Spin generalization of the Calogero-Moser system and the Matrix KP equation

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Abstract

The complete solutions of the spin generalization of the elliptic Calogero Moser systems are constructed. They are expressed in terms of Riemann theta-functions. The analoguous constructions for the trigonometric and rational cases are also presented.

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1 Introduction

The elliptic Calogero-Moser system [1],[2] is a system of N identical particles on a line interacting with each other via the potential $V(x) = \wp(x)$, where $\wp(x) = \wp(x|\omega, \omega')$ is the Weierstrass elliptic function with periods 2ω, $2\omega'$. This system (and its quantum version, as well) is a completely integrable system [3]. The complete solution of the elliptic Calogero–Moser model was constructed by algebro–geometrical methods in [5]. The degenerate cases where $V(x) = 1/\sinh^2 x$ or $V(x) = 1/x^2$ are also of interest, and admit nice interpretations as reductions of geodesic motions on symmetric spaces [3, 4]. The analogous interpretation for the elliptic case was recently given in [6].

In this work, we consider the spin generalization of the Calogero–Moser model, which was defined in [7]. Again this model exists in the elliptic, trigonometric and rational versions, each one being of its own interest. In particular the hidden symmetry of the model changes from a current algebra type in the rational case, to a yangian type in the trigonometric case [7, 8, 9]. Our main goal is to construct the action-angle type variables for these spin generalizations of the Calogero-Moser system, and to solve the equations of motion in terms of Riemann theta-functions. The algebro–geometric constructions of the solutions substantially differ in the three cases and we shall present them in parallel.

Let us consider the classical hamiltonian system of N particles on a line, with coordinates x_i and momenta p_i, and internal degrees of freedom described for each particle by a l-dimensional vector $a_i = (a_{i,\alpha})$ and a l-dimensional co-vector $b_i^+ = (b_i^\alpha)$, $\alpha = 1, \ldots, l$. The Hamiltonian has the form

$$H = \frac{1}{2}\sum_{i=1}^{N} p_i^2 + \frac{1}{2}\sum_{i\neq j}(b_i^+ a_j)(b_j^+ a_i)V(x_i - x_j). \tag{1.1}$$

where $(b_i^+ a_j)$ stands for the corresponding scalar product

$$(b_i^+ a_j) = (b_i^\alpha a_{i,\alpha}). \tag{1.2}$$

and the potential $V(x)$ is one of the functions $\wp(x), 1/\sinh^2 x$, or $1/x^2$. The non trivial Poisson brackets between the dynamical variables $x_i, p_i, b_i^\alpha, a_{i,\alpha}$ are

$$\{p_i, x_j\} = \delta_{ij}, \quad \{b_i^\beta, a_{j,\alpha}\} = -\delta_{i,j}\delta_\alpha^\beta. \tag{1.3}$$

The equations of motion have the form

$$\ddot{x}_i = \sum_{j\neq i}(b_i^+ a_j)(b_j^+ a_i)V'(x_i - x_j), \quad V'(x) = \frac{dV(x)}{dx}, \tag{1.4}$$

$$\dot{a}_i = -\sum_{j\neq i} a_j(b_j^+ a_i)V(x_i - x_j), \tag{1.5}$$

$$\dot{b}_i^+ = \sum_{j\neq i} b_j^+(b_i^+ a_j)V(x_i - x_j) \tag{1.6}$$

From (1.5,1.6) it follows that $(b_i^+ a_i)$ are integrals of motions. We restrict the system on the invariant submanifold

$$(b_i^+ a_i) = c = 2 \tag{1.7}$$

Remark. The reduction of the system (1.1) onto the invariant submanifold defined by the constraint (1.7) is a completely integrable hamiltonian system for any value of the constant c. Changing the value of c amounts to a rescaling of the time variable. In the following we shall assume $c = 2$ for definiteness.

Let us introduce the quantities

$$f_{ij} = (b_i^+ a_j). \tag{1.8}$$

The Poisson brackets (1.3) imply

$$\{f_{ij}, f_{kl}\} = \delta_{jk} f_{il} - \delta_{il} f_{kj}. \tag{1.9}$$

The Hamiltonian (1.1) in terms of these new variables has the form:

$$H = \frac{1}{2}\sum_{i=1}^{N} p_i^2 + \frac{1}{2}\sum_{i\neq j} f_{ij} f_{ji} V(x_i - x_j) \tag{1.10}$$

The system (1.10) with f_{ij} as dynamical variables satisfying the relations (1.9) was introduced in [7] and was called Euler–Calogero–Moser system. When $l < N$ the relations (1.8) give a parametrisation of special symplectic leaves of the system (1.9,1.10).

Let us count the number of non–trivial degrees of freedom. We start with $2N + 2Nl$ dynamical variables corresponding to the x_i, p_i, $a_{i,\alpha}$, b_i^α. The Hamiltonian (1.10) has a symmetry under rescaling:

$$a_i \to \lambda_i a_i, \quad b_i \to \frac{1}{\lambda_i} b_i \tag{1.11}$$

(notice that f_{ij} is non–invariant but $f_{ij}f_{ji}$ is invariant, and the Poisson brackets are also invariant). The corresponding moment is given by the collection of $b_i^+ a_i$ and we fix it to the values $b_i^+ a_i = 2$, which makes N conditions. The stabilizer of this moment consists in the whole group so that the reduced system is defined by N more constraints, e.g. $\sum_\alpha b_i^\alpha = 1$, leaving us with a phase space of dimension $2Nl$. Moreover the Hamiltonian and the symplectic structure are invariant under a further symmetry:

$$a_i \to W^{-1} a_i, \quad b_i^+ \to b_i^+ W \tag{1.12}$$

where W is any matrix in $GL(r, \mathbf{R})$ independent of the label i, preserving the above condition on the b_i's. This means that W must leave the vector $v = (1, \cdots, 1)$ invariant. Hence this group is of dimension $l^2 - l$. Fixing the momentum, $\mathcal{P}$, gives $l^2 - l$ conditions. The stabilizer of a generic momentum is trivial: this is because such a generic element can be diagonalized as $\mathcal{P} = m^{-1}\Lambda m$. Its stabilizer under the adjoint action consists of the matrices of the form $g = m^{-1} D m$ with D diagonal. The condition $gv = v$ translates into $Dmv = mv$ wich implies generically $D = 1$, i.e., $g = 1$. We have proved the

Proposition 1.1 *The dimension of the reduced phase space $\mathcal{M}$ is*

$$\dim \mathcal{M} = 2\left[Nl - \frac{l(l-1)}{2}\right] \tag{1.13}$$

Our method for the construction of the solutions of system (1.1) is a generalization of the approach that was used for the classical Calogero-Moser system. In [10] a remarkable connection between the Calogero-Moser system and the motion of poles of the rational and the elliptic solutions of KdV equation was found. It turned out that the corresponding relation becomes an isomorphism in the case of the rational or the elliptic solutions of the Kadomtsev-Petviashvili equation. In [11] and [12] this isomorphism in the rational case was used in opposite directions.

In [12] using the known solutions of the rational Calogero-Moser system, the rational solutions of KP equation were constructed.

In [11] the construction of rational solutions for various partial differential equations admitting a zero-curvature representation was proposed. Applying this result to the KP equation yielded an alternative way to solve the Calogero-Moser system. This approach was generalized in [5] where the action-angle variables for the elliptic Calogero-Moser system were constructed and the exact formula for elliptic solutions of KP equation was obtained. (Further developments in the theory of so-called elliptic solitons are presented in the special issue of Acta Applicandae Mathematicae **35** (1994) dedicated to the memory of J.L. Verdier).

2 Relation to the matrix KP equation.

The zero-curvature representation for the KP equation

$$\frac{3}{4} u_{yy} = (u_t - \frac{3}{2} u u_x + \frac{1}{4} u_{xxx})_x \tag{2.1}$$

has the form ([13],[14])

$$[\partial_y - L, \partial_t - M] = 0, \tag{2.2}$$

where

$$L = \partial_x^2 - u(x,y,t), \quad M = \partial_x^3 - \frac{3}{2} u \partial_x + w(x,y,t). \tag{2.3}$$

In this scalar case ($l = 1$), assuming that u is an elliptic function of the variable x, the comparison of singular terms in the expansion of the right and left hand sides of (2.1) near the poles of u gives directly that:

1.) Any elliptic (in the variable x) solution of KP equation has the form

$$u(x, y, t) = 2\sum_{i=1}^{N} \wp(x - x_i(y, t)) + const, \tag{2.4}$$

2.) The dependence of the poles $x_i(y, t)$ with respect to the variable y coincide with the elliptic Calogero-Moser system and their dependence with respect to the variable t is described by the "third integral" of this system.

Let us consider the same equations (2.2) in the case when operators (2.3) have matrix ($l \times l$) coefficients. They are equivalent to the system

$$w_x = \frac{3}{4}u_y, \quad w_y = u_t - \frac{3}{4}(uu_x + u_x u) + \frac{1}{4}u_{xxx} - [u, w], \tag{2.5}$$

that we call the matrix KP equation.

In the matrix case we don't know the complete classification of all elliptic solutions of (2.5). It turns out that the system (1.1) is isomorphic to the special elliptic solutions of matrix KP equation having the form

$$u(x, y, t) = \sum_{i=1}^{N} \rho_i(y, t)\wp(x - x_i(y, t)), \tag{2.6}$$

$$w(x, y, t) = \sum_{i=1}^{N} (A_i(y, t)\zeta(x - x_i(y, t)) + B_i(y, t)\wp(x - x_i(y, t))). \tag{2.7}$$

where ρ_i is a rank-one matrix-function depending on y, t

$$\rho_i = a_i b_i^+, \quad \text{i.e.} \quad \rho_{i,\alpha}^{\beta} = a_{i,\alpha} b_i^{\beta}. \tag{2.8}$$

The precise relation is provided by the:

Theorem 2.1 *Let us introduce the functions*

$$V(x) = \wp(x), \quad \Phi(x, z) = \frac{\sigma(z - x)}{\sigma(z)\sigma(x)} e^{\zeta(z)x}. \tag{2.9}$$

The equations

$$\left(\partial_t - \partial_x^2 + \sum_{i=1}^{N} a_i(t) b_i^+(t) V(x - x_i(t))\right)\Psi = 0 \tag{2.10}$$

$$\Psi^+\left(\partial_t - \partial_x^2 + \sum_{i=1}^{N} a_i(t) b_i^+(t) V(x - x_i(t))\right) = 0 \tag{2.11}$$

(where we define $\Psi^+\partial \equiv -\partial\Psi^+$) have solutions Ψ, Ψ^+ of the form

$$\Psi = \sum_{i=1}^{N} s_i(t, k, z)\Phi(x - x_i(t), z) e^{kx + k^2 t}, \tag{2.12}$$

$$\Psi^+ = \sum_{i=1}^{N} s_i^+(t, k, z)\Phi(-x + x_i(t), z) e^{-kx - k^2 t}, \tag{2.13}$$

where s_i and s_i^+ are l-dimensional vector $s_i = (s_{i,\alpha})$ and co-vector $s_i^+ = (s_i^\alpha)$, respectively, if and only if $x_i(t)$ satisfy the equations (1.4) and the vectors a_i, b_i^+ satisfy the constraints (1.7) and the system of equations

$$\dot{a}_i = -\sum_{j \neq i} a_j (b_j^+ a_i) V(x_i - x_j) - \lambda_i a_i, \tag{2.14}$$

$$\dot{b}_i^+ = \sum_{j\neq i} b_j^+(b_i^+ a_j)V(x_i - x_j) + \lambda_i b_i^+ \tag{2.15}$$

where $\lambda_i = \lambda_i(t)$ *are scalar functions.*

Remark 1. The system (1.4,2.14,2.15) is "gauge equivalent" to the system (1.4-1.6). This means that if (x_i, a_i, b_i^+) satisfy the equations (1.4,2.14,2.15) then x_i and the vector-functions

$$\hat{a}_i = a_i q_i,\ \hat{b}_i^+ = b_i q_i^{-1},\quad q_i = \exp(\int^t \lambda_i(t)dt) \tag{2.16}$$

are solutions of the system (1.4-1.6).

Remark 2. In the scalar case the ansatz (2.10) was introduced in [5]. Its particular form was inspired by the well-known formula for the solution of the Lamé equation:

$$\big(\frac{d^2}{dx^2} - 2\wp(x)\big)\Phi(x,z) = \wp(z)\Phi(x,z). \tag{2.17}$$

Proof. Inserting equation (2.12) into equation (2.10) we find the condition:

$$\begin{aligned} A \quad \equiv \quad & \sum_{i=1}^{N} \Bigg\{ \dot{s}_i \Phi(x - x_i, z) - (\dot{x}_i + 2k)s_i\Phi'(x - x_i, z) - s_i\Phi''(x - x_i, z) \\ + \quad & \sum_{j=1}^{N} a_j (b_j^+ s_i)\wp(x - x_j)\Phi(x - x_i, z) \Bigg\} = 0 \end{aligned}$$

where $\Phi' = \partial_x \Phi$ and so on.

The vanishing of the triple pole $(x - x_i)^{-3}$ gives the condition:

$$a_i(b_i^+ s_i) = 2s_i \tag{2.18}$$

Using this condition and the Lamé equation (2.17) we can identify the double pole $(x - x_i)^{-2}$. Its vanishing gives the condition:

$$s_i(\dot{x}_i + 2k) + \sum_{j\neq i} a_i b_i^+ s_j \Phi(x_i - x_j, z) = 0 \tag{2.19}$$

We finally identify the residue of the simple pole and obtain the condition:

$$\dot{s}_i + \left(\sum_{j\neq i} a_j b_j^+ \wp(x_i - x_j) - \wp(z) \right) s_i + a_i \sum_{j\neq i} (b_i^+ s_j)\Phi'(x_i - x_j, z) = 0. \tag{2.20}$$

Inserting now equations (2.18,2.19,2.20) into the expression of A one sees that A vanishes identically due to the functional equation:

$$\Phi'(x,z)\Phi(y,z) - \Phi(x,z)\Phi'(y,z) = (\wp(y) - \wp(x))\Phi(x+y,z) \tag{2.21}$$

We have shown that the function ψ given by eq. (2.12) satisfies equation (2.10) if and only if the conditions (2.18,2.19,2.20) are fulfilled.

Equation (2.18) implies that the vector s_i is proportional to the vector a_i. Hence:

$$s_{i,\alpha}(t,k,z) = c_i(t,k,z)a_{i,\alpha}(t) \tag{2.22}$$

Moreover from (2.18) it follows that the constraints (1.7) should be fulfilled.

Equation (2.19) can then be rewritten as a matrix equation for the vector $C = (c_i)$:

$$(L(t,z) + 2kI)C = 0 \tag{2.23}$$

where the Lax matrix $L(t,z)$ with spectral parameter z is given by:

$$L_{ij}(t,z) = \dot{x}_i \delta_{ij} + (1-\delta_{ij}) f_{ij} \Phi(x_i - x_j, z). \tag{2.24}$$

We can rewrite equation (2.20) as:

$$\dot{a}_i = -\lambda_i a_i - \sum_{j \neq i} a_j (b_j^+ a_i) \wp(x_i - x_j) \tag{2.25}$$

where we have defined:

$$\lambda_i = \frac{\dot{c}_i}{c_i} - \wp(z) + \sum_{j \neq i} (b_i^+ a_j) \Phi'(x_i - x_j, z) \frac{c_j}{c_i}$$

But this last equation can be rewritten:

$$(\partial_t + M)C = 0 \tag{2.26}$$

where the second element M of the Lax pair is given by:

$$M_{ij}(t,z) = (-\lambda_i - \wp(z))\delta_{ij} + (1-\delta_{ij}) f_{ij} \Phi'(x_i - x_j, z). \tag{2.27}$$

The same arguments show that the existence of a solution Ψ^+ of the form (2.13) implies (cancellation of the triple pole):

$$s_i^\alpha = c_i^+ b_i^\alpha, \tag{2.28}$$

and the covector $C^+ = (c_i^+)$ satisfies the equation (cancellation of the double pole):

$$C^+(L(z) + 2k) = 0. \tag{2.29}$$

Finally looking at the simple pole one gets:

$$\dot{b}_i^+ = \lambda_i^+ b_i^+ + \sum_{j \neq i} (b_i^+ a_j) b_j^+ \wp(x_i - x_j) \tag{2.30}$$

with a new scalar λ_i^+ given by:

$$\lambda_i^+ = -\frac{\dot{c}_i^+}{c_i^+} - \wp(z) + \sum_{j \neq i} \frac{c_j^+}{c_i^+} (b_j^+ a_i) \Phi'(x_j - x_i)$$

The equations (2.25,2.30) are compatible with $f_{ii} = b_i^+ a_i = 2$ only when $\lambda_i^+ = \lambda_i$. Finally we can rewrite the definition of λ_i^+ as:

$$\partial_t C^+ - C^+ M = 0. \tag{2.31}$$

To end the proof of the theorem we have to establish that the $x_i(t)$ satisfy equation (1.4). For this we exploit the compatibility conditions between eq. (2.23,2.26) and between eq. (2.29, 2.31) which read respectively:

$$(\dot{L} + [M,L])C = 0 \quad C^+(\dot{L} + [M,L]) = 0$$

Computing $\dot{L}+[M,L]$ we see that the off-diagonal elements vanish identically due to equations (2.14,2.15) while the diagonal elements are precisely the equations of motion of the x_i. The computation uses again equation (2.21) and we have therefore shown the Lax form of the equations of motion:

$$\dot{L} = [L,M] \tag{2.32}$$

■

Remark. In [1] it was proved that Lax equation (2.32) with the matrices L and M given by the formulae (2.24, 2.27) (with $f_{ij} = 2, \lambda_i = 0$) is equivalent to the equations of motion of the Calogero-Moser system if and only if the functional equation (2.21) is fulfilled. In [1] the particular solutions of the functional equation corresponding to the values $z = \omega_l$ was found. The proof of this equation for arbitrary values of the spectral parameter z was given in [5].

Let us comment on the trigonometric and rational limits of the above formulae. The trigonometric limit is obtained when one of the periods $\omega \to \infty$. We choose the other one as $i\pi$. In this limit the function Φ becomes:

$$\Phi(x, z) = (\coth x - \coth z)\, e^{x \coth z}$$

The exponential factor in Φ comes from the factor $\exp(\zeta(z)x)$ in the elliptic case which is necessary to induce the double periodicity of Φ in z. In the trigonometric case however it can be absorbed into a redefinition of k and s_i of the form:

$$k \to k - \coth z \qquad s_i \to s_i \exp\left(x_i(t)\coth z + 2kt\coth z - t\coth^2 z\right)$$

and similarly for the dual quantities. In the following we shall therefore remove this exponential factor in the definition of Ψ. The definitions of the functions $V(x)$ and $\Phi(x, z)$ become

$$V(x) = \frac{1}{\sinh^2(x)}, \quad \Phi(x, z) = \coth x - \coth z, \tag{2.33}$$

With these new functions, the above theorem remains valid, but due to the redefinition of the function $\Phi(x, z)$, the expression of the Lax matrices is slightly modified and reads:

$$L_{ij}(t, z) = (\dot{x}_i - 2\coth z)\delta_{ij} + (1 - \delta_{ij})f_{ij}\Phi(x_i - x_j, z) \tag{2.34}$$

$$M_{ij}(t) = -\lambda_i\delta_{ij} - (1 - \delta_{ij})f_{ij}V(x_i - x_j) \tag{2.35}$$

The rational limit is obtained straightforwardly from the trigonometric limit by sending the second period $\omega' \to \infty$. The functions $V(x)$ and $\Phi(x, z)$ become

$$V(x) = \frac{1}{x^2}, \quad \Phi(x, z) = \frac{1}{x} - \frac{1}{z}, \tag{2.36}$$

and of course $\coth z \to 1/z$ in eq (2.34).

Notice that as compared to the elliptic case there is a decoupling between the spectral parameter z and the x_i's in the Lax matrix (2.34).

Part I

The Direct Problem

3 The spectral curve.

Due to equation (2.23) the parameters k and z are constrained to obey:

$$R(k, z) \equiv \det(2kI + L(t, z)) = 0 \tag{3.1}$$

This defines a curve Γ which is time–independent due to the Lax equation (2.32). This curve plays a fundamental role in the subsequent analysis. Its properties are different in the elliptic, trigonometric and rational cases. Remark moreover that Γ is invariant under the symmetries (1.11,1.12).

Proposition 3.1 *In the elliptic case we have:*

$$R(k, z) = \sum_{i=0}^{N} r_i(z)k^i \tag{3.2}$$

where the $r_i(z)$ are elliptic functions of z, independent of t, having the form:

$$r_i(z) = I_i^0 + \sum_{s=0}^{N-i-2} I_{i,s}\partial_z^s\wp(z). \tag{3.3}$$

In a neighbourhood of $z=0$ the function $R(k,z)$ can be represented in the form:

$$R(k,z) = 2^N\prod_{i=1}^{N}(k+\nu_i z^{-1}+h_i(z)), \tag{3.4}$$

where $h_i(z)$ are regular functions of z and

$$\nu_i = 1, \quad i > l. \tag{3.5}$$

Proof. The matrix elements (2.24) are double periodic functions of the variable z having an essential singularity at $z=0$, but the functions $r_i(z)$ are meromorphic because $L(t,z)$ can be represented in the form

$$L(t,z) = G(t,z)\tilde{L}(t,z)G^{-1}(t,z), \;\; G_{ij} = \delta_{ij}\exp(\zeta(z)x_i(t)), \tag{3.6}$$

where $\tilde{L}_{ij}(t,z)$ are meromorphic functions of the variable z in a neighbourhood of the point $z=0$. In fact we have:

$$\tilde{L}(t,z) = -\frac{1}{z}(F(t)-2I)+O(z^0) \tag{3.7}$$

where $F(t)$ is the matrix of elements $f_{ij}(t)$. Therefore $r_i(z)$ are elliptic functions having poles of degree $N-i$ at most at the point $z=0$. Hence they can be represented in the form (3.3) as a linear combination of the function $\wp(z)$ and its derivatives. The coefficients $I_i^0, I_{i,s}$ of this expansion are the integrals of motion of the system (1.1). Each set of given values of these integrals defines an algebraic curve Γ.

Since around $z=0$ the function $r_i(z)$ has a pole of order $N-i$, a factorization of the form (3.4) holds. Due to equation (3.7) the coefficients $-2\nu_i$ in eq.(3.4) are the eigenvalues of the matrix $F-2I$. From eq. (1.8) we see that F is of rank l, hence the eigenvalue $\nu_i=1$ has multiplicity $N-l$. Moreover the corresponding $(N-l)$-dimensional subspace of eigenvectors $C=(c_1,\ldots,c_N)$ is defined by the equations

$$\sum_{j=1}^{N} c_j a_{j,\alpha} = 0, \quad \alpha = 1,\ldots,l. \tag{3.8}$$

∎

Remark. The conditions (3.5) imply a full set of linear relations on the integrals I_i^0, I_{is} of the system (1.1). Let us take any polynomial (in k) $R(k,z)$ of the from (3.2) with $r_i(z)$ of the form (3.3). It depends on $N(N+1)/2$ parameters I_i^0, I_{is}. Let us introduce the variable $\tilde{k} = k + z^{-1}$. Then the polynomial in this variable $\tilde{R}(\tilde{k},z) = R(\tilde{k}-z^{-1},z)$ for a generic set of variables I_i^0, I_{is} can be represented in the form

$$\tilde{R}(\tilde{k},z) = \sum_{i=0}^{N}\tilde{R}_i(\tilde{k})z^{-i} + \mathcal{R}(z,\tilde{k}), \tag{3.9}$$

where $\tilde{R}_i$ are polynomials in $\tilde{k}$ of degree $\deg\tilde{R}_i = N-i$ and $\mathcal{R}(z,\tilde{k}) = O(z)$ is a regular series in z with coefficients that are polynomials in $\tilde{k}$ of degree $N-1$. The conditions (3.5) imply that

$$\tilde{R}_i(\tilde{k}) = 0, \quad i > l. \tag{3.10}$$

The coefficients of $\tilde{R}_i$ are linear combinations of the parameters I_i^0, I_{is}. Therefore, (3.10) is equivalent to a set of $(N-l)(N-l+1)/2$ linear equations on these parameters. The total number of independent parameters is therefore equal to $Nl - l(l-1)/2$ which is exactly half the dimension of the reduced phase space.

In the trigonometric and rational cases the parametrization of the corresponding spectral curve is even more explicit.

Proposition 3.2 *In the trigonometric case we have:*

$$R(k,z) = R_0(k) + \coth\, z R_1(k) + \cdots + \coth^l z R_l(k) \tag{3.11}$$

where the $R_m(k)$ are polynomials in k of degree $\deg_k R_m = N - m$ and

$$R(k, z=-\infty) = R(k+2, z=+\infty) \tag{3.12}$$

In a neighbourhood of $z=0$ the function $R(k,z)$ can be factorized in the form of eq.(3.4) where now $\nu_i = 0, \quad i > l$.

Proof. The matrix $L(t,z)$ depends on z only through the term $\coth zF$. Since F is of rank l, $R(k,z)$ is of the form (3.11). To prove the relation (3.12) it is enough to remark that:

$$L(t,-\infty) + 2kI = e^{2X}\left(L(t,+\infty) + 2(k+2)I\right)e^{-2X} \tag{3.13}$$

with $X = \mathrm{Diag}\,(x_i(t))$. The conditions $\nu_i = 0, i > l$ follow from the fact that around $z=0$ we now have:

$$L(t,z) = -\frac{1}{z}F + O(z^0) \tag{3.14}$$

■

Proposition 3.3 *In the rational case we have:*

$$R(k,z) = R_0(k) + \frac{1}{z}R_1(k) + \cdots + \frac{1}{z^l}R_l(k) \tag{3.15}$$

where the $R_m(k)$ are polynomials in k of degree $\deg_k R_m = N - m$ and

$$R_1(k) = -\frac{\mathrm{d}R_0(k)}{\mathrm{d}k} \tag{3.16}$$

In a neighbourhood of $z=0$ the function $R(k,z)$ can be factorized in the form of eq.(3.4) where now $\nu_i = 0, \quad i > l$.

Proof. Since

$$[X, L(t,\infty)] = F - 2I \tag{3.17}$$

we have:

$$\begin{aligned} L(t,z) + 2kI &= L(t,\infty) - \frac{1}{z}[X, L(t,\infty)] + 2(k - \frac{1}{z})I & (3.18)\\ &= (I - \frac{1}{z}X)(L(t,\infty) + 2(k-\frac{1}{z})I)(I - \frac{1}{z}X)^{-1} + O(\frac{1}{z^2}) & (3.19) \end{aligned}$$

hence $R(k,z) = R_0(k - \frac{1}{z}) + O(\frac{1}{z^2})$ so that $R_1 = -R_0'$. ■

As a consequence we can count the number of parameters entering the spectral curve. Each R_m depends on $N - m + 1$ parameters, but relations (3.12) or (3.16) remove N parameters and the leading term of R_0 is already given so that we get $Nl - l(l-1)/2$ parameters, which is exactly half the dimension of the reduced phase space. These parameters can be identified with the action variables of our model and are in involution since there exists an r–matrix for L [9].

We now compute the genus of the spectral curve Γ.

Proposition 3.4 *For generic values of the action variables the genus of the spectral curve is given by:*

$$\text{Elliptic case}: \quad g = Nl - \frac{l(l+1)}{2} + 1 \tag{3.20}$$

$$\begin{matrix}\text{Trigonometric}\\ \text{and rational cases}\end{matrix}: \quad g = N(l-1) - \frac{l(l+1)}{2} + 1 \tag{3.21}$$

Proof. Equation (3.1) allows to present the compact Riemann surface Γ as an N-sheeted branched covering of the base curve of the variable z, i.e., the completed plane in the trigonometric and rational cases and the torus in the elliptic case. The sheets are the N roots in k. By the Riemann–Hurwitz formula we have $2g-2 = N(2g_0-2)+\nu$ where g_0 is the genus of the base curve, i.e. $g_0 = 0$ in the trigonometric and rational cases, $g_0 = 1$ in the elliptic case. Here ν is the number of branch points, i.e. the number of values of z for which $R(k,z)$ has a double root in k. This is the number of zeroes of $\partial_k R(k,z)$ on the surface $R(k,z) = 0$. But $\partial_k R(k,z)$ is a meromorphic function on the surface, hence it has as many zeroes as poles. The poles are located above $z = 0$ or $k = \infty$ which is the same, and are easy to count.

Let P_i be the points of Γ lying on the different sheets over the point $z = 0$. In the neighbourhood of P_i the function k has the expansion

$$k_i = -\nu_i z^{-1} - h_i(z). \tag{3.22}$$

Hence, the function $\partial R/\partial k$ in the neighbourhood of P_i has the form

$$\partial R/\partial k = 2^N \prod_{j\neq i}((\nu_j-\nu_i)z^{-1} + (h_j(z)-h_i(z))). \tag{3.23}$$

From this we see that on each of the l sheets $(k_i(z),z)$ $(i = 1,\cdots,l)$ we have one pole of order $(N-1)$. On each of the $(N-l)$ sheets $(k_i(z),z)$ $(i = l+1,\cdots,N)$ we have one pole of order l. Finally $\nu = l(N-1)+(N-l)l$ in either case. Inserting this value in the Riemann–Hurwitz theorem yields the result. ■

4 Analytic properties of the eigenvectors of the Lax matrix.

For a generic point P of the curve Γ, i.e. for the pair $(k,z) = P$, which satisfies the equation (3.2), there exists at time $t = 0$ a unique eigenvector $C(0,P)$ of the matrix $L(0,z)$ normalized by the condition $c_1(0,P) = 1$. In fact the un–normalized components $c_i(0,P)$ can be taken as $\Delta_i(0,P)$ where $\Delta_i(0,P)$ are suitable minors of the matrix $L(0,z)+2kI$, and are thus holomorphic functions on Γ outside the points above $z = 0$. After normalizing the first component, all the other coordinates $c_j(0,P)$ are meromorphic functions on Γ, outside the points P_i above $z = 0$. The poles of $c_j(0,P)$ are the zeroes on Γ of the first minor of the matrix $L(0,z)+2kI$, i.e., they are defined by the system of the equation (3.2) and the equation

$$\det(2k\delta_{ij} + L_{ij}(0,z)) = 0, \quad i,j > 1. \tag{4.1}$$

Thus the position of these poles only depend on the initial data.

In the trigonometric and rational cases nothing particular happens above $z = 0$. In the elliptic case however one has to be careful because of the essential singularity.

Proposition 4.1 *In the elliptic case, in the neighbourhood of the point P_i the coordinate $c_j(0,P)$ has the form*

$$c_j(0,P) = (c_j^{(i)}(0) + O(z))\exp\left[\zeta(z)(x_j(0)-x_1(0))\right], \tag{4.2}$$

where $c_j^{(i)}(t)$ is the eigenvector of the matrix $F(t)$ corresponding to the non–zero eigenvalue $2(1-\nu_i)$ i.e.,

$$\sum_{j=1}^{N} f_{kj}(t)c_j^{(i)}(t) = 2(1-\nu_i)c_k^{(i)}(t). \tag{4.3}$$

Proof. From equation (3.6), we have $C(0,P) = G(0,z)\tilde{C}(0,P)$, where $\tilde{C}(0,P)$ is an eigenvector of $\tilde{L}(0,z)$. Using equation (3.7), we have $\tilde{C}(0,P) = \tilde{C}^{(i)} + O(z)$ where $\tilde{C}^{(i)}$ is an eigenvector of $(F-2I)$. Therefore we have $c_j(0,P) = (c_j^{(i)}(0)+O(z))\exp(\zeta(z)x_j(0))$. Normalizing $c_1(0,P) = 1$ yields the result. ■

We can now compute the number of poles of C on Γ. This number is the same in all cases, although its relation to the genus of Γ differs in the elliptic and other cases.

Proposition 4.2 *The number of poles of $C(0,P)$ is:*

$$m = Nl - \frac{l(l+1)}{2} = g-1 \qquad \text{Elliptic case} \tag{4.4}$$

$$m = Nl - \frac{l(l+1)}{2} = g+N-1 \quad \text{Trigonometric and rational cases} \tag{4.5}$$

Proof. Let us introduce the function W of the complex variable z defined by:

$$W(z) = (\mathrm{Det}\,|c_i(M_j)|)^2$$

where the M_j's are the N points above z. It is well–defined on the base curve since the Det^2 does not depend on the order of the M_j's.

In the trigonometric and rational cases it is a meromorphic function, hence has the same number of zeroes and poles. In the elliptic case it has an essential singularity at $z=0$ of the form $\exp 2\zeta(z)\sum(x_i(0)-x_1(0))$. This does not affect the property that the number of poles is equal to the number of zeroes. Clearly W has a double pole for values of z such that there exists above z a point M at which $C(M)$ has a simple pole.

We show that $W(z)$ has a simple zero for values of z corresponding to a branch–point of the covering, hence $m=\nu/2$.

First notice that $W(z)$ only vanishes on branch–points, where there are at least two identical columns. Indeed, let $M_i=(k_i,z)$ be the N points above z. Then the $C(M_i)$ are the eigenvectors of $L(z)$ corresponding to the eigenvalues $-2k_i$ hence are linearly independent when all the k_i's are different. Therefore $W(z)$ cannot vanish at such a point. Let us assume now that z corresponds to a branch point, which is generically of order 2. At such a point $W(z)$ has a simple zero. Indeed let ξ be an analytical parameter on the curve around the branch point. The covering projection $M\to z$ gets expressed as $z=z_0+z_1\xi^2+O(\xi^3)$. The determinant vanishes to order ξ^1 hence W vanishes to order ξ^2, but this is precisely proportional to $z-z_0$. ∎

At this point the analysis of the elliptic and the trigonometric and rational cases begin to differ substantially. We treat them separately.

4.1 The elliptic case

In this case we compute the time evolution of the above eigenvectors.

Proposition 4.3 *The coordinates $c_j(t,P)$ of the vector-function $C(t,P)$ are meromorphic functions on Γ except at the points P_i. Their poles $\gamma_1,\dots,\gamma_{g-1}$ do not depend on t. In the neighbourhood of P_i they have the form*

$$c_j(t,P) = c_j^{(i)}(t,z)\exp\left(\zeta(z)(x_j(t)-x_1(0))+\mu_i(z)t\right), \tag{4.6}$$

where $c_j^{(i)}(t,z)$ are regular functions of z

$$c_j^{(i)}(t,z) = c_j^{(i)}(t) + O(z) \tag{4.7}$$

and

$$\mu_i(z) = (1-2\nu_i)z^{-2} - 2h_i(0)z^{-1} + O(z^0). \tag{4.8}$$

Proof. The fundamental matrix $S(t,z)$ of solutions to equation

$$(\partial_t + M(t,z))S(t,z) = 0, \quad S(0,z) = 1, \tag{4.9}$$

is a holomorphic function of the variable z for $z\neq 0$. At $z=0$ however it has an essential singularity.

We have $L(t,z)=S(t,z)L(0,z)S^{-1}(t,z)$. Therefore the vector $C(t,z)=S(t,z)C(0,z)$ is the common solution to (2.26) and to the equation

$$(L(t,z)+2kI)C(t,P) = 0, \quad P=(k,z)\in\Gamma. \tag{4.10}$$

Since $S(t,z)$ is regular for $z \neq 0$ we see that $C(t,P)$ has the same poles as $C(0,P)$.

Let us consider the vector $\tilde{C}(t,P)$ defined as

$$C(t,P) = G(t,z)\tilde{C}(t,P), \tag{4.11}$$

where $G(t,z)$ is the same as in (3.6). This vector is an eigenvector of the matrix $\tilde{L}(t,z)$ and satisfies the equation

$$(\partial_t + \tilde{M}(t,z))\tilde{C}(t,P) = 0, \quad \tilde{M} = G^{-1}\partial_t G + G^{-1}MG. \tag{4.12}$$

From (2.24,2.27) it follows that

$$\tilde{M}(t,z) = -z^{-2}I + z^{-1}\tilde{L}(t,z) + O(z^0). \tag{4.13}$$

It follows from this relation that around P_i

$$\partial_t\tilde{C}(t,z) = (\tilde{\mu}_i(t,z) + O(z^0))\tilde{C}(t,P)$$

where

$$\tilde{\mu}_i(t,z) = z^{-2} + 2k_i(z)z^{-1} = (1-2\nu_i)z^{-2} - 2h_i(0)z^{-1} + O(z^0). \tag{4.14}$$

From this, we deduce that around P_i, we have

$$\tilde{C}(t,P) = e^{\mu_i(z)t}\hat{C}(t,P)$$

where the vector $\hat{C}(t,P)$ is regular around P_i. Multiplying by $G(t,z)$ and normalizing $c_1(0,P) = 1$, we get the result. ∎

4.2 The trigonometric and rational cases

In these cases M is constant on the curve and we can choose:

$$C(t,P) = S(t)C(0,P). \tag{4.15}$$

where $S(t)$ is defined as above and is independent of the point of the curve. Hence $C(t,P)$ is a meromorphic vector with the same poles as $C(0,P)$. Moreover since $C(0,P)$ is regular above $z = 0$ the same is true for $C(t,P)$.

However, there appear new points at infinity playing the major role.

In the trigonometric case, we have two series of such points above $z = \pm\infty$. Let us denote these points by $Q_j(k = \chi_j, z = -\infty),\ j = 1,\cdots,N$ and $T_j(k = \chi_j + 2, z = +\infty),\ j = 1,\cdots,N$, (the k-coordinate of T_j is $\chi_j + 2$ because of equation (3.12)).

In the rational case, we have only one series of such points above $z = \infty$. We denote them by $Q_j(k = \chi_j, z = \infty),\ j = 1,\cdots,N$.

In the trigonometric case the base curve is in fact a cylinder, i.e., a sphere with two marked points, while in the rational case it is a sphere with only one marked point.

We study the solutions of the equation

$$L(t,P)C(t,P) \equiv (L(t,z) + 2kI)\,C(t,P) = 0$$

around these points.

Proposition 4.4 *In the trigonometric case, the eigenvectors at the points Q_j and T_j are related by:*

$$C(t,T_j) = \mu_j e^{-4(\chi_j+1)t} e^{-2X} C(t,Q_j) \tag{4.16}$$

In the rational case, at the point Q_j, we have:

$$\partial_k C(t,Q_j) = -(X + 2\chi_j t - \mu_j)C(t,Q_j). \tag{4.17}$$

The parameters μ_j are constants and $X = \mathrm{Diag}\,(x_i(t))$. Moreover with the normalization $c_1(0,P) = 1$ all the μ_j's are equal to $e^{2x_1(0)}$ or $x_1(0)$ respectively.

Proof. Let us prove first eq.(4.16). From equation (3.13) we see that:

$$C(t,T_j) = \mu_j(t)e^{-2X}C(t,Q_j)$$

To compute $\mu_j(t)$ we exploit the Lax equation $\dot{C} = -MC$ at the points Q_j, T_j using the fact that M is independent of the point on Γ. Using the relation:

$$e^{-2X}M(t)e^{2X} = M(t) + 2L(t,+\infty) - 2\dot{X} + 4I$$

we find $\dot{\mu}_j = -(4\chi_j + 4)\mu_j$.

The proof of eq.(4.17) is slightly more complicated. First of all, around a point Q_j, the curve has the equation:

$$R_0(k) - \frac{1}{z}R_0'(k) + O(\frac{1}{z^2}) = 0$$

implying

$$\left.\frac{1}{z^2}\frac{dz}{dk}\right|_{Q_j} = -1 \Longrightarrow \frac{1}{z} = (k-\chi_j) + O(k-\chi_j)^2 \tag{4.18}$$

hence k is an analytic parameter around Q_j.

Next we consider the equation: $\left[L(t,\infty) + 2kI - \frac{1}{z}F\right]C(t,P) = 0$. It gives:

$$L(t,Q_j)\partial_k C(t,Q_j) \quad = \quad (F-2I)C(t,Q_j). \tag{4.19}$$

To solve this equation, we remark that by virtue of equation (3.17), we have

$$L(t,Q_j)(-XC(t,Q_j)) = [X, L(t,Q_j)]C(t,Q_j) = (F-2I)C(t,Q_j)$$

therefore the general solution of eq. (4.19) is of the form

$$\partial_k C(t,Q_j) = -XC(t,Q_j) + \mu_j(t)C(t,Q_j). \tag{4.20}$$

To find the functions $\mu_j(t)$, we use the evolution equation $\dot{C} = -MC$, which implies:

$$\dot{\mu}_j C(t,Q_j) = (\dot{X} - [X,M])C(t,Q_j)$$

but it is straightforward to check that $\dot{X} - [X,M] = L(t,\infty)$. Therefore $\dot{\mu}_j = -2\chi_j$. Applying equation (4.17) for $i=1$ at $t=0$ we have $c_1 = 1$, $\partial_k c_1 = 0$ hence all the μ_j's are equal to $x_1(0)$. ∎

Similarly for the covector C^+ we have:

Proposition 4.5 *At the point Q_j, we have in the trigonometric case:*

$$C^+(t,T_j) = \mu_j^+ e^{4(\chi_j+1)t}C^+(t,Q_j)e^{2X} \tag{4.21}$$

and in the rational case we have:

$$\partial_k C^+(t,Q_j) = C^+(t,Q_j)(X + 2\chi_j t + \mu_j^+). \tag{4.22}$$

Moreover with the normalization $c_1^+(0,P) = 1$ all the μ_j^+'s are equal to $e^{-2x_1(0)}$, or $-x_1(0)$ in the rational case.

5 The analytic properties of Ψ and Ψ^+.

In this section we encode the previous results on the eigenvectors of the Lax matrix into analyticity properties of Ψ and Ψ^+. We treat separately the three cases.

5.1 The elliptic case.

Theorem 5.1 *The components $\Psi_\alpha(x,t,P)$ of the solution $\Psi(x,t,P)$ to the nonstationary matrix Schrödinger equation (2.10) are defined on the N-sheeted covering Γ of the initial elliptic curve. They are meromorphic on Γ outside l points $P_i,\ i=1,\ldots,l$. For general initial conditions the curve Γ is smooth, its genus equals $g = Nl - \frac{l(l+1)}{2} + 1$ and the Ψ_α have $(g-1)$ poles $\gamma_1,\ldots,\gamma_{g-1}$ which do not depend on the variables x,t. In a neighbourhood of $P_i,\ i=1,\ldots,l$, the function Ψ_α has the form:*

$$\Psi_\alpha(x,t,P) = (\chi_0^{\alpha i} + \sum_{s=1}^{\infty} \chi_s^{\alpha i}(x,t)z^s)e^{\lambda_i(z)x+\lambda_i^2(z)t}\ \Psi_1(0,0,P), \tag{5.1}$$

where

$$\lambda_i(z) = z^{-1} + k_i(z) = (1-\nu_i)z^{-1} - h_i(0) + O(z) \tag{5.2}$$

and $\chi_0^{\alpha i}$ are constants independent of t.

Proof. We recall the relation between the function Ψ and the eigenvectors of the Lax matrix.

$$\Psi(x,t,P) = \sum_{j=1}^{N} s_j(t,P)\Phi(x-x_j(t),z)e^{kx+k^2t}, \quad s_j(t,P) = c_j(t,P)a_j(t)$$

It is obvious that the $(g-1)$ poles γ_k of the c_i's are time–independent poles of Ψ. To study the behaviour of Ψ above $z=0$ we use the expansion of Φ at $z=0$:

$$\Phi(x,z) = \left(-\frac{1}{z} + \zeta(x) + O(z)\right)e^{\zeta(z)x} \tag{5.3}$$

and the expansion of the eigenvectors, equation (4.6). We get:

$$\Psi_\alpha = \sum_{j=1}^{N}\left(-\frac{1}{z} + \zeta(x-x_j(t)) + O(z)\right)a_{j,\alpha}(t)c_j^{(i)}(t,z)e^{\lambda_i(z)x+\lambda_i^2(z)t}e^{-z^{-1}x_1(0)}. \tag{5.4}$$

On the $(N-l)$ branches $i>l$ we see that $\lambda_i(z)$ is regular at $z=0$ due to eq. (3.5) so that Ψ has no essential singularity at the P_i for $i>l$ apart from the irrelevant constant factor $\exp(-z^{-1}x_1(0))$. Even more there is no pole at these points. This is because $\sum_j a_j^\alpha(t)c_j^{(i)}(t,z) = O(z)$ due to equation (3.8).

It only remains to prove that the leading term of the expansion of the first factor in the right hand side of (5.1) does not depend on t. (It does not depend on x because the singular part of Φ at $z=0$ does not depend on x.) The substitution of the right hand side of (5.1) with $\chi_0^{\alpha j} = \chi_0^{\alpha j}(t)$ into the equation

$$(\partial_t - \partial_x^2 + u(x,t))\Psi(x,t,P) = 0, \tag{5.5}$$

gives that

$$u = 2(\partial_x\chi_1)\Lambda\chi_0^{-1} - (\partial_t\chi_0)\chi_0^{-1}, \tag{5.6}$$

where χ_s is a matrix with entries $\chi_s^{\alpha j}$ and Λ is a diagonal matrix $\Lambda^{\alpha j} = (1-\nu_j)\delta^{\alpha j}$. From (5.4) it follows that χ_1 has the form

$$\chi_1 = \sum_{i=1}^{N} R_i(t)\zeta(x-x_i(t)). \tag{5.7}$$

Therefore, for a potential of the form $u = \sum \rho_i(t)\wp(x-x_i(t))$, the equality (5.7) implies that

$$\rho_i = -2R_i(t)\Lambda\chi_0^{-1}, \quad (\partial_t\chi_0)\chi_0^{-1} = 0. \tag{5.8}$$

Dividing eq.(5.4) by the normalization factor $\Psi_1(0,0,P)$, which plays no role in the fact that $\Psi(x,t,P)$ satisfies the differential equation (2.10) we get the final result. One can express χ_0 in terms of the $c_i^{(j)}(t)$ defined in eq.(4.3)

$$\chi_0^{\alpha j} = \sum_{i=1}^{l} c_i^{(j)}(0)a_{i,\alpha}(0) \tag{5.9}$$

$\blacksquare$

The same arguments show that:

Theorem 5.2 *The components $\Psi^{+,\alpha}(x,t,P)$ of the solution $\Psi^{+}(x,t,P)$ to the nonstationary matrix Schrödinger equation (2.11) are defined on the same curve Γ. They are meromorphic on Γ outside the l punctures $P_i,\ i=1,\dots,l$. In general $\Psi^{+,\alpha}$ have $(g-1)$ poles $\gamma_1^+,\dots,\gamma_{g-1}^+$ which do not depend on the variables x,t. In a neighbourhood of $P_i,\ i=1,\dots,l$, the function $\Psi^{+,\alpha}$ has the form:*

$$\Psi^{+,\alpha}(x,t,P)=(\chi_0^{+,\alpha i}+\sum_{s=1}^{\infty}\chi_s^{+,\alpha i}(x,t)z^s)e^{-\lambda_i(z)x-\lambda_i^2(z)t}\ \Psi^{+,1}(0,0,P), \tag{5.10}$$

where the $\chi_0^{+,\alpha j}$ are constants.

Remark. Theorem 5.1 states, in particular, that the solution Ψ of equation (2.10) is (up to normalization) a Baker-Akhiezer vector-function ([15]). In the next section we show that this function is uniquely defined by the curve Γ, its poles γ_s, the matrix χ_0 and the value $x_1(0)$. All these values are defined by the initial Cauchy data and do not depend on t. At the same time it is absolutely necessary to emphasize that part of them depend on the choice of the normalization point t_0 that we choose as $t_0=0$. Let us be more accurate. Any point of the phase space $\{x_i,p_i,a_i,b_i^+|(b_i^+,a_i)=2\}$ defines the matrix L with the help of formulae (2.24). The characteristic equation (3.2) defines an algebraic curve Γ. The equation (4.1) defines a set of $g-1$ points γ_s on Γ. Therefore, we may define a map

$$\{x_i,p_i,a_i,b_i^+|(b_i^+,a_i)=2\}\longmapsto\{\Gamma,\quad D\in J(\Gamma)\}, \tag{5.11}$$

$$D=\sum_{s=1}^{g-1}A(\gamma_s)+x_1U^{(1)}. \tag{5.12}$$

where $A:\Gamma\to J(\Gamma)$ is an Abel map and $U^{(1)}$ is a vector depending on Γ, only (see (6.9)). The coefficients of the equation (3.2) are integrals of the hamiltonian system (1.1). As we shall see in the next section the second part of data (5.11) define angle-type variables, i.e. the vector D(t) evolves linearly $D(t)=D(t_0)+(t-t_0)U^{(2)}$ if a point in phase space evolves according to the equations (1.4–1.6). These equations have the obvious symmetries:

$$a_i,b_i^+\to\lambda_i a_i,\lambda_i^{-1}b_i^+,\quad a_i,b_i^+\to W^{-1}a_i,b_i^+W, \tag{5.13}$$

where q_i are constants and W is an arbitrary constant matrix. In the next section we prove that to the data Γ,D one can associate a unique point in the phase space reduced under the symmetry (5.13).

5.2 The trigonometric and rational cases.

Theorem 5.3 *The components $\Psi_\alpha(x,t,P)$ of the solution $\Psi(x,t,P)$ to the nonstationary matrix Schrödinger equation (2.10) are defined on an N-sheeted covering Γ of the completed complex plane. They are meromorphic on Γ outside l points $P_i,\ i=1,\dots,l$. For general initial conditions the curve Γ is smooth, its genus equals $g=N(l-1)-\frac{l(l+1)}{2}+1$ and the Ψ_α have $(g+N-1)$ poles $\gamma_1,\dots,\gamma_{g+N-1}$ which do not depend on the variables x,t. In a neighbourhood of $P_i,\ i=1,\dots,l$, the function Ψ_α has the form:*

$$\Psi_\alpha(x,t,P)=(\chi_0^{\alpha i}z^{-1}+\sum_{s=1}^{\infty}\chi_s^{\alpha i}(x,t)z^{s-1})e^{k_i(z)x+k_i^2(z)t}, \tag{5.14}$$

where

$$k_i(z)=-\nu_i z^{-1}-h_i(0)+O(z) \tag{5.15}$$

and $\chi_0^{\alpha i}$ are constants.

In the trigonometric case, at the points T_j, and Q_j above $z=\pm\infty$ we have:

$$\Psi(x,t,T_j)=\mu_j\Psi(x,t,Q_j) \tag{5.16}$$

In the rational case, at the points Q_j above $z=\infty$ we have:

$$\partial_k\Psi(x,t,Q_j)=\mu_j\Psi(x,t,Q_j) \tag{5.17}$$

where the μ_j's are defined in eqs.(4.16,4.17).

Proof. In the trigonometric case, the result follows immediatly from equation (4.16).

In the rational case, we use eq.(2.12) with $s_i(t,P) = a_i(t)c_i(t,P)$, we get:

$$\partial_k \Psi = \sum_i a_i \left\{ -\frac{1}{z} c_i x - \frac{1}{z} \partial_k c_i + (1 + \frac{1}{z^2}\partial_k z - \frac{2kt}{z}) c_i + \frac{\partial_k c_i + (x_i + 2kt)c_i}{x - x_i} \right\} e^{kx+k^2 t}.$$

When $z = \infty$ we have $1 + \frac{1}{z^2}\partial_k z = 0$ and:

$$\partial_k \Psi|_{Q_j} = \left(\sum_i a_i \frac{\partial_k c_i + (x_i + 2kt)c_i}{x - x_i} \right) e^{kx+k^2 t} \Big|_{Q_j}.$$

The result follows from equation (4.17). ∎

Similarly we have for Ψ^+:

Theorem 5.4 *The components $\Psi^{+,\alpha}(x,t,P)$ of the solution $\Psi^+(x,t,P)$ to the nonstationary matrix Schrödinger equation (2.11) are defined on an N-sheeted covering Γ of the completed complex plane. They are meromorphic on Γ outside l points $P_i,\ i = 1,\dots,l$ and have $(g+N-1)$ poles $\gamma_1^+,\dots,\gamma_{g+N-1}^+$ which do not depend on the variables x,t. In a neighbourhood of $P_i,\ i = 1,\dots,l$, the function $\Psi^{+,\alpha}$ has the form:*

$$\Psi^{+,\alpha}(x,t,P) = (\chi_0^{+,\alpha i} z^{-1} + \sum_{s=1}^{\infty} \chi_s^{+,\alpha i}(x,t) z^{s-1}) e^{-k_i(z)x - k_i^2(z)t}, \tag{5.18}$$

and $\chi_0^{+,\alpha i}$ are constants. In addition at the points $T_j,\ Q_j$ above $z = \pm\infty$ we have in the trigonometric case:

$$\Psi^+(T_j) = \mu_j^+ \Psi^+(Q_j) \tag{5.19}$$

and in the rational case we have at the points Q_j above $z = \infty$:

$$\partial_k \Psi^+(x,t,Q_j) = \mu_j^+ \Psi^+(x,t,Q_j) \tag{5.20}$$

where the μ_j^+'s are defined in equation (4.21,4.22).

Remark. Obviously one can multiply the functions Ψ_α by a meromorphic function $f(P)$ on Γ without affecting the Schrödinger equation (2.10). We have already used this property in equation (5.1) to factor out $\Psi_1(0,0,P)$. The $m+l$ poles of the resulting Baker–Akhiezer function are now in arbitrary position. In the trigonometric and rational cases we can use the same feature in order to match the normalizations used in the elliptic case. Let us define $f(P)$ with $m+l$ zeroes at the points $\gamma_1,\cdots,\gamma_m$ and $P_1,\cdots,P_l$, N poles at the points $Q_1,\cdots,Q_N$ and $l-1$ poles $\gamma'_1,\cdots,\gamma'_{l-1}$ at some arbitrary prescribed positions. This defines a divisor of degree g, and such a function f is uniquely determined. It has g extra poles $\gamma'_l,\cdots,\gamma'_{g+l-1}$. The function

$$\psi' = f\psi \tag{5.21}$$

has now $m+l$ poles at the points Q_j and γ'_k, and satisfies the differential equation (2.10). In the following sections we will use ψ' (denoted ψ) and γ'_k (denoted γ_k).

Part II

The inverse Problem

6 The elliptic case

6.1 The Baker–Akhiezer functions.

At the begining of this section we present the necessary information on the finite-gap theory [15].

Theorem 6.1 *Let Γ be a smooth algebraic curve of genus g with fixed local coordinates $w_i(P)$ in neighbourhoods of l punctures $P_i,\ w_i(P_i)=0,\ i=1,\dots,l$. Then for each set of $g+l-1$ points $\gamma_1,\dots,\gamma_{g+l-1}$ in a general position there exists a unique function $\psi_\alpha(x,t,P)$ such that*

1^0. *The function ψ_α of the variable $P\in\Gamma$ is meromorphic outside the punctures and has at most simple poles at points γ_s (if all of them are distinct);*

2^0. *In the neighbourhood of the puncture P_j it has the form*

$$\psi_\alpha(x,t,P)=e^{w_j^{-1}x+w_j^{-2}t}\left(\delta_{\alpha j}+\sum_{s=1}^{\infty}\xi_s^{\alpha j}(x,t)w_j^s\right),\quad w_j=w_j(P). \tag{6.1}$$

Proof. The existence follows from the explicit formula given below in terms of Riemann Theta functions. Uniqueness results from the Riemann–Roch theorem applied to the ratio of two such functions. ∎

We now give a fundamental formula expressing the Baker–Akhiezer functions in terms of Riemann theta functions. According to the Riemann-Roch theorem for any divisor $D=\gamma_1+\cdots+\gamma_{g+l-1}$ in general position there exists a unique meromorphic function $h_\alpha(P)$ such that the divisor of its poles coincides with D and such that

$$h_\alpha(P_j)=\delta_{\alpha j}. \tag{6.2}$$

Using the results recalled in Appendix B, this function may be written as follows:

$$h_\alpha(P)=\frac{f_\alpha(P)}{f_\alpha(P_\alpha)};\quad f_\alpha(P)=\theta(A(P)+Z_\alpha)\frac{\prod_{j\neq\alpha}\theta(A(P)+R_j)}{\prod_{i=1}^{l}\theta(A(P)+S_i)}, \tag{6.3}$$

where

$$R_j=-\mathcal{K}-A(P_j)-\sum_{s=1}^{g-1}A(\gamma_s),\quad j=1,\dots,l \tag{6.4}$$

$$S_i=-\mathcal{K}-A(\gamma_{g-1+i})-\sum_{s=1}^{g-1}A(\gamma_s), \tag{6.5}$$

and

$$Z_\alpha=Z_0-A(P_\alpha),\quad Z_0=-\mathcal{K}-\sum_{i=1}^{g+l-1}A(\gamma_i)+\sum_{j=1}^{l}A(P_j). \tag{6.6}$$

Let $d\Omega^{(i)}$ be the unique meromorphic differential holomorphic on Γ outside the punctures $P_j,\ j=1,\dots,l$, which has the form

$$d\Omega^{(i)}=d(w_j^{-i}+O(w_j)) \tag{6.7}$$

near the punctures and is normalized by the conditions (see Appendix B for the definition of the cycles $a_k^0,\ b_k^0$)

$$\oint_{a_k^0}d\Omega^{(i)}=0. \tag{6.8}$$

It defines a vector $U^{(i)}$ with coordinates

$$U_k^{(i)}=\frac{1}{2\pi i}\oint_{b_k^0}d\Omega^{(i)} \tag{6.9}$$

Theorem 6.2 *The components of the Baker-Akhiezer function $\psi(x,t,P)$ are equal to*

$$\psi_\alpha(x,t,P)=h_\alpha(P)\frac{\theta(A(P)+U^{(1)}x+U^{(2)}t+Z_\alpha)\theta(Z_0)}{\theta(A(P)+Z_\alpha)\theta(U^{(1)}x+U^{(2)}t+Z_0)}e^{(x\Omega^1(P)+t\Omega^{(2)}(P))} \tag{6.10}$$

$$\Omega^{(i)}(P)=\int_{q_0}^{P}d\Omega^{(i)} \tag{6.11}$$

Proof. It is enough to check that the function defined by the formula (6.10) is well-defined (i.e. it does not depend on the path of integration between q_0 and P) and has the desired analytical properties. (The ratio of the product of theta-functions at $P = P_\alpha$ equals 1 due to (6.6).) ∎

From the exact theta-function formula (6.10) it follows that ψ may be represented in the form

$$\psi(x,t,P) = r(x,t)\hat{\psi}(x,t,P), \tag{6.12}$$

where $\hat{\psi}(x,t,P)$ is an entire function of the variables x, t and $r(x,t)$ is meromorphic function.

We now give the definition of the dual Baker–Akhiezer function. For any set of $g+l-1$ points in general position there exists a unique meromorphic differential $d\Omega$ that has poles of the form

$$d\Omega = \frac{dw_j}{w_j^2} + \frac{\lambda_j dw_j}{w_j} + O(1)dw_j \tag{6.13}$$

at the punctures P_j, and equals zero at the points γ_s

$$d\Omega(\gamma_s) = 0. \tag{6.14}$$

Besides γ_s this differential has $g+l-1$ other zeros that we denote γ_s^+.

The dual Baker-Akhiezer function is the unique function $\psi^+(x,t,P)$ with coordinates $\psi^{+,\alpha}(x,t,P)$ such that

1^0. The function $\psi^{+,\alpha}$ as a function of the variable $P \in \Gamma$ is meromorphic outside the punctures and has at most simple poles at the points γ_s^+ (if all of them are distinct);

2^0. In the neighbourhood of the puncture P_j it has the form

$$\psi^{+,\alpha}(x,t,P) = e^{-w_j^{-1}x - w_j^{-2}t}\left(\delta_{\alpha j} + \sum_{s=1}^{\infty} \xi_s^{+,\alpha j}(x,t) w_j^s\right). \tag{6.15}$$

Let $h_\alpha^+(P)$ be the function that has poles at the points of the dual divisor $\gamma_1^+, \ldots, \gamma_{g+l-1}^+$ and is normalized by (6.2) (i.e. $h_\alpha^+(P_j) = \delta_{\alpha j}$). It can be written in the form (6.3) in which γ_s are replaced by γ_s^+. It follows from the definition of the dual divisors that

$$\sum_{s=1}^{g+l-1} A(\gamma_s) + \sum_{s=1}^{g+l-1} A(\gamma_s^+) = K_0 + 2\sum_{j=1}^{l} A(P_j), \tag{6.16}$$

where K_0 is the canonical class (i.e. the equivalence class of of the divisor of zeros of a holomorphic differential).

Theorem 6.3 *The components of the dual Baker-Akhiezer function $\psi^+(x,t,P)$ are equal to*

$$\psi_\alpha^+(x,t,P) = h_\alpha^+(P) \frac{\theta(A(P) - U^{(1)}x - U^{(2)}t + Z_\alpha^+)\theta(Z_0^+)}{\theta(A(P) + Z_\alpha^+)\theta(U^{(1)}x + U^{(2)}t - Z_0^+)} e^{-(x\Omega^1(P) + t\Omega^{(2)}(P))}, \tag{6.17}$$

where

$$Z_0^+ = Z_0 - 2\mathcal{K} - K_0, \quad Z_\alpha^+ = Z_0^+ - A(P_\alpha). \tag{6.18}$$

The above results are valid for any curve Γ. We now consider a more specific setting which will correspond to the elliptic model.

Theorem 6.4 *Let Γ be a smooth algebraic curve defined by an equation of the form*

$$R(k,z) = k^N + \sum_{i=0}^{N} r_i(z) k^i = 0, \tag{6.19}$$

where $r_i(z)$ are elliptic functions, holomorphic outside the point $z = 0$, such that the covering $P \to z$ has no branching points over $z = 0$ (i.e. the function $k(P)$ has N simple poles $P_1, \ldots, P_N$ on Γ which are preimages of $z = 0$). Let us assume also that the residues ν_j of $k(P)$ at the poles defined by the expansion of $R(z, k)$ near $z = 0$

$$R(k, z) = \prod_{i=1}^{N} (k + \nu_i z^{-1} + O(z^0)), \tag{6.20}$$

satisfy (3.5), $\nu_j = 1$, $j > l$. Then there exists a function $\varphi_i(P)$ on Γ such that the Baker–Akhiezer function ψ corresponding to the curve Γ and the local parameters $w_j = (k_j(z) + \zeta(z))^{-1}$ at the puncture P_j obeys:

$$\psi(x + 2\omega_i, t, P) = \varphi_i(P)\psi(x, t, P). \tag{6.21}$$

Proof. Consider the functions

$$\varphi_i(P) = \exp(2(k(P) + \zeta(z))\omega_i - 2\eta_i z), \quad i = 1, 2, \tag{6.22}$$

where $2\omega_i$ are periods of the elliptic base curve and $\eta_i = \zeta(\omega_i)$. From the monodromy properties of ζ-function and relation $2(\eta_1\omega_2 - \eta_2\omega_1) = \pi i$ it follows that $\varphi(P)$ is a well-defined function on the curve Γ. It is holomorphic outside the points $P_1, \ldots, P_l$. In the neighbourhood of P_j it has the form

$$\varphi_i(P) = (1 + O(z)) \exp(2(k_j(z) + \zeta(z))\omega_i). \tag{6.23}$$

Let $\psi(x, t, P)$ be the Baker-Akhiezer vector function corresponding to $\Gamma, P_j, w_j(P)$ and to any divisor D of the degree $g + l - 1$. Then equation (6.21) follows from the fact that the right and left hand sides have the same analytical properties. ■

Corollary 6.1 *The vector Baker–Akhiezer function $\psi(x, t, P)$ with components ψ_α, $\alpha = 1, \cdots, l$ can be written in the form:*

$$\psi(x, t, P) = \sum_{i=1}^{m} s_i(t, P)\Phi(x - x_i(t), z)e^{kx + k^2 t}, \quad P = (k, z), \tag{6.24}$$

Proof. Let $x_i(t), i = 1, \ldots, m$, be the set of poles of the function $\psi(x, t, P)$ (as a function of the variable x) in the fundamental domain of the lattice with periods $2\omega, 2\omega'$. It follows from (6.12) that they do not depend on P. Let us assume that these poles are simple. Then their exist vectors $s_i(t, P)$ such that the function

$$\mathcal{F}(x, t, P) = \psi(x, t, P) - \sum_{i=1}^{m} s_i(t, P)\Phi(x - x_i(t), z)e^{kx + k^2 t}$$

is *holomorphic* in x in this fundamental domain. This function has the same monodromy properties (6.21) as the function ψ. Any non-trivial function satisfying (6.21) has at least one pole in the fundamental domain. Hence, $\mathcal{F} = 0$. ■

Let us remark that for the above specific curve Γ, the Baker–Akhiezer function is exactly of the form postulated in equation (2.12). The same arguments show that the dual Baker-Akhiezer function has the form (2.13).

6.2 The potential.

The following theorem is a particular case of a general statement [15].

Theorem 6.5 *Let $\psi(x, t, P)$ be a vector-function with above-defined co-ordinates $\psi_\alpha(x, t, P)$. Then it satisfies the equation*

$$(\partial_t - \partial_x^2 + u(x, t))\psi(x, t, P) = 0, \tag{6.25}$$

where the entries of the matrix-function u are equal to:

$$u^{\alpha i}(x,t) = 2\partial_x \xi_1^{\alpha i}(x,t). \tag{6.26}$$

The potentials $u(x,t)$ corresponding to some Baker-Akhiezer vector-function are called algebro-geometrical or finite-gap potentials.

Proof. This follows directly from the fact that $(\partial_t - \partial_x^2)\psi_\alpha$ has the same analytic properties as ψ but for the normalizations at the P_i's, so can be expanded on the ψ_β with coefficients $-u^{\alpha\beta}$. ■

Theorem 6.6 *The dual Baker-Akhiezer function satisfies the equation:*

$$\psi^+(x,t,P)(\partial_t - \partial_x^2 + u(x,t)) = 0, \tag{6.27}$$

where $u(x,t)$ is the same as in (6.25).

Proof. To show that the potentials are the same, we consider the form $\psi_\alpha \psi^{+\beta} d\Omega$ where $d\Omega$ is defined by equation (6.13) and the conditions (6.14). This is a meromorphic 1–form on Γ with poles only at the P_j's. Around P_j we have:

$$\psi_\alpha \psi^{+\beta} d\Omega = \left[\frac{\delta_{\alpha j}\delta_{\beta j}}{w_j^2} + (\delta_{\alpha j}\delta_{\beta j}\lambda_j + \delta_{\alpha j}\xi_1^{+\beta_j} + \delta_{\beta j}\xi_1^{\alpha j})\frac{1}{w_j} + O(1) \right] dw_j$$

Since the sum of residues must vanish we get:

$$\xi_1^{+\beta\alpha} + \xi_1^{\alpha\beta} = -\lambda_\alpha \delta_{\alpha\beta}$$

This implies the result since $u^{\alpha\beta} = 2\partial_x \xi_1^{\alpha\beta}$, $u^{+\alpha\beta} = -2\partial_x \xi_1^{+\beta\alpha}$ and λ_α is independent of x. ■

In general position the algebro-geometrical potentials are quasi-periodic functions of all arguments. In [5] the necessary conditions on the algebraic geometrical data $\{\Gamma, P_1, \ldots, P_l, w_1(P), \ldots, w_l(P)\}$ were found in order that the corresponding potentials be elliptic functions of the variable x. Here we have

Proposition 6.1 *Let Γ be a specific curve as in Theorem (6.4). Then the algebro-geometrical potential $u(x,t)$ corresponding to the curve Γ, to the points $P_1, \ldots, P_l$, and to the local coordinates $w_j(z) = (k_j(z) + \zeta(z))^{-1}$ is an elliptic function. In general position this potential has the form*

$$u(x,t) = \sum_{i=1}^{N} a_i(t) b_i^+(t) \wp(x - x_i(t)) \tag{6.28}$$

Proof. The potential is elliptic because of equation (6.21). Since the Baker–Akhiezer function has the form (6.24) u has only double poles. Hence it is of the form: $u = \sum \rho_i(t)\wp(x - x_i(t))$. We show that the matrices $\rho_i(t)$ are of rank one. It follows from (6.10) that the poles $x = x_i(t)$ of the Baker-Akhiezer function correspond to the solutions of the equation:

$$\theta(U^{(1)}x + U^{(2)}t + Z) = 0. \tag{6.29}$$

From (6.6) it follows that for such a pair $(x_i(t),\ t)$ the first factor in the numerator of formula (6.10) vanishes at P_α. At a point P_β, $\beta \neq \alpha$, it is the function $h_\alpha(P)$ which vanishes. Therefore, the residue $\psi^0_{\alpha,i}(t,P)$ of $\psi_\alpha(x,t,P)$ at $x = x_i(t)$, as a function of the variable P, has the following analytical properties:

1^0. It is a meromorphic function outside the punctures P_j and has the same set of poles as ψ;

2^0. In a neighbourhood of the puncture P_j it has the form

$$\psi^0_{\alpha,i}(t,P) = \exp(w_j^{-1} x_i(t) + w_j^{-2} t) O(w_j). \tag{6.30}$$

Hence, it has the same analytical properties as the Baker–Akhiezer function but with one difference. Namely, the regular factor of its expansion at *all* the punctures vanishes. For generic x, t there are no such function. For special pairs $(x = x_i(t), t)$ such a function $\psi_{i0}(t, P)$ exists and is *unique* up to a constant (in P) factor (it is unique in a generic case when $x_i(t)$ is a simple root of (6.29)). Therefore, the components of the Baker-Akhiezer function can be represented in the form

$$\psi_\alpha(x, t, P) = \frac{\phi_\alpha(t)\ \psi_{i0}(t, P)}{x - x_i(t)} + O((x - x_i(t))^0). \tag{6.31}$$

The last equality implies that the residue $\tilde{\rho}_i(t)$ of the matrix $\xi_1(x, t)$ with entries $\xi_1^{\alpha j}(x, t)$

$$\xi_1(x, t) = \frac{\tilde{\rho}_i(t)}{x - x_i(t)} + O((x - x_i(t))^0) \tag{6.32}$$

is of rank 1. It follows from (6.26) that $\rho_i(t) = -2\tilde{\rho}_i(t)$. Hence this matrix is of rank one too, and therefore has the form (6.28). ∎

Let us examine now the effect of replacing the divisor by an equivalent one. Let $D = \gamma_1 + \cdots + \gamma_{g+l-1}$ and $D^{(1)} = \gamma_1^{(1)} + \cdots + \gamma_{g+l-1}^{(1)}$ be two equivalent divisors (i.e. there exists a meromorphic function $h(P)$ on Γ such that D is a divisor of its poles and $D^{(1)}$ is a divisor of its zeros). Then for the corresponding Baker-Akhiezer vector-functions $\psi(x, t, P)$ and $\psi^{(1)}(x, t, P)$ the equality

$$H\psi(x, t, P) = \psi^{(1)}(x, t, P)h(P), \tag{6.33}$$

is valid. Here H is a diagonal matrix

$$H^{\alpha j} = h(P_j)\delta^{\alpha j}. \tag{6.34}$$

The proof of (6.33) follows from the uniqueness of the Baker-Akhiezer functions, because the left and the right hand sides of (6.33) have the same analytical properties.

Corollary 6.2 *The algebraic-geometrical potentials $u(x, t)$ and $u^{(1)}(x, t)$ corresponding to $\Gamma, P_j, w_j(P)$ and to equivalent effective divisors D and $D^{(1)}$, respectively, are gauge equivalent*

$$u^{(1)}(x, t) = Hu(x, t)H^{-1}, H^{\alpha j} = h_j\delta^{\alpha j}. \tag{6.35}$$

Corollary 6.3 *A curve Γ in a general position satisfies the conditions of theorem 6.2 if and only if it is the spectral curve (3.2) of a Lax matrix L defined by the formula (2.24) where x_i, p_i are arbitrary constants and f_{ij} are defined by vectors a_i, b_i^+, satisfying (1.7), with the help of (1.8). The corresponding vectors are defined uniquely up to the transformation (5.13)*

Notice that the Baker–Akhiezer function $\Psi_\alpha(x, t, P)/\Psi_1(0, 0, P)$ appearing in equation (5.1) is related to the normalized Baker–Akhiezer function $\psi_\alpha(x, t, P)$ appearing in equation (6.1) by:

$$\frac{\Psi_\alpha(x, t, P)}{\Psi_1(0, 0, P)} = \sum_\beta \chi_0^{\alpha\beta}\psi_\beta(x, t, P)$$

This induces a similarity transformation on the potential, and we have:

Corollary 6.4 *If $a_i(t), b_i(t), x_i(t)$ are a solution of the equation of motion of the hamiltonian system (1.1) then*

$$\sum_{i=1}^N a_i(t)b_i^+(t)\wp(x - x_i(t)) = \chi_0 u(x, t)\chi_0^{-1}, \tag{6.36}$$

where $u(x, t)$ is the algebro–geometrical potential corresponding to the data that are described in theorem 6.2 and to the normalized Baker–Akhiezer functions.

Corollary 6.5 *The correspondence*

$$a_i(t),\ b_i^+(t), x_i(t) \longmapsto \{\Gamma,\ [D]\}, \tag{6.37}$$

where $[D]$ is a equivalence class of the divisor D (i.e. the corresponding point of the Jacobian) is an isomorphism up to the equivalence (5.13).

The curve Γ is a time independent. At the same time $[D]$ depends on the choice of the normalisation point $t_0 = 0$. From the exact formulae for the solutions (see below) it follows that the dependence of $D(t_0)$ is linear onto the Jacobian.

6.3 Reconstruction formulae.

Theorem 6.7 *Let Γ be a curve that is defined by the equation of the form (3.2) and $D = \gamma_1, \dots, \gamma_{g+l-1}$ be a set of points in general position. Then the formulas*

$$\theta(U^{(1)}x_i(t) + U^{(2)}t + Z_0) = 0, \tag{6.38}$$

$$a_{i,\alpha}(t) = Q_i^{-1}(t)h_\alpha(q_0)\frac{\theta(U^{(1)}x_i(t) + U^{(2)}t + Z_\alpha)}{\theta(Z_\alpha)}. \tag{6.39}$$

$$b_i^\alpha(t) = Q_i^{-1}(t)h_\alpha^+(q_0)\frac{\theta(U^{(1)}x_i(t) + U^{(2)}t - Z_\alpha^+)}{\theta(Z_\alpha^+)}, \tag{6.40}$$

where

$$Q_i^2(t) = \frac{1}{2}\sum_{\alpha=1}^{l} h_\alpha^+(q_0)h_\alpha(q_0)\frac{\theta(U^{(1)}x_i(t) + U^{(2)}t - Z_\alpha)\theta(U^{(1)}x_i(t) + U^{(2)}t - Z_\alpha^+)}{\theta(Z_\alpha)\theta(Z_\alpha^+)} \tag{6.41}$$

define the solutions of the system (1.4, 2.14, 2.15). Any solution of the system (1.1) may be obtained from the solutions (6.38-6.40) with the help of symmetries (5.13).

Remark. (6.38) may be reformulated as follows: the vector $U^{(1)}$ defines an imbedding of the elliptic curve into the Jacobian $J(\Gamma)$. Therefore the restriction of the theta function onto the corresponding elliptic curve is a product of the elliptic σ-Weiestrass functions, i.e. of the vector

$$\theta(U^{(1)}x + U^{(2)}t + Z_0) = const\prod_{i=1}^{N}\sigma(x - x_i(t)). \tag{6.42}$$

It should be mentioned that exactly the same formula was obtained in [5] for the solutions of the elliptic Calogero-Moser system. The difference for various spins l is encoded in the set of corresponding curves.

From (2.12,2.22) it follows that the vector $a_i(t)$ is proportional to the singular part of the expansion of $\psi(x,t,P)$ near $x = x_i(t)$ and up to the scalar factor this singular part does not depend on P. Therefore, using the formula (6.10) for $P = q_0$ we obtain (6.39). ■

7 The rational and trigonometric cases.

7.1 Baker–Akhiezer functions

Let Γ be the smooth algebraic curve of genus g defined by the equation (3.11) or (3.15). The function $k(P)$ has l simple poles above $z = 0$, denoted by $P_1, \cdots, P_l$. In the vicinity of P_i one has $k = k_i(z) \equiv -\nu_i/z - h_i(z)$ with h_i regular. We take $1/k_i(z)$ as local parameter around P_i. We shall consider the space of Baker–Akhiezer functions with $(g+l-1)$ poles γ_k and N poles at the points Q_j above $z = -\infty$, which are of the form $\exp(kx + k^2t)$ times a meromorphic function. The space of such functions is of dimension $N + l$, and imposing the N conditions (5.16) or (5.17) one ends up with a space of dimension

l. One can define a basis ψ_α of this space by choosing the normalization in the neighbourhood of the points P_j above $z = 0$ with local parameters (z, k_j) as:

$$\psi_\alpha(P, x, t) = \left(\delta_{\alpha j} + \sum_{s=1}^{\infty} \frac{\xi_s^{\alpha j}(x,t)}{k_j^s}\right) e^{k_j x + k_j^2 t} \tag{7.1}$$

We have now at our disposal $g + l - 1$ parameters γ_k, and the N coefficients μ_j which add up to $m + l$, i.e. half of the dimension of the phase space, just as in the elliptic case.

One can construct directly the Baker functions ψ_α from the above analyticity properties. One first chooses a convenient basis to expand them.

There exists a unique function $g_j(P)$ with $g + l - 1$ poles γ_k, one pole at Q_j with residue 1 (i.e. of the form $1/(k - \chi_j)$) and l zeroes at the P_α's. Then we can write:

$$\psi_\alpha(P, x, t) = \left(h_\alpha(P) + \sum_{j=1}^{N} g_j(P) r_{j\alpha}(x,t)\right) e^{kx + k^2 t} \tag{7.2}$$

where $h_\alpha(P)$ is the function defined in eq.(6.3), so that conditions (7.1) are fulfilled.

It remains only to express the N conditions (5.16 or 5.17). This yields the

Theorem 7.1 *The components of the Baker–Akhiezer function $\psi(x, t, P)$ are given by:*

$$\psi_\alpha(x, t, P) = \frac{\mathrm{Det}\begin{pmatrix} h_\alpha(P) & g_j(P) \\ h_\alpha(T_i) & \Theta_{ij}(x,t) \end{pmatrix}}{\mathrm{Det}\,(\,\Theta_{ij}(x,t)\,)} e^{kx + k^2 t} \tag{7.3}$$

where Θ is the matrix with elements in the trigonometric case:

$$\Theta_{ii} = -\sigma_i e^{-2x - 4(\chi_i + 1)t} + g_i(T_i), \quad \Theta_{ij} = g_j(T_i) \quad i \neq j. \tag{7.4}$$

In the rational case one has to replace $h_\alpha(T_i)$ by $h_\alpha(Q_i)$ in (7.3) and to define:

$$\Theta_{ii} = x + 2\chi_i t - \sigma_i + g_i^{(1)}, \quad \Theta_{ij} = g_j(Q_i) \quad i \neq j \tag{7.5}$$

where $g_j(P) = 1/(k - \chi_j) + g_j^{(1)} + O(k - \chi_j)$.

Proof. We first express the conditions on ψ arising from the the conditions (5.16) or (5.17) on ψ/f where f is the meromorphic function introduced in equation (5.21). As a matter of fact near the point Q_j in the trigonometric case we have:

$$\psi_\alpha(x, t, P) = \frac{R_{j\alpha}^{(-1)}(x,t)}{k - \chi_j} + O((k - \chi_j)^0)$$

while around T_j we have:

$$\psi_\alpha(x, t, P) = R_{j\alpha}^{(0)}(x,t) + O(k - \chi_j)$$

The relations (5.16) take the form:

$$R_{j\alpha}^{(0)}(x,t) = \sigma_j R_{j\alpha}^{(-1)}(x,t) \quad \text{with } \sigma_j = \mu_j \frac{f(T_j)}{f_j^{(0)}} \tag{7.6}$$

where around Q_j the function f appearing in equation (5.21) has the corresponding expansion:

$$f(P) = \frac{f_j^{(0)}}{k - \chi_j} + f_j^{(1)} + O(k - \chi_j)$$

and σ_j is independent of x and t. In the rational case near Q_j we have:

$$\psi_\alpha(P,x,t) = \frac{R^{(-1)}_{j\alpha}(x,t)}{k-\chi_j} + R^{(0)}_{j\alpha}(x,t) + O(k-\chi_j)$$

and the conditions (5.17) on ψ are equivalent to:

$$R^{(0)}_j(x,t) = \sigma_j R^{(-1)}_j(x,t) \quad \text{with } \sigma_j = \mu_j + \frac{f^{(1)}_j}{f^{(0)}_j} \tag{7.7}$$

Using the expression (7.2) of ψ these conditions take the form (in the rational case T_j is replaced by Q_j below):

$$\sum_k \Theta_{jk}(x,t) r_{k\alpha} = -h_\alpha(T_j) \tag{7.8}$$

Solving this linear system with Cramer's rule yields the result. ∎

Proposition 7.1 *The Baker–Akhiezer function given in (7.3) can be written in the form:*

$$\psi = \sum_i^N s_i(t,k,z)\Phi(x-x_i(t),z)e^{kx+k^2t} \tag{7.9}$$

Proof. Let us give the proof in the trigonometric case. In the rational case, the proof is similar and even simpler. From eq.(7.3) we see that one can write

$$\begin{aligned}\psi_\alpha(x,t,P) &= \left(h_\alpha(P) - \sum_{i=1}^N \frac{2e^{-2x_i} s_{i,\alpha}(t,P)}{e^{-2x}-e^{-2x_i}}\right) e^{kx+k^2t} \\ &= \left(h_\alpha(P) + \sum_{i=1}^N s_{i,\alpha}(t,P) + \sum_{i=1}^N s_{i,\alpha}(t,P)\coth(x-x_i)\right) e^{kx+k^2t} \end{aligned}\tag{7.10}$$

We have to show that

$$h_\alpha(P) = -(1+\coth z)\sum_{i=1}^N s_{i,\alpha}(t,P)$$

But the function $h_\alpha(P)/(1+\coth z)$ vanishes at the points P_i above $z=0$, and has poles at the points Q_j above $z=-\infty$. Hence, we can write

$$\frac{h_\alpha(P)}{1+\coth z} = \sum_{j=1}^N \frac{h_\alpha(Q_j)}{\alpha_j} g_j(P)$$

where the constants α_j are defined by $1+\coth z = \alpha_j\,(k-\chi_j) + O(k-\chi_j)^2$ around Q_j. Using this formula at $P=T_i$ we get in particular

$$h_\alpha(T_i) = 2\sum_{j=1}^N \frac{h_\alpha(Q_j)}{\alpha_j} g_j(T_i) \tag{7.11}$$

On the other hand, using eq.(7.10), we find

$$2\sum_{i=1}^N s_{i,\alpha}(t,P) = \psi_\alpha(x,t,P)e^{-kx-k^2t}|_{x=+\infty} \;-\; \psi_\alpha(x,t,P)e^{-kx-k^2t}|_{x=-\infty} = \frac{\mathrm{Det}\begin{pmatrix} 0 & g_j(P) \\ h_\alpha(T_i) & g_j(T_i)\end{pmatrix}}{\mathrm{Det}\,(\,g_j(T_i)\,)}$$

Expanding the determinant in the numerator along the first line, and using eq.(7.11) to evaluate $h_\alpha(T_j)$, we get

$$\sum_{i=1}^{N} s_{i,\alpha}(P,t) = -\sum_{j=1}^{N} \frac{h_\alpha(Q_j)}{\alpha_j} g_j(P)$$

which is what we had to prove. ■

One can also give an explicit formula for $s_{i,\alpha}(t,P)$. Since $\mathrm{Det}\,\Theta(x_i,t)=0$ one can write a linear dependency relation:

$$\Theta_{k1} = \sum_{j=2}^{N} \lambda_j^{(i)}(t)\Theta_{kj}, \quad \forall k$$

and we see that the residue $s_{i,\alpha}(t,P)$ in eq. (7.10) is given by ($\lambda_1^{(i)}=-1$):

$$s_{i,\alpha}(t,P) = \left\{ \frac{\sum_{k=1}^{N} \lambda_k^{(i)}(t) g_k(P)}{\prod_{j=1}^{N}(2\sigma_j e^{-4(\chi_j+1)t-x_i-x_j}) \prod_{j\neq i} \sinh(x_i-x_j)} \right\} \mathrm{Det}\,\Theta_\alpha^{(i)}(t) \tag{7.12}$$

Here $\Theta_\alpha^{(i)}$ is obtained from $\Theta(x,t)$ by taking $x=x_i$ and replacing the first column by $h_\alpha(T_j)$. This equation has to be compared with eq.(2.22). In the rational case we find a similar and simpler formula, including the same factor $\Theta_\alpha^{(i)}(t)$.

As in the elliptic case we need the dual Baker–Akhiezer function ψ^+ and for this we introduce the differential $d\Omega$ with poles of order 2 at the punctures P_j's such that $d\Omega = dw_j/w_j^2 + O(1/w_j)dw_j$ and vanishing on the $g+l-1$ points γ_k. Let γ_k^+ the $g+l-1$ other zeroes of $d\Omega$.

Let $h^{+,\alpha}(P)$ be the unique function with poles at the γ_k^+'s and such that $h^{+,\alpha}(P_j)=\delta_{\alpha j}$. In the trigonometric case we introduce the function $g_j^+(P)$ with $g+l-1$ poles γ_k^+, one pole at T_j with residue 1 (i.e. of the form $1/(k-\chi_j-2)$), and l zeroes at the P_j's. Then we define the dual Baker–Akhiezer function:

$$\psi^{+,\alpha}(P,x,t) = \left(h^{+,\alpha}(P) + \sum_{j=1}^{N} g_j^+(P) r_j^{+,\alpha}(x,t) \right) e^{-kx-k^2t}. \tag{7.13}$$

and such that relations of the type (7.7) are satisfied with some coefficients σ_j^+. We choose σ_j^+ as

$$\sigma_j^+ = -\sigma_j \frac{d\Omega(T_j)}{d\Omega(Q_j)}$$

where the form $d\Omega$ is expressed on dk at Q_j and T_j. With this choice the sum of the residues of $\psi^{+,\alpha}\psi_\beta d\Omega$ at Q_j and T_j vanishes. This condition ensures that the potential reconstructed from ψ^+ is the same as the one reconstructed from ψ.

Notice that the roles of Q_j and T_j are interchanged in the definitions of ψ and ψ^+.

A similar analysis holds in the rational case.

Theorem 7.2 *The components of the Baker–Akhiezer function $\psi^+(x,t,P)$ are given by:*

$$\psi^{+,\alpha}(P,x,t) = \frac{\mathrm{Det}\begin{pmatrix} h^{+,\alpha}(P) & g_j^+(P) \\ h^{+,\alpha}(Q_i) & \Theta_{ij}^+(x,t) \end{pmatrix}}{\mathrm{Det}\,(\,\Theta_{ij}^+(x,t)\,)} e^{-kx-k^2t} \tag{7.14}$$

where Θ^+ is the matrix with elements:

$$\Theta_{ii}^+ = -\sigma_i^+ e^{2x+4(\chi_i+1)t} + g_i^+(Q_i), \quad \Theta_{ij}^+ = g_j^+(Q_i). \tag{7.15}$$

In the rational case one has to define:

$$\Theta_{ii}^+ = -x - 2\chi_i t - \sigma_i^+ + g_i^{(1)+}, \quad \Theta_{ij}^+ = g_j^+(Q_i). \tag{7.16}$$

7.2 The potential.

Theorem 7.3 *The vector Baker–Akhiezer function $\psi(x,t,P)$ is a solution of the equation $(\partial_t - \partial_x^2 + u(x,t))\psi = 0$ where the potential u is given by: $u(x,t) = \sum \rho_i(t) V(x - x_i(t))$ and $\rho_i(t)$ is an $l \times l$ matrix of rank 1.*

Proof. The usual argument from the unicity of the Baker–Akhiezer function shows that $(\partial_t - \partial_x^2)\psi$ is of the form $-u(x,t)\psi$ with $u = 2\partial_x \xi_1^{\alpha j}(x,t)$. From equation (7.9) it is clear that $u(x,t)$ is of the form $\sum \rho_i(t) V(x - x_i(t))$. To compute ρ_i let us expand around P_β

$$g_i(P) = \frac{g_i^\beta}{k_\beta} + O(\frac{1}{k_\beta^2}) \quad \text{and} \quad h_\alpha(P) = \delta_{\alpha\beta} + \frac{h_\alpha^\beta}{k_\beta} + O(\frac{1}{k_\beta^2})$$

and $s_{i,\alpha}(t,P)$ given in eq. (7.12). We find $\rho_{i,\alpha}^\beta = a_{i,\alpha} b_i^\beta$ where:

$$a_{i,\alpha} = \frac{1}{Q_i(t)} \mathrm{Det}\, \Theta_\alpha^{(i)}(t) \quad b_i^\beta = -2Q_i(t) \left\{ \frac{\sum_{k=1}^N \lambda_k^{(i)}(t) g_k^\beta}{\prod_{j=1}^N (2\sigma_j e^{-4(\chi_j+1)t - x_i - x_j}) \prod_{j\neq i} \sinh(x_i - x_j)} \right\}$$

∎

Alternatively one could use the dual Baker function ψ^+. It satisfies a Schrödinger equation with the same potential u than ψ. This is because the sum of the residues of the form $\psi^{+\alpha}\psi_\beta\Omega$ at the points Q_j and T_j vanishes, so that the same argument as in the elliptic case applies. This shows that Θ and Θ^+ have the same eigenvalues $-x_i(t)$ and gives alternative formulae for a_i^α and b_i^β, in particular:

$$b_i^\alpha = \frac{1}{Q_i^+(t)} \mathrm{Det}\, \Theta^{+,\alpha(i)}(t)$$

The normalizations $Q_i(t)$ and $Q_i^+(t)$ are as usual determined by the conditions $f_{ii} = 2$ and $\sum_\alpha b_i^\alpha = 1$.

This implies that $x_i(t), a_i(t), b_i(t)$ are the solutions of the trigonometric or rational model. Note that the curve is necessarily of the form of the spectral curve of the Calogero model.

7.3 Reconstruction formulae.

To construct the functions g_j one can take advantage of the fact that we know on Γ the function $1/(k-\chi_j)$ which vanishes at the l punctures P_α and has a pole with residue 1 at Q_j. It has $l-1$ other poles at some well–defined points $\delta_k^{(j)}$. The function $g_j(P)(k-\chi_j)$ has $g+l-1$ poles γ_k and $l-1$ zeroes $\delta_k^{(j)}$. By Riemann–Roch theorem this function $F_j(P)$ is uniquely determined by these data and the normalization condition $F_j(Q_j) = 1$. One can give an expression in terms of theta functions as in eq.(6.3). Then

$$g_j(P) = \frac{F_j(P)}{k - \chi_j}$$

In the standard Calogero-Moser model, we have $l = 1$, and $F_j = 1$.

Let us summarize the results:

Theorem 7.4 *Let Γ be a curve that is defined by the equation of the form (3.11) or (3.15) and $D = \gamma_1, \ldots, \gamma_{g+l-1}$ be a set of points in general position. Then the formulas*

$$\mathrm{Det}\ \Theta(x_i(t), t) = 0 \tag{7.17}$$

$$a_{i,\alpha}(t) = \frac{1}{Q_i(t)} \mathrm{Det}\ \Theta_\alpha^{(i)}, \quad b_i^\alpha(t) = \frac{1}{Q_i^+(t)} \mathrm{Det}\ \Theta^{+,\alpha(i)} \tag{7.18}$$

where $Q_i(t)$ and $Q_i^+(t)$ are determined by the conditions $f_{ii} = 2$ and $\sum_\alpha b_i^\alpha = 1$, define the solutions of the system (1.4, 2.14, 2.15). Here Θ is an $N \times N$ matrix with elements given in equations (7.4) in the trigonometric case and (7.5) in the rational case. Moreover $\Theta_\alpha^{(i)}$ is obtained from Θ by replacing its first column by $h_\alpha(T_j)$, $j = 1, \cdots, N$. Similarly for $\Theta^{+,\alpha\ (i)}$. Any solution of the system (1.1) may be obtained from these solutions taking into account the symmetries of the system.

Appendix A

The Weierstrass σ function of periods $2\omega_1, 2\omega_2$ is the entire function defined by

$$\sigma(z) = z \prod_{m,n\neq 0} \left(1 - \frac{z}{\omega_{mn}}\right) \exp\left[\frac{z}{\omega_{mn}} + \frac{1}{2}\left(\frac{z}{\omega_{mn}}\right)^2\right] \tag{7.19}$$

with $\omega_{mn} = 2m\omega_1 + 2n\omega_2$. The functions ζ and $\wp$ are

$$\zeta(z) = \frac{\sigma'(z)}{\sigma(z)} \ , \qquad \wp(z) = -\zeta'(z), \tag{7.20}$$

The $\wp$-function is doubly periodic, and the σ-function and ζ-functions transform according to

$$\zeta(z + \omega_l) = \zeta(z) + \eta_l, \quad \sigma(z + \omega_l) = -\sigma(z) e^{\eta_l (z + \frac{\omega_l}{2})}$$

where

$$2(\eta_1\omega_2 - \eta_2\omega_1) = i\pi$$

These functions have the symmetries

$$\sigma(-z) = -\sigma(z) \ , \qquad \zeta(-z) = -\zeta(z) \ , \qquad \wp(-z) = \wp(z). \tag{7.21}$$

Their behaviour at the neighbourhood of zero is

$$\sigma(z) = z + O(z^5) \ , \qquad \zeta(z) = z^{-1} + O(z^3) \ , \qquad \wp(z) = z^{-2} + O(z^2). \tag{7.22}$$

Setting

$$\Phi(x, z) = \frac{\sigma(z - x)}{\sigma(x)\ \sigma(z)} e^{\zeta(z)x} \tag{7.23}$$

it is easy to check that

$$\Phi(-x, z) = -\Phi(x, -z) \ , \qquad \frac{d}{dx}\Phi(x, z) = \Phi(x, z)\ [\zeta(z + x) - \zeta(x)]. \tag{7.24}$$

The function $\Phi(x, z)$ is a double-periodic function of the variable z

$$\Phi(x, z + 2\omega_l) = \Phi(x, z), \tag{7.25}$$

and has the expansion of the form

$$\Phi(x, z) = (-z^{-1} + \zeta(x) + O(z)) e^{\zeta(z)x} \tag{7.26}$$

at the point $z = 0$. As a function of x it has the following monodromy properties

$$\Phi(x + 2\omega_l, z) = \Phi(x, z) \exp 2(\zeta(z)\omega_l - \eta_l z). \tag{7.27}$$

and has a pole at the point $x = 0$

$$\Phi(x, z) = x^{-1} + O(x), \tag{7.28}$$

Choosing the periods $\omega_1 = \infty$ and $\omega_2 = i\frac{\pi}{2}$ we obtain the hyperbolic case

$$\sigma(z) \to \sinh(z) \exp\left(-\frac{z^2}{6}\right) \ , \qquad \zeta(z) \to \coth(z) - \frac{z}{3} \ , \qquad \wp(z) \to \frac{1}{\sinh^2(z)} + \frac{1}{3} \tag{7.29}$$

and

$$\Phi(x, z) \to \frac{\sinh(z - x)}{\sinh(z)\sinh(x)} e^{x \coth z} \tag{7.30}$$

In the rational limit, we have

$$\sigma(z) \to z, \quad \zeta(z) \to \frac{1}{z}, \quad \wp(z) \to \frac{1}{z^2}, \quad \Phi(x,z) \to \left(\frac{1}{x} - \frac{1}{z}\right) e^{\frac{x}{z}}$$

Appendix B

Let us recall briefly some facts we need about Riemann's theta functions.

First there is an embedding of the Riemann surface Γ into its Jacobian $J(\Gamma)$ by the Abel map.

Let a_i^0, b_i^0 be a basis of cycles on Γ with canonical matrix of intersections $a_i^0 \cdot a_j^0 = b_i^0 \cdot b_j^0 = 0,\ a_i^0 \cdot b_j^0 = \delta_{ij}$. In a standard way it defines a basis of normalized holomorphic differentials $\omega_j(P)$

$$\oint_{a_j^0} \omega_i = \delta_{ij}. \tag{7.31}$$

The matrix of b-periods of these differentials

$$B_{ij} = \oint_{b_i^0} \omega_j \tag{7.32}$$

defines the Riemann theta-function

$$\theta(z_1, \ldots, z_g) = \sum_{m \in Z^g} e^{2\pi i(m,z) + \pi i(Bm,m)}. \tag{7.33}$$

on the torus $J(\Gamma)$ which is called the Jacobian variety.

$$J(\Gamma) = C^g / \mathcal{B} \tag{7.34}$$

The lattice $\mathcal{B}$ is generated by the basic vectors $e_i \in C^g$ and by the vectors $B_j \in C^g$ with coordinates B_{ij}.

The theta function has remarkable automorphy properties with respect to this lattice: for any $l \in \mathbf{Z}^g$ and $z \in \mathbf{C}^g$

$$\begin{aligned} \theta(z+l) &= \theta(z) \\ \theta(z+Bl) &= \exp[-i\pi(Bl,l) - 2i\pi(l,z)]\theta(z) \end{aligned} \tag{7.35}$$

Let us choose a point $q_0 \in \Gamma$. Then the vector $A(P)$ with coordinates

$$A_k(P) = \int_{q_0}^{P} \omega_k \tag{7.36}$$

defines the Abel map

$$A : \Gamma \longmapsto J(\Gamma) \tag{7.37}$$

which is an embedding of Γ into $J(\Gamma)$.

The fundamental theorem of Riemann expresses the intersection of the image of this embedding with the zero set of the theta function.

Theorem. *Let $Z = (Z_1, \cdots, Z_g) \in \mathbf{C}^g$ arbitrary. Either the function $\theta(A(P) - Z)$ vanishes identically for $P \in \Gamma$ or it has exactly g zeroes $P_1, \cdots, P_g$ such that:*

$$A(P_1) + \cdots + A(P_g) = Z - \mathcal{K} \tag{7.38}$$

where $\mathcal{K}$ is the so–called vector of Riemann's constants, depending on the curve Γ and the point q_0 but independent of Z.

From this one can prove the Jacobi theorem, that is any point in the Jacobian $J(\Gamma)$ is of the form $(A(P_1), \cdots, A(P_g))$ for some divisor of degree g on Γ.

This allows to find a formula for a function that has g poles at points $\delta_1, \ldots, \delta_g$ and an additional pole at point Q^+. The dimension of the space of such functions is two. Of course it contains the constant. We choose the second basic function by the condition that it vanishes at some fixed point Q^-.

Let Z, Z^+, Z^-, Z^0 be vectors that are defined by formulae:

$$\begin{aligned}
Z &= \sum_{s=1}^{g} A(\delta_s) + \mathcal{K} \\
Z^+ &= Z - A(\delta_1) + A(Q^+) = A(Q^+) + \sum_{s=2}^{g} A(\delta_s) + \mathcal{K} \\
Z^- &= Z - A(\delta_1) + A(Q^-) \\
Z^- + Z^0 &= Z + Z^+
\end{aligned}$$

Let us define the function

$$f(P) = \frac{\theta(A(P) - Z^-)\theta(A(P) - Z^0)}{\theta(A(P) - Z)\theta(A(P) - Z^+)} \tag{7.39}$$

From the Jacobi theorem it follows that two factors in the denominator vanish at the points $\delta_1, \ldots, \delta_g$ and $Q^+, \delta_2, \ldots, \delta_g$ respectively. Similarly the two factors in the numerator vanish at $Q^+, \delta_2, \ldots, \delta_g$ and g other points.

The zeroes at $\delta_2, \ldots, \delta_g$ cancel between the numerator and the denominator, thereby leaving us with the correct divisor of zeroes and poles. It remains to show that the function f is well–defined on Γ. This is because, due to the definition of Z^0 the automorphy factors of the theta functions in equation (7.35) cancel between the numerator and the denominator when P describes b–cycles on Γ.

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