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The τ-function of the universal Whitham hierarchy, matrix models and topological field theories

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Abstract

The universal Witham hierarchy is considered from the point of view of topological field theories. The τ-function for this hierarchy is defined. It is proved that the algebraic orbits of Whitham hierarchy can be identified with various topological matter models coupled with topological gravity.

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1 Introduction

The last two years breakthrough in low-dimensional string theory is one of most exiting results in modern mathematical and theoretical physics. In [1] remarkable connections between the non-perturbative theory of two-dimensional gravity coupled with various matter fields [2] and the theory of integrable KdV-type systems were found that had led to complete solvability of double-scaling limit of the matrix-models that are used to simulate fluctuating triangulated Riemann surfaces. Shortly after that Witten [3] presented some evidence for a relationship between random surfaces and the algebraic topology of moduli spaces of Riemann surfaces with punctures. His approach involved a particular field theory, known as topological gravity [5]. Further developments of his approach (especially, Kontsevich's proof [6] of Witten's conjecture [3, 4] on the coincidence of the generating function for intersection numbers of moduli spaces with τ-function of the KdV hierarchy) have shown that two-dimensional topological gravity is the corestone of this new subject of mathematical physics that includes: two-dimensional quantum field theories, intersection theory on the moduli spaces of Riemann surfaces with punctures, integrable hierarchies with special Virasoro-type constraints, matrix integrals, random surfaces and so on.

In this paper we continue our previous attempts [7, 8] to find a right place in this range of disciplines for the Whitham theory which is a most interesting part of perturbation theory of KdV-type integrable hierarchies. They were stimulated by results of [9], where correlation functions for topological minimal models were found. It turned out that calculations [9] of perturbed primary rings for A_n-models can be identified with the construction of a particular solution of the first n "flows" of the dispersionless Lax hierarchy (semi-classical limit of the usual Lax hierarchy). It provided a possibility to include corresponding deformations of primary chiral rings into a hierarchy of inifinite number of commuting "flows". The calculations [9] of partition function for perturbed A_n models gave an impulse for introduction in [7] a τ-function for dispersionless Lax equations. The truncated version of Virasoro constraints for the corresponding τ-function were proved. Their comparision with [3] shows that they are necessary conditions for identification of "generators" of "higher" flows with gravitational descendants of primary fields after coupling model with gravity. For $n > 2$ they are not sufficient. (The problem is the same as for τ-function of multi-matrix models. As it was shown in [10] the τ-function of in multi-matrix models satisfies and uniquelly defined with the help of higher W-constraints.) In section 4 we prove the truncated version of W-constraints for τ-function of dispersionless Lax equations. Therefore, the full dispersionless Lax hierarchy really can be identified with topological A_n minimal model coupled with gravity.

The results of [7] were generalized in [8, 11] for higher genus case. In [8] it was shown that self-similar solutions of the Whitham equations on the moduli space of genus g Riemann surfaces are related with "multi-cut" solutions of loop-equations for matrix models. In [11] the generalization of topological Landau-Ginsburg models on Riemann surfaces of special type were proposed and their primary rings and correlation function were found. In [11] the Hamiltonian formulation [12] of the Whitham averaging procedure was used. With its help it was proved that "coupling" constants for primary fields of such models give a system of *global* flat coordinates on the moduli space of corresponding curves. In [13] using the Hamiltonian approach to the Whitham theory the integrability of general Witten-Dijgraagh-Verlinder-Verlinder (WDVV)

Two- and three-points correlation functions

$$\langle \phi_\alpha \phi_\beta \rangle = \eta_{\alpha\beta}, \quad c_{\alpha\beta\gamma} = \langle \phi_\alpha \phi_\beta \phi_\gamma \rangle \tag{1.1}$$

of any topological field theory with N primary fields $\phi_1, \ldots, \phi_N$ define an associative algebra

$$\phi_\alpha \phi_\beta = c^\gamma_{\alpha\beta} \phi_\gamma, \ c^\gamma_{\alpha\beta} = c_{\alpha\beta\mu} \eta^{\gamma\mu}, \ \eta_{\alpha\mu} \eta^{\mu\beta} = \delta^\beta_\alpha. \tag{1.2}$$

with a unit ϕ_1

$$\eta_{\alpha\beta} = c_{1\alpha\beta}. \tag{1.3}$$

It turns out that there exists N parametric deformation of the theory such that "metric" $\eta_{\alpha\beta}$ is a constant and three-point correlators are given by the derivatives of free energy $F(t)$ of the deformed theory

$$c_{\alpha\beta\gamma}(t) = \partial_{\alpha\beta\gamma} F(t), \quad \eta_{\alpha\beta} = \partial_{1\alpha\beta} F(t) = const. \tag{1.4}$$

The associativity conditions of algebra (1.2) with structure constants (1.4) are equivalent to a system of partial differential equations on F (WDVV equations). In [13] "spectral transform" for these equations were proposed. It proves their integrability, however (as it seems for us) the explicit representation of all corresponding models remains an open problem.

In section 5 we show that each "algebraic" orbit of the universal Whitham hierarchy gives an exact solutions of WDVV equations. Moreover, a generalization of W-constraints for corresponding τ-functions, that are proved in section 4, provide some evidence that the universal Whitham hierarchy can be considered as a universal (at tree-level) topological field theory coupled with gravity.

In this introduction we present a definition of the Whitham hierarchy in a most general form. All "integrable" partial differential equations, that are considered in the framework of the "soliton" theory, are equivalent to compatibility conditions of auxiliary linear problems. The general algebraic-geometrical construction of their exact periodic and quasi-periodic solutions was proposed in [14]. There the concept of the Baker-Akhiezer functions were introduced. (The analytical properties of the Baker-Akhiezer functions are the generalization of properties of the Bloch solutions of the finite-gap Sturm-Liouville operators, which were found in a serious of papers by Novikov, Dubrovin, Matveev and Its [15]).

The "universal" set of algebraic-geometrical data is as follows. Consider the space $\hat{M}_{g,N}$ of smooth algebraic curves Γ_g of genus g with local coordinates $k_\alpha^{-1}(P)$ in neighbourhoods of N punctures P_α , $(k_\alpha^{-1}(P_\alpha)=0)$

$$\hat{M}_{g,N}=\{\Gamma_g, P_\alpha, k_\alpha^{-1}(P),\ \alpha=1,\ldots,N\}. \tag{1.5}$$

This space is a natural bundle over the moduli space $M_{g,N}$ of smooth algebraic curves Γ_g of genus g with N punctures

$$\hat{M}_{g,N}=\{\Gamma_g, P_\alpha, k_\alpha^{-1}(P)\}\longmapsto M_{g,N}=\{\Gamma_g, P_\alpha\}. \tag{1.6}$$

For each set of data (1.5) and each set of g points $\gamma_1\ldots,\gamma_g$ on Γ_g in a general position (or, equivalently, for a point of the Jacobian $J(\Gamma_g)$) the algebraic-geometrical construction gives a quasi-periodic solution of some integrable PNDE. (For given non-linear integrable equation the corresponding set of data have to be specified. For example, the solutions of the Kadomtsev-Petviashvili (KP) hierarchy corresponds to the case $N=1$. The solutions of the two-dimensional Toda lattice corresponds to the case $N=2$.)

The data (1.5) are "integrals" of the infinite "hierarchy" of integrable non-linear differential equations, that can be represented as a set of commuting "flows" on a phase space. Let t_A be a set of all corresponding "times". In the framework of the "finite-gap" (algebraic-geometrical) theory of integrable equations each time t_A is coupled with a meromorphic differential $d\Omega_A(P|\mathcal{M}),\ \mathcal{M}\in\hat{M}_{g,N}$

$$t_A\longmapsto d\Omega_A(P|\mathcal{M}) \tag{1.7}$$

that is "responsible" for the flow. ($d\Omega_A(P|\mathcal{M})$ is a differential with respect to the variable $P\in\Gamma$ depending on the data (1.5) as on external parameters.)

In [16] the algebraic-geometrical perturbation theory for integrable non-linear (soliton) equations was developed. It was stimulated by the application of the Whitham approach for (1+1) integrable equation of the KdV type [18]. As usual in the perturbation theory, "integrals" of an initial equation become functions of the "slow" variables εt_A (ε is a small parameter). "The Whitham equations" is a name for equations which describe "slow" variation of "adiabatic integrals". (We would like to emphasize that algebraic-geometrical approach represents only one side of the Whitham theory. In [12] a deep differential-geometrical structure that is assosiated with the Whitham equations were developed.)

Let $\Omega_A(k,T)$ be a set of holomorphic functions of the variable k (which is defined in some complex domain D), depending on a finite or infinite number of variables $t_A,\ T=\{t_A\}$. (We preserve the same notation t_A for slow variables εt_A because we are not going to consider "fast" variables in this paper.) Let us introduce on the space with coordinates (k,t_A) a one-form

$$\omega=\sum_A\Omega_A(k,T)dt_A. \tag{1.8}$$

Its full external derivative equals

$$\delta\omega = \sum_A \delta\Omega_A(k,T) \wedge dt_A, \tag{1.9}$$

where

$$\delta\Omega_A = \partial_k\Omega_A dk + \sum_B \partial_B\Omega_A dt_B,\ \partial_k = \partial/\partial k,\ \partial_A = \partial/\partial t_A. \tag{1.10}$$

The following equation

$$\delta\omega \wedge \delta\omega = 0 \tag{1.11}$$

we shall call by definition *the Whitham hierarchy.*

The "algebraic" form (1.11) of the Whitham equations is equivalent to a A, B, C

$$\sum_{\{A,B,C\}} \varepsilon^{\{A,B,C\}}\partial_A\Omega_B\partial_k\Omega_C = 0 \tag{1.12}$$

(summation in (1.12) is taken over all permutations of indecies A, B, C a $\varepsilon^{\{A,B,C\}}$ is a sign of permutation).

The equations (1.11) are invariant with respect to an invertable change of variable

$$k = k(p,T),\ \partial_p k \neq 0. \tag{1.13}$$

Let us fix some index A_0 and denote the corresponding function by

$$p(k,T) = \Omega_{A_0}(k,T). \tag{1.14}$$

At the same time we introduce the special notation for the corresponding "time"

$$t_{A_0} = x. \tag{1.15}$$

After that all Ω_A can be cosidered as the functions of new variable p, $\Omega_A = \Omega_A(p,T)$. The equations (1.12) for $A, B, C = A_0$ have the form

$$\partial_A\Omega_B - \partial_B\Omega_A + \{\Omega_A, \Omega_B\} = 0, \tag{1.16}$$

where $\{f, g\}$ denotes the usual Poisson bracket on the space of functions of two variables x, p

$$\{f, g\} = \partial_x f\partial_p g - \partial_x g\partial_p f. \tag{1.17}$$

The Whitham equations were obtained in [16] in the form (1.12). In [17] it was noticed that they can be represented in the algebraic form (1.11). (We would like to mention here the papers [19] where it was shown that algebraic form of the Whitham equations leads directly to semiclassical limit of "strings" equations.)

The Whitham equations in the form (1.12) are equations on the set of func real content. It's necessary to show that they do define correct systems of equations on the spaces $\hat{M}_{g,N}$. For zero-genus case ($g = 0$) it will be done in the next section. In the same section a construction ([16])of exact solutions of the zero-genus hierarchy corresponding to its "algebraic" orbits are presented. The key element of the scheme [16] is a construction of a potential $S(p,T)$ and a "connection" $E(p,T)$ such that after change of variable

$$p = p(E,T),\ \Omega_A(E,T) = \Omega_A(p(E,T),T) \tag{1.18}$$

the following equalities

$$\Omega_A(E,T) = \partial_A S(E,T). \tag{1.19}$$

are valid.

In section 3 the τ-function for the Whitham equations on the spaces $\hat{M}_{0,N}$ is introduced. For all genera (the case $g > 0$ is considered in section 7) τ-function can be represented in the following "field theory" form

$$\tau = \int_\Gamma \bar{d}S \wedge dS. \tag{1.20}$$

Important remark. The integral (1.20) does not equal to zero, because $S(p,T)$ is holomorphic on Γ except at the punctures P_α and some contours, where it has "jumps". Therefore, the integral over Γ equals to a sum of the residues at P_α and the contour integrals of the corresponding one-form.

The τ-function is a function of the variables t_A, $\tau = \tau(T)$. As it will be shown in the section 3 it contains a full information about the corresponding solutions $\Omega_A(p,T)$ of the Whitham equations. For $g > 0$ in τ a part of geometry of moduli spaces is incoded.

In section 4 zero-genus Virasoro and W-constraints for τ-function are proved. In section 5 the primary chiral rings corresponding to algebraic orbits of Whitham hierarchy are considered. The last section is preceded by section 6 where using ideas of [19] a "direct transform" for general Whitham hierarchy is discussed. It turns out that the existence of a potential S is not a characteristic property of the construction of solutions. In a hidden form

All results that are proved for genus-zero Whitham hierarchy in the first five sections are generalized for arbitrary genus case in section 7. We present same way as in genus zero case but requier greater lenght due to pure technical

2 Whitham hierarchy. Zero genus case

In zero genus case a point of the "phase space" $\hat{M}_{g=0,N}$ is a set of points $p_\alpha,\ \alpha = 1,\ldots,N$, and a set of formal local coordinates $k_\alpha^{-1}(p)$

$$k_\alpha(p) = \sum_{s=-1}^{\infty} v_{\alpha,s}(p-p_\alpha)^s \tag{2.1}$$

("formal local coordinate" means that r.h.s of (2.1) is considered as a formal series without any assumption on its convergency). Hence, $\hat{M}_{0,N}$ is a set of sequences

$$\hat{M}_{0,N} = \{p_\alpha,\ v_{\alpha,s},\ \alpha = 1,\ldots,N,\ s = -1,0,1,2,\ldots\} \tag{2.2}$$

The Whitham equations define a dependence of points of $\hat{M}_{0,N}$ with respect to the variables t_A where the set of indecies $\mathcal{A}$ is as follows

$$\mathcal{A} = \{A = (\alpha, i),\ \alpha = 1, \ldots, N,\ i = 1, 2, \ldots \text{ and for } i = 0, \alpha \neq 1\}. \qquad (2.3)$$

As it was explained in the introduction we can fix one of the points p_α with the help of a appropriate change of the variable p. Let us choose: $p_1 = \infty$.

Introduce meromorphic functions $\Omega_{\alpha,i}(p)$ for $i > 0$ with the help of the following conditions:

$\Omega_{\alpha,i}(p)$ has a pole only at p_α and coincides with the singular part of an expansion of $k_\alpha^i(p)$ near this point, i.e.

$$\Omega_{\alpha,i}(p) = \sum_{s=1}^{i} w_{\alpha,i,s}(p - p_\alpha)^{-s} = k_\alpha^i(p) + O(1),$$

$$\Omega_{\alpha,i}(\infty) = 0,\ \alpha \neq 1. \qquad (2.4)$$

$$\Omega_{1,i}(p) = \sum_{s=1}^{i} w_{1,i,s} p^s = k_1^i(p) + O(k_1^{-1}). \qquad (2.5)$$

These polynomials can be written in the form of the Cauchy integrals

$$\Omega_{\alpha,i}(p, T) = \frac{1}{2\pi i} \oint_{C_\alpha} \frac{k_\alpha^i(z_\alpha, T) dz_\alpha}{p - z_\alpha}. \qquad (2.6)$$

Here C_α is a small cycle around the point p_α.

The functions $\Omega_{\alpha,i=0}(p),\ \alpha \neq 1$ are equal to

$$\Omega_{\alpha,0}(p) = -\ln(p - p_\alpha). \qquad (2.7)$$

Remark. The asymmetry of the definitions of $\Omega_{\alpha,i}$ reflects our intention to choose the index $A_0 = (1,1)$ as "marked" index.

The coefficients of $\Omega_{\alpha,i>0}(p)$ are polynomial functions of $v_{\alpha,s}$. Therefore, the Whitham equations (1.10) (or (1.16)) can be rewritten as equations on $\hat{M}_{0,N}$. But still it have to be shown that they can be considered as a correctly defined system.

Theorem 2.1 *The zero-curvature form (1.16) of the Whitham hierarchy in zero-genus case is equivalent to the Sato-form that is a compatible system of evolution equations*

$$\partial_A k_\alpha = \{k_\alpha, \Omega_A\}. \qquad (2.8)$$

Proof. Consider the equations (1.16) for $B = (\alpha, j > 0)$. From the definition of Ω_A it follows that

$$\partial_A k_\alpha^j - \{k_\alpha^j, \Omega_A\} = \partial_B \Omega_A - \{\Omega_A, \Omega_B^-\} + O(1). \qquad (2.9)$$

Here and below we use the notation

$$\begin{aligned} \Omega_A^- &= \Omega_A - k_\alpha^i, \qquad \text{for } A = (\alpha, i > 0), \\ \Omega_{\alpha,0}^- &= \Omega_{\alpha,0} - \ln k_\alpha. \end{aligned} \qquad (2.10)$$

Hence,

$$\partial_A k_\alpha^j - \{k_\alpha^j, \Omega_A\} = 0(k^{i-1}). \tag{2.11}$$

Therefore,

$$\partial_A k_\alpha - \{k_\alpha, \Omega_A\} = 0(k^{i-j}). \tag{2.12}$$

The limit of (2.12) for $j \to \infty$ proves (2.8). The inverse statement that (2.8) is a correctly defined system can be proved in a standart way. So we shall skip it.

Let us demonstrate a few examples.

Example 1. Khokhlov-Zabolotskaya hierarchy.

The Khokhlov-Zabolotskaya hierarchy is the particular $N = 1$ case of our considerations. Any local coordinate $K^{-1}(p)$ near infinity ($p_1 = \infty$)

$$K(p) = p + \sum_{s=1}^{\infty} v_s p^{-s} \tag{2.13}$$

defines a set of polynomials:

$$\Omega_i(p) \ = \ [K^i(p)]_+, \tag{2.14}$$

here $[...]_+$ denotes a non-negative part of Laurent series. For example,

$$\Omega_2 = k^2 + u, \quad \Omega_3 = k^3 + \frac{3}{2}uk + w, \quad \text{where } u = 2v_1, \quad w = 3v_2. \tag{2.15}$$

If we denote $t_2 = y,\ t_3 = t$ then the equation (1.16) for $A = 2,\ B = 3$ gives

$$w_x = \frac{3}{4}u_y, \quad w_y = u_t - \frac{3}{2}uu_x, \tag{2.16}$$

from which the dispersionless KP (dKP) equation (which is also called by the Khokhlov-Zabolotskaya equation) follows :

$$\frac{3}{4}u_{yy} + (u_t - \frac{3}{2}uu_x)_x = 0. \tag{2.17}$$

The Khokhlov-Zabolotskaya equation is a partial differential equation and though it has not pure evolution form, one can expect that its solutions have to be uniquelly defined by their Cauchy data $u(x, y, t = 0)$, that is a function of two variables x, y. Up to now it is not clear if this two-dimensional equation can be considered as the third equivalent form of the Whitham hierarchy (we remind that formally solutions of the hierarchy (2.8) depend on a infinite number of functions of one variable).

Example 2. Longwave limit of 2d Toda Lattice.

The hierarchy of longwave limit of two-dimensional Toda equation are the particular $N = 2$ case of our considerations. There are two local parameters. One of them is near the infinity $p_1 = \infty$ and one is near a point $p_2 = a$. They depend on two set of the variables $t_{\alpha,s}$, $\alpha = 1, 2$, $s = 1, 2, \ldots$ and also on the variable t_0. We shall present here only the basic two-dimentional equation of this hierarchy (an analogue of Khokhlov-Zabolotskaya equation).

Consider three variables $t = t_0, x = t_{1,1}, y = t_{2,1}$. The corresponding functions are

$$\Omega_0 = \ln(p - a), \quad \Omega_{1,1} = p, \quad \Omega_{2,1} = \frac{v}{p - a}. \tag{2.18}$$

Their substitution into the zero-curvature equation (1.16) gives

$$v_x = a_t v, \quad v_t + a_y = 0, \quad w_t = 0. \tag{2.19}$$

From (2.19) it follows

$$\partial^2_{xy}\phi + \partial^2_t e^{\phi} = 0, \text{ where } \phi = \ln v. \tag{2.20}$$

This is a longwave limit of the 2d Toda lattice equation

$$\partial^2_{xy}\varphi_n = e^{\varphi_{n-1}-\varphi_n} - e^{\varphi_n - \varphi_{n+1}} \tag{2.21}$$

corresponding to the solutions that are slow functions of the discrete variable n, which is replaced by the continuous variable t. The equation (2.20) has arised independently in general relativity, in the theory of wave can be found in [22] where a representation of solutions of (2.20) in terms of convergent series were proposed.

Example 3. N-layer solutions of the Benny equation.

This example corresponds to a general $N + 1$ points case, but we consider only one zero-curvature equation. Let us choose three functions

$$\Omega_1 = p, \quad \Omega_2 = p + \sum_1^N \frac{v_i}{p - p_i}, \quad \Omega_3 = p^2 + u, \tag{2.22}$$

which are coupled with the variables x, y, t, respectively. (In our standart notations they are

$$x = t_{1,1}, \quad y = \sum_{\alpha=1}^{N+1} t_{\alpha,1}, \quad t = t_{1,2}.) \tag{2.23}$$

The zero-curvature equation (1.16) gives the system

$$p_{it} - (p_i^2)_x + u_x = 0, v_{it} = 2(v_i p_i)_x, u_y - u_x + 2\sum_i v_{ix} = 0. \tag{2.24}$$

Solutions of this system that do not depend on y are N-layer solutions of the Benny equation. As it was noticed in [20] the corresponding system

$$p_{it} - (p_i^2)_x + u_x = 0, \quad v_{it} = 2(v_i p_i)_x, \quad u = 2\sum_i v_i \tag{2.25}$$

is a classical limit of the vector non-linear Schrödinger equation

$$i\psi_{it} = \psi_{i,xx} + u\psi_i, \quad u = \sum_i |\psi_i|^2. \tag{2.26}$$

(Using this observation in [20] the integrals of (2.25) were found.)

In the second part of this section we consider "algebraic" orbits of the genus-zero Whitham equations. By definition they are specified with the help of the constraint: *there exists a meromorphic solution $E(p,T)$ of the equations*

$$\partial_A E = \{E, \Omega_A\}, \tag{2.27}$$

such that

$$\{E(p,T), k_\alpha(p,T)\} = 0. \tag{2.28}$$

The last equality implies that there exist functions $f_\alpha(E)$ of one variable such that

$$k_\alpha(p,T) = f_\alpha(E(p,T)). \tag{2.29}$$

In order not to be lost in too general setting right at the begining, let us beging with an example.

Example. Lax reductions (N=1)

Consider solutions of the dKP hierarchy such that some power of local parameter (2.13) is a polynomial in p, i.e.

$$E(p,T) = p^n + u_{n-2}p^{n-2} + ... + u_0 = k_1^n(p,T). \tag{2.30}$$

The relation (2.30) implies that only a few first coefficients of the local parameter are independent. All of them are polynomials with respect to the cofficients u_i of the polynomial $E(p,T)$. The corresponding solutions of dKP hierarchy can be described in terms of dispersionless Lax equations

$$\partial_i E(p,T) = \{E(p,T), \Omega_i(p,T)\}, \tag{2.31}$$

where

$$\Omega_i(p,T) = [E^{i/n}(p,T)]_+ \tag{2.32}$$

(as before, $[\ldots]_+$ denotes a non-negative part of corresponding Laurent series). These solutions of KP hierarchy can be also characterized by the property that they do not depend on the variables $t_n, t_{2n}, t_{3n}, \ldots$. We are going to construct an analoque of the "hodograph" transform for the solution of this equations. It is a generalization and effectivization of a scheme [21], where "hodograph-type" transform were proposed for hydrodinamic-type diagonalizable Hamiltonian systems ([12]).

Let us introduce a generating function

$$S(p) = \sum_{i=1}^{\infty} t_i \Omega_i(p) = \sum_{i=1}^{\infty} t_i K^i + O(K^{-1}), \tag{2.33}$$

where Ω_i are given by (2.14) and $K = E^{1/n}$. (If there is only a finite number of t_i that are not equal to zero, then $S(p)$ is a polynomial.) The coefficients of S are linear functions of t_i and polynomials in u_i. We introduce a dependence of u_j on the variables t_i with the help of the following algebraic equations:

$$\frac{dS}{dp}(q_s) = 0, \tag{2.34}$$

where q_s are zeros of the polynomial

$$\frac{dE}{dp}(q_s) \ = \ 0. \tag{2.35}$$

Remark. It is not actually necessary to solve the equation (2.35) in order to find q_s. We can choose $q_s,\ s = 1, \ldots, n-1$ and u_0 as a new set of unknown functions, due to the equality

$$\frac{dE}{dp} = np^{n-1} + (n-2)u_{n-2}p^{n-3} + \ldots + u_1 = \prod_{s=1}^{n}(p - q_s), \tag{2.36}$$

$$q_n = -\sum_{s=1}^{n-1} q_s.$$

Let us prove that if the dependence of $E = E(p, T)$ with respect to the variables t_i is defined from (2.34), then

$$\partial_i S(E, T) = \Omega_i(E, T). \tag{2.37}$$

Consider the function $\partial_i S(E, T)$. From (2.33) it follows that

$$\partial_i S(E) = K^i + O(K^{-1}) = \Omega_i(E) + O(K^{-1}). \tag{2.38}$$

Hence, it is enough to prove that $\partial_i S(E)$ is a polynomial in p, because by definition Ω_i is the only polynomial in p such that

$$\Omega_i(p) = K^i + O(K^{-1}). \tag{2.39}$$

The function $\partial_i S(E, T)$ is holomorphic everywhere except may be at $q_s(T)$. In a neighbourhood of $q_s(T)$ a local coordinate is

$$(E - E_s(T))^{1/2}, \quad E_s(T) = E(q_s(T)) \tag{2.40}$$

(if q_s is a simple root of (2.35)). Hence, a priori S has the expansion

$$S(E,T) = \alpha_s(T) + \beta_s(T)(E - E_s(T))^{1/2} + \dots \tag{2.41}$$

and the derivative $\partial_i S(E,T)$ might be singular at the points q_s. But the defining relations (2.34) imply that $\beta_s = 0$. Therefore, $\partial_i S(E)$ is regular everywhere except at the infinity and, hence, is a polynomial. The equations (2.37) are proved.

Let us present this scheme in another form. For each polynomial $E(p)$ of the form (2.30) and each formal series

$$Q(p) = \sum_{j=1}^{\infty} b_j p^j \tag{2.42}$$

the formula

$$t_i = \frac{1}{i}\mathrm{res}_\infty (K^{-i}(p)Q(p)dE(p)). \tag{2.43}$$

defines the variables

$$t_k = t_k(u_i, b_j), \quad i = 0, \dots, n-2, \quad j = 0, \dots. \tag{2.44}$$

as functions of the coefficients of E, Q. Consider the inverse functions

$$u_i = u_i(t_1, \dots), \ b_j = b_j(t_1, \dots). \tag{2.45}$$

Remark. In order to be more precise let us consider a case when Q is a polynomial, i.e. $b_j = 0, \quad j > m$. From (2.43) it follows that $t_k = 0, \quad k > n+m-1$. Therefore, we have $n+m-1$ "times" $t_k, \ k = 1, \dots, n+m-1$ that are linear functions of $b_j, \ j = 1, \dots, m$ and polynomials in $u_i, \ i = 0, \dots, n-2$. So, locally the inverse functions (2.45) are well-defined.

Theorem 2.2 *The functions $u_i(T)$ are solutions of dispersionless Lax equation (2.31). Any other solutions of (2.31) are obtained from this particular one with the help of translations, i.e. $\tilde{u}(t_i) = u(t_i - t_i^0)$.*

Consider now general N case. Let $E(p)$ be a meromorphic functions with a pole of order n at infinity and with poles of orders n_α at points negative $n_\alpha \neq 0$, which means that E has a zero of the order $-n_\alpha$, can be considered as well, but we are not going to do it here in

$$E = p^n + u_{n-2}p^{n-2} + ... + u_0 + \sum_{\alpha=2}^{M} \sum_{s=1}^{n_\alpha} v_{\alpha,s}(p - p_\alpha)^{-s}. \tag{2.46}$$

Consider a linear space of such functions, i.e. the space of sets

$$\mathcal{N}(n_\alpha) = \{u_i, i = 0, \dots, n-2; \quad v_{\alpha,s}, s = 1, \dots, n_\alpha)\}, \tag{2.47}$$

$$\alpha = 1, \ldots, M, \quad n_1 = n$$

If $N \leq M$ the function $E(p)$ defines the local coordinates at the points p_α with the help of formula

$$k_\alpha^{n_\alpha}(p) = E(p), \alpha = 1, \ldots, N. \tag{2.48}$$

Therefore, $\mathcal{N}(n_\alpha)$ can be identified with a subspace of $\hat{M}_{0,N}$

$$\mathcal{N} \subset \hat{M}_{0,N}. \tag{2.49}$$

Theorem 2.3 *The subspaces $\mathcal{N}_M(n,\ n_\alpha, \alpha = 1, \ldots, M)$ are invariant with respect to the Whitham equations on $\hat{M}_{0,N}$ that coincide with the flows*

$$\partial_A E(p, T) = \{E(p, T), \Omega_A(p, T)\}, \tag{2.50}$$

where k_α and Ω_A are defined with the help of formulae (2.48,2.6), respectively. General solutions of (2.50) are given in an implicite form with the help of the following algebraic equations

$$\frac{dS}{dp}(q_s, T) = 0, \quad \text{where} \quad S(p, T) = \sum_A (t_A - t_A^0)\Omega_A(p, T \tag{2.51}$$

which have to be fulfilled for all zeros q_s of the function

$$\frac{dE}{dp}(q_s) = 0. \tag{2.52}$$

The proof is the same as the proof of the previous theorem. Its main step is a proof that from the defining relations (**??**) it follows that

$$\partial_A S(E, T) = \Omega_A(E, T). \tag{2.53}$$

Definition. The particular solutions of the Whitham hierarchy corresponding to the algebraic orbits (2.50) and for which $t_A^0 = 0$ will be

An alternative formulation of this theorem can be done in the following form. Let $Q(p)$ be a meromorphic function with poles at the points p_α, i.e.

$$Q(p) = \sum_{j=1}^{\infty} b_{1,j} p^j + \sum_{\alpha=2}^{M} \sum_{j=1}^{\infty} b_{\alpha,j}(p - p_\alpha)^{-j}. \tag{2.54}$$

The formulae

$$\begin{aligned} t_{\alpha,i} &= \frac{1}{i}\mathrm{res}_\alpha(k_\alpha^{-i}(p)Q(p)dE(p))), \quad i > 0; \\ t_{\alpha,0} &= \mathrm{res}_\alpha(Q(p)dE(p)) \end{aligned} \tag{2.55}$$

defines "times" $t_{\alpha,i}$ as functions of the coefficients of $Q(p)$ and $E(p)$ (which has the form (2.46)). Consider the inverse functions

$$v_{\alpha,s} = v_{\alpha,s}(t_{\beta,i}), \quad b_{\alpha,j} = b_{\alpha,j}(t_{\beta,i}). \tag{2.56}$$

(we remind that index $\alpha = 1$ corresponds to the infinity $p_1 = \infty$).

Corollary 2.1 *The inverse functions $v_{\alpha,s}(t_{\beta,i})$ are solutions of Whitham equations. In particular, (for all N)*

$$u(x,y,t) = \frac{2}{n} u_{n-2}(t_{1,1} = x, t_{2,1} = y, t_{3,1} = t, \ldots) \quad (2.57)$$

is a solution of Khokhlov-Zabolotskay equation (2.17) and

$$\phi(x,y,t) = \frac{1}{n_2} \ln v_{2,n_2}(x = t_{1,1}, y = t_{2,1}, t = t_{2,0}, \ldots) \quad (2.58)$$

is a solution of longwave limit of 2d Toda lattice equation (2.20).

(We would like to emphasize that that in the formulae (2.57,2.58) all "times" except of the first ones are parameters.)

Inverse functions (2.56) define the dependence on the variables t_A of the functions $E(p,T)$ and $Q(p,T)$. The differential of the potential $S(p,T)$ equals

$$dS(p,T) = Q(p,T) dE(p,T). \quad (2.59)$$

From (2.53) it follows that

$$\partial_A Q(E,T) = \frac{d\Omega_A(E,T)}{dE}. \quad (2.60)$$

In particular,

$$\partial_x Q(E,T) = \frac{dp(E,T)}{dE}. \quad (2.61)$$

The derivatives with fixed E and p are related with each other with the help of the following relation

$$\partial_A f(p,T) = \partial_A f(E,T) + \frac{df}{dE} \partial_A E(p,T). \quad (2.62)$$

Using (2.61) and (2.62) we obtain

Corollary 2.2 *The functions $Q(p,T)$ and $E(p,T)$ satisfy the quasi-classical "string equation"*

$$\{Q, E\} = 1. \quad (2.63)$$

(This corollary was prompted by [19] and we shall return to it at greater lenght in section 6.)

3 τ-function

In the previous section the "algebraic" solutions of the Whitham hierarchy in zero-genus case were constructed. It was shown that for their "potentials"

$$S(p,T)=\sum_A (t_A-t_A^0)\Omega_A(p,T) \tag{3.1}$$

(where t_A^0 are corresponding constants) the following equalities

$$\Omega_B(k_\alpha,T)=\partial_B S(k_\alpha,T), \quad B=(\beta,i) \tag{3.2}$$

are fulfilled. In this section we define with the help of $S(p,T)$ the τ-functions corresponding to algebraic solutions of the Whitham equations.

The τ-function of the universal Whitham hierarchy (in zero genus case) would be by definition

$$\log\tau(T)=F(T),$$

$$F=\frac{1}{2}\sum_{\alpha=1}^{N}\mathrm{res}_\alpha\Big(\sum_{i=1}^{\infty}\tilde t_{\alpha,i}k_\alpha^i dS(p,T)\Big)+\tilde t_{\alpha,0}s_\alpha(T), \tag{3.3}$$

$$\tilde t_{\alpha,i}=t_{\alpha,i}-t_{\alpha,i}^0,$$

where res_α denotes a residue at the point p_α and s_α is the coefficient of the expansion

$$S(p,T)=\sum_{\alpha=1}^{N}\sum_{i=1}^{\infty}\tilde t_{\alpha,i}k_\alpha^i+t_{\alpha,0}\mathrm{ln}k_\alpha+s_\alpha+O(k^{-1})). \tag{3.4}$$

Here and below we use the notaion

$$t_{1,0}=-\sum_{\alpha=2}^{N}t_{\alpha,0}. \tag{3.5}$$

The τ-function can be rewritten in more compact form. Let us make a cuts connecting the point $p_1=\infty$ with the points p_α. After that we can choose a branch of the function $S(p,T)$. The coefficient s_α equals

$$s_\alpha=\frac{1}{2\pi i}\oint_{\sigma_\alpha}\ln(p-p_\alpha)dS, \tag{3.6}$$

where σ_α is a contour around the corresponding cut. The function S has jumps on the cuts. Its $\bar\partial$-derivative is a sum of delta-functions and its derivatives at the points p_α and one-dimentional delta-functions on cuts. Therefore, (3.3) can be represented in such a form:

$$F=\int \bar d S\wedge dS. \tag{3.7}$$

(The integral in (3.7) is taken over the hole complex plane of the variable p. It is non-zero because $S(p,T)$ is holomorphic outside punctures and cuts,

Theorem 3.1 *For the above defined τ-function the following equalities are fulfilled:*

$$\partial_{\alpha,i}F(T) = \text{res}_\alpha(k_\alpha^i dS(p,T)), \quad i > 0, \tag{3.8}$$

$$\partial_{\alpha,0}F(T) = s_\alpha. \tag{3.9}$$

Proof. Let us consider the derivative for $A = (\alpha, i > 0)$. It equals

$$2\partial_A F = \text{res}_\alpha(k_\alpha^i dS) + \sum_{\beta=1}^{N}\sum_{j=1}^{\infty} \text{res}_\beta(\tilde{t}_{\beta,i}k_\beta^j d\Omega_A) + \tilde{t}_{\beta,0}\Omega_A(p_\beta). \tag{3.10}$$

We use in(3.10) the equality

$$\partial_A s_\beta = \Omega_A(p_\beta). \tag{3.11}$$

From

$$\sum_{\alpha=1}^{N} \text{res}_\alpha(\Omega_A d\Omega_B) = 0 \tag{3.12}$$

it follows that

$$\text{res}_\beta(k_\beta^j d\Omega_{\alpha,i}) = \text{res}_\alpha(k_\alpha^i d\Omega_{\beta,j}), \quad j > 0. \tag{3.13}$$

Besides this,

$$\Omega_A(p_\beta) = \text{res}_\beta(\Omega_A d\ln(p - p_\beta)) = \text{res}_\alpha(k_\alpha^i d\Omega_{\beta,0}). \tag{3.14}$$

The substitution of (3.13,3.14) into (3.10) proves (3.8). The proof of (3.9) is absolutely analoqous.

The formulae (3.8,3.9) show that an expansion of $S(p,T)$ at the point p_α has the form

$$S(p,T) = \sum_{\alpha=1}^{N}\sum_{i=1}^{\infty} \tilde{t}_{\alpha,i}k_\alpha^i + \tilde{t}_{\alpha,0}\ln k_\alpha + \partial_{\alpha,0}F + \sum_{j=1}^{\infty}\frac{1}{j}\partial_{\alpha,j}F k_\alpha^{-j}. \tag{3.15}$$

From (3.8,3.9) it follows:

Corollary 3.1 *The second derivatives of F are equal to*

$$\partial^2_{A,B}F(T) = res_\alpha(k_\alpha^i d\Omega_B), \ A = (\alpha, i > 0), \tag{3.16}$$

$$\partial_{\alpha,0}\partial_{\beta,0}F(T) = \ln(p_\alpha - p_\beta). \tag{3.17}$$

Hence, an expansion of the non-positive part (2.10)of $\Omega_A(p)$ at the point p_β has the form

$$\Omega_A^- = \partial_{\beta,0}F + \sum_{j=0}^{\infty}\frac{1}{j}(\partial_A\partial_{\beta,j}F)k_\beta^j. \tag{3.18}$$

Therefore, the τ-function that depends on "times", only, contains a complete information on the functions Ω_A.

4 Truncated Virasoro and W-constraints

In section 3 it was shown that any solution of the Whitham equations ($g = 0$) corresponding to an algebraic orbit can be obtained from "homogeneous" solution with the help of translations $\tilde{t}_A = t_A - t_A^0$. In this section we consider τ-functions of homogeneous solutions, only.

The truncated Virasoro constraints for the τ-function of the dispersionless an invariance of residues with respect to a change of variables. The same approach can be applied for the general N-case, also. In this paper we use another way that was inspired by the $N \to \infty$ limit of the loop-equations for the one-matrix model (a review of recent developments of the loop-equations technique can be found in [23]).

The function $E(p)$ of the form (2.46) represents the complex plane of the variable p as D-sheet branching covering of the complex plane of the variable E, $D = \sum_\alpha n_\alpha$. The zeros q_s of $dE(q_s) = 0$ are branching points of the covering. Hence, any function $f(p)$ can be consider are zeros of the equation

$$E(p_i) = E. \tag{4.1}$$

The symmetric combination of the values $f(p_i)$

$$\tilde{f}(E) = \sum_{i=1}^{D} f(p_i) \tag{4.2}$$

is a single-valued function of E. Let us apply this argument for the function $Q^K(p)$, where

$$Q(p) = \frac{dS(p)}{dE(p)}; \tag{4.3}$$

$S(p)$ is the potential of the homogeneous solution of the Whitham equations. The defining algebraic relations (**??**) imply that $Q(p)$ is holomorphic outside the punctures p_α (that are "preimages" of the infinity $E(p_\alpha) = \infty$). Therefore, the function

$$\tilde{Q}^K(E) = \sum_{i=1}^{N} Q^K(p_i), \tag{4.4}$$

is an enteir function of the variable E. In other words the Laurent expansion

$$\mathrm{res}_\infty(\sum_{i=1}^{N} Q^K(p_i)E^{m+1}dE) = 0, \quad m = -1, 0, 1, \dots\ . \tag{4.5}$$

The residue (4.5) at the infinite of the complex plane of E is equal to a sum of the residues at the points p_α, i.e.

$$\sum_{i=1}^{N} \mathrm{res}_\alpha((Q^K(k_\alpha)E^{m+1}dE) = 0, \quad E = k_\alpha^{n_\alpha}. \tag{4.6}$$

From (3.15) it follows that the function $Q(p)$ has the expansion

$$Q(k_\alpha) = \frac{1}{n_\alpha}\sum_{i=1}^{\infty} i t_{\alpha,i} k_\alpha^{i-n_\alpha} + t_{\alpha,0}k_\alpha^{-n_\alpha} + \sum_{j=1}^{\infty}\partial_{\alpha,j}F k_\alpha^{-j-n_\alpha}. \tag{4.7}$$

at the point p_α. The substitution of (4.7) into (4.6) for $K = 1$ gives obvious identities:

$$\sum_{\alpha=1}^{N} t_{\alpha,0} = 0, \quad m = -1,$$

$$\sum_{\alpha=1}^{N} \partial_{\alpha,mn_\alpha}F = 0, \quad m = 0, 1, \ldots. \tag{4.8}$$

(For Lax reductions $N = 1$ the equalities (4.8) imply that F does not depend on $t_n, t_{2n}, \ldots$.) For $K > 1$ the relations (4.6) lead to highly nontrivial equations. For example, the case $K = 2$ corresponds to

Theorem 4.1 *The τ-function of the homogeneous solution of the Whitham equations (corresponding to the orbit $\mathcal{N}(n_\alpha)$) is a solution of the equations*

$$\sum_{\alpha=1}^{N}\frac{1}{n_\alpha}\Big(\sum_{i=n_\alpha+1}^{\infty} i t_{\alpha,i}\partial_{\alpha,i-n_\alpha}F + n_\alpha t_{\alpha,0}t_{\alpha,n_\alpha} + \frac{1}{2}\sum_{j=1}^{n_\alpha-1} j(n_\alpha - j)t_{\alpha,j}t_{\alpha,n_\alpha-j}\Big) = 0; \tag{4.9}$$

$$\sum_{\alpha=1}^{N}\frac{1}{n_\alpha}\sum_{i=1}^{\infty} i t_{\alpha,i}\partial_{\alpha,i}F + \frac{1}{2}t_{\alpha,0}^2 = 0; \tag{4.10}$$

$$\sum_{\alpha=1}^{N}\frac{1}{n_\alpha}\Big(\sum_{i=1}^{\infty} i t_{\alpha,i}\partial_{\alpha,i+mn_\alpha}F + \frac{1}{2}\sum_{j=1}^{mn_\alpha-1}\partial_{\alpha,j}F\partial_{\alpha,mn_\alpha-j}F\Big) = 0, \quad m = 1, \ldots. \tag{4.11}$$

The equalities (4.5) for any K can be written in the form

$$\sum_{\alpha=1}^{N} n_\alpha^{1-K}\sum_{I,J}[i_1]t_{\alpha,i_1}\cdots[i_s]t_{\alpha,i_s}\partial_{\alpha,j_{s+1}}F\cdots\partial_{\alpha,j_K}F = 0, \tag{4.12}$$

where the second sum is taken over all sets of indecies $I = \{i_k\},\ J = \{j_k\}$ such that

$$\sum_{k=1}^{s} i_k = \sum_{k=s+1}^{K} j_k - (m + K - 2)n_\alpha, \quad m > -1. \tag{4.13}$$

and $[i]$ denotes

$$[i] = i, \quad \text{if } i \neq 0; \quad [0] = 1. \tag{4.14}$$

For $N = 1$ the equations (4.12) coincide with a nonlinear part of the W_K constraints.

At the end of this section we present the truncated Virasoro constraints for $N = 1$ and $n = 2$ in the form of the planar limit of the loop-equations for one-matrix hermitian model.

Consider the negative part of $Q(k)$ $(N=1, n=2)$

$$-\mathcal{W}_0 = Q^-(k) = \frac{1}{2}\sum_{i=1}^{\infty} t_1 k^{-1} + \sum_{j=1}^{\infty} \partial_{2j-1} F k^{-2j-1} \tag{4.15}$$

and introduce

$$V(k) = \sum_{i=1}^{\infty} \tilde{t}_{2i} k^{2i}, \tag{4.16}$$

where

$$\tilde{t}_{2i} = \frac{2i+1}{2i} t_{2i+1}. \tag{4.17}$$

Then

$$Q(k) = V'(k) - \mathcal{W}_0 \tag{4.18}$$

and (4.6) is equivalent to the equation

$$\oint_C \frac{V'(\xi)\mathcal{W}_0(\xi)}{k-\xi} d\xi = \frac{1}{2}\mathcal{W}_0^2, \tag{4.19}$$

where C is a small contour around the infinity. (4.19) is a planar limit (see [23])

$$\oint_C \frac{V'(\xi)\mathcal{W}(\xi)}{k-\xi} d\xi = \frac{1}{2}\mathcal{W}^2 + \sum_{i=1}^{\infty} k^{-2i-1} \frac{\partial \mathcal{W}}{\partial t_{2i}}. \tag{4.20}$$

In (4.20) $\mathcal{W}(k)$ is the Wilson loop-correlator that by definition is equal to

$$\mathcal{W} = \langle \mathrm{tr} \frac{1}{k-X} \rangle = \int \mathrm{tr} \frac{1}{k-X} e^{-\mathrm{tr} V(X)} dX, \tag{4.21}$$

where X is a hermitian $M \times M$ matrix.

Remark. As it was shown in [24] the double-scaling limit of $n-1$ matrix chain model is related with the n-th reduction of the KP-equation. Dispersionless Lax equations (2.30,2.31) are their classical limit. Therefore, the negative part $\mathcal{W}_0$ of the series (4.7) for $N=1$ and arbitrary n has to be related with the planar limit of some Wilson-type correlators for multi-matrix models. Therefore, higher "loop-equations" (corresponding to $K>2$) have to be fulfilled for them. It should be interesting to find a direct way to produce the corresponding equations in the framework of the multimatrix models.

5 Primary rings of the topological field theories

Topological minimal models were introduced in [25] and were considered in [26]. They are a twisted version of the discrete series of $N=2$ superconformal Landau-Ginzburg (LG) models. A large class of the $N=2$ superconformal LG models has been studied in [27]. It was shown, that a finite number of states are topological, which means that their operator products have no singularities. These states form a closed ring $\mathcal{R}$, which is called a primary chiral ring. It can be expressed in terms of the superpotential $E(p_i)$ of the corresponding model

$$\mathcal{R} = \frac{C[p_i]}{dE = 0}, \quad dW = \frac{\partial E}{\partial p_i} dp_i. \tag{5.1}$$

In topological models these primary states are the only local physical excitations.

In [9], it was shown that correlation functions of primary chiral fields can be expressed in terms of perturbed superpotentials $E(p_i, t_1, t_2, ...)$. For A_{n-1} model the unperturbed superpotential has the form:

$$E_0 = p^n. \tag{5.2}$$

The coefficients of a perturbed potential

$$E(p) = p^n + u_{n-2}p^{n-2} + ... + u_0 \tag{5.3}$$

can be considered as the coordinates on the space of deformed topological minimal models. In [9] the dependence of u_i on the coordinates $t_1, \ldots, t_{n-1}$ that are "coupled" with primary fields ϕ_i were found. It was shown that the deformation of the ring $\mathcal{R}$

$$\mathcal{R}(t_1, \ldots, t_{n-1}) = C[p]/(dE(p, t_1, \ldots, t_{n-1}) = 0) \tag{5.4}$$

is a potential deformation of the Fröbenius algebra (in the sense that was explained in Introduction).

In this section we consider an application of the general Whitham equations on $\hat{M}_{0,N}$ to the theory of potential deformations of the Fröbenius algebras. They are based on the following formula for the third logariphmic derivatives of the τ-function. Let $E(p, T)$ be a homogeneous solution of the Whitham equations (2.50) corresponding to an algebraic orbit $\mathcal{N}(n_\alpha)$, i.e. $E(p)$ has the form (2.46)

$$E = p^n + u_{n-2}p^{n-2} + ... + u_0 + \sum_{\alpha=2}^{M} \sum_{s=1}^{n_\alpha} v_{\alpha,s}(p - p_\alpha)^{-s}.$$

The formulae (2.55,2.56) define the dependence $E(p)$ and the "dual" function $Q(p)$ with respect to the variables t_A.

Theorem 5.1 *The third logariphmic derivatives of the τ-function of the homogeneous solution of the Whitham equation corresponding to an algebraic orbit $\mathcal{N}(n_\alpha)$ are equal to*

$$\partial^3_{ABC} F = \sum_{q_s} \text{res}_{q_s} \left(\frac{d\Omega_A d\Omega_B d\Omega_C}{dQ \ dE}\right), \tag{5.5}$$

where q_s are zeros of the differential $dE(q_s) = 0$.

Proof. Let us suppose that $A = (\alpha, i > 0)$. (The case when A, B, C are equal to $(\alpha, 0), (\beta, 0), (\gamma, 0)$ can be considered in the same way.) From (3.8) it follows that

$$\partial_C \partial^2_{AB} F = \mathrm{res}_\alpha (k^i_\alpha d\partial_C \Omega_B) = -\mathrm{res}_\alpha (\partial_C \Omega_B d\Omega_A). \tag{5.6}$$

Here the derivative $\partial_C \Omega_A(E, T)$ is taken for the fixed E. As it was explained in section 2 the function $E(p, T)$ is a "good" coordinate except at the points $q_s(T)$ where local coordinates have the form (2.40). Hence, at the point $q_s(T)$ the function Ω_A has the expansion

$$\Omega_B(E, T) = w_{B,0}(T) + w_{B,1}(T)(E - E_s(T))^{1/2} + \cdots, \tag{5.7}$$

$$E_s(T) = E(q_s(T), T).$$

A sum of all residues of a meromorphic differential equals zero. Therefore,

$$\partial^3_{ABC} F = \sum_{q_s} \mathrm{res}_{q_s} (\partial_C \Omega_B d\Omega_A). \tag{5.8}$$

From (5.7) we have that in a neighbourhood of the point q_s

$$\partial_C \Omega_B = -\partial_C E_s \frac{d\Omega_B}{dE} + O(1). \tag{5.9}$$

Therefore,

$$\mathrm{res}_{q_s} (\partial_C \Omega_B d\Omega_A) = -\mathrm{res}_{q_s} (\partial_C E_s \frac{d\Omega_A d\Omega_B}{dE}). \tag{5.10}$$

From (2.50) it follows

$$\partial_C E_s = \partial_x E_s \frac{d\Omega_C}{dp}(q_s). \tag{5.11}$$

The string equation (2.63) implies

$$\partial_x E_s = -\frac{dp}{dQ}(q_s). \tag{5.12}$$

Substitution of (5.11) and (5.12) into (5.10) proves the theorem.

For each algebraic orbit $\mathcal{N}(n_\alpha)$ let us define a "small phase" space (see motivation in [3]). It will be a space of times t_a with the indecies a belonging to a subset $\mathcal{A}_{sm}$

$$\mathcal{A}_{sm} = \{(\alpha, i) | \alpha = 1, i = 1, \ldots, n-1; \alpha = 2, \ldots, N, i = 0, \ldots, n_\alpha\}. \tag{5.13}$$

Let us fix all other times t_A

$$t_{1,n} = 0, \ t_{1,n+1} = \frac{n}{n+1}, \ t_{1,i} = 0, \ i > n+1;$$

$$t_{\alpha,i} = 0, \ \alpha = 2, \ldots, N, \ i > n_\alpha. \tag{5.14}$$

Comparision with (4.7) shows that in this case

$$Q(p) = p \tag{5.15}$$

in the formula (2.55). In other words t_a as a functions of $u_i,\ v_{\alpha,s}$ are given by the formulae

$$\begin{aligned} t_{\alpha,i} &= \frac{1}{i}\text{res}_\alpha(k_\alpha^{-i}(p)pdE(p))), \quad i > 0; \\ t_{\alpha,0} &= \text{res}_\alpha(pdE(p)) \end{aligned} \tag{5.16}$$

Inverse functions define a dependence of the coefficients of E with respect to the variables t_a

Corollary 5.1 *Let*

$$F(t_a) = F(t_a, t_{1,n+1} = \frac{n}{n+1}, t_A = 0,\ A \notin \mathcal{A}_{sm}) \tag{5.17}$$

be the restriction of $F = \ln\tau$ on the affine space that is $\mathcal{A}_{sm}$ shifting by $t_{1,n+1} = \frac{n}{n+1}$. Then

$$\partial^3_{abc}F = \sum_{q_s} \text{res}_{q_s}(\frac{d\Omega_a d\Omega_b d\Omega_c}{dp\ dE}). \tag{5.18}$$

Let us summarize the results. Each meromorphic function $E(p)$ of the form (2.46) defines a factor ring

$$\mathcal{R}_E = \hat{\mathcal{R}}/(dE = 0) \tag{5.19}$$

of the ring $\hat{\mathcal{R}}$ of all meromorphic functions that are regular at the zeros q_s of the differential dE. The formula

$$\langle f, g\rangle = \sum_{q_s} \text{res}_{q_s}(\frac{f(p)g(p)}{E_p}dp),\ \ f(p),\ g(p) \subset \hat{\mathcal{R}}, \tag{5.20}$$

defines a non-degenerate scalar product on $\mathcal{R}_E$. The scalar product (5.20) supplies $\mathcal{R}_E$ by the structure of the Fröbenius algebra. In the basis

$$\phi_a = \frac{d\Omega_a}{dp} \tag{5.21}$$

the scalar product has the form

$$\langle \phi_a \phi_b\rangle = \eta_{ab} = \frac{[i][j]}{n_\alpha}\delta_{\alpha,\beta}\delta_{i+j,n_\alpha}, \tag{5.22}$$

where $[i]$ is the same as in (4.14). Our last statement is that the formulae (5.16) define in an implicite form the potential deformations of these Fröbenius algebras.

The case $N = 1$ covers the results of [9]. As it was mentioned in the Introduction an integrability of WDVV equations was proved in [13]. The results of this section can be considered as a explicit construction of their particular solutions.

The process of a "coupling" the ring $\mathcal{R}_E$ with topological gravity corresponds to the process of "switching on" of *all* times of the Whitham hierarchy. It follows from the recurrent formula for the third derivatives of τ-function. First of all let us present a formula

$$\Omega_{\alpha,i>0}dE = \frac{n_\alpha}{i+n_\alpha}d\Omega_{\alpha,i+n_\alpha} + \sum_{b=(\beta,j)\in\mathcal{A}_{sm}} \frac{n_\beta}{[n_\beta - j][j]}(\partial^2_{A,b}F)d\Omega_{\beta,n_\beta-j}. \tag{5.23}$$

It can be proved in a following way. The right and the left hand sides of (5.23) are holomorphic ounside the punctures. Hence, it is enough to compair their expansions at the points p_α. The coefficients of an expansion of Ω_A are given by the second derivatives of F (3.18). Therefore, (5.23) is fulfilled.

Let us denote for each $a = (\alpha, i > 0) \in \mathcal{A}_{sm}$ the fields

$$\frac{d\Omega_{\alpha,pn_\alpha+i}}{dQ} = \sigma_p(\phi_a). \tag{5.24}$$

then the substitution of (5.23) into (5.8) proves the recurrent formular for the correlation function for the gravitational decendants [3]

$$\langle\sigma_p(\phi_a)\sigma_B\sigma_C\rangle = \langle\sigma_{p-1}(\phi_a)\phi_b\rangle\eta^{bc}\langle\phi_c\sigma_B\sigma_C\rangle, \tag{5.25}$$

where σ_B, σ_C are any other states. (The integrability of general decendant equations were proved in [13].)

Remark. This paper had been already written when author got a preprint [28] where the Frobenious algebras and their "small phase" deformations corersponding to the Whitham hierarchy for the multi-puncture case

6 Generating form of the Whitham equations

In this short section (or rather a long remark) we would like to clarify our construction of the algebraic solutions of the Whitham equations and the definition of the corresponding τ-function. It was stimulated by the papers [19] where using our approach ([7]) the τ-function for longwave limit of 2d Toda lattice were introduced.

Let $\Omega_A(k,T)$ be a solution of the general zero-curvature equation (1.16)

$$\partial_A\Omega_B - \partial_B\Omega_A + \{\Omega_A, \Omega_B\} = 0, \tag{6.1}$$

They are a compatibility condition for the equation

$$\partial_A E = \{E, \Omega_A\}, \tag{6.2}$$

Therefore, an arbitrary function $E(p,x)$ defines (at least locally) the corresponding solution $E(p,T)$ of (6.2), $E(p,x) = E(p, t_{A_0} = x, t_A = 0, A \neq A_0)$. In the domain where $\partial_p E(p,T) \neq 0$ we can use a variable E as a new coordinate, $p = p(E,t)$. From (2.62) it follows that in the new coordinate (6.1) are equivalent to the equations

$$\partial_A \Omega_B(E,T) = \partial_B \Omega_A(E,T). \tag{6.3}$$

Hence, there exists a potential $S(E,T)$ such that

$$\Omega_A(E,T) = \partial_A S(E,T). \tag{6.4}$$

Using this potential the one-form ω (1.8) can reperesented as

$$\omega = \delta S(E,T) - Q(E,T)dE, \tag{6.5}$$

where

$$Q(E,T) = \frac{\partial S(E,T)}{\partial E}. \tag{6.6}$$

Hence,

$$\delta\omega = \delta E \wedge \delta Q. \tag{6.7}$$

The formulae (2.59-2.62) that are valid in general case prove that the functions E and Q as function of two variables p, x satisfy the classical string equation

$$\{Q, E\} = 1. \tag{6.8}$$

They show that

$$\partial_A Q = \{Q, \Omega_A\}. \tag{6.9}$$

A set of pairs of functions $Q(p,x), E(p,x)$ satisfying the string equation is a group with respect to the composition, i.e. if $Q(p,x), E(p,x)$ and $Q_1(p,x), E_1(p,x)$ are solutions of (6.8) then the functions

$$\tilde{Q}(p,x) = Q_1(Q(p,x), E(p,x); \tilde{E}(p,x) = E_1(Q(p,x), E(p,x)) \tag{6.10}$$

are solution of (6.8) as well. The Lie algebra of this group is the algebra $SDiff(T^2)$ of two-dimensional vector-fields preserving an area. The action of this algebra on potential, τ-function (and so on) in the framework of the longwave limit of $2d$ Toda lattice was considered in [19].

The previous formulae can be used in the inverse direction. Let $E(p,x)$ and $Q(p,x)$ be any solution of the equation (6.8). Using them as Cauchy data for the equations (6.2,6.9 we define the functions $E(p,T)$, $Q(p,T)$ that satisfy (6.8) for all T. After that the potential $S(p,T)$ can be found with the help of formula

$$S(p,T) = \int^p Q(p,T)dE(p,T). \tag{6.11}$$

Let us revise from this general point of view the definition of the τ-function corresponding to the solutions of the Whitham equations on $\hat{M}_{0,N}$. As it was shown in the theorem 2.1 the local parameters k_α by themselves are solutions of the equations

(6.2). Therefore, they define a set of local potentials $S_\alpha(k_\alpha)$ such that the relation (3.2)

$$\Omega_B(k_\alpha, T) = \partial_B S_\alpha(k_\alpha, T), \quad B = (\beta, i).$$

are fulfilled. On the other hand let us consider the solutions $E(p,T), Q(p,T)$ of the equations (6.2) with the initial data

$$E(p,x) = p, \quad Q(p,x) = x. \tag{6.12}$$

They are the holomorphic outside the punctures $p_\alpha(T)$. Hence, there exists a "global" differential $dS_0(p,T)$ that is holomorphic outside the punctures $p_\alpha(T)$, also. Let us define a one-form on the space with the coordinates t_A

$$\delta \log \tau = \frac{1}{2}\sum_{\alpha=1}^{N}(\sum_{i=1}^{\infty}\mathrm{res}_\alpha(k_\alpha^i dS_\alpha(p,T))dt_{\alpha,i} + \frac{1}{2\pi i}\oint_{\sigma_\alpha}\ln(p-p_\alpha)dS_0(p,T)dt_{\alpha,0}), \tag{6.13}$$

where σ_α is a contour around the cut connecting $p_1 = \infty$ and p_α. It is easy to check with the help of the formulae (3.11-3.14) that $\delta \log \tau$ is a closed form. Therefore, locally there exist a τ-function. What are advantages of the algebraic solutions ?

As it was shown in section 3 for algebraic solutions there exist constants t_A^0 such that a sum

$$S(p,T) = \sum_A (t_A - t_A^0)\Omega_A(p,T) \tag{6.14}$$

is a "global" potential coinciding in a neibourhoods of p_α with the local potentials S_α. This provides the possibility to define explicitle with the help of formula (3.3) the τ-function but not only its full external differential (6.13).

7 Arbitrary genus case

1. Definition. The moduli space $\hat{M}_{g,N}$ is "bigger" then $\hat{M}_{0,N}$. In the approach in which "times" of the Whitham hierarchy are considered as a new system of coordinates on the phase space it is natural to expect that there should be more "times" in the Whitham hierachy on $\hat{M}_{g,N}$. We shall increase their number in a few steps. But at the begining let us consider the *basic* Whitham hierachy in the form that has arisen as a result of the averaging procedure for the algebraic-geometrical solutions of two-dimensional integrable equations. In this hierarchy there are the same set of "times" (2.3) and this is the only part of universal Whitham hierarchy on $\hat{M}_{g,N}$ that has a smooth degeneration to the zero-genus hierarchy.

Let Γ_g be a smooth algebraic curve of genus g with local coordinates $k_\alpha^{-1}(P)$ in neighbourhoods of N punctures P_α, $(k_\alpha^{-1}(P_\alpha) = 0)$. Let us introduce meromorphic differentials

1^0. $d\Omega_{\alpha,i>0}$ is holomorphic outside P_α and has the form

$$d\Omega_{\alpha,i} = d(k_\alpha^i + O(k_\alpha^{-1})) \tag{7.1}$$

in a neighbourhood of P_α;

2^0. $\Omega_{\alpha,0},\ \alpha\neq 1$ is a differential with simple poles at P_1 and P_α with residues 1 and -1, respectively

$$d\Omega_{\alpha,0}=dk_\alpha(k_\alpha^{-1}+O(k_\alpha^{-1}))$$
$$d\Omega_{\alpha,0}=-dk_1(k_1^{-1}+O(k_1^{-1})); \tag{7.2}$$

3^0. The differentials $d\Omega_A$ are uniquelly normalized by the condition that *all* their periods are real, i.e.

$$\mathrm{Im}\oint_c d\Omega_A=0,\quad c\in H_1(\Gamma_g,Z). \tag{7.3}$$

The normalization (7.1) does not depend on the choice of basic cycles on Γ_g. Therefore, $d\Omega_A$ is indeed defined by data $\hat{M}_{g,N}$.

Below, for the simplification of formulae we consider the complexcification of the Whitham hierarchy on $\hat{M}_{g,N}$ that is a hierarchy on the moduli space

$$\hat{M}^*_{g,N}=\{\Gamma_g,P_\alpha,k_\alpha^{-1}(P),\quad a_i,b_i\in H_1(\Gamma_g,Z)\}, \tag{7.4}$$

where $a_i,\ b_i$ is a canonical basis of cycles on Γ_g, i.e. cycles with the intersection matrix of the form $a_ia_j=b_ib_j=0,\quad a_ib_j=\delta_{i,j}$. In this case the differentials $d\Omega_A$ should be normalized by the usual conditions

$$\oint_{a_i} d\Omega_A=0,\quad i=1,\ldots,g. \tag{7.5}$$

Both types of hierarchies can be considered absolutely in parallel way.

Now we are going to show that generating equations (1.11) in which Ω_A are integrals of the above-defined differentials is equivalent to a set of commuting evolution equations on $\hat{M}^*_{g,N}$ (or $\hat{M}_{g,N}$, respectively). Let fix one point P_1 and choose as a"marked" index $A_0=(1,1)$. The multi-valued function

$$p(P)=\Omega_{1,1}(P)=\int^P d\Omega_{1,1},\quad P\in\Gamma_g \tag{7.6}$$

can be used as a coordinate on Γ everywhere except for the points Π_s, where $dp(\Pi_s)=0$. The parameters (2.2), i.e.

$$\{p_\alpha=p(P_\alpha),\ v_{\alpha,s},\ \alpha=1,\ldots,N,\ s=-1,0,1,2,\ldots\}$$

and additional parameters

$$\pi_s=p(\Pi_s),\ s=1,\ldots,2g, \tag{7.7}$$

$$U_i^p=\oint_{b_i} dp,\ i=1,\ldots,g \tag{7.8}$$

are a full system of local coordinates on $\hat{M}^*_{g,N}$.

Theorem 7.1 *The zero curvature-form (1.20) of the Whitham hierarchy on $\hat{M}^*_{g,N}$ is equivalent to the compatible system of evolution equations*

$$\partial_A k_\alpha(p,T) = \{k_\alpha(p,T), \Omega_A(p,T)\}, \tag{7.9}$$

$$\partial_A U_i^p = \partial_x U_i^A, \quad \text{where} \quad U_i^A = \oint_{b_i} d\Omega_A, \tag{7.10}$$

$$\partial_A \pi_s = \partial_A p(\Pi_s) = \partial_x \Omega_A(\Pi_s). \tag{7.11}$$

In [11] where the application of the Whitham equations for generalized Landau-Ginsburg madels were considered for the first time it was noticed that the construction of solutions of the Whitham equations that were proposed by author in [16] can be reformulated in the form that actually includes a new "additional" flows commuting with basic ones (7.11). It have to be mentioned that only g of them are universal. Let us introduce a set of g new times $t_{h,1}, \ldots, t_{h,g}$ that are coupled with normalized holomorphic dufferentials $d\Omega_{h,k}$

$$\oint_{a_i} d\Omega_{h,k} = \delta_{i,k}, \quad i,k = 1, \ldots, g\ . \tag{7.12}$$

Theorem 7.2 *The basic Whitham hierarchy (7.11) is compatible with the system that is defined by the same equations but with new "hamiltonians" $d\Omega_{h,k}$.*

The proof of the both theorems does not differ seriously from the usual consideration in the Sato approach and we shall skip them.

2. Algebraic orbits and exact solutions. Let us introduce a finite-dimensional subspaces of $\hat{M}^*_{g,N}$ that are invariant with respect to the Whitham hierarchy. Consider a normalized meromorphic differential dE of the second kind (i.e. dE has not residues at any point of Γ_g) that has poles of orders $n_\alpha + 1$ at the points P_α. (Normalized means that

$$\oint_{a_i} dE = 0. \tag{7.13}$$

for hierarchy on $\hat{M}^*_{g,N}$ and that dE has real periods for hierarchy on $\hat{M}_{g,N}$.) The integral $E(p)$ of this differntial has the expansions of the form

$$E(p) = p^n + u_{n-2}p^{n-2} + ... + u_0 + O(p^{-1}), \tag{7.14}$$

$$E(p) = \sum_{s=1}^{n_\alpha} v_{\alpha,s}(p - p_\alpha)^{-s} + O(1) \tag{7.15}$$

at the point P_1 and the points $P_\alpha,\ \alpha \neq 1$, respectively. The formula (2.48), i.e.

$$k_\alpha^{n_\alpha}(p) = E(p), \alpha = 1, \ldots, N.$$

defines local coordinates k_α^{-1} in neighbourhoods of P_α. Therefore, we have defined the embedding of the moduli space $\mathcal{N}_g(n_\alpha)$ of curves with fixed normalized meromorphic differential dE into $\hat{M}^*_{g,N}$

$$\mathcal{N}_g(n_\alpha) \subset \hat{M}^*_{g,N}. \tag{7.16}$$

The dimension of this subspace equals

$$D = \dim \mathcal{N}_g(n_\alpha) = 3g - 2 + \sum_{\alpha=1}^{N} (n_\alpha + 1). \tag{7.17}$$

There are two systems of local coordinates on $\mathcal{N}_g(n_\alpha)$. The first system is given by the coefficients of the expansions (7.14, 7.15)

$$\{u_i, i = 0, \ldots, n-2; \quad p_\alpha, \quad v_{\alpha,s}, s = 1, \ldots, n_\alpha)\}, \tag{7.18}$$

and by the variables (7.7,7.8), i.e.

$$\pi_s = p(\Pi_s), \ s = 1, \ldots, 2g; \ \ U_i^p = \oint_{b_i} dp, \ i = 1, \ldots, g.$$

The second system is given by the following parameters

$$U_i^E = \oint_{b_i} dE, \quad i = 1, \ldots, g, \tag{7.19}$$

$$E_s = E(q_s), \quad \text{where} \ \ dE(q_s) = 0, \quad s = 1, \ldots, D - g. \tag{7.20}$$

Using the first system of coordinates it easy to show that

Theorem 7.3 *The restriction of the Whitham hierarchy on $\mathcal{N}_g(n_\alpha)$ is given by the compatible system of equations (2.50)*

$$\partial_A E(p, T) = \{E(p, T), \Omega_A(p, T)\}.$$

(We whould like to remind that now besides $t_{\alpha,i}$ the set of "times" t_A includes the times $t_{h,k}$ that are

Let dH_i be a normalized differential that is defined on the cycle a_i, i.e

$$\oint_{a_i} dH_i = 0. \tag{7.21}$$

For each set $H = \{dH_i\}$ of such differentials there exists a unique differential dS_H such that:

dS_h is holomorphic on Γ_g except for the cycles a_i where it has "jumps" that are equal to

$$dS_H^+(P) - dS_H^-(P) = dh_i(P), \quad P \in a_i, \tag{7.22}$$

$$\oint_{a_i} dS = 0. \tag{7.23}$$

Theorem 7.4 *For any solution of the Whitham equations on $\mathcal{N}_g(n_\alpha)$ there exist constants t_A^0 and constant differentials dh_i (i.e. they do not depend on T) such that this solution is given in an implicite form with the help of equations*

$$\frac{dS}{dp}(q_s, T) = 0, \tag{7.24}$$

$$S(p,T) = \sum_A (t_A - t_A^0)\Omega_A(p,T) + dS_h. \tag{7.25}$$

The relations (7.24) implies that

$$dS = QdE, \tag{7.26}$$

where $Q(p)$ is holomorphic on Γ_g outside the punctures P_α and has "jumps"

$$Q^+(E) - Q^-(E) = \frac{dH_i(E)}{dE}, \quad E \in a_i, \tag{7.27}$$

on cycles a_i.

In this section we consider the solutions of the Whitham hierarchy corresponding to the constant jumps, only, i.e.

$$dH_i(P) = t_{Q,k} dE(P). \tag{7.28}$$

In that case dQ is a single-valued differential on Γ_g. Let us present an alternative formulation of the construction of such solutions.

Consider the moduli space

$$\widetilde{\mathcal{N}}_g(n_\alpha) = \{\Gamma_g, \quad dQ, \quad dE\} \tag{7.29}$$

of curves with fixed canonical basis of cycles, with fixed normalized meromorphic differential dE havong poles of orders $n_\alpha + 1$ at points P_α and with fixed holomorphic outside the punctures normalized differential $dQ(P)$.

The coordinates on this space are the variables (7.7,7.8,7.18)

$$\{\pi_s, \quad U_i^p, \quad u_i, \quad p_\alpha, \quad v_{\alpha,s},\}$$

and the coefficients of singular terms in the expansion

$$Q(p) = \sum_{j=1}^{\infty} b_{1,j} p^j + O(p^{-1}),$$

$$Q(p) = \sum_{j=1}^{\infty} b_{\alpha,j} (p - p_\alpha)^{-j} + O(p - p_\alpha). \tag{7.30}$$

The formulae (2.55), i.e.

$$t_{\alpha,i} = \frac{1}{i}\text{res}_\alpha(k_\alpha^{-i}(p)Q(p)dE(p))), \quad i>0;$$
$$t_{\alpha,0} = \text{res}_\alpha(Q(p)dE(p)) \tag{7.31}$$

and the formulae

$$t_{h,i} = \oint_{a_i} dS, \quad i=1,\ldots,g, \quad dS = QdE, \tag{7.32}$$

$$t_{Q,i} = -\oint_{b_i} dE, \quad t_{E,i} = \oint_{b_i} dQ, \quad i=1,\ldots,g. \tag{7.33}$$

defines times t_A as functions on the space $\widetilde{\mathcal{N}}_g(n_\alpha)$

The differntials $d\Omega_{E,i},\ d\Omega_{Q,i}$ that are couple with times $t_{E,i},\ t_{Q,i}$ are uniquelly defined with the help of following analytical

1^0. The differentials $d\Omega_{E,i},\ d\Omega_{Q,i}$ are holomorphic on the curve Γ_g everywhere except for the a-cycles, where they have "jumps". Their boundary values on a_j cycle satisfy the relations

$$d\Omega^+_{E,i} - d\Omega^-_{E,i} = \delta_{i,j}dE,$$
$$d\Omega^+_{Q,i} - d\Omega^-_{Q,i} = \delta_{i,j}dQ; \tag{7.34}$$

2^0.

$$\oint_{a_j} d\Omega_{E,i} = \oint_{a_j} d\Omega_{Q,i} = 0, \ j=1,\ldots,g. \tag{7.35}$$

In the same way as it was done in section 2 it can be shown that the number of "times" is equal to the dimension of $\widetilde{\mathcal{N}}_g(n_\alpha)$. Therefore, the "times" t_A can be considered as new coordinates on $\widetilde{\mathcal{N}}_g(n_\alpha)$, i.e.

$$\Gamma_g = \Gamma_g(T), \ dQ = dQ(T), \ dE = dE(T). \tag{7.36}$$

Theorem 7.5 *For the differential*

$$dS(E,T) = Q(E,T)dE \tag{7.37}$$

the following equalities

$$\partial_A S(E,T) = \Omega_A(E,T), \tag{7.38}$$

are fulfilled.

Remark. From the definition of times (7.31,7.32,7.33) it follows that

$$dS = \sum_{\alpha=1}^{N}\sum_{i=0}^{\infty} t_{\alpha,i}d\Omega_{\alpha,i} + \sum_{k=1}^{g} t_{h,k}d\Omega_{h,k} + t_{E,k}\Omega_{E,k}. \tag{7.39}$$

We shall give here a brief sketch of the proof (7.38) for $A=(Q,k)$, only, because for all other A the proof is essentially the same as for the proof of the theorem

2.2. Consider the derivative $\partial_{Q,k}S(E,T)$. From the definition (7.37) it follows that $\partial_{Q,k}S(E,T)$ is holomorphic everywhere except for the cycle a_k. On different sides of these cycle the coordinates are E^- and $E^+ = E^- - t_{Q,k}$. Hence, taking the derivative of the equality

$$Q(E^- - t_{Q,k}) - Q(E^-) = t_{E,k}, \;\; E^- \in a_k, \tag{7.40}$$

we obtain

$$\partial_{Q,k}S^+ - \partial_{Q,k}S^- = \frac{dQ}{dE}. \tag{7.41}$$

Therefore, $\partial_{Q,k}S(E,T) = \Omega_{Q,k}$.

Corollary 7.1 *The integrals $E(p,T)$ and $Q(p,T)$ as functions of the variable $p = \Omega_{1,1}$ satisfy the Whitham equations (2.50) and the classical string equation (2.63).*

(In both theorems a set of times t_A includes all the times $t_{\alpha,i}, t_{h,i}, t_{E,i}, t_{Q,i}$.

3. τ-function. The τ-function of the particular solution of the Whitham equation on $\widetilde{\mathcal{N}}_g(n_\alpha)$ that was constructed above are defined by the formula

$$\ln \tau(T) = F(T),$$

$$F = F_0(T) + \frac{1}{4\pi i}\sum_{k=1}^{g} \oint_{a_k^-} t_{E,k} E dS - \oint_{b_k} t_{h,k} dS + t_{h,k} t_{E,k} E_k, \tag{7.42}$$

where $F_0(T)$ is given by (3.3), i.e.

$$F_0 = \frac{1}{2}\sum_{\alpha=1}^{N} \operatorname{res}_\alpha \left(\sum_{i=1}^{\infty} t_{\alpha,i} k_\alpha^i dS(p,T)\right) + t_{\alpha,0} s_\alpha(T),$$

(the first integral in (7.42) is taken over the left side of the a_k cycle and $E_k = E(P_k)$ where P_k is the intersection point of a_k and b_k cycles).

Remark. The differential dS is discontinuous. Therefore, its integral over b_k-cycle depends on the choice of the cycle. The last term in (7.42) restors an invariance (i.e. F depends on the homology class of cycles, only).

Theorem 7.6 *For the above-defined τ-function the equalities (3.8, 3.9) are fulfilled. Besides this,*

$$\partial_{h,k}F = \frac{1}{2\pi i}\left(t_{E,k}E_k - \oint_{b_k} dS\right), \tag{7.43}$$

$$\partial_{E,k}F = \frac{1}{2\pi i}\left(\oint_{a_k} E dS\right), \tag{7.44}$$

$$\partial_{Q,k}F = \frac{1}{4\pi i}\left(\oint_{a_k} Q dS - 2t_{E,k}t_{h,k}\right). \tag{7.45}$$

The proof of all these equalities is analogous to the proof of (3.8, 3.9) and use different types of identities that can be proved with the help of usual considerations of contour integrals.

Corollary 7.2 *For $A = (\alpha, i)$ the second derivatives $\partial^2_{A,B}F$ are given by the formulae (3.16,3.17). Besides this,*

$$\partial^2_{(h,k);A}F = \frac{1}{2\pi i}(E_k\delta_{(E,k);A} + Q_k\delta_{(Q,k);A} - \oint_{b_k} d\Omega_A), \tag{7.46}$$

$$\partial^2_{(E,k);A} = \frac{1}{2\pi i}(\oint_{a_k} Ed\Omega_A), \tag{7.47}$$

$$\partial^2_{(Q,k);A}F = \frac{1}{2\pi i}(\oint_{a_k} Qd\Omega_A - \partial_A(t_{E,k}t_{h,k})). \tag{7.48}$$

We would like to mention that in particular the formula (7.46) gives a matrix of b-periods of normalized holomorphic differentials on Γ_g

$$\partial^2_{(h,i);(h,j)}F = -\oint_{b_i} d\Omega_{h,j}. \tag{7.49}$$

(for the particular case this relation for the first time was obtained in [11]).

Theorem 7.7 *The third derivatives of F(T) are equal to*

$$\partial^3_{ABC}F = \sum_{q_s} \operatorname{res}_{q_s}(\frac{d\Omega_A d\Omega_B d\Omega_C}{dQ\ dE}) + \eta_{ABC}, \tag{7.50}$$

where

$$\eta_{ABC} = 0 \quad \text{if} \quad A, B, C \neq (Q, k),$$
$$\eta_{AB(Q,k)} = \frac{1}{2\pi i}\oint_{a_k} \frac{d\Omega_A d\Omega_B}{dE} \quad \text{if} \quad A, B \neq Q, k).$$

4. Virasoro constraints. In this subsection we present "$L_0,\ L_{-1}$" constraints for the τ-function of the homogenious solution of the Whitham hierarchy on $\widetilde{\mathcal{N}}_g(n_\alpha)$.

Consider the differential Q^2dE. It is holomorphic on Γ_g outside the punctures and cycles a_k where it has jumps

$$(Q^2dE)^+ - (Q^2dE)^- = 2t_{E,k}QdE = 2t_{E,k}dS. \tag{7.51}$$

Therefore,

$$\sum_{\alpha=1}^{N} \operatorname{res}_\alpha(Q^2dE) + \frac{1}{\pi i}\sum_{k=1}^{g} t_{E,k}t_{h,k} = 0. \tag{7.52}$$

The expansion of Q near the puncture P_α has a form (4.7). Its substitution into (7.52) gives

$$\sum_{\alpha=1}^{N}\frac{1}{n_\alpha}(\sum_{i=n_\alpha+1}^{\infty} it_{\alpha,i}\partial_{\alpha,i-n_\alpha}F + n_\alpha t_{\alpha,0}t_{\alpha,n_\alpha} + \frac{1}{2}\sum_{j=1}^{n_\alpha-1} j(n_\alpha - j)t_{\alpha,j}t_{\alpha,n_\alpha-j})$$

$$+\frac{1}{2\pi i}\sum_{k=1}^{g} t_{E,k}t_{h,k} = 0. \tag{7.53}$$

In the same way the consideration of the differential Q^2EdE proves an analogue of L_{-1} constraint:

$$\sum_{\alpha=1}^{N}\frac{1}{n_\alpha}\sum_{i=1}^{\infty} it_{\alpha,i}\partial_{\alpha,i}F + \frac{1}{2\pi i}\sum_{k=1}^{g} t_{E,k}\partial_{E,k}F + \frac{1}{2}t_{\alpha,0}^2 = 0; \tag{7.54}$$

Remark. In order to obtain higher "$L_{n>0}$" Virasoro constraints one has introduce p-gravitational decendants of the "fields" $d\Omega_{E,k}$ that are holomorphic differentials on Γ except

5. Landau-Ginzburg type models on Riemann surfaces In this subsection we present a generalisation of the results of section 5 for the case of Riemann surfaces of an arbitrary genus. Let us consider a genus g Riemann surface Γ_g with fixed canonical basis of cycles and with fixed meromorphic normalized differential dE, i.e. a point of the moduli space $\mathcal{N}_g(n_\alpha)$. The same formulae (5.19,5.20), as in genus zero case, define a Fröbenius algebra $\mathcal{R}_{\Gamma_g,dE}$

$$\mathcal{R}_{\Gamma_g,dE} = \hat{\mathcal{R}}/(dE=0), \tag{7.55}$$

where $\hat{\mathcal{R}}$ is a ring of all meromorphic functions that are regular at the zeros q_s of the differential dE. The formula

$$\langle f,g\rangle = \sum_{q_s}\text{res}_{q_s}(\frac{f(p)g(p)}{E_p}dp),\ \ f(p),\ \ g(p)\subset\hat{\mathcal{R}}, \tag{7.56}$$

defines a non-degenerate scalar product on $\mathcal{R}_{\Gamma_g,dE}$.

For any g a "small phase" space is a space of times t_a with indecies $a\in\mathcal{A}_{sm}^g$, where $\mathcal{A}_{sm}^g$ is a union of $\mathcal{A}_{sm}$ (that was defined in section 5)

$$\mathcal{A}_{sm} = \{a=(\alpha,i)|\alpha=1,\ i=1,\ldots,n-1;\ \alpha=2,\ldots,N,\ i=0,\ldots,n_\alpha\}$$

and indecies $(h,k),(E,k)$. In the basis

$$\phi_a = \frac{d\Omega_a}{dp} \tag{7.57}$$

the scalar product has the form:

$$\langle\phi_a\phi_b\rangle = \eta_{ab} = \frac{[i][j]}{n_\alpha}\delta_{\alpha,\beta}\delta_{i+j,n_\alpha},\quad a,b\in\mathcal{A}_{sm} \tag{7.58}$$

$$\langle\phi_{E,k}\phi_{h,s}\rangle = \delta_{k,s} \tag{7.59}$$

otherwise zero (here $[i]$ is the same as in (4.14).

Let us consider the Whitham "times" t_a that were defined in (7.31,7.32,7.33) for the choice $dQ = dp$, i.e.

$$
\begin{aligned}
t_{\alpha,i} &= \frac{1}{i}\mathrm{res}_\alpha(k_\alpha^{-i}(p)pdE(p))), \quad \alpha, i > 0 \in \mathcal{A}_{sm}; \\
t_{\alpha,0} &= \mathrm{res}_\alpha(pdE(p)); && (7.60)\\
t_{h,k} &= \oint_{a_k} pdE, \quad , k = 1, \ldots, g; && (7.61)\\
t_{p,i} &= -\oint_{b_k} dE, \quad t_{E,k} = \oint_{b_k} dp, \quad k = 1, \ldots, g. && (7.62)
\end{aligned}
$$

Theorem 7.8 *A Jacobian of the map*

$$
\mathcal{N}_g(n_\alpha) \longmapsto \{t_a, \ a \in \mathcal{A}_{sm}^g\} \tag{7.63}
$$

is nonvanishing anywhere (i.e. the map (7.63) is a non-remified covering of some domain of the complex space with coordinates t_a).

.

Let us fix the values $t_{p,k} = t_{p,k}^0$ and consider the restriction of $F = \ln\tau$ on the affine space that is $\mathcal{A}_{sm}^g$ shifting by

$$
t_{1,n+1} = \frac{n}{n+1}, \quad t_{p,k} = t_{p,k}^0.
$$

Then from the statement of the theorem 7.7 it follows that

$$
\partial_{abc}^3 F = \sum_{q_s} \mathrm{res}_{q_s}(\frac{d\Omega_a d\Omega_b d\Omega_c}{dp\ dE}), \ a, b, c \in \mathcal{A}_{sm}^g. \tag{7.64}
$$

Corollary 7.3 *The dependence of Fröbenius algebrae corresponding to $\mathcal{N}_g(n_\alpha)$ with respect to the coordinates t_a, $\ a \in \mathcal{A}_{sm}^g$ is a potential deformation.*

In [11] the particular case of this statement was proved. It corresponds to the Whitham hierarchy on moduli space of genus g curves with fixed function $E(P)$ having a pole of order n at a point P_1. (This moduli space is a subspace of $\mathcal{N}_g(n)$ that is specified by the conditions $t_{p,k} = 0$.) The differential-geometrical interpretation of Whitham coordinates that was proposed in [11] is valid in general case as well.

Let us denote a subspace of $\mathcal{N}_g(n_\alpha)$ corresponding to fixed values of $t_{p,k} = t_{p,k}^0$ by $\mathcal{N}_g(n_\alpha|t_{p,k}^0)$. A system of local coordinates on its open submanifold $\mathcal{D}$ is given by (7.20), i.e.

$$
E_s = E(q_s), \quad \text{where} \ \ dE(q_s) = 0, \quad s = 1, \ldots, D - g = 2g - 2 + \sum_{\alpha=1}^{N}(n_\alpha + 1).
$$

Submanifold $\mathcal{D}$ can be defined as a submanifold on which the values E_s are distinct. The formula

$$ds^2 = \sum_{s=1}^{D-g} \text{res}_{q_s} (\frac{dpdp}{dE})(dE_s)^2 \tag{7.65}$$

defines a metric on $\mathcal{D} \subset \mathcal{N}_g(n_\alpha|t^0_{p,k})$. The scalar products of the vector-fields $\partial_a = \frac{\partial}{\partial t_a}$ with respect to the metric (7.65) have the form:

$$\langle \partial_a \partial_b \rangle = \eta_{ab} = \frac{[i][j]}{n_\alpha} \delta_{\alpha,\beta} \delta_{i+j,n_\alpha}, \quad a, b \in \mathcal{A}_{sm} \tag{7.66}$$

$$\langle \partial_{E,k} \partial_{h,s} \rangle = \delta_{k,s}, \tag{7.67}$$

othewise zero. The proof of (7.66,7.67) is based on the formula (5.11)

$$\partial_A E_s = \frac{d\Omega_A}{dp}. \tag{7.68}$$

and formulae (7.58,7.59). Consequently, in the Whitham coordinates $t_a,\ a \in \mathcal{A}^g_{sm}$ the metric (7.65) has a constant coefficients (i.e. ds^2 is a flat metric and t_a are flat coordinates). In [11] the formulae (7.66,7.67) and the following two main arguments were used for the proof that the functions $\{t_a, a \in \mathcal{A}_{sm}, t_{E,k}, t_{h,k}\}$ define a system of coordinates everywhere on the moduli space of genus g curves with fixed function $E(P)$ having a pole of order n at a point P_1. First of all, the functions $t_a, a \in \mathcal{A}_{sm}, t_{E,k}, t_{h,k}$ are holomorphic functions on $\mathcal{N}_g(n_\alpha|t^0_{p,k})$. The second argument that had been used is: the "dual" metric

$$d\hat{s}^2 = \sum_{s=1}^{D-g} \text{res}_{q_s} (\frac{dpdp}{dE})^{-1} (\frac{\partial}{\partial E_s})^2 \tag{7.69}$$

on the cotangent bundle can be extended as a *smooth* on the whole moduli space. This very powerfull statement is a particular case of general results that were obtained by Novikov and Dubrovin in the framework of their Hamiltonian approach for the Whitham theory [12]. We would like to mention that the last argument can be replaced by the corollary of the theorem 7.1, because the Sato-form (7.9,7.10,7.11) shows that *vector-fields* ∂_a are *smooth* on the whole moduli space.

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