ELSEVIER

Physica D 87 (1995) 14–19

PHYSICA D

Linear operators with self-consistent coefficients and rational reductions of KP hierarchy

I. Krichever

Landau Institute for Theoretical Physics, Kosygina str. 2, 117940 Moscow, Russia

Abstract

A new general type of reductions of the hierarchy of Kadomtsev-Petviashvili equation is established. These reductions are equivalent to the Lax equations for pseudo-differential operators of the form $L_1^{-1}L_2$, where L_1, L_2 are ordinary differential operators with coefficients that are functions of variables $t_1 = x, t_2, \ldots$. It is shown that besides the KP hierarchy they are invariant with respect to the "rational" symmetries as well.

1. Introduction

All non-linear partial differential equations that are considered within the framework of the soliton theory are equivalent to the compatibility conditions of an overdetermined system of auxiliary linear problems. For spatially two-dimensional equations one of the most general type of the corresponding representations has the form ([1])

$$[\partial_y - L, \partial_t - A] = 0, \tag{1}$$

where L and A are differential operators with respect to the variable x

$$L = \sum_{i=1}^{n} u_i(x,y,t)\partial_x^i, \quad A = \sum_{j=1}^{m} v_j(x,y,t)\partial_x^j \tag{2}$$

with coefficients that are scalar or matrix functions of the variables x, y, t.

The basic example of these equations is the Kadomtsev-Petviashvili (KP) equation

$$\tfrac{3}{4}u_{yy} + (u_t - \tfrac{3}{2}uu_x - \tfrac{1}{4}u_{xxx})_x = 0 \tag{3}$$

that has the operator representation (1) with

$$L = \partial_x^2 + u(x,y,t),$$
$$A = \partial_x^3 + \tfrac{3}{2}u\partial_x + w(x,y,t). \tag{4}$$

If the coefficients of the operators (2) are independent of the variable y, then Eq. (1) has the form

$$\partial_t L = [A, L] \tag{5}$$

The non-linear Schrödinger equation

$$i\psi_t - \psi_{xx} + \alpha|\psi|^2\psi = 0 \tag{6}$$

has Lax representation (5) where L is a Dirac operator. In [2] it was shown that algebraic-geometrical or finite-gap solutions of this equation can be constructed within the framework of the approach that is based on the consideration of (6) as a *linear equation with self-consistent potential.* The corresponding linear equation is a non-stationary Schrödinger equation

$$(i\partial_t - \partial_x^2 + u(x,t))\psi = 0 \tag{7}$$

and the "self-consistency" condition has the form

0167-2789/95/$09.50
SSDI 0167-2789(95)00146-8

$$u = \alpha|\psi|^2. \tag{8}$$

Numerous physical models that describe interactions of long and short waves also have the form of the linear equation (7) with various self-consistency conditions. For example,

$$u_t + u_x = |\psi|^2_x \quad [3], \tag{9}$$

$$u_{tt} - u_{xx} + \alpha u_{xxxx} + \beta(u^2)_{xx} = |\psi|^2_{xx} \quad [4]. \tag{10}$$

The general construction of exact solutions of the non-stationary Schrödinger equation (7) with various types of self-consistency conditions was advanced in [5] (its realization for the construction of soliton solutions was presented in [6]).

In [7] this approach has been extended for non-linear σ-models

$$\partial_+\partial_-\Psi + \langle\partial_+\Psi, \partial_-\Psi\rangle\Psi = 0,$$
$$\partial_\pm = \frac{\partial}{\partial x_\pm}, \quad x_\pm = \tfrac{1}{2}(\tau \pm \sigma), \tag{11}$$

where $\Psi(x_+, x_-)$ is a vector-function and $\langle .,. \rangle$ stands for a scalar product. Eq. (11) is a linear wave equation

$$(\partial_+\partial_- + u(x_+, x_-))\Psi = 0, \tag{12}$$

with self-consistency condition

$$u = \langle \partial_-\Psi, \partial_-\Psi \rangle. \tag{13}$$

The basic idea of this construction may be very briefly presented as follows. The first step of the scheme is constructing an "integrable" linear equation. For example, for the non-linear Schrödinger equation that is the construction of the potentials $u(x,t)$ such that a family of solutions $\psi = \psi(x,t,Q)$ parametrized by points Q of the auxiliary *spectral* Riemann surface Γ is known for the linear equation (20). The next step is to select among these potentials the potentials such that there exists a point Q on the corresponding Riemann surface such that

$$u(x,t) = \alpha\psi(x,t,Q)\psi^+(x,t,Q), \tag{14}$$

where $\psi^+(x,t,Q)$ are solutions of the formal adjoint linear equation

$$(\partial_t + \partial_x^2 - u(x,t))\psi^+(x,t,Q) = 0. \tag{15}$$

All self-consistency conditions that are considered in this scheme are corollaries of the residue theorem, because bilinear combinations of the functions $\psi(x,t,Q)$ and $\psi^+(x,t,Q)$ (for example $\psi(x,t,Q)\psi^+(x,t,Q)$) or its derivatives are *meromorphic* functions of the spectral parameter Q. Therefore, the sum of all residues of the differentials that are a product of the corresponding bilinear combination and a constant meromorphic differential $d\Omega(Q)$ (for example

$$\tilde{d}\Omega(x,t,Q) = \psi(x,t,Q)\psi^+(x,t,Q)d\Omega(Q)) \tag{16}$$

is equal to zero. The right-hand sides of the self-consistency conditions (8),(9),(10),(13) are equal to the sum of residues of the corresponding differentials at finite points of the spectral Riemann surface. The left-hand sides are residues at "the infinities" of the spectral parameter.

These *analytical* arguments have been used in various forms for different schemes in the inverse problem method (see [8]). They show that in all these cases the non-linear Schrödinger equation can be considered as a *reduction* of the KP hierarchy, because the linear equation (7) is one of the Lax operators for the KP equation (after the redefinition of the variable $y \to it$). The algebraic nature of this reduction has been clarified in [9,10]. The main goal of this report is to present the general rational reductions of the KP hierarchy [11] which, as it seems to the author, contain all the integrable equations that have Lax representation (5) with matrix coefficients and all the integrable equations that can been considered as linear systems with self-consistent potentials.

2. Rational reductions of the KP hierarchy.

The KP hierarchy in its original form is a system of non-linear equations on the coefficients $v_{i,n}$ of the infinite set of linear differential operators

$$L_n = \partial_x^n + \sum_{i=0}^{n-2} v_{i,n}(t_1, t_2, \ldots)\partial_x^i. \tag{17}$$

These equations are equivalent to the operator equations

$$[\partial_m - L_m, \partial_n - L_n] = 0, \quad \partial_n = \partial/\partial t_n. \tag{18}$$

In [12] the KP hierarchy has been defined as a system of commuting evolution equations on a space of infinite sequences $u_i(x), i = 1, 2, \ldots,$ of functions of one variable. The corresponding equations have the Lax representation

$$\partial_n \mathcal{L} = [\mathcal{L}_+^n, \mathcal{L}], \tag{19}$$

where

$$\mathcal{L} = \partial_x + \sum_{i=1}^{\infty} u_i \partial_x^{-i} \tag{20}$$

is a pseudo-differential operator. (Here and below D_+ stands for the differential part of the pseudo-differential operator D.) The equivalence of these two definitions of the KP hierarchy was proved in [13].

The basic type of reductions that has been considered in the theory of KP hierarchy is the reduction on the stationary points of one of the flows of the hierarchy (or linear combination of such flows). These submanifolds are characterized by the property that the n-th power of the corresponding pseudo-differential operator $\mathcal{L}$ is a differential operator, i.e.

$$\mathcal{L}^n = \mathcal{L}_+^n = L = \partial_x^n + \sum_{i=1}^{n-2} w_i(x). \tag{21}$$

The coefficients of $\mathcal{L}$ are differential polynomials in the coefficients w_i of the differential operator L, that parametrize the corresponding invariant subspace of the KP hierarchy.

Let $\mathcal{K}_{m,n}$ be a manifold of pseudo-differential operators $\mathcal{L}$ such that

$$\mathcal{L}^n = L_1^{-1} L_2, \tag{22}$$

where L_1 and L_2 are co-prime differential operators of degrees m and $(n+m)$, respectively. (Two differential operators are called co-prime if their kernels do not intersect.) The coefficients of these operators

$$L_1 = \partial_x^k + \sum_{i=1}^{m-1} w_{1,i} \partial_x^i,$$

$$L_2 = \partial_x^{n+k} + \sum_{j=1}^{n+m-1} w_{2,j} \partial_x^j \tag{23}$$

are parameters that define the corresponding point of $\mathcal{K}_{m,n}$. The normalization of $\mathcal{L}$, such that in the right-hand side of (20) there is no free term, is equivalent to the only relation on the coefficients of L_1 and L_2

$$w_{1,m-1} = w_{2,n+m-1}. \tag{24}$$

The coefficients of the pseudo-differential operators which belong to $\mathcal{K}_{m,n}$, are differential polynomials with respect to $w_{1,i}$ and $w_{2,j}$.

A priori, the definition of $\mathcal{K}_{m,n}$ depends on the order of factors in the right-hand side of (22). But we are going to show that the corresponding submanifold does not depend on this order and depends on the degrees of the numerator and denominator of the non-commutative fraction only.

Lemma 1. For any co-prime differential operators L_3 and L_4 of degrees m and $n + m$ there exist unique normalized differential operators L_1 and L_2 of degrees m and $n + m$, respectively, such that

$$L_1^{-1} L_2 = L_4 L_3^{-1}. \tag{25}$$

(A differential operator is called normalized if its leading coefficient equals 1.)

Remark. This result is well-known. The exact formulae for the coefficients of L_1 and L_2 may be found in [14]. Nevertheless, we are going to present the proof of it in a way that will be useful for us later.

Proof. Let $\mathcal{O}_i$, $i = 3, 4$, be the kernel of L_i, i.e.

$$y(x) \in \mathcal{O}_i : L_i y(x) = 0. \tag{26}$$

The dimension of $\mathcal{O}_4$ equals $(n + m)$. From the lemma's assumption it follows that the image of this space $L_3(\mathcal{O}_4)$ has the same dimension. Therefore, there exists a unique normalized differential operator L_2 of degree $n + m$ such that

$$L_2 y(x) = 0, \quad y(x) \in L_3(\mathcal{O}_4). \tag{27}$$

Let us define the operator L_1 of degree n by the equality

$$L_1 y(x) = 0, \quad y(x) \in L_4(\mathcal{O}_3). \tag{28}$$

From (27),(28) it follows that the differential operators L_2L_3 and L_1L_4 of degree $(2n+m)$ have the same kernel $\mathcal{O}_3 + \mathcal{O}_4$. Therefore, they are equal to each other. The equality

$$L_1L_4 = L_2L_3. \tag{29}$$

is equivalent to (25). The lemma is proved.

In the same way one can prove the inverse statement that for any co-prime differential operators L_1, L_2 there exist unique normalized differential operators L_3, L_4 such that (25) is valid.

Theorem 1. For any n and m the space $\mathcal{K}_{m,n}$ is invariant with respect to the KP hierarchy (19).

Proof. The equality (19) for $\mathcal{L} \in \mathcal{K}_{m,n}$ is equivalent to the equality

$$\partial_i(L_1^{-1}L_2) = [(L_1^{-1}L_2)_+^{i/n}, L_1^{-1}L_2]. \tag{30}$$

From (30) it follows that

$$\begin{aligned}(\partial_i L_2)L_3 - (\partial_i L_1)L_4 \\ = L_1(L_1^{-1}L_2)_+^{i/n}L_4 - L_2(L_1^{-1}L_2)_+^{i/n}L_3,\end{aligned} \tag{31}$$

where L_3 and L_4 are differential operators such that (25) is fulfilled.

For the proof of the theorem it is enough to show that for given L_1 and L_2 Eq. (31) uniquely defines operators $(\partial_i L_1)$ and $(\partial_i L_2)$ of degrees $(m-1)$ and $(n+m-1)$, respectively.

Let $\mathcal{D}$ be an operator that is defined by the right-hand side of (31). The coefficients of this operator are differential polynomials with respect to the coefficients of the operators L_1 and L_2, i.e. $\mathcal{D} = \mathcal{D}(L_1, L_2)$. The differential part of the operator $(L_1^{-1}L_2)^{i/n}$ can be replaced in (30) by its integral part

$$(L_1^{-1}L_2)_-^{i/n} = (L_1^{-1}L_2)^{i/n} - (L_1^{-1}L_2)_+^{i/n}. \tag{32}$$

Therefore,

$$\mathcal{D} = L_2(L_1^{-1}L_2)_-^{i/n}L_3 - L_1(L_1^{-1}L_2)_-^{i/n}L_4. \tag{33}$$

The latter equality proves that the degree of the differential operator $\mathcal{D}$ is less or equal to $2n+m-1$.

Lemma 2. For any differential operator $\mathcal{D}$ of the degree $2n+m-1$ and for any co-prime differential operators L_3 and L_4 of degrees m and $n+m$ there exist unique differential operators A_1 and A_2 of degrees m and $n+m$ such that

$$A_2L_3 - A_1L_4 - \mathcal{D} = 0. \tag{34}$$

Proof. Differential operator A_1 of degree m is uniquely defined by its action on any m-dimensional linear space. Therefore, it can be defined by the equality

$$A_1 y(x) = \mathcal{D}y(x), y(x) \in L_4(\mathcal{O}_3). \tag{35}$$

The analogous equality

$$A_2 y(x) = \mathcal{D}y(x), y(x) \in L_3(\mathcal{O}_4) \tag{36}$$

defines the operator A_2. From (34),(35) it follows that the differential operator that is defined by the left-hand side of (34) has degree $2n+m-1$ and its kernel has a dimension bigger or equal to $2n+m$. Hence, this operator is equal to zero. The lemma is proved.

The operators A_i that are defined due to this lemma for L_3, L_4 and $\mathcal{D}$ given by equalities (25) and (33), depend on the operators L_1 and L_2, only. I.e. $A_i = A_i(L_1, L_2)$. Therefore, the equalities

$$\partial_i L_1 = A_1(L_1, L_2), \quad \partial_i L_2 = A_2(L_1, L_2) \tag{37}$$

do define the evolution of the operators L_1 and L_2. The theorem is proved.

When discussing this result with the author, T. Shiota proposed the exact form of the equations that define the evolution of the operators L_1 and L_2.

Theorem 2. The restriction of the KP hierarchy on $\mathcal{K}_{m,n}$ is equivalent to the equations

$$\partial_i L_1 = L_1(L_1^{-1}L_2)_+^{i/n} - (L_2L_1^{-1})_+^{i/n}L_1, \tag{38}$$

$$\partial_i L_2 = L_2(L_1^{-1}L_2)_+^{i/n} - (L_2L_1^{-1})_+^{i/n}L_2, \tag{39}$$

Proof. From Theorem 1 it follows that (30) correctly defines the evolution of L_1 and L_2. The equality (30) is a corollary of the equalities (38),(39). Therefore, for the proof of the theorem it is enough to verify that the right-hand sides of (38),(39) are differential operators of degrees not greater than $m-1$ and $n+m-1$. The last statement follows from the equalities

$$L_1^{-1}(L_2L_1^{-1})^{i/n}L_1 = (L_1^{-1}L_2)^{i/n},$$
$$L_2^{-1}(L_2L_1^{-1})^{i/n}L_2 = (L_1^{-1}L_2)^{i/n} \tag{40}$$

which imply that (38),(39) are equivalent to the equations

$$\partial_i L_1 = (L_2L_1^{-1})_-^{i/n}L_1 - L_1(L_1^{-1}L_2)_-^{i/n}, \tag{41}$$

$$\partial_i L_2 = (L_2L_1^{-1})_-^{i/n}L_2 - L_2(L_1^{-1}L_2)_-^{i/n}. \tag{42}$$

The theorem is proved.

Remark. The formulae (41),(42) are similar to the formulae for the evolution of the factors $L = L_1L_2$, where L_i are differential operators or some special kinds of pseudo-differential operators, which have been obtained in [15].

Example. In [10,9] it was proved that the hierarchy of the non-linear Schrödinger equation can be obtained as a reduction of the KP hierarchy. This is the particular case of the above-proved theorem. In the notations of this work the corresponding reduction is the reduction onto the space $\mathcal{K}_{1,2}$.

The coefficients of the pseudo-differential operator $\mathcal{L}$ of the form

$$\mathcal{L} = (\partial_x + v)^{-1}(\partial_x^2 + v\partial_x + w) = \partial_x + \sum_{i=1}^{\infty} u_i\partial_x^{-i} \tag{43}$$

are differential polynomials with respect to the functions v and w. They are recurrently defined from the equalities

$$u_1 = w,\ u_{i+1} + u_{ix} + vu_i = 0, \quad i > 1. \tag{44}$$

Let us consider the second flow of the KP hierarchy (19), i.e. the equation that defines the dependence of $\mathcal{L}$ with respect to the variable t_2. For $n = 2$ the operator $\mathcal{L}_+^2$ equals $\partial_x^2 + 2u_1$. The corresponding equations for the first two coefficients of $\mathcal{L}$ become the closed system of two equations for two unknown functions due to (44) for $i = 1,2$

$$\partial_2 u_1 = u_{1xx} + 2u_{2x}, \quad u_2 = -u_{1x} + vu_1, \tag{45}$$

$$\begin{aligned}\partial_2 u_2 &= u_{2xx} + 2u_{3x} + 2u_1u_{1x}\\ &= u_{2xx} - 2(u_{2x} + vu_2)_x + 2u_1u_{1x}.\end{aligned} \tag{46}$$

Let us define the two new unknown functions r and q with the help of the relations

$$u_1 = rq, \quad v = -\frac{r_x}{r}. \tag{47}$$

Eqs. (45),(46) are equivalent to the system

$$\partial_2 r = r_{xx} + rqr, \quad \partial_2 q = -q_{xx} + qrq, \tag{48}$$

which for $r = \pm q^*$ and $t_2 = it$ implies the non-linear Schrödinger equation

$$ir_t - r_x x \mp |r|^2 r = 0. \tag{49}$$

The restriction of the KP hierarchy onto the spaces $\mathcal{K}_{m,n}$ is the quantization of the corresponding algebraic orbits of the dispersionless KP hierarchy that was introduced in [16]. In [16] it was found also that algebraic orbits of the dispersionless KP hierarchy have additional symmetries. The next theorem shows that these additional symmetries can be quantized.

Theorem 3. The restriction of the KP hierarchy onto the invariant spaces $\mathcal{K}_{m,n}$ is compatible with the equations

$$\mathcal{L}_T = [A_{-i}, \mathcal{L}], \tag{50}$$

where

$$A_{-i} = M_1^{-1}M_2, \tag{51}$$

M_1 is a normalized differential operator of degree i, and M_2 is a differential operator of degree $i-1$.

Proof. Eq. (50) for $\mathcal{L} = L_1^{-1}L_2$ is equivalent to the equation

$$(\partial_T L_2)L_3 - (\partial_T L_1)L_4 = L_1A_{-i}L_4 - L_2A_{-i}L_3, \tag{52}$$

where L_3, L_4 are the operators for which the equality (25) is fulfilled. The pseudo-differential operator

A_i has a negative degree. Therefore, (52) is a well-defined evolution system iff the pseudo-differential operator $\mathcal{D}$ in its right-hand side is a differential operator. For the generic A_i the operator $\mathcal{D}$ can be uniquely represented in the form

$$\mathcal{D} = D_1 D_2^{-1}, \tag{53}$$

where D_1, D_2 are differential operators and D_2 has degree $2i$. Therefore, the condition that D_1 is divisible by D_2 is equivalent to the system of $2i$ ordinary differential equations for $2i$ unknown coefficients of the operators M_1 and M_2. The theorem is proved.

Remark 1. The compatibility conditions of two symmetries of the form (51)

$$\left[\frac{\partial}{\partial \tau_1} - M_1^{-1}M_2, \frac{\partial}{\partial \tau_2} - M_3^{-1}M_4\right] = 0 \tag{54}$$

were considered in the recent paper [17] with no relation to the symmetries of rational reductions of the KP hierarchy.

Remark 2. We would like to mention that the symmetries of the form (50),(51) exist for the usual Lax reductions of the KP hierarchy, as well. For example, for the KdV equation (i.e. for $\mathcal{L}^2 = L = \partial_x^2 + u$) and $i = 1$ the equation

$$L_\tau = u_\tau = [(v\partial_x + w)^{-1}, L] \tag{55}$$

is equivalent to the equations

$$u_\tau w^2 + v(u_\tau w)_x = w_{xx} - vu_x,$$
$$v(u_\tau v)_x + 2u_\tau vw = v_{xx} + 2w_x, \tag{56}$$

$$u_\tau v^2 = 2v_x. \tag{57}$$

Let w be equal to

$$w = \tfrac{1}{2}v_x, \tag{58}$$

then from (56),(57) it follows

$$u = \tfrac{1}{2}\phi_{xx} - \tfrac{1}{4}\phi_x^2, \tag{59}$$

where ϕ is defined form the relation

$$v = e^\phi. \tag{60}$$

Eqs. (59) and (57) imply that

$$\partial_\tau(\tfrac{1}{2}\phi_{xx} - \tfrac{1}{4}\phi_x^2) = 2\phi_x e^{-\phi} \tag{61}$$

is equivalent to the Lax equation (55) with the condition (58).

References

[1] V.E. Zakharov and A.B. Shabat, A scheme for integration of nonlinear equations of mathematical physics by the method of inverse scattering problem. I, Funct. Anal. Appl. 8 12 (1974) 226–235.
[2] I. Cherednik, Funkts. Anal. Pril. 12 (1978) 45–52.
[3] N. Yajima and M. Oikawa, Formation and interaction of sonic-Langmuir solitons, Prog. Theor. Phys. 56 (1976) 1719–1739.
[4] V.G. Makhankov, On stationary solutions of the Schrödinger equation with a self-consistent potential satisfying Boussinesq's equation, Phys. Lett. A 50 (1974) 42–44.
[5] I. Krichever, Spectral theory of nonstationary Schrödinger operators and nonstationary Peierls model, Funkts. Anal. Pril. 20 (1986) 42–54.
[6] B. Dubrovin, I. Krichever, T. Malanuyk and V. Makhankov, Exact solutions of non-stationary Schrödinger equation with self-consistent potentials, Phys. Elem. Part. Nucl. 19 3 (1988) 579–621.
[7] I. Krichever, Algebraic-geometrical two-dimensional operators with self-consistent potentials, Funkts. Anal. Pril. 28 1 (1994) 26–40.
[8] S. Manakov.
[9] A. Orlov, Vertex operators, $\bar{\partial}$-problems, symmetries, variational identities and Hamiltonian formalism for 2+1 integrable systems, in: Plasma Theory and Nonlinear and Turbulent Processes in Physics (World Scientific, Singapore, 1988).
[10] F. Topan, $N = 1, 2$ super-NLS hierarchies as super-KP coset reductions, preprint ENSLAPP-L-467/94, Paris.
[11] I. Krichever, General rational reductions of the KP hierarchy, Funkts. Anal. Pril. (1995).
[12] M. Sato, Soliton equations and universal Grassmann manifold, Math. Lect. Notes Ser. 18 (Sophia University, Tokyo, 1984).
[13] T. Takebe, Representation theoretical meaning for the initial value problem for the Toda lattice hierarchy, Lett. Math. Phys. 21 (1991).
[14] O. Ore, Theory of non-commutative polynomials, Ann. Math. 34 (1933) 480–508.
[15] V.G. Drinfeld and V.V. Sokolov, Lie algebras and equations of KdV-type, in: Sovremennye problemy matematiki 24 (1984), Itogi nauki i tehniki, 81–180.
[16] I. Krichever, Tau-function of the universal Whitham hierarchy and topological field theories, Commun. Pure Appl. Math. 47 (1994) 1–40.
[17] V.E. Zakharov, Dispersionless limit of integrable systems in 2+1 dimensions, in: N. Ercolani, I. Gabitov, D. Levermore and D. Serre, eds., Singular limits of Dispersive Waves (NATO ARW series, New York 1994).