1$.

The relation (1.13) (in the particular case $\mathcal{H} \cong \mathbf{C}^2$) was the starting point for Baxter in his solution of the 8-vertex and XYZ models [12]. In the context of the quantum inverse scattering method [13] the equivalent condition (1.14) is more customary. It defines local vacua of the (gauge-transformed) L-operator. A generalization of that solution to the higher spin XYZ model was given by Takebe in [14], where, in particular, the generalized vacuum vectors were constructed. However, the vacuum curve itself was implicit in that work.

In Sect. 2 we recollect the main formulas related to the Sklyanin algebra and its representations. As we shall see in Sect. 3, the vacuum curves corresponding to finite-dimensional representations of the principal analytic series [2] of the Sklyanin algebra are completely reducible, i.e. their equations have the form

$$P(U,V) = \prod_{k=1}^{M} P_k(U,V) = 0$$

with some polynomials P_k. If the dimension of $\mathcal{H}$ is even, then $2M = \dim \mathcal{H}$ and all irreducible components are elliptic curves isomorphic to the vacuum curve of the R-matrix R (1.3) found by Baxter. If the dimension of $\mathcal{H}$ is odd, then $2M = \dim \mathcal{H} + 1$ and all but one irreducible components are isomorphic to the elliptic curve, while the last component is rational. We would like to mention, in passing, that one of the variables in the new realization of the Sklyanin algebra arises just as the index k marking irreducible components of the vacuum curve. The other one is, like in [3], a uniformization parameter on any one of the elliptic components.

The origin of this realization can be traced back to the structure of the vacuum curve of the operator Λ defined by the left (or right) hand side of eq. (1.2). By definition, Λ

is an operator in the tensor product $\mathcal{H} \otimes \mathbf{C}^2 \otimes \mathbf{C}^2$. Let us consider it as an operator in $\hat{\mathcal{H}} \otimes \mathbf{C}^2$, where $\hat{\mathcal{H}} = \mathcal{H} \otimes \mathbf{C}^2$ is product of the first two factors. Proceeding in the way explained above, we may assign a vacuum curve to this operator. As it was shown in [11], this curve is a result of a "composition" of vacuum curves of the operators L and R. This curve is completely reducible, too. Besides, all but one components of this latter curve are *two-fold degenerate*, i.e., the equation of the curve has the form

$$\hat{P}(U,V) = \hat{P}_0(U,V) \left[\prod_{i=1}^{N} \hat{P}_i(U,V) \right]^2 = 0$$

The multiplicity of components of the composed vacuum curve "mixes" different components of the vacuum curve of the L-operator and – in terms of action to the vacuum vectors – leads to shifts in the index k which appear in the two-variable realization of the Sklyanin generators. A detailed study of action of the $L(u)$ to the vacuum vectors is given in Sect. 4.

At last, in Sect. 7 we make some remarks on the trigonometric degeneration of the construction presented in Sects. 4-6. This is related to representations of the quantum algebra $U_q(sl(2))$. In particular, our approach provides a new type of representations of this algebra which is a q-analogue of representations of the $sl(2)$ algebra by vector fields on the two-dimensional sphere.

2 The elliptic L-operator and the Sklyanin algebra

In this section we collect main formulas related to the Sklyanin algebra and its representations.

Consider the elliptic L-operator (1.7):

$$\begin{aligned} L(u) &= \sum_{a=0}^{3} W_a(u) S_a \otimes \sigma_a \\ &= \begin{pmatrix} W_0(u)S_0 + W_3(u)S_3 & W_1(u)S_1 - iW_2(u)S_2 \\ W_1(u)S_1 + iW_2(u)S_2 & W_0(u)S_0 - W_3(u)S_3 \end{pmatrix} \end{aligned} \tag{2.1}$$

with $W_a(u)$ given by (1.4). In the sequel we write $\theta_a(x) \equiv \theta_a(x|\tau)$, $\theta_a(x|\frac{\tau}{2}) \equiv \bar{\theta}_a(x)$ for brevity. The operators S_a obey Sklyanin algebra (1.5) with the structure constants

$$J_{\alpha\beta} = -(-1)^{\alpha-\beta} \frac{\theta_1^2(\eta)\theta_{\gamma+1}^2(\eta)}{\theta_{\alpha+1}^2(\eta)\theta_{\beta+1}^2(\eta)} \tag{2.2}$$

We remind that $\{\alpha, \beta, \gamma\}$ is any cyclic permutation of $\{1, 2, 3\}$. Note that the coefficients W_a in (2.1) satisfy the algebraic relations

$$(W_\alpha^2 - W_\beta^2) = J_{\alpha\beta}(W_\gamma^2 - W_0^2) \tag{2.3}$$

Any two of them are independent and define an elliptic curve $\mathcal{E}_0 \subset \mathbf{C}P^3$ as intersection of the two quadrics. The spectral parameter u uniformizes this curve, i.e., (2.3) is identically satisfied under the substitutions (1.4), (2.2). Another useful form of the structure constants is

$$J_{\alpha\beta} = \frac{J_\beta - J_\alpha}{J_\gamma}\,, \qquad J_\alpha = \frac{\theta_{\alpha+1}(2\eta)\theta_{\alpha+1}(0)}{\theta^2_{\alpha+1}(\eta)} \tag{2.4}$$

(see (1.6)). The algebra has two independent central elements:

$$\Omega_0 = \sum_{a=0}^{3} S_a^2\,, \qquad \Omega_1 = \sum_{\alpha=1}^{3} J_\alpha S_\alpha^2 \tag{2.5}$$

In some formulas below it is more convenient to deal with the renormalized generators:

$$S_a = (i)^{\delta_{a,2}}\theta_{a+1}(\eta)\mathcal{S}_a \tag{2.6}$$

The relations (1.5) can be rewritten in the form

$$\begin{aligned} (-1)^{\alpha+1} I_{\alpha 0}\mathcal{S}_\alpha\mathcal{S}_0 &= I_{\beta\gamma}\mathcal{S}_\beta\mathcal{S}_\gamma - I_{\gamma\beta}\mathcal{S}_\gamma\mathcal{S}_\beta \\ (-1)^{\alpha+1} I_{\alpha 0}\mathcal{S}_0\mathcal{S}_\alpha &= I_{\gamma\beta}\mathcal{S}_\beta\mathcal{S}_\gamma - I_{\beta\gamma}\mathcal{S}_\gamma\mathcal{S}_\beta \end{aligned} \tag{2.7}$$

where

$$I_{ab} = \theta_{a+1}(0)\theta_{b+1}(2\eta)$$

The central elements in the renormalized generators read

$$\Omega_0 = \theta_1^2(\eta)\mathcal{S}_0^2 + \sum_{\alpha=1}^{3}(-1)^{\alpha+1}\theta^2_{\alpha+1}(\eta)\mathcal{S}_\alpha^2\,, \qquad \Omega_1 = \sum_{\alpha=1}^{3}(-1)^{\alpha+1} I_{\alpha\alpha}\mathcal{S}_\alpha^2 \tag{2.8}$$

Sklyanin's realization [2] of the algebra by difference operators has the form (1.8). The parameter ℓ is "spin" of the representation. When $\ell \in \frac{1}{2}\mathbf{Z}_+$, these operators have a finite-dimensional invariant subspace $\mathcal{T}^+_{4\ell}$ of *even* θ-functions of order 4ℓ, i.e., the space of entire functions $F(z)$, $z \in \mathbf{C}$, such that $F(-z) = F(z)$ and

$$\begin{aligned} F(z+1) &= F(z) \\ F(z+\tau) &= \exp(-4\ell\pi i\tau - 8\ell\pi i z)F(z) \end{aligned} \tag{2.9}$$

This is the representation space of a $(2\ell+1)$-dimensional irreducible representation (of series a)) of the Sklyanin algebra[2]. In the spin-ℓ represenation of series a), the central elements take the values

$$\begin{aligned} \Omega_0 &= 4\theta_1^2((2\ell+1)\eta) \\ \Omega_1 &= 4\theta_1(2\ell\eta)\theta_1(2(\ell+1)\eta) \end{aligned} \tag{2.10}$$

In terms of shift operators $T_\pm \equiv \exp(\pm\eta\partial_z)$ (1.8) can be rewritten for the renormalized generators (2.6) in a little bit simpler form:

$$\mathcal{S}_a = \frac{\theta_{a+1}(2z - 2\ell\eta)}{\theta_1(2z)}T_+ - \frac{\theta_{a+1}(-2z - 2\ell\eta)}{\theta_1(2z)}T_- \tag{2.11}$$

[2]There are three other series of irreducible representations [2], [15], [3] which we do not discuss here.

Plugging the difference operators (1.8) into (2.1), one can represent the L-operator in a "factorized" form [16] which is especially convenient in the computations. Introduce the matrix

$$\hat{\Phi}(u;z) = \begin{pmatrix} \bar{\theta}_4(z-\ell\eta+\frac{u}{2}) & \bar{\theta}_3(z-\ell\eta+\frac{u}{2}) \\ \\ \bar{\theta}_4(z+\ell\eta-\frac{u}{2}) & \bar{\theta}_3(z+\ell\eta-\frac{u}{2}) \end{pmatrix} \tag{2.12}$$

Then it holds

$$L(u)F(z) = 2\theta_1(u+2\ell\eta)\hat{\Phi}^{-1}(u+4\ell\eta;z) \begin{pmatrix} F(z+\eta) & 0 \\ \\ 0 & F(z-\eta) \end{pmatrix} \hat{\Phi}(u;z) \tag{2.13}$$

where elements of the L-operator are assumed to act to the function $F(z)$ according to (1.8).

3 The vacuum curve

Our goal in this section is to find an explicit parametrization of the vacuum curve and vacuum vectors of the elliptic L-operator (2.1) in a finite-dimensional representation of the Sklyanin algebra.

In practical computations, it will be more convenient to write 2-dimensional vectors like $|U\rangle$ from the left of the L-operator rather than from the right, so the basic equation (1.13) acquires the form

$$\langle U|L(u)\,X = \langle V|\,Y \tag{3.1}$$

Here $\langle U| = (U_1, U_2)$, $\langle V| = (V_1, V_2)$ are *covectors*. It is easy to see that this leads to the definition of the vacuum curve, $\det(\langle U|L|V^{\perp}\rangle) = 0$, equivelent to the one given in the Introduction.

Let the operators S_a in (2.1) be realized as in (1.8) and let them act in the finite-dimensional subspace $\mathcal{T}^{+}_{4\ell}$ (2.9). Then X and Y in (3.1) are functions of z belonging to the space $\mathcal{T}^{+}_{4\ell}$. In this section it is convenient to normalize $\langle U|$ and $\langle V|$ by the condition that the second components are equal to 1: $\langle U| = (U,\,1)$, $\langle V| = (V,\,1)$.

In this notation the equation of the vacuum curve (1.15) acquires the form

$$\det K(U,V) = 0 \tag{3.2}$$

where

$$K(U,V) = (U-V)W_0S_0 + (1-UV)W_1S_1 + i(1+UV)W_2S_2 + (U+V)W_3S_3 \tag{3.3}$$

In other words, we have to find a relation on U and V under that the operator $K(U,V)$ has an eigenvector (belonging to $\mathcal{T}^{+}_{4\ell}$) with zero eigenvalue (a "zero mode"). One can write $K(U,V)$ in an equivalent form

$$K(U,V) = \sum_{a=0}^{3} \zeta_a W_a S_a$$

where the new variables ζ_a are subject to the constraint

$$\zeta_1^2 + \zeta_2^2 + \zeta_3^2 = \zeta_0^2 \tag{3.4}$$

Theorem 3.1 *The vacuum curve of the L-operator (2.1) for $\ell \in \mathbf{Z}_+ + \frac{1}{2}$ splits into a union of $\ell + \frac{1}{2}$ elliptic curves isomorphic to $\mathcal{E}_0$ (2.3), for $\ell \in \mathbf{Z}_+$ the vacuum curve splits into ℓ components isomorphic to $\mathcal{E}_0$ and one rational component. The equation defining the vacuum curve in the coordinates U, V has the form*

$$P_\ell(U,V) = 0, \qquad \ell \in \mathbf{Z}_+ + \frac{1}{2} \tag{3.5}$$

$$(U-V)^{-1} P_\ell(U,V) = 0, \qquad \ell \in \mathbf{Z}_+ \tag{3.6}$$

where

$$P_\ell(U,V) = \prod_{n=0}^{[\ell]} (U^2 + V^2 - 2\Delta_{\ell-n} UV + \Gamma_{\ell-n}(1 + U^2V^2))$$

($[\ell]$ denotes the integer part of ℓ), $\Gamma_0 = 0$, $\Delta_0 = 1$ and the u-independent constants Γ_k, Δ_k for $k \geq 1$ are defined below in (3.24), (3.25).

Remark Note that at integer ℓ the polynomial $P_\ell(U,V)$ is divisible by $(U-V)^2$, so the left hand side of (3.6) is a polynomial.

Proof. Let us consider first a linear combination of the generators S_a with arbitrary coefficients:

$$K = \sum_{a=0}^{3} y_a S_a \tag{3.7}$$

According to (1.8), the equation $KX = 0$ is equivalent to

$$\frac{s(-z-\ell\eta)}{s(z-\ell\eta)} = \frac{X(z+\eta)}{X(z-\eta)} \tag{3.8}$$

where

$$s(z) = \sum_{a=0}^{3} (i)^{\delta_{a,2}} y_a \theta_{a+1}(\eta)\theta_{a+1}(2z) \tag{3.9}$$

belongs to the space $\mathcal{T}_4$ of θ-functions of order 4 with the following monodromy properties: $s(z+1) = s(z)$, $s(z+\tau) = \exp(-4\pi i\tau - 8\pi i z)s(z)$. Note that $\dim \mathcal{T}_4 = 4$ and $s(z)$ has 4 zeros in the fundamental domain of the lattice formed by 1 and τ. The function $s(z)$ can be parametrized by its zeros:

$$s(z) = \prod_{i=1}^{4} \theta_1(z - a_i), \qquad \sum_{i=1}^{4} a_i = 0 \tag{3.10}$$

(up to an inessential factor). Functions $\theta_{a+1}(2z)$ form a basis in the space $\mathcal{T}_4$. Expanding (3.10) with respect to this basis, we get:

$$s(z) = 2\sum_{a=0}^{3} (-1)^a \theta_{a+1}(2z) \prod_{i=2}^{4} \theta_{a+1}(-a_1 - a_i) \tag{3.11}$$

Let us first consider the case $\ell \geq 1$. Since $X(z)$ has 4ℓ zeros in the fundamental domain, the equality (3.8) is possible only if the functions $X(z+\eta)$ and $X(z-\eta)$ have $4\ell-4$ common zeros. Such a cancellation of zeros of the numerator and the denominator in the r.h.s. of (3.8) takes place if zeros of $X(z)$ are arranged into "strings", i.e.,

$$X(z) = \prod_{i=1}^{4} \prod_{j=0}^{m_i-1} \theta_1(z - z_i - 2j\eta) \tag{3.12}$$

with the condition

$$\sum_{i=1}^{4} m_i = 4\ell\,, \quad m_i \geq 0 \tag{3.13}$$

At this stage we do not impose any other restrictions; in particular, it is not implied that $X(z)$ is even. The number m_i is called length of the string. So, if all $m_i > 0$, there are four strings in (3.12) with total length 4ℓ. If some of m_i equal zero, the number of strings is less than four. Let $X(z)$ be given by (3.12), then

$$\frac{X(z+\eta)}{X(z-\eta)} = \prod_{i=1}^{4} \frac{\theta_1(z - z_i + \eta)}{\theta_1(z - z_i - (2m_i - 1)\eta)} \tag{3.14}$$

Identifying zeros of the left hand side of (3.8) with zeros of (3.14), we get

$$z_i + a_i = -(\ell - 1)\eta\,, \quad i = 1, \ldots, 4. \tag{3.15}$$

Taking these relations into account, we then identify poles of (3.8) and (3.14) and conclude that (unordered) sets of points $(z_i + (2m_i - 1)\eta)$ and $(-z_i + \eta)$, $i = 1, \ldots, 4$ must coincide. In order to describe all the possibilities, denote by P any permutation of indices (1234) and consider systems of linear equations

$$z_i + z_{P(i)} = -2(m_i - 1)\eta\,, \quad i = 1, \ldots, 4 \tag{3.16}$$

for each P. Solutions to these systems consistent with the condition

$$\sum_{i=1}^{4} z_i = -4(\ell - 1)\eta \tag{3.17}$$

which follows from (3.10) and (3.15), yield all possible values of z_i. Depending on the choice of P rank of the linear system (3.16) may equal 4, 3 or 2, i.e., the number of free parameters in the solutions may be respectively 0, 1 or 2. Since coefficients of the function $s(z)$ with respect to the basis $\theta_a(2z)$ are already constrained by one relation (3.4) (describing the embedding $\mathbf{C}P^1 \times \mathbf{C}P^1 \subset \mathbf{C}P^3$), the vacuum curve corresponds to the case of minimal rank. It is easy to see that system (3.16) has rank 2 for the following three permutations of (1234): (2143), (3412), (4321); otherwise its rank is greater than 2. Solving (3.16) for these three permutations, we arrive at the following result.

The function $s(z)$ has the form

$$s(z) \equiv s^{(m)}(z) = \theta_1(z - \mu_1)\theta_1(z - \mu_2)\theta_1(z + \mu_1 + 2(\ell - m)\eta)\theta_1(z + \mu_2 - 2(\ell - m)\eta) \tag{3.18}$$

where $m = 0, 1, \ldots, [\ell]$ and μ_1, μ_2 are free parameters. The corresponding "zero mode" of K is given by the formula [3]

$$
\begin{aligned}
X^{(m)}(z) \;\; = \;\; & \prod_{j=1}^{m} \theta_1\left(z+\mu_1+(\ell+1-2j)\eta\right)\theta_1\left(z-\mu_1-(\ell+1-2j)\eta\right) \\
& \prod_{j=1}^{2\ell-m} \theta_1\left(z+\mu_2+(\ell+1-2j)\eta\right)\theta_1\left(z-\mu_2-(\ell+1-2j)\eta\right)
\end{aligned}
\tag{3.19}
$$

Note that $X^{(m)}(z)$ is even function.

Let us set $k = \ell - m$, $k = 0, 1, \ldots, \ell$ for $\ell \in \mathbf{Z}_+$ and $k = \frac{1}{2}, \frac{1}{2}, \ldots, \ell$ for $\ell \in \mathbf{Z}_+ + \frac{1}{2}$ and identify $\mu_1 + \mu_2 = u$ (the spectral parameter), $\mu_2 - \mu_1 = 2\zeta + 2k\eta$, where ζ is a uniformizing parameter on the vacuum curve. It follows from (1.4), (3.9), (3.11) and (3.17) that the coefficients ζ_a in (3.3) are given by the formula

$$
\zeta_a = \frac{2(i)^{\delta_{a,2}}\theta_{a+1}(2k\eta)\theta_{a+1}(2\zeta)}{\bar{\theta}_3(\zeta+n\eta)\bar{\theta}_3(\zeta-k\eta)} \tag{3.20}
$$

They satisfy homogeneous quadratic equations

$$
\begin{aligned}
& \zeta_1^2 + \zeta_2^2 + \zeta_3^2 = \zeta_0^2 \, , \\
& \sum_{\alpha=1}^{3} \zeta_\alpha^2 \frac{\theta_{\alpha+1}^2(0)}{\theta_{\alpha+1}^2(2k\eta)} = 0
\end{aligned}
\tag{3.21}
$$

which provide purely algebraic description of the vacuum curve. Rescaling the coordinates, $\zeta_a = \xi_a \theta_{a+1}(2k\eta)$ (for $k \neq 0$), one may represent (3.21) in the form independent of k:

$$
\begin{aligned}
& \sum_{\alpha=1}^{3} \xi_\alpha^2 \theta_{\alpha+1}^2(0) = 0 \, , \\
& \sum_{\alpha=1}^{3} \xi_{3-\alpha}^2 \theta_{\alpha+1}^2(0) = 0
\end{aligned}
\tag{3.22}
$$

It is straightforward to see that the equations (2.3) defining the elliptic curve $\mathcal{E}_0$ can be transformed to the same form. This means that the vacuum curve is reducible: it splits into a union of components isomorphic to $\mathcal{E}_0$ (corresponding to nonzero values of k).

In the coordinates U, V the system (3.21) is equivalent to a single equation of degree 4:

$$
U^2 + V^2 - 2\Delta_k UV + \Gamma_k(1 + U^2V^2) = 0 \tag{3.23}
$$

where

$$
\Gamma_k = \frac{\theta_1^2(2k\eta)\theta_4^2(2k\eta)}{\theta_2^2(2k\eta)\theta_3^2(2k\eta)} = \left(\frac{\bar{\theta}_1(2k\eta)}{\bar{\theta}_2(2k\eta)}\right)^2 \tag{3.24}
$$

[3]These functions coincide with "intertwining vectors" from the paper [14] found there by a different method.

$$\begin{aligned}\Delta_k &= \frac{\theta_3^2(0)}{\theta_4^2(0)}\left(\frac{\theta_4^2(2k\eta)}{\theta_3^2(2k\eta)}+\frac{\theta_1^2(2k\eta)}{\theta_2^2(2k\eta)}\right) \\ &= \frac{\bar\theta_2^2(0)\bar\theta_3(2k\eta)\bar\theta_4(2k\eta)}{\bar\theta_2^2(2k\eta)\bar\theta_3(0)\bar\theta_4(0)} \end{aligned} \tag{3.25}$$

For $k=\frac{1}{2}$ this equation after trivial redefinitions coincides with the equation of the vacuum curve for spin $\frac{1}{2}$ derived for the first time by Baxter [12]. So formulas (3.21)-(3.25) are valid for $\ell=\frac{1}{2}$ as well (in this case k takes only one value $\frac{1}{2}$).

The case $k=0$ (4 strings of length ℓ each) needs a separate consideration. In this case $s(z)$ is an even function, so $\zeta_0=0$ and the corresponding component of the vacuum curve is a rational curve (the cone). Therefore, for integer ℓ the vacuum curve has a rational component (corresponding to $k=0$) given by the equation $U=V$. ■

Remark At $\tau=0$ the elliptic L-operator (2.1) degenerates into a trigonometric one. It corresponds to the higher spin XXZ model. Its vacuum curve was studied in [17]. This curve can be obtained from our formulas (3.23)-(3.25) in the limit $\tau\to 0$, $\eta\to 0$ provided $\eta'\equiv\eta/\tau$ is finite: $\Gamma_k=0$, $\Delta_k=\cos(4\pi k\eta')$. Eq. (3.23) turns into $(U-e^{4\pi ik\eta'}V)(U-e^{-4\pi ik\eta'}V)=0$, so each elliptic component (of degree 4) splits into two rational component (of degree 2 each). The equation for the whole vacuum curve acquires the form

$$\prod_{n=-[\ell]}^{[\ell]}\left(U-e^{4\pi in\eta'}V\right)=0$$

Other types of rational degenerations are also possible [18]. Their detailed classification is not discussed here.

We call the "copy" of the elliptic curve $\mathcal{E}_0$ defined by eq. (3.23) with $k=\ell$ *the highest component* of the vacuum curve. The results of [3] suggest that it plays a distinguished role in representations of the Sklyanin algebra.

Let us summarize the results of this section and prepare some formulas which will be extensively used in the sequel. The vacuum curve of the elliptic spin-ℓ L-operator consists of $[\ell+\frac{1}{2}]$ components which are marked by $k=0,1,\ldots,\ell$ for integer ℓ and $k=\frac{1}{2},\frac{3}{2},\ldots,\ell$ for half-integer ℓ. Each elliptic component is uniformized by the variable ζ:

$$U=\frac{\bar\theta_4(\zeta+k\eta)}{\bar\theta_3(\zeta+k\eta)}, \qquad V=\frac{\bar\theta_4(\zeta-k\eta)}{\bar\theta_3(\zeta-k\eta)}$$

(at $k=0$ it degenerates into a rational component). In the sequel it will be more convenient to pass to the homogeneous coordinates and work with the following two-dimensional covectors:

$$\langle\zeta|\equiv\left(\bar\theta_4(\zeta),\,\bar\theta_3(\zeta)\right), \qquad \langle-\zeta|=\langle\zeta| \tag{3.26}$$

The vacuum vectors are given by (3.19), which we rewrite in terms of ζ, k:

$$
\begin{aligned}
X_k^\ell(z,\zeta) \;\; = \;\; & \prod_{j=1}^{\ell-k}\theta_1\Big(z-\zeta+\frac{u}{2}+(\ell-k+1-2j)\eta\Big)\,\theta_1\Big(z+\zeta-\frac{u}{2}-(\ell-k+1-2j)\eta\Big) \\
& \prod_{j=1}^{\ell+k}\theta_1\Big(z+\zeta+\frac{u}{2}+(\ell+k+1-2j)\eta\Big)\,\theta_1\Big(z-\zeta-\frac{u}{2}-(\ell+k+1-2j)\eta\Big)
\end{aligned} \tag{3.27}
$$

We call X_ℓ^ℓ *the highest vacuum vector*. The formula (3.27) still has sense for negative values of k, too, extending $X_k^\ell(z,\zeta)$ to $-\ell \le k \le \ell$. From now on we assume that k varies in this region. Let us point out the following simple properties of vacuum vectors:

$$X_k^\ell(-z,\zeta) = X_k^\ell(z,\zeta)$$

$$X_k^\ell(z,-\zeta) = X_{-k}^\ell(z,\zeta)$$

$$X_k^\ell(z+\eta,\zeta) = X_k^\ell(z,\zeta+\eta)\frac{\theta_1(z-\zeta+\frac{u}{2}-(k-\ell)\eta)\theta_1(z-\zeta-\frac{u}{2}+(k+\ell)\eta)}{\theta_1(z-\zeta+\frac{u}{2}+(k-\ell)\eta)\theta_1(z-\zeta-\frac{u}{2}-(k+\ell)\eta)}$$

$$X_k^\ell(z,\zeta+\eta) = X_{k+1}^\ell(z,\zeta)\frac{\theta_1(z-\zeta+\frac{u}{2}-(\ell-k)\eta)\theta_1(z+\zeta-\frac{u}{2}+(\ell-k)\eta)}{\theta_1(z+\zeta+\frac{u}{2}-(\ell+k)\eta)\theta_1(z-\zeta-\frac{u}{2}+(\ell+k)\eta)}$$

In the next section we study how $L(u)$ acts to the vacuum vectors.

4 Action of $L(u)$ to the vacuum vectors

By a straightforward computation we obtain:

$$\langle\zeta+k\eta|\, L(u)X_k^\ell(z,\zeta) = 2\theta_1(u-2\ell\eta)\langle\zeta-k\eta|\, Y_k^\ell(z,\zeta+\eta) \tag{4.1}$$

where

$$
\begin{aligned}
Y_k^\ell(z,\zeta) \;\; = \;\; & \prod_{j=0}^{\ell-k-1}\theta_1\Big(z-\zeta+\frac{u}{2}+(\ell-k+1-2j)\eta\Big)\,\theta_1\Big(z+\zeta-\frac{u}{2}-(\ell-k+1-2j)\eta\Big) \\
& \prod_{j=1}^{\ell+k}\theta_1\Big(z+\zeta+\frac{u}{2}+(\ell+k+1-2j)\eta\Big)\,\theta_1\Big(z-\zeta-\frac{u}{2}-(\ell+k+1-2j)\eta\Big)
\end{aligned} \tag{4.2}
$$

It is easy to see that

$$Y_k^\ell(z,-\zeta) = Y_{-k}^\ell(z,\zeta+2\eta)$$

Indicating explicitly the vacuum vector dependence on the spectral parameter u, we get for any $-\ell \le k \le \ell$:

$$Y_k^\ell(z,\zeta+\eta,u) = X_k^\ell(z,\zeta,u+2\eta) \tag{4.3}$$

and

$$X_{-k}^\ell(z,\zeta,u) = X_k^\ell(z,\zeta,-u)$$

Below we supress the u-dependence of vacuum vectors if it is not misleading.

For the highest vacuum vectors we have

$$Y_\ell^\ell(z,\zeta) = X_\ell^\ell(z,\zeta)$$

Therefore, at $k=\ell$ one can rewrite eq. (4.1) in the form

$$\langle\zeta \pm \ell\eta|\, L(u) X_{\pm\ell}^\ell(z,\zeta) = 2\theta_1(u-2\ell\eta)\langle\zeta \mp \ell\eta|\, X_{\pm\ell}^\ell(z,\zeta\pm\eta) \tag{4.4}$$

The equation with the lower sign is obtained by changing $\zeta \to -\zeta$ and using the above listed properties of the vacuum vectors. From (3.27) we see that

$$X_\ell^\ell\Big(z,\, \zeta - \frac{u}{2},\, u\Big) = X_{-\ell}^\ell\Big(z,\, \zeta + \frac{u}{2},\, u\Big) \tag{4.5}$$

This property allows us to convert (4.4) into a closed system of equations for the vector X_ℓ^ℓ only. Indeed, let us substitute $\zeta \to \zeta \mp \frac{u}{2}$ in the first (second) equality in (4.4) and after that make use of (4.5). In this way we get the following system of equations:

$$\begin{aligned}
\langle\zeta - \tfrac{u}{2} + \ell\eta|\, L(u) X_\ell^\ell(z,\zeta-\tfrac{u}{2}) &= 2\theta_1(u-2\ell\eta)\langle\zeta - \tfrac{u}{2} - \ell\eta|\, X_\ell^\ell(z,\zeta-\tfrac{u}{2}+\eta)\\
\langle\zeta + \tfrac{u}{2} - \ell\eta|\, L(u) X_\ell^\ell(z,\zeta-\tfrac{u}{2}) &= 2\theta_1(u-2\ell\eta)\langle\zeta + \tfrac{u}{2} + \ell\eta|\, X_\ell^\ell(z,\zeta-\tfrac{u}{2}-\eta)
\end{aligned} \tag{4.6}$$

In our paper [3] it is shown that the representation (1.8) in the variable ζ follows from solution to this system. Therefore, we can say that these representations are realized in the space of functions on the highest component of the vacuum curve.

Now let us turn to other components of the vacuum curve. It is possible to express action of the L-operator in terms of the vectors $X_k^\ell(z,\zeta)$ only. A straightforward computation leads to the following result:

$$\begin{aligned}
&\Big\langle \zeta \pm w + k\eta \Big|\, L(u-2w) X_k^\ell(z,\zeta,u)\\
=\;& 2\,\frac{\theta_1(2(\zeta\mp\ell\eta))\theta_1(u-2w\mp 2k\eta)}{\theta_1(2(\zeta-k\eta))}\Big\langle \zeta\pm w - k\eta\Big| X_k^\ell(z,\zeta\pm\eta,u)\\
\pm\;& 2\,\frac{\theta_1(2(k\mp\ell)\eta)\theta_1(2\zeta\mp u\pm 2w)}{\theta_1(2(\zeta-k\eta))}\Big\langle \zeta\mp w - k\eta\Big| X_{k\pm1}^\ell(z,\zeta,u)
\end{aligned} \tag{4.7}$$

Here w is an arbitrary parameter. The formulas with upper and lower signs are connected by the transformation $\zeta \to -\zeta$ and $k \to -k$. Equalities of this type are sometimes called "intertwining relations" or "vertex-face correspondence". In the particular case $\ell = \frac{1}{2}$ they were suggested by Baxter [12] and recently generalized to any half-integer values of ℓ by Takebe [14].

Putting w equal to 0 and comparing these formulas with (4.1), we get the following linear relations between vectors Y_k^ℓ and X_k^ℓ:

$$Y_k^\ell(z,\zeta+\eta,u) \;=\; \frac{\theta_1(2(\zeta\mp\ell\eta))\theta_1(u\mp 2k\eta)}{\theta_1(2(\zeta-k\eta))\theta_1(u-2\ell\eta))}\,X_k^\ell(z,\zeta\pm\eta,u)$$

$$\pm\;\frac{\theta_1(2(k\mp\ell)\eta)\theta_1(2\zeta\mp u)}{\theta_1(2(\zeta-k\eta))\theta_1(u-2\ell\eta))}\,X_{k\pm1}^\ell(z,\zeta,u) \tag{4.8}$$

The two choices of signs in the r.h.s. provide a 4-term linear relation between vectors X_k^ℓ among themselves. Moreover, taking into account the property (4.3), one can rewrite (4.8) in a more symmetric form

$$\begin{aligned}
&\theta_1(2(\zeta-k\eta))\theta_1(u-2\ell\eta)\,X_k^\ell(z,\zeta,u+2\eta)\\
=\;&\theta_1(2(\zeta-\ell\eta))\theta_1(u-2k\eta)\,X_k^\ell(z,\zeta+\eta,u)\\
+\;&\theta_1(2(k-\ell)\eta)\theta_1(2\zeta-u)\,X_{k+1}^\ell(z,\zeta,u)\\
=\;&\theta_1(2(\zeta+\ell\eta))\theta_1(u+2k\eta)\,X_k^\ell(z,\zeta-\eta,u)\\
-\;&\theta_1(2(k+\ell)\eta)\theta_1(2\zeta+u)\,X_{k-1}^\ell(z,\zeta,u)
\end{aligned} \tag{4.9}$$

Note that the three variables ζ, k and u after a proper rescaling enter symmetrically inspite of their very different nature.

5 Difference operators on the vacuum curve

Formulas (4.7) give rise to some distinguished difference operators in *two* variables – ζ and k – related to representations of the Sklyanin algebra. Consider a particular case of the first equation (4.7) (with upper sign) at $u=0$, $w=-\frac{u}{2}$. From (3.27) it follows that $X_k^\ell(z,\zeta,0)$ is *even function of* ζ. Note that it automatically implies $X_{-k}^\ell(z,\zeta,0)=X_k^\ell(z,\zeta,0)$. Therefore, the substitution $\zeta\to-\zeta$ provides us with another equation for the same vector $X_k^\ell(z,\zeta,0)$. Together with the first one they form a closed system:

$$\begin{aligned}
&\Big\langle\zeta\mp\frac{u}{2}\pm k\eta\Big|\,L(u)X_k^\ell(z,\zeta,0)\\
=\;&2\,\frac{\theta_1(2(\zeta\mp\ell\eta))\theta_1(u-2k\eta)}{\theta_1(2(\zeta\mp k\eta))}\Big\langle\zeta\mp\frac{u}{2}\mp k\eta\Big|X_k^\ell(z,\zeta\pm\eta,0)\\
+\;&2\,\frac{\theta_1(2(k-\ell)\eta)\theta_1(2\zeta\mp u)}{\theta_1(2(\zeta\mp k\eta))}\Big\langle\zeta\pm\frac{u}{2}\mp k\eta\Big|X_{k+1}^\ell(z,\zeta,0)
\end{aligned} \tag{5.1}$$

which is an extension of the simpler system (4.6) on the highest component to the whole vacuum curve.

Let X_i, ${}^i\bar X$, $i=1,2,\ldots,2\ell+1$, be dual bases in the space $\mathcal{T}_{4\ell}^{+}$ of theta-functions and its dual $\mathcal{T}_{4\ell}^{+,*}$, respectively, i.e. $({}^i\bar X,\,X_j)=\delta_{ij}$, where $(\;,\;)$ denotes pairing of the spaces

$\mathcal{T}_{4\ell}^{+,*}$ and $\mathcal{T}_{4\ell}^{+}$. The linear space $\mathcal{F}_{\zeta,k}$ of functions spanned by

$$ {}^{i}X_k^\ell(\zeta) \equiv \Big({}^{i}\bar{X},\, X_k^\ell(z,\zeta,0)\Big) $$

which are functions of ζ, k but not of z, plays a central role in what follows. Let $\mathcal{A}$ be any difference operator in z. One can translate its action to the space $\mathcal{F}_{\zeta,k}$ according to the definition

$$ \Big(\mathcal{A}\circ {}^{i}X_k^\ell\Big)(\zeta) = \Big({}^{i}\bar{X},\, \mathcal{A}X_k^\ell(z,\zeta,0)\Big) \tag{5.2} $$

This action is extendable to the whole space $\mathcal{F}_{\zeta,k}$ by linearity.

Note the composition rule

$$ \mathcal{A}\circ(\mathcal{A}'\circ F) = (\mathcal{A}'\mathcal{A})\circ F \tag{5.3} $$

for any $F\in\mathcal{F}_{\zeta,k}$, i.e. the order of the operators $\mathcal{A}$, $\mathcal{A}'$ with respect to the action $\circ$ must be reversed. Indeed, let us write

$$ X_k^\ell(z,\zeta) = \sum_i {}^{i}X_k^\ell(\zeta)X_i(z) $$

and define matrix elements of an operator $\mathcal{A}$ to be

$$ (\mathcal{A})_i^j = \Big({}^{i}\bar{X},\, \mathcal{A}X_i(z)\Big) $$

Then we have

$$ \mathcal{A}\circ {}^{j}X_k^\ell = \sum_i (\mathcal{A})_i^j\; {}^{i}X_k^\ell\,, \qquad \mathcal{A}'\circ {}^{j}X_k^\ell = \sum_i (\mathcal{A}')_i^j\; {}^{i}X_k^\ell\,, $$

$$ \begin{aligned} (\mathcal{A}'\mathcal{A})\circ {}^{j}X_k^\ell &= \Big({}^{j}\bar{X},\, \mathcal{A}'\mathcal{A}X_k^\ell\Big) \\ &= \sum_{i,l} (\mathcal{A}')_l^j(\mathcal{A})_i^l\; {}^{i}X_k^\ell \\ &= \mathcal{A}\circ(\mathcal{A}'\circ {}^{j}X_k^\ell) \end{aligned} $$

After these preliminaries we can make a convolution of the system (5.1) with the basis covectors ${}^{i}\bar{X}$ and rewrite it in terms of the action $\circ$ as a relation between functions in the space $\mathcal{F}_{\zeta,k}$:

$$ \begin{aligned} & \Big\langle \zeta\mp\frac{u}{2}\pm k\eta \Big|\, L(u)\circ X_k^\ell(\zeta) \\ = \;& 2\,\frac{\theta_1(2(\zeta\mp\ell\eta))\theta_1(u-2k\eta)}{\theta_1(2(\zeta\mp k\eta))}\Big\langle \zeta\mp\frac{u}{2}\mp k\eta \Big| X_k^\ell(\zeta\pm\eta) \\ + \;& 2\,\frac{\theta_1(2(k-\ell)\eta)\theta_1(2\zeta\mp u)}{\theta_1(2(\zeta\mp k\eta))}\Big\langle \zeta\pm\frac{u}{2}\mp k\eta \Big| X_{k+1}^\ell(\zeta) \end{aligned} \tag{5.4} $$

(here and below $X_k^\ell(\zeta) \in \mathcal{F}_{\zeta,k}$). These relations form a system of four linear equations for the four functions $(S_a \circ X_k^\ell)(\zeta)$ entering the left hand side. To make this clear, it is useful to rewrite the system in a more explicit form:

$$\begin{pmatrix} \bar\theta_4(\zeta-\frac{u}{2}+k\eta) & \bar\theta_3(z-\frac{u}{2}+k\eta) \\ \bar\theta_4(z+\frac{u}{2}-k\eta) & \bar\theta_3(z+\frac{u}{2}-k\eta) \end{pmatrix} \begin{pmatrix} L_{11}(u)\circ X_k^\ell(\zeta) & L_{12}(u)\circ X_k^\ell(\zeta) \\ L_{21}(u)\circ X_k^\ell(\zeta) & L_{22}(u)\circ X_k^\ell(\zeta) \end{pmatrix}$$

$$= 2 \begin{pmatrix} W_{11}\bar\theta_4(\zeta-\frac{u}{2}-k\eta)X_k^\ell(\zeta+\eta) & W_{11}\bar\theta_3(z-\frac{u}{2}-k\eta)X_k^\ell(\zeta+\eta) \\ W_{22}\bar\theta_4(z+\frac{u}{2}+k\eta)X_k^\ell(\zeta-\eta) & W_{22}\bar\theta_3(z+\frac{u}{2}+k\eta)X_k^\ell(\zeta-\eta) \end{pmatrix}$$

$$+2 \begin{pmatrix} W_{12}\bar\theta_4(\zeta+\frac{u}{2}-k\eta) & W_{12}\bar\theta_3(z+\frac{u}{2}-k\eta) \\ W_{21}\bar\theta_4(z-\frac{u}{2}+k\eta) & W_{21}\bar\theta_3(z-\frac{u}{2}+k\eta) \end{pmatrix} X_{k+1}^\ell(\zeta)$$

where W_{ij} are elements of the matrix

$$W = \begin{pmatrix} \dfrac{\theta_1(2(\zeta-\ell\eta))\theta_1(u-2k\eta)}{\theta_1(2(\zeta-k\eta))} & \dfrac{\theta_1(2(k-\ell)\eta)\theta_1(2\zeta-u)}{\theta_1(2(\zeta-k\eta))} \\[2ex] \dfrac{\theta_1(2(k-\ell)\eta)\theta_1(2\zeta+u)}{\theta_1(2(\zeta+k\eta))} & \dfrac{\theta_1(2(\zeta+\ell\eta))\theta_1(u-2k\eta)}{\theta_1(2(\zeta+k\eta))} \end{pmatrix}$$

Solving this system, we obtain the following action rules of the generators S_a to functions of two variables:

$$(S_0\circ X_k^\ell)(\zeta) = \frac{\theta_1(\eta)}{\theta_1(2\zeta)}\Big(\theta_1(2\zeta-2\ell\eta)X_k^\ell(\zeta+\eta)+\theta_1(2\zeta+2\ell\eta)X_k^\ell(\zeta-\eta)\Big) \tag{5.5}$$

$$\begin{aligned}(S_\alpha\circ X_k^\ell)(\zeta) \;=\; & -(i)^{\delta_{a,2}}\frac{\theta_{\alpha+1}(\eta)}{\theta_1(2\zeta)}\left(\frac{\theta_1(2\zeta-2\ell\eta)\theta_{\alpha+1}(2\zeta-2k\eta)}{\theta_1(2\zeta-2k\eta)}X_k^\ell(\zeta+\eta)\right. \\ & - \left.\frac{\theta_1(2\zeta+2\ell\eta)\theta_{\alpha+1}(2\zeta+2k\eta)}{\theta_1(2\zeta+2k\eta)}X_k^\ell(\zeta-\eta)\right) \\ & + (i)^{\delta_{a,2}}\frac{2\theta_{\alpha+1}(\eta)\theta_{\alpha+1}(2k\eta)\theta_{\beta+1}(2\zeta)\theta_{\gamma+1}(2\zeta)\theta_1(2(k-\ell)\eta)}{\theta_{\beta+1}(0)\theta_{\gamma+1}(0)\theta_1(2\zeta-2k\eta)\theta_1(2\zeta+2k\eta)}X_{k+1}^\ell(\zeta)\end{aligned} \tag{5.6}$$

Note that the formula for S_0 remains the same as in Sklyanin's realization while the other generators are substantially different. They are non-trivial operators in two variables rather than one.

Remark At $k=\ell$ the last term in the r.h.s. of (5.6) disappears and we come back to Sklyanin's formulas in the variable ζ. However, the last three generators differ from the ones in (1.8) by a sign. This sign can be explained if we recall that the action $\circ$ has "contravariant" composition rule (5.3) that amounts to the formal transpositon of the commutation relations. Commutation relations of the Sklyanin algebra imply that the transposition of all generators is equivalent (up to an automorphism of the algebra) to changing signs of S_1, S_2 and S_3. Hence formulas (5.5), (5.6) at $k=\ell$ are indeed equivalent to (1.8).

Do the difference operators (5.5), (5.6) obey the (transposed) Sklyanin algebra? The answer is *no* since, as it is clear from (5.6), the highest shifts in k do not cancel in the commutation relations (2.7). The matter is that functions $X_k^\ell(\zeta)$ are not arbitrary functions of two variables. By construction, they belong to the space $\mathcal{F}_{\zeta,k}$. This implies an additional condition that follows from (4.8) at $u=0$ after convolution with the basis covectors ${}^i\bar{X}$:

$$\begin{aligned}&\frac{\theta_1(2\zeta-2\ell\eta)}{\theta_1(2\zeta)}X_k^\ell(\zeta+\eta)+\frac{\theta_1(2\zeta+2\ell\eta)}{\theta_1(2\zeta)}X_k^\ell(\zeta-\eta)\\ =\;&\frac{\theta_1(2k\eta-2\ell\eta)}{\theta_1(2k\eta)}X_{k+1}^\ell(\zeta)+\frac{\theta_1(2k\eta+2\ell\eta)}{\theta_1(2k\eta)}X_{k-1}^\ell(\zeta)\end{aligned}\tag{5.7}$$

This equality means that S_0 has a "dual" realization as a difference operator in the variable k which has the same form.

Investigating commutation properties of the difference operators standing in (5.5), (5.6), it is convenient to modify them in two respects. First, let us change signs of S_α, thus coming back to the "covariant" action (see the remark above). Second, it is natural to disregard the origin of the k, ℓ as (half) integer numbers and allow them to take arbitrary complex values. Namely, let us set $x=k\eta$, $K_\pm=\exp(\pm\eta\partial_x)$, $T_\pm=\exp(\pm\eta\partial_\zeta)$. In this notation our difference operators in two variables ζ, x are

$$\mathcal{D}_0=\frac{\theta_1(2\zeta-2\ell\eta)}{\theta_1(2\zeta)}T_+ +\frac{\theta_1(2\zeta+2\ell\eta)}{\theta_1(2\zeta)}T_-\tag{5.8}$$

$$\begin{aligned}\mathcal{D}_\alpha\;=\;&\frac{\theta_1(2\zeta-2\ell\eta)\theta_{\alpha+1}(2\zeta-2x)}{\theta_1(2\zeta)\theta_1(2\zeta-2x)}T_+-\frac{\theta_1(2\zeta+2\ell\eta)\theta_{\alpha+1}(2\zeta+2x)}{\theta_1(2\zeta)\theta_1(2\zeta+2x)}T_-\\ -\;&\frac{2\theta_{\alpha+1}(2x)\theta_{\beta+1}(2\zeta)\theta_{\gamma+1}(2\zeta)\theta_1(2x-2\ell\eta)}{\theta_{\beta+1}(0)\theta_{\gamma+1}(0)\theta_1(2\zeta-2x)\theta_1(2\zeta+2x)}K_+\end{aligned}\tag{5.9}$$

The last line can be transformed as follows:

$$\begin{aligned}\mathcal{D}_\alpha\;=\;&\frac{\theta_1(2\zeta-2\ell\eta)}{\theta_1(2\zeta)}\frac{\theta_{\alpha+1}(2\zeta-2x)}{\theta_1(2\zeta-2x)}T_+-\frac{\theta_1(2\zeta+2\ell\eta)}{\theta_1(2\zeta)}\frac{\theta_{\alpha+1}(2\zeta+2x)}{\theta_1(2\zeta+2x)}T_-\\ -\;&\frac{\theta_1(2x-2\ell\eta)}{\theta_1(2x)}\left(\frac{\theta_{\alpha+1}(2\zeta-2x)}{\theta_1(2\zeta-2x)}-\frac{\theta_{\alpha+1}(2\zeta+2x)}{\theta_1(2\zeta+2x)}\right)K_+\end{aligned}\tag{5.10}$$

In the next section we show that these operators do form a non-standard realization of the Sklyanin algebra on a subspace of functions of two variables.

6 Representations of the Sklyanin algebra

Let us consider the operator

$$
\begin{aligned}
\nabla &= \frac{\theta_1(2\zeta-2\ell\eta)}{\theta_1(2\zeta)}\,T_+ + \frac{\theta_1(2\zeta+2\ell\eta)}{\theta_1(2\zeta)}\,T_- \\
&- \frac{\theta_1(2x-2\ell\eta)}{\theta_1(2x)}\,K_+ - \frac{\theta_1(2x+2\ell\eta)}{\theta_1(2x)}\,K_-
\end{aligned} \tag{6.1}
$$

The condition (5.7) is then $\nabla X_k^\ell(\zeta)=0$. The main statement of this section is:

Theorem 6.1 *For any complex parameter ℓ the operators $\mathcal{D}_a$ (5.8), (5.9) form a representation of the Sklyanin algebra (2.7) in the invariant subspace of functions of two variables $X=X(\zeta,x)$ such that $\nabla X=0$. The values of the central elements (2.5) in this representation are*

$$
\begin{aligned}
\Omega_0 &= 4\theta_1^2((2\ell+1)\eta) \\
\Omega_1 &= 4\theta_1(2\ell\eta)\theta_1(2(\ell+1)\eta)
\end{aligned} \tag{6.2}
$$

First of all let us show that the space of solutions to the equation $\nabla X=0$ is invariant under action of the operators $\mathcal{D}_a$.

Lemma 6.1 *The following commutation relations hold:*

$$
\nabla\mathcal{D}_a = \mathcal{D}'_a\nabla \tag{6.3}
$$

where $\mathcal{D}'_0=\mathcal{D}_0$ and

$$
\begin{aligned}
\mathcal{D}'_\alpha &= \frac{\theta_1(2\zeta-2\ell\eta)}{\theta_1(2\zeta)}\,T_+\,\frac{\theta_{\alpha+1}(2\zeta-2x)}{\theta_1(2\zeta-2x)} - \frac{\theta_1(2\zeta+2\ell\eta)}{\theta_1(2\zeta)}\,T_-\,\frac{\theta_{\alpha+1}(2\zeta+2x)}{\theta_1(2\zeta+2x)} \\
&- \frac{\theta_1(2x-2\ell\eta)}{\theta_1(2x)}\,K_+\left(\frac{\theta_{\alpha+1}(2\zeta-2x)}{\theta_1(2\zeta-2x)} - \frac{\theta_{\alpha+1}(2\zeta+2x)}{\theta_1(2\zeta+2x)}\right)
\end{aligned} \tag{6.4}
$$

Corollary 6.1 *The condition*

$$
\nabla X = 0
$$

is invariant under action of the operators $\mathcal{D}_a$.

The lemma can be proved by straightforward though quite long computations. Let us show how to reduce them to a reasonable amount. Our strategy is to begin with the case $\ell=0$. For brevity we use the notation

$$
b_\alpha(\zeta) = \frac{\theta_{\alpha+1}(2\zeta)}{\theta_1(2\zeta)} \tag{6.5}
$$

At $\ell=0$ the operators ∇, $\mathcal{D}_a$, $\mathcal{D}'_a$ take the form

$$
\nabla = T_+ + T_- - K_+ - K_- \tag{6.6}
$$

$$\mathcal{D}_0 = \mathcal{D}'_0 = T_+ + T_- \tag{6.7}$$

$$\mathcal{D}_\alpha = b_\alpha(\zeta - x)T_+ - b_\alpha(\zeta + x)T_- - \Big(b_\alpha(\zeta - x) - b_\alpha(\zeta + x)\Big)K_+ \tag{6.8}$$

$$\mathcal{D}'_\alpha = T_+ b_\alpha(\zeta - x) - T_- b_\alpha(\zeta + x) - K_+\Big(b_\alpha(\zeta - x) - b_\alpha(\zeta + x)\Big) \tag{6.9}$$

It is not too difficult to verify the relations (6.3) by the direct substitution of these operators. Note that in this case the specific form of the function $b_\alpha(\zeta)$ given by (6.5) is irrelevant. This proves the lemma at $\ell = 0$.

The relation $\nabla\mathcal{D}_0 = \mathcal{D}'_0\nabla$ for general values of ℓ is obvious. To prove the other ones in the case $\ell \neq 0$, we modify the shift operators as

$$\tilde{T}_\pm = c_\ell(\pm\zeta)T_\pm\,, \qquad \tilde{K}_\pm = c_\ell(\pm x)K_\pm \tag{6.10}$$

where

$$c_\ell(\zeta) = \frac{\theta_1(2\zeta - 2\ell\eta)}{\theta_1(2\zeta)} \tag{6.11}$$

These $\tilde{T}_\pm$, $\tilde{K}_\pm$ possess the same commutation relations as $T_\pm$, $K_\pm$ *except* for the properties $T_+T_- = T_-T_+ = 1$ (and similarly for $K_\pm$). The latter are substituted by

$$\tilde{T}_\pm\tilde{T}_\mp = \rho_\ell(\pm\zeta)\,, \qquad \tilde{K}_\pm\tilde{K}_\mp = \rho_\ell(\pm x) \tag{6.12}$$

where

$$\rho_\ell(\zeta) \equiv c_\ell(\zeta)c_\ell(-\zeta - \eta) = \frac{\theta_1(2\zeta - 2\ell\eta)\theta_1(2\zeta + 2(\ell+1)\eta)}{\theta_1(2\zeta)\theta_1(2\zeta + 2\eta)} \tag{6.13}$$

Clearly, the operators (6.6)-(6.9) with the substitutions $T_\pm \to \tilde{T}_\pm$, $K_\pm \to \tilde{K}_\pm$ convert into the corresponding operators for $\ell \neq 0$. Moreover, it is easy to see that the same is true for the operator parts of the both sides of eq. (6.3). Only the c-number contributions get modified. Collecting them together, we come to the relation

$$\nabla\mathcal{D}_\alpha = \mathcal{D}'_\alpha\nabla + f_\alpha(\zeta, x)$$

where the function f_α is given by

$$f_\alpha = (\rho_\ell(x) - \rho_\ell(\zeta))b_\alpha(\zeta + x + \eta) + (\rho_\ell(-x) - \rho_\ell(\zeta))b_\alpha(\zeta - x + \eta) + (\zeta \to -\zeta)$$

It is easy to verify that this function has no singularities and has the same monodromy properties in ζ as $b_\alpha(\zeta)$. (This time the explicit form of the functions b_α is of course crucial.) Therefore, $f_\alpha(\zeta, x) = 0$ and the assertion is proved.

Now let us turn to the commutation relations of the Sklyanin algebra. It is more convenient to deal with the relations in the form (2.7). We set

$$\begin{aligned} G_\alpha &\equiv I_{\beta\gamma}\mathcal{D}_\beta\mathcal{D}_\gamma - I_{\gamma\beta}\mathcal{D}_\gamma\mathcal{D}_\beta + (-1)^\alpha I_{\alpha 0}\mathcal{D}_\alpha\mathcal{D}_0 \\ \bar{G}_\alpha &\equiv I_{\gamma\beta}\mathcal{D}_\beta\mathcal{D}_\gamma - I_{\beta\gamma}\mathcal{D}_\gamma\mathcal{D}_\beta + (-1)^\alpha I_{\alpha 0}\mathcal{D}_0\mathcal{D}_\alpha \end{aligned} \tag{6.14}$$

$$\tilde{\Omega}_0 = \theta_1^2(\eta)\mathcal{D}_0^2 + \sum_{\alpha=1}^{3}(-1)^{\alpha+1}\theta_{\alpha+1}^2(\eta)\mathcal{D}_\alpha^2$$

$$\tilde{\Omega}_1 = \sum_{\alpha=1}^{3}(-1)^{\alpha+1}I_{\alpha\alpha}\mathcal{D}_\alpha^2 \tag{6.15}$$

As is clear from (2.7), (2.8), if $\mathcal{D}_\alpha$ were generators of the Sklyanin algebra, one would have $G_\alpha = \bar{G}_\alpha = 0$, $\tilde{\Omega}_0 = \Omega_0$, $\tilde{\Omega}_1 = \Omega_1$.

Lemma 6.2 *For any $\ell \in \mathbf{C}$ it holds*

$$G_\alpha = \lambda_\alpha(\zeta, x)c_\ell(x)K_+\nabla$$

$$\bar{G}_\alpha = -\lambda_\alpha(\zeta, -x-\eta)c_\ell(x)K_+\nabla \tag{6.16}$$

$$\tilde{\Omega}_0 = 4\theta_1^2((2\ell+1)\eta) + \lambda'(\zeta, x)c_\ell(x)K_+\nabla$$

$$\tilde{\Omega}_1 = 4\theta_1(2\ell\eta)\theta_1(2(\ell+1)\eta) + \lambda_0(\zeta, x)c_\ell(x)K_+\nabla \tag{6.17}$$

where

$$\lambda_a(\zeta, x) = -2(-1)^a b_a(\zeta)b_a(x)\frac{\theta_1(2x)\theta_1(2x+2\eta)\theta_1(2\zeta)\theta_1(2\zeta+2\eta)}{\theta_1(2\zeta-2x)\theta_1(2\zeta+2x+2\eta)} + (\zeta \to -\zeta) \tag{6.18}$$

$$\lambda'(\zeta, x) = -2\,\frac{\theta_1(2x)\theta_1(2x+2\eta)\theta_1^2(2\zeta+\eta)}{\theta_1(2\zeta-2x)\theta_1(2\zeta+2x+2\eta)} + (\zeta \to -\zeta) \tag{6.19}$$

Again, the computations can be essentially simplified by dealing with the case $\ell = 0$ first and making the substitutions (6.10) after that. Then only c-number contributions need some additional attention. A few identities used in the computation are given in the Appendix.

It follows from (6.16) that in the space of solutions to the equation $\nabla X = 0$ we have $G_\alpha = \bar{G}_\alpha = 0$ that proves the theorem.

Let us mention the relation

$$\mathcal{D}'_\alpha = -\Xi\,\check{\mathcal{D}}_\alpha^\dagger\,\Xi^{-1} \tag{6.20}$$

where † means transposition of the operator,

$$\begin{aligned}\check{\mathcal{D}}_\alpha &= b_\alpha(\zeta-x)c_\ell(-\zeta-\eta)T_+ - b_\alpha(\zeta+x)c_\ell(\zeta-\eta)T_- \\ &- \Big(b_\alpha(\zeta-x) - b_\alpha(\zeta+x)\Big)c_\ell(-x-\eta)K_+\end{aligned}$$

and Ξ is the operator changing the sign of x: $\Xi f(x) = f(-x)$, $\Xi^2 = 1$. Note that the operators $\check{\mathcal{D}}_\alpha$ differ from $\mathcal{D}_\alpha$ by a "gauge" transformation of the form $U(\zeta, x)(...)U^{-1}(\zeta, x)$ with a function $U(\zeta, x)$. This transformation acts as follows:

$$c_\ell(\mp\zeta-\eta)T_\pm \;\longrightarrow\; c_\ell(\pm\zeta)T_\pm$$

$$c_\ell(\mp x-\eta)K_\pm \;\longrightarrow\; c_\ell(\pm x)K_\pm$$

and takes $\nabla^\dagger$, $\mathcal{D}_0^\dagger$ to ∇, $\mathcal{D}_0$ respectively. This fact allows us to reduce the computation of $\bar{G}_\alpha$ to that of G_α.

Remark The theorem remains true for operators

$$\tilde{\mathcal{D}}_a = \mathcal{D}_a + g_a(\zeta, x)\nabla$$

where g_a are arbitrary functions. Using this remark, one can write down many other equivalent realizations of the Sklyanin algebra in two variables. In particular, it is possible to "symmetrize" the $\mathcal{D}_a$, i.e. to include in it all the four shift operators $T_\pm$, $K_\pm$ in a symmetric fashion. We do not know whether it is possible to choose g_a in such a way that the r.h.s. of (6.16) would vanish.

7 Remarks on the trigonometric limit

The construction of this paper admits several trigonometric degenerations. Let us outline the simplest one – when the Sklyanin algebra degenerates into the quantum algebra $U_q(sl(2))$ [19]-[22]. In this section a *multiplicative* parametrization is more convenient than the *additive* one used in the previous sections. To ease the comparison with the previous formulas, we denote variables like u, z, etc by the same letters having in mind, however, that shifts are now "multiplicative":

$$T_\pm f(\zeta) = f(q^{\pm 1}\zeta)\,, \qquad K_\pm f(x) = f(q^{\pm 1}x)$$

The L-operator reads

$$L(u) = \begin{pmatrix} uA - u^{-1}D & (q-q^{-1})C \\ (q-q^{-1})B & uD - u^{-1}A \end{pmatrix} \tag{7.1}$$

where A, B, C, D are generators of the $U_q(sl(2))$:

$$\begin{aligned} &AB = qBA\,, \quad BD = qDB\,, \quad DC = qCD\,, \quad CA = qAC \\ &AD = 1\,, \qquad BC - CB = \frac{A^2 - D^2}{q - q^{-1}} \end{aligned} \tag{7.2}$$

The central element is

$$\Omega = \frac{q^{-1}A^2 + qD^2}{(q-q^{-1})^2} + BC \tag{7.3}$$

The standard realization of this algebra by difference operators has the form

$$\begin{aligned} &(AF)(z) = F(qz)\,, \qquad (DF)(z) = F(q^{-1}z) \\ &(BF)(z) = \frac{z}{q-q^{-1}}\Big(q^\ell F(q^{-1}z) - q^{-\ell}F(qz)\Big) \\ &(CF)(z) = \frac{z^{-1}}{q-q^{-1}}\Big(q^\ell F(qz) - q^{-\ell}F(q^{-1}z)\Big) \end{aligned} \tag{7.4}$$

The invariant subspace of the spin-ℓ representation is spanned by $z^{-\ell}, \dots, z^{\ell}$. The analogue of the formula (2.13) has the form

$$L(u)F(z) = (q^{\ell}u - q^{-\ell}u^{-1})\hat{\Phi}^{-1}(uq^{2\ell}; z) \begin{pmatrix} q^{-\ell}F(qz) & 0 \\ 0 & q^{\ell}F(q^{-1}z) \end{pmatrix} \hat{\Phi}(u; z) \qquad (7.5)$$

with the matrix

$$\hat{\Phi}(u; z) = \begin{pmatrix} 1 & \bar{q}^{\ell}u^{-1}z^{-1} \\ 1 & q^{-\ell}uz^{-1} \end{pmatrix}$$

The vacuum vectors are (cf. (3.27))

$$X_k^{\ell}(z, \zeta) = z^{-\ell} \prod_{j=1}^{\ell-k} \Big(z - \zeta u^{-1} q^{\ell-k+1-2j}\Big) \prod_{j=1}^{\ell+k} \Big(z - \zeta u q^{\ell+k+1-2j}\Big) \qquad (7.6)$$

where ζ parametrizes the (rational) vacuum curve.

Skipping all the intermediate steps (which are parallel to the elliptic case), we turn right to the representation of the $U_q(sl(2))$ obtained in this way. One arrives at the following realization of the quantum algebra by difference operators in two variables ζ, x:

$$\begin{aligned} A &= q^{-\ell}T_{+}\,, \qquad D = q^{\ell}T_{-} \\ B &= \frac{\zeta}{q - q^{-1}}\Big(q^{\ell}xT_{-} - q^{-\ell}x^{-1}T_{+} - (q^{-\ell}x - q^{\ell}x^{-1})K_{+}\Big) \\ C &= \frac{\zeta^{-1}}{q - q^{-1}}\Big(q^{-\ell}xT_{+} - q^{\ell}x^{-1}T_{-} - (q^{-\ell}x - q^{\ell}x^{-1})K_{+}\Big) \end{aligned} \qquad (7.7)$$

This is to be compared with (5.8), (5.9). An easy computation shows that these operators obey the algebra (7.2) *without any additional conditions*! However, in our present set-up there *is*, too, an analogue of the operator ∇:

$$\nabla = q^{-\ell}T_{+} + q^{\ell}T_{-} - \frac{q^{-\ell}x - q^{\ell}x^{-1}}{x - x^{-1}}\, K_{+} - \frac{q^{\ell}x - q^{-\ell}x^{-1}}{x - x^{-1}}\, K_{-} \qquad (7.8)$$

and the condition $\nabla X = 0$. Now its role is to restrict the functional space to the representation space of spin ℓ. Indeed, computing the central element (7.3), we get:

$$\Omega = \frac{q^{2\ell+1} + q^{-2\ell-1}}{(q - q^{-1})^2} - \frac{(q^{-\ell}x - q^{\ell}x^{-1})(qx - q^{-1}x^{-1})}{(q - q^{-1})^2}\, K_{+}\nabla \qquad (7.9)$$

The invariance of the space of solutions to the equation $\nabla X = 0$ is then obvious.

At last, let us discuss continuum limit ($q \to 1$) of the obtained formulas. In this limit we arrive at representations of the algebra $sl(2)$. In fact there are several different limits

$q \to 1$. The most interesting one reads

$$
\begin{aligned}
s_0 &= \zeta\partial_\zeta - \ell \\
s_+ &= \tfrac{1}{2}\zeta\Big((x+x^{-1})(2\ell-\zeta\partial_\zeta)-(x^2-1)\partial_x\Big) \\
s_- &= \tfrac{1}{2}\zeta^{-1}\Big((x+x^{-1})\zeta\partial_\zeta-(x^2-1)\partial_x\Big)
\end{aligned}
\tag{7.10}
$$

where s_0, $s_\pm$ are standard generators of the $sl(2)$. After the change of variables $\zeta = e^{i\varphi}$, $x = i\cot(\theta/2)$ eqs. (7.10) acquire the form

$$
\begin{aligned}
s_0 &= -i\partial_\varphi - \ell \\
s_+ &= e^{i\varphi}(i\partial_\theta - \cot\theta\,\partial_\varphi + 2i\ell\cot\theta) \\
s_- &= e^{-i\varphi}(i\partial_\theta + \cot\theta\;\partial_\varphi)
\end{aligned}
\tag{7.11}
$$

At $\ell = 0$ this is the well known realization of sl_2 by vector fields on the two-dimensional sphere. The representation (7.7) at $\ell = 0$ is its q-deformation, with the q-deformed variables being $e^{i\varphi}$ and $\cot(\theta/2)$. Another q-deformed version of (7.11) (at $\ell = 0$), where $e^{i\varphi}$ and $e^{i\theta}$ are "discretized", has been suggested in [23].

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Appendix

We use the following definition of the θ-functions:

$$
\begin{aligned}
\theta_1(z|\tau) &= \sum_{k\in\mathbf{Z}} \exp\left(\pi i\tau(k+\frac{1}{2})^2 + 2\pi i(z+\frac{1}{2})(k+\frac{1}{2})\right), \\
\theta_2(z|\tau) &= \sum_{k\in\mathbf{Z}} \exp\left(\pi i\tau(k+\frac{1}{2})^2 + 2\pi iz(k+\frac{1}{2})\right), \\
\theta_3(z|\tau) &= \sum_{k\in\mathbf{Z}} \exp\left(\pi i\tau k^2 + 2\pi izk\right), \\
\theta_4(z|\tau) &= \sum_{k\in\mathbf{Z}} \exp\left(\pi i\tau k^2 + 2\pi i(z+\frac{1}{2})k\right)
\end{aligned}
\tag{A1}
$$

Throughout the paper we write $\theta(z|\tau) = \theta(z)$, $\theta(z|\frac{\tau}{2}) = \bar{\theta}(z)$.

The list of identities used in the computations is given below.

$$
\begin{aligned}
&\bar{\theta}_4(x)\bar{\theta}_3(y)+\bar{\theta}_4(y)\bar{\theta}_3(x)=2\theta_4(x+y)\theta_4(x-y)\\
&\bar{\theta}_4(x)\bar{\theta}_3(y)-\bar{\theta}_4(y)\bar{\theta}_3(x)=2\theta_1(x+y)\theta_1(x-y)\\
&\bar{\theta}_3(x)\bar{\theta}_3(y)+\bar{\theta}_4(y)\bar{\theta}_4(x)=2\theta_3(x+y)\theta_3(x-y)\\
&\bar{\theta}_3(x)\bar{\theta}_3(y)-\bar{\theta}_4(y)\bar{\theta}_4(x)=2\theta_2(x+y)\theta_2(x-y)
\end{aligned}
\tag{A2}
$$

$$
\begin{aligned}
&\bar{\theta}_4(z-x+\ell\eta)\theta_1(z+y+x-(k+\ell)\eta)\theta_1(z-y+x+(k-\ell)\eta)\\
-\;&\bar{\theta}_4(z+x-\ell\eta)\theta_1(z-y-x+(k+\ell)\eta)\theta_1(z+y-x-(k-\ell)\eta)\\
=\;&\bar{\theta}_4(y-k\eta)\theta_1(2z)\theta_1(2x-2\ell\eta)
\end{aligned}
\tag{A3}
$$

(and the same with the change $\bar{\theta}_4 \to \bar{\theta}_3$)

$$
\begin{aligned}
&\theta_{\alpha+1}(2\zeta-u+2x)\theta_1(2\zeta-2x)\theta_1(2\zeta+u)\\
-\;&\theta_{\alpha+1}(2\zeta+u-2x)\theta_1(2\zeta+2x)\theta_1(2\zeta-u)\\
=\;&\frac{\theta_1(4\zeta)\theta_{\alpha+1}(2x)}{\theta_{\alpha+1}(2\zeta)}\,\theta_1(u-2x)\theta_{\alpha+1}(u)\,,\qquad \alpha=1,2,3
\end{aligned}
\tag{A4}
$$

$$
\theta_1(2z)\theta_2(0)\theta_3(0)\theta_4(0)=2\theta_1(z)\theta_2(z)\theta_3(z)\theta_4(z)
\tag{A5}
$$

In the main text we use the notation

$$
\begin{aligned}
&I_{ab}=\theta_{a+1}(0)\theta_{b+1}(2\eta)\\
&b_\alpha(\zeta)=\frac{\theta_{\alpha+1}(2\zeta)}{\theta_1(2\zeta)}\\
&c_\ell(\zeta)=\frac{\theta_1(2\zeta-2\ell\eta)}{\theta_1(2\zeta)}\\
&\rho_\ell(\zeta)=c_\ell(\zeta)c_\ell(-\zeta-\eta)
\end{aligned}
$$

For verifying the comutation relations (6.3), (6.16) we need the identities

$$
I_{\gamma\beta}\theta_{\gamma+1}(x)\theta_{\beta+1}(x+2\eta)-I_{\beta\gamma}\theta_{\beta+1}(x)\theta_{\gamma+1}(x+2\eta)=(-1)^\alpha I_{\alpha 0}\theta_{\alpha+1}(x)\theta_1(x+2\eta)
\tag{A6}
$$

$$
b_\alpha(\zeta-x)-b_\alpha(\zeta+x)=2\,\frac{\theta_{\alpha+1}(2x)\theta_{\beta+1}(2\zeta)\theta_{\gamma+1}(2\zeta)\theta_1(2x)}{\theta_{\beta+1}(0)\theta_{\gamma+1}(0)\theta_1(2\zeta-2x)\theta_1(2\zeta+2x)}
\tag{A7}
$$

$$\rho_\ell(\zeta) - \rho_\ell(x) = \frac{\theta_1(2\ell\eta)\theta_1(2(\ell+1)\eta)\theta_1(2\zeta - 2x)\theta_1(2\zeta + 2x + 2\eta)}{\theta_1(2x)\theta_1(2x+2\eta)\theta_1(2\zeta)\theta_1(2\zeta+2\eta)} \tag{A8}$$

To find the r.h.s. of eqs. (6.17), we need the identities

$$\sum_{\alpha=1}^{3}(-1)^\alpha \theta_{\alpha+1}^2(\eta) b_\alpha(x) b_\alpha(x+\eta) = \theta_1^2(\eta) \tag{A9}$$

$$\sum_{\alpha=1}^{3}(-1)^\alpha \theta_{\alpha+1}(0)\theta_{\alpha+1}(2\eta) b_\alpha(x) b_\alpha(x+\eta) = 0 \tag{A10}$$

$$\begin{aligned} &\sum_{\alpha=1}^{3}(-1)^\alpha \theta_{\alpha+1}(0)\theta_{\alpha+1}(x)\theta_{\alpha+1}(y)\theta_{\alpha+1}(z) \\ = \quad & 2\theta_1\Big(\frac{x+y+z}{2}\Big)\theta_1\Big(\frac{x-y+z}{2}\Big)\theta_1\Big(\frac{x+y-z}{2}\Big)\theta_1\Big(\frac{x-y-z}{2}\Big) \end{aligned} \tag{A11}$$

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