IMRN International Mathematics Research Notices
1996, No. 10

Multidimensional Vector Addition Theorems and the Riemann Theta Functions

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This work is a continuation of the previous paper [5] by the authors. In [5] it was shown that the Baker-Akhiezer functions that are the main object in the algebraic-geometrical construction of exact solutions [6], [7] of solitonic equations define solutions of the following functional equation:

$$\sum_{k=0}^{N} c_k(x+y)\psi_k(x)\psi_k(y) = 1, \tag{1}$$

where $c_k(x)$, $\psi_k(x)$ are unknown functions of the scalar variable x. They have been called the vector analogs of the Cauchy equation. Note that the classic Cauchy equation is a particular case of (1) corresponding to $N = 0$, and in this case all solutions of (1) are exponential functions. It was proved that for $N = 1$ and $N = 2$ *all* the solutions of this equation are the Baker-Akhiezer functions corresponding to algebraic curves of genus 1 and 2, respectively. A starting point for the consideration of (1) in [5] was close connections of this equation with the theory of one-dimensional integrable systems of the Calogero-Moser system type.

The main goal of this note is to show that the multivariable Baker-Akhiezer functions give solutions of the following *multivariable* generalization of (1):

$$\sum_{k=0}^{N} c_k(x+y)\psi_k^1(x)\psi_k^2(y) = 1, \tag{2}$$

where c_k, ψ_k^1, ψ_k^2 are unknown functions of the *vector* variable $x = (x_1, \dots, x_g)$. The authors believe that Theorem 2 gives *all* analytical solutions of this equation.

We would like to emphasize that even in the case of a scalar argument, the equation (2) is a highly nontrivial generalization of (1). For $N = 0$ all the solutions of this

Received 7 March 1996. Revision received 12 April 1996.

equation are again exponential functions. A general analytic solution of (2) for $N = 1$ was constructed in [4]. Note that the particular case of (2) with a scalar argument and with $c_k = \text{const}$, $k > 0$, was considered in [1], [2], where it was shown that the theory of such functional equations is equivalent to the theory of ordinary differential equations with constant coefficients.

The problem of constructing general solutions to the equation (2) with the vector argument is considered by the authors as a background for the realization of a project that includes

(a) construction of multidimensional analogs of the Moser-Calogero system;
(b) construction of integrable partial differential equations;
(c) characterization of the Baker-Akhiezer functions in terms of the functional equations;
(d) characterization of the Jacobians of the algebraic curves (Riemann-Schottky problem) in terms of the addition-type theorems for the theta-functions).

The Riemann-Schottky problem was solved in [8] in terms of the Kadomtsev-Petviashvilii (KP) equation. This approach was proposed by S. P. Novikov. In [5] the authors conjectured that Jacobians of algebraic curves can be characterized with the help of equation (1). This conjecture was inspired by the following result that was proved in [5]. The theta-function corresponding to the generic g-dimensional Abelian variety satisfies the equation (1) with $N = 2^g$. It turns out that for the Jacobian varieties, the number of terms in the sum (1) becomes much less, $N = g$.

Let Γ be a nonsingular algebraic curve of genus g with a fixed point P_0 on it and with a fixed basis a_i, b_i of cycles on Γ with the canonical matrix of intersections, i.e., $a_i a_j = b_i b_j = 0$, $a_i b_j = \delta_{ij}$. The basis of normalized holomorphic differentials ω_i on Γ is defined with the help of the conditions

$$\oint_{a_i} \omega_i = \delta_{ij}. \tag{3}$$

The matrix

$$B_{ij} = \oint_{b_i} \omega_j \tag{4}$$

is called a matrix of b-periods of the curve Γ. It is symmetrical and has positively defined imaginary part. Each of these matrices defines an entire function of g variables (which is called the Riemann theta-function) with the help of the formula

$$\theta(z_1, \dots, z_g) = \sum_{m \in Z^g} \exp(2\pi i(z, m) + \pi i(Bm, m)). \tag{5}$$

(Here $m = (m_1, \dots, m_g)$ is an integer vector.) The Riemann theta-function has the following monodromy properties:

$$\theta(z + e_k) = \theta(z), \quad \theta(z + B_k) = \theta(z) \exp(-\pi i B_{kk} - 2\pi i z_k) \tag{6}$$

(e_k and B_k are vectors with the coordinates (δ_{ki}) and (B_{ki}), respectively). The vectors e_k, B_k generate a lattice in the linear complex space C^g. The corresponding factor-space is a g-dimensional torus $J(\Gamma)$ that is called the Jacobian variety of the algebraic curve Γ. The Abel map

$$A : \Gamma \longmapsto J(\Gamma) \tag{7}$$

is defined by the formula

$$A_k(Q) = \int_{Q_0}^{Q} \omega_k. \tag{8}$$

If vector Z equals

$$Z = K - \sum_{s=1}^{g} A(\gamma_s), \tag{9}$$

where K is a vector of the Riemann constants (it depends on the basis of cycles and initial point P_0 but does not depend on the points γ_s), then the function $\theta(A(Q) + Z)$ is either identically zero, or has exactly g zeros on Γ that coincide with γ_s, i.e.,

$$\theta(A(\gamma_s) + Z) = 0. \tag{10}$$

It is to be mentioned that the function $\theta(A(Q) + Z)$ is multivalued on Γ but, as it follows from (6), its zeros are well defined. It is single-valued on Γ^*, that is, obtained from Γ with the help of cuts along a-cycles.

Proposition 1. For a generic set of points $\gamma_1, \dots, \gamma_g$ the formula

$$\Phi(x, Q) = \frac{\theta(A(Q) + x + Z)\theta(A(P_0) + Z)}{\theta(A(Q) + Z)\theta(A(P_0) + x + Z)}, \tag{11}$$

where Z is given by (9), defines a unique function $\Phi(x, Q)$, $x = (x_1, \dots, x_g)$, $Q \in \Gamma$, such that

(1) Φ is a meromorphic single-valued function of the variable Q on Γ^* with at most simple poles at the points γ_s (if all of them are distinct);

(2) its boundary values $\Phi_j^{\pm}(x, Q)$, $Q \in a_j$ on different sides of cuts satisfy the relation

$$\Phi_j^{+}(x, Q) = e^{-2\pi i x_j} \Phi_j^{-}(x, Q), \; Q \in a_j; \tag{12}$$

(3) Φ is normalized by the condition

$$\Phi(x, P_0) = 1. \tag{13}$$

□

Remark. These types of functions were called "factorial functions" in [3]. They are very closely related to the Baker-Akhiezer functions. Namely, the Baker-Akhiezer function that is a single-valued function on Γ, depending on parameters $t = \{t_a\}$, may be represented in the form

$$\psi(t, Q) = \exp(\sum_a t_a \int_{Q_0}^{Q} d\Omega_a)\Phi(x, Q), \tag{14}$$

where $d\Omega_a$ are the normalized meromorphic differentials on Γ and the coordinates x_k of the vector x are equal to

$$x_k = \sum_a \frac{1}{2\pi i} t_a \oint_{b_k} d\Omega_a. \tag{15}$$

Proposition 2. For a generic set of points $\gamma_1, \ldots, \gamma_g$ the formula

$$C_k(x, Q) = \frac{\theta(A(Q) + Z + A(\gamma_k) - A(P_0))\theta(A(Q) + x + Z - A(\gamma_k) + A(P_0))}{\theta^2(A(Q) + Z)\theta(A(P_0) + x + Z)\theta(2A(\gamma_k) - A(P_0) + Z)}, \tag{16}$$

where Z is given by (9), defines a unique function $C_k(x, Q)$, $x = (x_1, \ldots, x_g)$, $Q \in \Gamma$, such that

(1) C_k is a meromorphic single-valued function of the variable Q on Γ^* with at most simple poles at the points $\gamma_s, s \neq k$, and has second-order pole of the form

$$C_k(x, Q) = \theta^{-2}(A(Q) + Z) + O(\theta^{-1}(A(Q) + Z)) \tag{17}$$

at the point γ_k;

(2) its boundary values $C_{k,j}^{\pm}(x, Q)$, $Q \in a_j$, on different sides of cuts satisfy the relation

$$C_{k,j}^{+}(x, Q) = e^{-2\pi i x_j} C_{k,j}^{-}(x, Q), \; Q \in a_j; \tag{18}$$

(3) we have

$$C_k(x, P_0) = 0. \tag{19}$$

□

Note that $\theta(A(Q) + Z)$ equals zero at the point γ_k, and therefore the equality (16) means that C_k has the pole of second order with a normalized leading coefficient. The proof of the proposition is straightforward and uses the identity (9) and monodromy properties of the theta-function, only. From (9) it follows that the first factor in the numerator has zeros at the point P_0 and at the points γ_s, $s \neq k$. At the same time the first factor in the denominator has poles of second order at all the points γ_s. Therefore, C_k has poles of first order at γ_s, $s \neq k$, and a pole of second order at the point γ_k. The difference of arguments of the theta-functions in factors containing $A(Q)$ in the numerator and the denominator is equal to x. That implies (18).

Theorem 1. Let $\phi_k(x)$ be a set of functions that are defined by the formula

$$\phi_k(x) = \frac{\theta(A(\gamma_k) + x + Z)\theta(A(P_0) + Z)}{\theta(A(P_0) + x + Z)}. \tag{20}$$

Then the equality

$$\Phi(x + y, Q) = \Phi(x, Q)\Phi(y, Q) - \sum_{k=1}^{g} C_k(x + y, Q)\phi_k(x)\phi_k(y) \tag{21}$$

is valid. □

Remark. Note that

$$\Phi(0, Q) = 1, \qquad \phi_k(0) = 0, \qquad k = 1, \ldots, g. \tag{22}$$

In order to prove the statement of the theorem, it is enough to check if the right-hand side has the same analytical properties (as a function of Q) that uniquely define the function $\Psi(x + y, Q)$.

First of all, all the terms in the right-hand side satisfy the boundary conditions on cuts that should be fulfilled for $\Phi(x+y, Q)$. From the definition of C_k and ϕ_k it follows that the right-hand side has no poles of second order at the points γ_s (though the first term has poles of second order at all these points). The equality (19) implies that the normalization condition (13) is fulfilled.

Note that after dividing of (21) on $\Phi(x+y,Q)$ we come to the equation of the form (2) with $N = g$ and $\psi_k^1 = \psi_k^2$. The generalization which is necessary to make in order to have the solutions of (2) is more or less obvious.

Theorem 2. Let $\Phi^1(x,Q)$ be a function that is defined by the formula (11) in which the vector Z is replaced by the vector

$$Z_1 = K - \sum_{k=1}^{g} A(\gamma_k^1), \tag{23}$$

where $\gamma_1^1, \dots, \gamma_g^1$ is a set of points in general position. Then the equation

$$\Phi^1(x+y,Q) = \Phi(x,Q)\Phi^1(y,Q) - \sum_{k=1}^{g} \widetilde{C}_k(x+y,Q)\phi_k(x)\phi_k^1(y), \tag{24}$$

where $\phi_k^1(y) = \Phi^1(y,\gamma_k)$ and the functions $\tilde{C}_k$ are given by the formulas

$$\widetilde{C}_k = H(x)\frac{\theta(A(Q)+Z_1+A(\gamma_k^1)-A(P_0))\theta(A(Q)+x+Z_1-A(\gamma_k)+A(P_0))}{\theta(A(Q)+Z_1+A(\gamma_k^1)-A(\gamma_k))\theta(A(Q)+Z_1)}, \tag{25}$$

$$H(x) = h\frac{\theta(A(\gamma_k)+Z_1)}{\theta(A(\gamma_k)+A(\gamma_k^1)+Z-A(P_0))\theta(A(P_0)+x+Z_1)}, \tag{26}$$

$$h = \lim_{Q\to\gamma_k}\frac{\theta(A(Q)+Z_1+A(\gamma_k^1)-A(\gamma_k))}{\theta(A(Q)+Z)}, \tag{27}$$

is fulfilled. □

Again, the proof is standard. First of all, let us note that the first factors in the denominator and numerator of (25) have zeros at the points γ_s^1, $s \neq k$ that cancel each other. Besides these zeros, the factor in the numerator has zero at the point P_0 and the factor in the denominator equals zero at the point γ_k. Therefore, the function $\widetilde{C}_k$ as a function of Q satisfies the same boundary conditions on a-cycles as all the other functions. It has $g+1$ poles at the points $\gamma_1^1, \dots, \gamma_g^1$ and at the point γ_k. It equals zero at the point P_0. These analytical properties define $\widetilde{C}_k$ uniquely up to a constant (in Q) factor. The normalization (26) provides that the right-hand side in (25) has no poles at the points γ_s.

Remark 1. We would like to mention that (24) implies, in particular, the well-known formula for a sum of squares of eigenfunctions of Hill's equation. Indeed, let Q_i, $i =$

$1, \ldots, g+1$ be a set of $g+1$ points of Γ and let s_i be the solution of a system of $g+1$ linear equations

$$\sum_{i=1}^{g+1} \Phi^1(0, Q_i) s_i = 1, \qquad \sum_{i=1}^{g+1} \widetilde{C}_k(0, Q_i) s_i = 0, \qquad k = 1, \ldots, g. \tag{28}$$

Then (24) implies

$$\sum_{i=1}^{g+1} s_i \Phi(x, Q_i) \Phi^1(-x, Q_i) = 1, \tag{29}$$

if Γ is a hyperelliptic curve given by the equation

$$y^2 = \prod_{i=1}^{2g+1} (E - E_j) \tag{30}$$

and the corresponding Baker-Akhiezer function (14) is a solution to Hill's equation. (The theta-functional formula for the Baker-Akhiezer function corresponding to finite-gap Hill's operators was proposed by Matveev and Its.) Let the vector Z_1 (23) be chosen such that $Z_1 = -Z$; then

$$\Phi(x, E_j) = \Phi^1(-x, E_j). \tag{31}$$

Therefore, if the set of points Q_i is a subset of branching points E_j of the hyperelliptic curve (30), then (29) implies

$$\sum_{i=1}^{g+1} s_i \psi^2(x, Q_i) = 1. \tag{32}$$

Remark 2. Note that the techniques used for the proof of Theorem 2 allow us to prove that the functions $\Phi^i(x, Q)$, $i = 1, \ldots, m$ given by the formula (11) for a set of vectors $Z = Z_i$ give solutions to the functional equations containing products of m functions. In particular, for $m = 3$ this generalized functional equation has the form

$$\Phi^1(x + y + z, Q) = \Phi^1(x, Q)\Phi^2(y, Q)\Phi^3(z, Q) - \sum_{k=1}^{2g} \hat{C}_k(x + y + z, Q)\phi_k^1(x)\phi_k^2(y)\phi_k^3(z). \tag{33}$$

Theorems 1 and 2 are fulfilled for all the values of x and y. It turns out that they are simplified drastically for special values of x and y.

Corollary 1. Let x and y be vectors of the form

$$x = \sum_{i=1}^{n} (A(\gamma_i) - A(Q_i)), \qquad y = \sum_{i=n+1}^{g} (A(\gamma_i) - A(Q_i)), \tag{34}$$

where $Q_i, i = 1, \dots, g$ is a set of arbitrary points on Γ; then

$$\Phi(x + y, Q) = \Phi(x, Q)\Phi(y, Q). \tag{35}$$

□

From (34) it follows that $\theta(A(Q) + x + Z)$ is equal to zero at the points $\gamma_{n+1}, \dots, \gamma_g$. At the same time, $\theta(A(Q) + y + Z)$ is equal to zero at the points $\gamma_1, \dots, \gamma_n$. Therefore, for such values of x and y,

$$\phi_k(x)\phi_k(y) = 0, \qquad k = 1, \dots, g. \tag{36}$$

In that case, the equality (21) becomes (35).

Important remark. The equality (35) has the form of a classical Cauchy functional equation and it is well known that all the solutions of this equation are exponential functions. We would like to emphasize that this is true if and only if the variables x and y belong to some linear space. In our case they belong to nonlinear submanifolds in C^g that are the symmetric powers of the curve.

Corollary 2. Let x be an arbitrary vector and y be a vector of the form

$$y = A(\gamma_1^1) - A(P), \qquad P \in \Gamma; \tag{37}$$

then

$$\Phi(x + y, Q) = \Phi(x)\Phi^1(y) + \tilde{C}_1(x + y)\phi_1(x)\phi_1^1(y). \tag{38}$$

□

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