Linear operators with self-consistent coefficients and rational reductions of KP hierarchy

I. Krichever
Landau Institute for Theoretical Physics, Kosygina str. 2, 117940 Moscow, Russia

Abstract

A new general type of reductions of the hierarchy of Kadomtsev-Petviashvili equation is established. These reductions are equivalent to the Lax equations for pseudo-differential operators of the form $L_1^{-1}L_2$, where $L_1, L_2$ are ordinary differential operators with coefficients that are functions of variables $t_1 = x, t_2, \ldots$. It is shown that besides the KP hierarchy they are invariant with respect to the "rational" symmetries as well.

1. Introduction

All non-linear partial differential equations that are considered within the framework of the soliton theory are equivalent to the compatibility conditions of an overdetermined system of auxiliary linear problems. For spatially two-dimensional equations one of the most general type of the corresponding representations has the form (1)

$$[\partial_x - L, \partial_t - A] = 0,$$

where $L$ and $A$ are differential operators with respect to the variable $x$

$$L = \sum_{i=1}^{n} u_i(x,y,t) \partial_x^i, \quad A = \sum_{j=1}^{m} v_j(x,y,t) \partial_x^j$$

with coefficients that are scalar or matrix functions of the variables $x, y, t$.

The basic example of these equations is the Kadomtsev-Petviashvili (KP) equation

$$\frac{3}{4} u_{yy} + (u_t - \frac{3}{2} u u_x - \frac{1}{4} u_{xxx})_x = 0$$

that has the operator representation (1) with

$$L = \partial_x^2 + u(x,y,t),$$
$$A = \partial_x^3 + \frac{3}{2} u \partial_x + w(x,y,t).$$

If the coefficients of the operators (2) are independent of the variable $y$, then Eq. (1) has the form

$$\partial_t L = [A, L]$$

The non-linear Schrödinger equation

$$i \psi_t - \psi_{xx} + \alpha |\psi|^2 \psi = 0$$

has Lax representation (5) where $L$ is a Dirac operator. In [2] it was shown that algebraic-geometrical or finite-gap solutions of this equation can be constructed within the framework of the approach that is based on the consideration of (6) as a linear equation with self-consistent potential. The corresponding linear equation is a non-stationary Schrödinger equation

$$(i \partial_t - \partial_x^2 + u(x,t)) \psi = 0$$

and the "self-consistency" condition has the form
Numerous physical models that describe interactions of long and short waves also have the form of the linear equation (7) with various self-consistency conditions. For example,

\[ u_t + u_x = |\psi|^2 \]  

(9)

\[ u_{tt} - u_{xx} + \alpha u_{xxxx} + \beta (u^2)_{xx} = |\psi|^2 \]  

(10)

The general construction of exact solutions of the non-stationary Schrödinger equation (7) with various types of self-consistency conditions was advanced in [5] (its realization for the construction of soliton solutions was presented in [6]).

In [7] this approach has been extended for non-linear \( \sigma \)-models

\[ \partial_+ \partial_- \Psi + \langle \partial_+ \Psi, \partial_- \Psi \rangle \Psi = 0, \]

(11)

where \( \Psi(x_+,x_-) \) is a vector-function and \( \langle \ldots \rangle \) stands for a scalar product. Eq. (11) is a linear wave equation

\[ (\partial_+ \partial_- + u(x_+,x_-))\Psi = 0, \]

(12)

with self-consistency condition

\[ u = \langle \partial_+ \Psi, \partial_- \Psi \rangle. \]

(13)

The basic idea of this construction may be very briefly presented as follows. The first step of the scheme is constructing an "integrable" linear equation. For example, for the non-linear Schrödinger equation that is the construction of the potentials \( u(x,t) \) such that a family of solutions \( \Psi = \psi(x,t,Q) \) parametrized by points \( Q \) of the auxiliary spectral Riemann surface \( \Gamma \) is known for the linear equation (20). The next step is to select among these potentials the potentials such that there exists a point \( Q \) on the corresponding Riemann surface such that

\[ u(x,t) = \alpha \psi(x,t,Q) \psi^+(x,t,Q), \]

(14)

where \( \psi^+(x,t,Q) \) are solutions of the formal adjoint linear equation

\[ (\partial_t + \partial_x^2 - u(x,t)) \psi^+(x,t,Q) = 0. \]

(15)

All self-consistency conditions that are considered in this scheme are corollaries of the residue theorem, because bilinear combinations of the functions \( \psi(x,t,Q) \) and \( \psi^+(x,t,Q) \) (for example \( \psi(x,t,Q) \psi^+(x,t,Q) \)) or its derivatives are meromorphic functions of the spectral parameter \( Q \). Therefore, the sum of all residues of the differentials that are a product of the corresponding bilinear combinations and a constant meromorphic differential \( d\Omega(Q) \) (for example

\[ d\Omega(x,t,Q) = \psi(x,t,Q) \psi^+(x,t,Q) d\Omega(Q) \]

(16)

is equal to zero. The right-hand sides of the self-consistency conditions (8),(9),(10),(13) are equal to the sum of residues of the corresponding differentials at finite points of the spectral Riemann surface. The left-hand sides are residues at "the infinities" of the spectral parameter.

These analytical arguments have been used in various forms for different schemes in the inverse problem method (see [8]). They show that in all these cases the non-linear Schrödinger equation can be considered as a reduction of the KP hierarchy, because the linear equation (7) is one of the Lax operators for the KP equation (after the redefinition of the variable \( y \rightarrow it \)). The algebraic nature of this reduction has been clarified in [9,10]. The main goal of this report is to present the general rational reductions of the KP hierarchy [11] which, as it seems to the author, contain all the integrable equations that have Lax representation (5) with matrix coefficients and all the integrable equations that can be considered as linear systems with self-consistent potentials.

2. Rational reductions of the KP hierarchy.

The KP hierarchy in its original form is a system of non-linear equations on the coefficients \( v_i,n \) of the infinite set of linear differential operators

\[ L_n = \partial_t^n + \sum_{i=0}^{n-2} v_{i,n}(t_1,t_2,\ldots) \partial_t^i. \]

(17)
These equations are equivalent to the operator equations
\[ [\partial_m - L_m, \partial_n - L_n] = 0, \quad \partial_n = \partial/\partial t_n. \quad (18) \]

In [12] the KP hierarchy has been defined as a system of commuting evolution equations on a space of infinite sequences \( u_i(x), i = 1, 2, \ldots \), of functions of one variable. The corresponding equations have the Lax representation
\[ \partial_n L = [L^\alpha_m, L], \quad (19) \]
where
\[ L = \partial_x + \sum_{i=1}^{\infty} u_i \partial_x^{-i} \quad (20) \]
is a pseudo-differential operator. (Here and below \( D_x \) stands for the differential part of the pseudo-differential operator \( D \).) The equivalence of these two definitions of the KP hierarchy was proved in [13].

The basic type of reductions that has been considered in the theory of KP hierarchy is the reduction on the stationary points of one of the flows of the hierarchy (or linear combination of such flows). These submanifolds are characterized by the property that the \( n \)-th power of the corresponding pseudo-differential operator \( L \) is a differential operator, i.e.
\[ L^n = L^n = L = \partial_x^n + \sum_{i=1}^{n-2} w_i(x). \quad (21) \]
The coefficients of \( L \) are differential polynomials in the coefficients \( w_i \) of the differential operator \( L \), that parametrize the corresponding invariant subspace of the KP hierarchy.

Let \( K_{m,n} \) be a manifold of pseudo-differential operators \( L \) such that
\[ L^n = L_1^{-1} L_2, \quad (22) \]
where \( L_1 \) and \( L_2 \) are co-prime differential operators of degrees \( m \) and \( n + m \), respectively. (Two differential operators are called co-prime if their kernels do not intersect.) The coefficients of these operators
\[ L_1 = \partial_x^k + \sum_{i=1}^{m-1} w_{1,i} \partial_x^i, \]
\[ L_2 = \partial_x^{n+k} + \sum_{j=1}^{n+m-1} w_{2,j} \partial_x^j \quad (23) \]
are parameters that define the corresponding point of \( K_{m,n} \). The normalization of \( L \), such that in the right-hand side of (20) there is no free term, is equivalent to the only relation on the coefficients of \( L_1 \) and \( L_2 \)
\[ w_{1,m-1} = w_{2,n+m-1}. \quad (24) \]
The coefficients of the pseudo-differential operators which belong to \( K_{m,n} \) are differential polynomials with respect to \( w_{1,i} \) and \( w_{2,j} \).

A priori, the definition of \( K_{m,n} \) depends on the order of factors in the right-hand side of (22). But we are going to show that the corresponding submanifold does not depend on this order and depends on the degrees of the numerator and denominator of the non-commutative fraction only.

**Lemma 1.** For any co-prime differential operators \( L_3 \) and \( L_4 \) of degrees \( m \) and \( n + m \) there exist unique normalized differential operators \( L_1 \) and \( L_2 \) of degrees \( m \) and \( n + m \), respectively, such that
\[ L_1^{-1} L_2 = L_4 L_3^{-1}. \quad (25) \]

(A differential operator is called normalized if its leading coefficient equals 1.)

**Remark.** This result is well-known. The exact formulae for the coefficients of \( L_1 \) and \( L_2 \) may be found in [14]. Nevertheless, we are going to present the proof of it in a way that will be useful for us later.

**Proof.** Let \( \mathcal{O}_i, \quad i = 3, 4 \), be the kernel of \( L_i \), i.e.
\[ y(x) \in \mathcal{O}_i : L_i y(x) = 0. \quad (26) \]
The dimension of \( \mathcal{O}_4 \) equals \( (n + m) \). From the lemma’s assumption it follows that the image of this space \( L_3(\mathcal{O}_4) \) has the same dimension. Therefore, there exists a unique normalized differential operator \( L_2 \) of degree \( n + m \) such that
\[ L_2 y(x) = 0, \quad y(x) \in L_3(\mathcal{O}_4). \quad (27) \]
Let us define the operator \( L_1 \) of degree \( n \) by the equality
\begin{equation}
L_1 y(x) = 0, \quad y(x) \in L_4(\mathcal{O}_3).
\end{equation}

From (27), (28) it follows that the differential operators \( L_2 L_3 \) and \( L_1 L_4 \) of degree \((2n + m)\) have the same kernel \( \mathcal{O}_3 + \mathcal{O}_4 \). Therefore, they are equal to each other. The equality

\begin{equation}
L_1 L_4 = L_2 L_3,
\end{equation}

is equivalent to (25). The lemma is proved.

In the same way one can prove the inverse statement that for any co-prime differential operators \( L_1, L_2 \) there exist unique normalized differential operators \( L_3, L_4 \) such that (25) is valid.

**Theorem 1.** For any \( n \) and \( m \) the space \( \mathcal{K}_{m,n} \) is invariant with respect to the KP hierarchy (19).

**Proof.** The equality (19) for \( L \in \mathcal{K}_{m,n} \) is equivalent to the equality

\begin{equation}
\partial_t (L_1^{-1} L_2) = [ (L_1^{-1} L_2)^{i/n}_+ , L_1^{-1} L_2 ].
\end{equation}

From (30) it follows that

\begin{equation}
(\partial_t L_2) L_3 - (\partial_t L_1) L_4 = L_1 (L_1^{-1} L_2)^{i/n}_+ L_4 - L_2 (L_1^{-1} L_2)^{i/n}_+ L_3,
\end{equation}

where \( L_3 \) and \( L_4 \) are differential operators such that (25) is fulfilled.

For the proof of the theorem it is enough to show that for given \( L_1 \) and \( L_2 \) Eq. (31) uniquely defines operators \( (\partial_t L_1) \) and \( (\partial_t L_2) \) of degrees \( (m - 1) \) and \( (n + m - 1) \), respectively.

Let \( \mathcal{D} \) be an operator that is defined by the right-hand side of (31). The coefficients of this operator are differential polynomials with respect to the coefficients of the operators \( L_1 \) and \( L_2 \), i.e. \( \mathcal{D} = \mathcal{D}(L_1, L_2) \). The differential part of the operator \( (L_1^{-1} L_2)^{i/n} \) can be replaced in (30) by its integral part

\begin{equation}
(L_1^{-1} L_2)^{i/n}_+ = (L_1^{-1} L_2)^{i/n}_- - (L_1^{-1} L_2)^{i/n}_+.
\end{equation}

Therefore,

\begin{equation}
\mathcal{D} = L_2 (L_1^{-1} L_2)^{i/n}_- L_3 - L_1 (L_1^{-1} L_2)^{i/n}_- L_4.
\end{equation}

The latter equality proves that the degree of the differential operator \( \mathcal{D} \) is less or equal to \( 2n + m - 1 \).

**Lemma 2.** For any differential operator \( \mathcal{D} \) of the degree \( 2n + m - 1 \) and for any co-prime differential operators \( L_3 \) and \( L_4 \) of degrees \( m \) and \( n + m \) there exist unique differential operators \( A_1 \) and \( A_2 \) of degrees \( m \) and \( n + m \) such that

\begin{equation}
A_2 L_3 - A_1 L_4 - \mathcal{D} = 0.
\end{equation}

**Proof.** Differential operator \( A_1 \) of degree \( m \) is uniquely defined by its action on any \( m \)-dimensional linear space. Therefore, it can be defined by the equality

\begin{equation}
A_1 y(x) = \mathcal{D} y(x), \quad y(x) \in L_4(\mathcal{O}_3).
\end{equation}

The analogous equality

\begin{equation}
A_2 y(x) = \mathcal{D} y(x), \quad y(x) \in L_3(\mathcal{O}_4)
\end{equation}
defines the operator \( A_2 \). From (34), (35) it follows that the differential operator that is defined by the left-hand side of (34) has degree \( 2n + m - 1 \) and its kernel has a dimension bigger or equal to \( 2n + m \). Hence, this operator is equal to zero. The lemma is proved.

The operators \( A_i \) that are defined due to this lemma for \( L_3, L_4 \) and \( \mathcal{D} \) given by equalities (25) and (33), depend on the operators \( L_1 \) and \( L_2 \), only. I.e. \( A_i = A_i(L_1, L_2) \). Therefore, the equalities

\begin{equation}
\partial_t L_1 = A_1(L_1, L_2), \quad \partial_t L_2 = A_2(L_1, L_2)
\end{equation}
do define the evolution of the operators \( L_1 \) and \( L_2 \). The theorem is proved.

When discussing this result with the author, T. Shiota proposed the exact form of the equations that define the evolution of the operators \( L_1 \) and \( L_2 \).

**Theorem 2.** The restriction of the KP hierarchy on \( \mathcal{K}_{m,n} \) is equivalent to the equations

\begin{equation}
\partial_t L_1 = L_1 (L_1^{-1} L_2)^{i/n}_+ - (L_2 L_1^{-1} L_2)^{i/n}_+ L_1,
\end{equation}

\begin{equation}
\partial_t L_2 = L_2 (L_1^{-1} L_2)^{i/n}_+ - (L_2 L_1^{-1} L_2)^{i/n}_+ L_2,
\end{equation}

**Proof.** From Theorem 1 it follows that (30) correctly defines the evolution of \( L_1 \) and \( L_2 \). The equality (30)
is a corollary of the equalities (38), (39). Therefore, for the proof of the theorem it is enough to verify that the right-hand sides of (38), (39) are differential operators of degrees not greater than \( m - 1 \) and \( n - m - 1 \). The last statement follows from the equalities

\[
\begin{align*}
L_1^{-1} (L_2 L_1^{-1})^{i/n} L_1 & = (L_1^{-1} L_2)^{i/n}, \\
L_2^{-1} (L_2 L_1^{-1})^{i/n} L_2 & = (L_1^{-1} L_2)^{i/n}
\end{align*}
\]

which imply that (38), (39) are equivalent to the equations

\[
\begin{align*}
\partial_t L_1 = (L_2 L_1^{-1})^{i/n} L_1 - L_1 (L_1^{-1} L_2)^{i/n}, \\
\partial_t L_2 = (L_2 L_1^{-1})^{i/n} L_2 - L_2 (L_1^{-1} L_2)^{i/n}.
\end{align*}
\]

The theorem is proved.

**Remark.** The formulae (41), (42) are similar to the formulae for the evolution of the factors \( L = L_1 L_2 \), where \( L_i \) are differential operators or some special kinds of pseudo-differential operators, which have been obtained in [15].

**Example.** In [10, 9] it was proved that the hierarchy of the non-linear Schrödinger equation can be obtained as a reduction of the KP hierarchy. This is the particular case of the above-proved theorem. In the notations of this work the corresponding reduction is the reduction onto the space \( \mathcal{K}_{2,1,2} \).

The coefficients of the pseudo-differential operator \( \mathcal{L} \) of the form

\[
\mathcal{L} = (\partial_x + v)^{-1}(\partial_x^2 + v \partial_x + w) = \partial_x + \sum_{i=1}^{\infty} u_i \partial_x^{-i}
\]

are differential polynomials with respect to the functions \( v \) and \( w \). They are recurrently defined from the equalities

\[
u_1 = w, \ u_{i+1} + u_i + \nu u_i = 0, \ i > 1.
\]

Let us consider the second flow of the KP hierarchy (19), i.e., the equation that defines the dependence of \( \mathcal{L} \) with respect to the variable \( t_2 \). For \( n = 2 \) the operator \( \mathcal{L}^2 \) equals \( \partial_x^2 + 2u_1 \). The corresponding equations for the first two coefficients of \( \mathcal{L} \) become the closed system of two equations for two unknown functions due to (44) for \( i = 1, 2 \)

\[
\begin{align*}
\partial_t u_1 & = u_{1xx} + 2u_{2x}, \ u_2 = -u_{1x} + \nu u_1, \\
\partial_t u_2 & = u_{2xx} + 2u_{3x} + 2u_1 u_1 \\
& = u_{2xx} - 2(u_{2x} + \nu u_2) + 2u_1 u_1.
\end{align*}
\]

Let us define the two new unknown functions \( r \) and \( q \) with the help of the relations

\[
u_1 = rq, \ \nu = -\frac{r_x}{r},
\]

Eqs. (45), (46) are equivalent to the system

\[
\begin{align*}
\partial_t r & = r_{xx} + rqr, \\
\partial_t q & = q_{xx} + qrq,
\end{align*}
\]

which for \( r = \pm q^* \) and \( t_2 = it \) implies the non-linear Schrödinger equation

\[
irr - r_{xx} + |r|^2 r = 0.
\]

The restriction of the KP hierarchy onto the spaces \( \mathcal{K}_{m,n} \) is the quantization of the corresponding algebraic orbits of the dispersionless KP hierarchy that was introduced in [16]. In [16] it was found also that algebraic orbits of the dispersionless KP hierarchy have additional symmetries. The next theorem shows that these additional symmetries can be quantized.

**Theorem 3.** The restriction of the KP hierarchy onto the invariant spaces \( \mathcal{K}_{m,n} \) is compatible with the equations

\[
\mathcal{L}_T = [A_{-i}, \mathcal{L}],
\]

where

\[
A_{-i} = M_1^{-1} M_2,
\]

\( M_1 \) is a normalized differential operator of degree \( i \), and \( M_2 \) is a differential operator of degree \( i - 1 \).

**Proof.** Eq. (50) for \( \mathcal{L} = L_1^{-1} L_2 \) is equivalent to the equation

\[
(\partial_x L_2) L_3 - (\partial_x L_1) L_4 = L_1 A_{-i} L_4 - L_2 A_{-i} L_3,
\]

where \( L_3, L_4 \) are the operators for which the equality (25) is fulfilled. The pseudo-differential operator
$A_i$ has a negative degree. Therefore, (52) is a well-defined evolution system if the pseudo-differential operator $D$ in its right-hand side is a differential operator. For the generic $A_i$, the operator $D$ can be uniquely represented in the form

$$D = D_1 D_2^{-1}, \tag{53}$$

where $D_1$, $D_2$ are differential operators and $D_2$ has degree $2i$. Therefore, the condition that $D_1$ is divisible by $D_2$ is equivalent to the system of $2i$ ordinary differential equations for $2i$ unknown coefficients of the operators $M_1$ and $M_2$. The theorem is proved.

Remark 1. The compatibility conditions of two symmetries of the form (51)

$$\left[ \frac{\partial}{\partial r_1} - M_{1}^{-1} M_2 \frac{\partial}{\partial r_2} - M_{3}^{-1} M_4 \right] = 0 \tag{54}$$

were considered in the recent paper [17] with no relation to the symmetries of rational reductions of the KP hierarchy.

Remark 2. We would like to mention that the symmetries of the form (50), (51) exist for the usual Lax reductions of the KP hierarchy, as well. For example, for the KdV equation (i.e. for $L^2 = L = \partial_x^2 + u$) and $i = 1$ the equation

$$L = u = [(v \partial_x + w)^{-1}, L] \tag{55}$$

is equivalent to the equations

$$u_r w^2 + u (u_r w)_x = w_{xx} - u u_x, \tag{56}$$

$$u (u_r v)_x + 2 u_r u v w = v_{xx} + 2 w_x, \tag{57}$$

$$u_r v^2 = 2 v_x. \tag{58}$$

Let $w$ be equal to

$$w = \frac{1}{2} v_x, \tag{59}$$

then from (56), (57) it follows

$$v = e^\phi \tag{60}$$

Eqs. (59) and (57) imply that

$$\partial_t \left( \frac{1}{2} \phi_{xx} - \frac{1}{4} \phi_x^2 \right) = 2 \phi_x e^{-\phi} \tag{61}$$

is equivalent to the Lax equation (55) with the condition (58).

References