VECTOR ADDITION THEOREMS AND BAKER—AKHIEZER FUNCTIONS

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Functional equations that arise naturally in various problems of modern mathematical physics are discussed. We introduce the concepts of an N-dimensional addition theorem for functions of a scalar argument and Cauchy equations of rank N for a function of a g-dimensional argument that generalize the classical functional Cauchy equation. It is shown that for $N=2$ the general analytic solution of these equations is determined by the Baker—Akhiezer function of an algebraic curve of genus 2. It is also shown that θ functions give solutions of a Cauchy equation of rank N for functions of a g-dimensional argument with $N \leq 2^g$ in the case of a general g-dimensional Abelian variety and $N \leq g$ in the case of a Jacobian variety of an algebra curve of genus g. It is conjectured that a functional Cauchy equation of rank g for a function of a g-dimensional argument is characteristic for θ functions of a Jacobian variety of an algebraic curve of genus g, i.e., solves the Riemann—Schottky problem.

In memory of M. K. Polivanov

1. INTRODUCTION

The classical functional Cauchy equation [1]

$$\psi(x+y) = \psi(x)\psi(y), \tag{1.1}$$

which arises in numerous problems, completely characterizes the exponential

$$\psi(x) = \exp(kx), \tag{1.2}$$

where k is a parameter. Equation (1.1) is one of the examples of so-called addition theorems:

$$F(f(x), f(y), f(x+y)) = 0. \tag{1.3}$$

The number of such examples is small. Indeed, in accordance with Weierstrass's theorem, if F is a polynomial of three variables, then in the class of analytic functions $f(x)$ only elliptic functions (i.e., functions associated with algebraic curves of genus $g=1$) and their degenerate forms possess an addition theorem.

By a vector addition theorem we shall understand an equation of the form

$$F(f(x), \phi(y), \psi(x+y)) = 0, \tag{1.4}$$

where $f(x)=(f_1(x), \ldots, f_N(x))$, $\phi(y)=(\phi_1(y), \ldots, \phi_N(y))$, $\psi(z)=(\psi_1(z), \ldots, \psi_N(z))$ are vector functions, and F is a function of $3N$ variables.

It should be noted that forms of such theorems can already be found in Abel's classical study [2], in which the following problem is considered:

To find three functions ϕ, f, and ψ that satisfy the equation

$$\psi(\alpha(x,y)) = F\big(x, y, \phi(x), \phi'(x), \ldots, f(y), f'(y), \ldots\big), \tag{1.5}$$

where α and F are given functions of an appropriate number of variables. In particular, in [2] this problem is solved for the equation

$$\psi(x+y) = \phi(x)f'(y) + f(y)\phi'(x). \tag{1.6}$$

Vector addition theorems important for modern applications were given by Frobenius and Stikelberger [3], who, for example, showed that the Weierstrass zeta function satisfies the functional equation

$$\big(\zeta(x)+\zeta(y)+\zeta(z)\big)^2 + \zeta'(x) + \zeta'(y) + \zeta'(z) = 0, \tag{1.7}$$

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where $x+y+z=0$.

The main aim of the present paper is to prove vector addition theorems characterizing so-called Baker—Akhiezer functions, which are defined on algebraic curves of arbitrary genus. Special cases of such functions were introduced at the end of the last century in the Clebsch—Gordan studies as a natural generalization of the exponential concept to the case of Riemann surfaces of arbitrary genus. Baker [4] noted the connection between such functions and the problem of classifying commuting ordinary linear differential operators [5]. Subsequently, the remarkable but, unfortunately, forgotten results of these studies were rediscovered and significantly developed in the theory of integrable equations of the type of the Korteweg—de Vries equation. A general definition of Baker—Akhiezer functions (many-point and depending on many variables) was given by one of the present authors [6,7]. The point of departure of [6,7] was the result of Novikov, Dubrovin, Matveev, and Its relating to the construction of periodic and quasiperiodic solutions of the Korteweg—de Vries, nonlinear Schrödinger, and sine—Gordon equations (for a review of these results, see [8,9]). Beginning with [6,7], the concept of Baker—Akhiezer functions became the foundation in the theory of algebro-geometric or finite-gap integration (the further development of which is presented in the reviews [10,11,12,13]).

As will be shown in the third section of the present paper, Baker—Akhiezer functions satisfy a functional equation that is a "vector analog" of the Cauchy equation (1.1). Before we give this, we transform Eq. (1.1).

The Cauchy equation is "rigid." The class of functions defined by it hardly changes if one weakens (1.1) and considers the equation

$$\phi(x+y)=\psi(x)\psi(y). \tag{1.8}$$

It follows from (1.8) that

$$\psi(x)=\exp\big(k(x+x_0)\big),\ \phi(x)=\exp\big(k(x+2x_0)\big). \tag{1.9}$$

Denoting the function $\phi^{-1}(x)$ by $c(x)$, we can represent Eq. (1.8) in the form

$$c(x+y)\psi(x)\psi(y)=1. \tag{1.10}$$

We shall call the functional equation

$$\sum_{k=0}^{N} c_k(x+y)\psi_k(x)\psi_k(y)=1 \tag{1.11}$$

on the vector functions $c(x)=(c_0(x), \ldots, c_N(x))$, $\psi(x)=(\psi_0(x), \ldots, \psi_N(x))$ the vector analog of Eq. (1.1).

Remark. Note that although formally all the functions c_k and ψ_k are unknowns in Eq. (1.11) the functions c_k can, as is readily seen, be explicitly expressed in terms of the functions ψ_k [see Eq. (3.12)]. Therefore, in what follows we shall, without loss of generality, call the vector function $\psi=(\psi_1, \ldots, \psi_N)$ the solution of Eq. (1.11).

Equation (1.11) arose as a result of attempts to generalize the addition formula for Baker—Akhiezer functions of genus 1. As was noted in [14], a special case of such functions gives solutions of the equation

$$\Phi(x+y)=\frac{\Phi'(x)\Phi(y)-\Phi(x)\Phi'(y)}{V(x)-V(y)}, \quad \Phi(x)\Phi(-x)=V(x)+\mathrm{const}. \tag{1.12}$$

The system (1.12) was first proposed in [15] in connection with the problem of constructing a Lax representation for the equations of motion of a system of pairwise interacting particles with two-body interaction potential given by the function $V(x)$. In [15] particular solutions of (1.12) were found for the case $V(x)=\wp(x)$ (where $\wp$ is the Weierstrass function). The solutions of (1.12) proposed in [14] made it possible to introduce in the Lax representation a "spectral parameter" and, as a consequence, construct theta-function formulas for the dynamics of the Moser—Calogero system. It was subsequently shown in [17] and [27] that the solutions constructed in [14] exhaust all solutions of this equation. The idea of reducing the problem of integrating dynamical systems to functional equations proved to be extremely fruitful. We mention [15—19], in which this idea led to new results both in the theory of dynamical systems and in the theory of functional equations.

Another nontrivial field of application of functional equations is the branch of the theory of algebraic topology associated with Hirzebruch genera. The separation of classical genera in terms of functional equations was already considered in Hirzebruch's pioneering study [20]. In [21], the functional equation (1.12) was used for the proof proposed there of the "rigidity" property of elliptic genera. (Elliptic genera were introduced by Ochanine [22]; the hypothesis of their "rigidity" was also put forward by Witten [23] and proved by Taubes [24,25].)

In [26] a universal solution of Abel's equation (1.6) was found and used to construct a cohomology theory corresponding to a general arithmetic genus ([20]). It was shown in [27] that the functional equation (1.12) is equivalent to the functional equation

$$\phi(u+v)=\frac{\phi(u)^2\xi_1(v)-\phi(v)^2\xi_1(u)}{\phi(u)\xi_2(v)-\phi(v)\xi_2(u)}. \tag{1.13}$$

As a consequence there were obtained the structures of an algebraic two-valued group on the Riemann sphere, and it was shown that the moduli space of such structures is a space of nondegenerate elliptic curves with marked points.

In Sec. 3, it will be shown that Baker—Akhiezer functions corresponding to algebraic curves of genus g give solutions of Eq. (1.11) for $N=g$. The explicit expressions for these functions in terms of theta functions of Riemann surfaces [7] (see Sec. 4) show that the corresponding solutions of Eq. (1.11) have, up to an exponential factor, the special form

$$\psi_k(x)=\frac{\Psi(Ux+A_k)}{\Psi(Ux+A_*)}, \tag{1.14}$$

where $\Psi(z_1, \ldots, z_g)$ is a function of a vector argument that is a theta function of an algebraic curve; $U=(U_1, \ldots, U_g)$, $A_k=(A_{k,1}, \ldots, A_{k,g})$ are g-dimensional vectors, $k=0, \ldots, N$ or $*$.

This makes it possible to introduce the concept of a "functional Cauchy equation of rank N" as a vector addition theorem of the following special form.

Definition. We shall say that a function $\Psi(z)$ of g-dimensional vector argument $z=(z_1, \ldots, z_g)$ is a solution of a Cauchy equation of rank N if there exist g-dimensional vectors U and A_*, A_0, A_1, $\ldots$, A_N such that

$$\sum_{k=0}^{N} c_k(x+y)\Psi(Ux+A_k)\Psi(Uy+A_k)=\Psi(Ux+A_*)\Psi(Uy+A_*). \tag{1.15}$$

Important Remark—Definition. In the general case, one and the same function $\Psi(z)$ can solve functional Cauchy equations of different ranks (by virtue of special choices of the vectors U, A_1, $\ldots$, A_N). We shall define the rank of such a function $\Psi(z)$ as the minimum N_* among the possible ranks N of the equations (1.5) that it satisfies, rk $\Psi=N_*$.

In Sec. 3 it will be shown that *all* solutions of Eq. (1.11) for the case $N=2$ are given by Baker—Akhiezer functions of genus 2. This shows that the vector analog of the Cauchy equation (1.11) for $N=2$ is equivalent to a functional Cauchy equation of rank 2.

In this connection, it is natural to pose the two following questions:

1. Are the functional equations (1.11) and (1.15) equivalent for $N>2$?

2. Is every solution of a Cauchy equation of rank N given by theta functions of Riemann surfaces?

As is shown in the final section of the paper, the answer to a rough form of the second question is negative. It turns out that the theta functions corresponding to a general Abelian variety of dimension g give solutions of a functional Cauchy equation of rank $N=2^g$.

Therefore, theta functions of a general g-dimensional Abelian variety determine functions of rank $\leq 2^g$. Taken together, the results of Secs. 3 and 4 show that in the case of Jacobian curves the rank of the corresponding functions does not exceed g. This enables us to formulate the following conjecture:

A theta function has a rank not exceeding g if and only if it is constructed from a matrix of b periods of holomorphic differentials on a Riemann surface of genus g.

This conjecture can be reformulated as follows:

Equation (1.15) *with $N=g$ for a function of g-dimensional argument is characteristic for the theta functions of Jacobian varieties of algebraic curves, i.e., it solves the Riemann—Schottky problem.*

Support for this conjecture is provided by the connection noted above between addition theorems and the theory of integrable systems and the proof in [28] of Novikov's hypothesis that the theta-function formulas for solutions of the Kadomtsev—Petviashvili equation are characteristic for Jacobian varieties, i.e., solve the Riemann—Schottky problem.

2. PRELIMINARIES

Let Γ be a nonsingular algebraic curve of genus g with marked points and local coordinates $k_\alpha^{-1}(Q)$, $k_\alpha^{-1}(P_\alpha)=0$, $\alpha=1, \ldots, l$. We fix a set of polynomials $q_\alpha(k)$. As was shown in [6,7], for any generic set of points $\gamma_1, \ldots, \gamma_g$ there exists a unique (up to proportionality) function $\psi(x, Q)$ which

1) is meromorphic on Γ outside the points P_α and has not more than simple poles at the points γ_s (if they are all distinct) and

2) in the neighborhood of the point P_α has the form

$$\psi(x,Q) = \exp\left(xq_\alpha(k_\alpha)\right)\left(\sum_{s=0}^{\infty}\xi_{s,\alpha}(x)k^{-s}\right), \qquad k_\alpha = k_\alpha(Q). \tag{2.1}$$

Choosing a point P_0, we normalize ψ by

$$\psi(x, P_0) = 1. \tag{2.2}$$

Remark. The Baker—Akhiezer function $\psi(x, Q)$ is determined by its algebraic properties with respect to the variable Q. It depends on the variable x and on the coefficients of the polynomials q_α as on external parameters. In ψ we shall not indicate its dependence on q_α, retaining only the dependence on x, which is important in what follows.

In [7], an explicit theta-function formula for ψ was proposed. Let a_i, b_i be a basis of cycles on Γ with canonical matrix of intersections: $a_i \circ a_j = b_i \circ b_j = 0$, $a_i \circ b_j = \delta_{ij}$. We define a basis of holomorphic differentials ω_i on Γ, normalized by the conditions

$$\oint_{a_i} \omega_j = \delta_{ij}. \tag{2.3}$$

The matrix

$$B_{ij} = \oint_{b_j} \omega_i \tag{2.4}$$

is called the matrix of b periods of the curve Γ. It is symmetric and has positive-definite imaginary part. Any such matrix defines an entire function of g variables (called a Riemann theta function) in accordance with

$$\theta(z_1, \ldots, z_g) = \sum_{m \subset Z^g} \exp\left(2\pi i(z,m) + \pi i(Bm,m)\right) \tag{2.5}$$

[here $m=(m_1, \ldots, m_g)$ is an integer vector]. A Riemann theta function possesses the following translational properties:

$$\theta(z+e_k) = \theta(z), \qquad \theta(z+B_k) = \theta(z)\exp\left(-\pi i B_{kk} - 2\pi i z_k\right) \tag{2.6}$$

(e_k and B_k are the vectors with coordinates $\{\delta_{ki}\}$ and $\{B_{ki}\}$). The vectors e_k and B_k generate a lattice in C^g, the factor with respect to which is a g-dimensional torus $J(\Gamma)$ called the Jacobian of the curve Γ. The Abel mapping is the mapping

$$A: \Gamma \to J(\Gamma) \tag{2.7}$$

defined by

$$A_k(Q) = \int_{P_0}^{Q} \omega_k. \tag{2.8}$$

If we define the vector Z:

$$Z = K - \sum_{s=1}^{g} A(\gamma_s) \tag{2.9}$$

(where K is the vector of Riemann constants, which depend on the choice of the basis cycles and the initial point P_0), then the function $\theta(A(Q)+z)$ has precisely g zeros on Γ that coincide with the points γ_s,

$$\theta(A(\gamma_s) + Z) = 0. \tag{2.10}$$

Note that the function $\theta(A(Q)+Z)$ itself is multiply valued on Γ, but, as follows from (2.6), its zeros are correctly defined. We introduce normalized differentials $d\Omega_\alpha$ such that

1) the differential $d\Omega_\alpha$ is holomorphic outside P_α, at which it has a pole of the form

$$d\Omega_\alpha = dq(k_\alpha) + O(k_\alpha^{-1}); \tag{2.11}$$

2)

$$\oint_{Q_\alpha} d\Omega_\alpha = 0.$$

Conditions 1 and 2 define $d\Omega_\alpha$ uniquely. We denote by $2\pi i U_\alpha$ the vector of its b periods:

$$2\pi i U_{\alpha,k} = \oint_{b_k} d\Omega_\alpha. \tag{2.12}$$

As was shown in [7], the Baker—Akhiezer function has the form

$$\psi(x, Q) = \exp\left(x \int_{P_0}^{Q} d\Omega_\alpha\right) \cdot \phi(x, Q), \tag{2.13}$$

where

$$\phi(x, Q) = \frac{\theta(A(Q) + xU + Z)\theta(A(P_0) + Z)}{\theta(A(Q) + Z)\theta(A(P_0) + xU + Z)}, \tag{2.14}$$

$$U = \sum_{\alpha=1}^{l} U_\alpha. \tag{2.15}$$

To conclude this section, we give one further proposition, which will be needed in what follows. For any positive divisor $D=\Sigma n_i Q_i$, we consider the linear space $L(x, D)$ of functions that outside the points P_α have poles at the points Q_i of multiplicity not higher than n_i and in the neighborhood of P_α have the form (2.1). For divisors of general position of degree $d=\Sigma n_i \geq g$ the dimension of this space is

$$\dim L(x, D) = d - g + 1. \tag{2.16}$$

3. ADDITION FORMULAS

For any set of non-negative integers $S=\{n_0 < n_1 < \ldots < n_g\}$ and set of functions $\{f_0, \ldots, f_g\}$ we define the "generalized Wronskians"

$$W_S(f_0, \ldots, f_g) = \det M_S, \tag{3.1}$$

where the elements of the matrix M_S are

$$M_S^{i,j} = \partial_-^{n_i}(f_j(x)f_j(y)), \tag{3.2}$$

$$\partial_- = \partial/\partial x - \partial/\partial y. \tag{3.3}$$

The main result of this section is the following theorem.

THEOREM 3.1. *For any set of $g+1$ points $Q_0, \ldots, Q_g$ of generic position on Γ the functions*

$$\psi_k(x) = \psi(x, Q_k) \tag{3.4}$$

[*where $\psi(x, Q)$ is a Baker—Akhiezer function*] *satisfy the equations*

$$W_S(\psi_0, \psi_1, \ldots, \psi_g) = 0 \tag{3.5}$$

for sets S that do not contain zero (i.e., if $S=\{0 < n_0 < n_1 \ldots < n_g\}$).

Proof. We consider the function $W_S(x, y, Q_0)=W_S(\psi_0, \psi_1, \ldots, \psi_g)$ as a function of the variable Q_0, i.e., we fix all $Q_1, \ldots, Q_g$ and vary the point Q_0. It follows from the definition of a Baker—Akhiezer function that outside the points P_α the function $W_S(x, y, Q_0)$ is meromorphic and has poles of order not higher than the second at the points $\gamma_1, \ldots, \gamma_g$. In the neighborhood of the point P_α, it has the form

$$W_S(x, y, Q_0) = \exp\left((x+y)q_\alpha(k_\alpha)\right)\left(\sum_{n=0}^{\infty} W_{S,n}(x, y)k^{-n}\right). \tag{3.6}$$

Therefore

$$W_S(x, y, Q_0) \in L(x+y, D = 2\gamma_1 + \cdots + 2\gamma_g). \tag{3.7}$$

It follows from (2.16) that for fixed x and y the dimension of the space of such functions is $g+1$.

The function $W_S(x, y, Q_0)$ vanishes at the points Q_k:

$$W_S(x, y, Q_0 = Q_k) = 0. \tag{3.8}$$

In addition, if $n_0 > 0$, then from the normalization conditions (2.2) we have

$$W_S(x, y, P_0) = 0. \tag{3.9}$$

The vanishing of $W_S(x, y, Q_0)$ at $g+1$ fixed points of generic position implies that the function is equal to zero everywhere. This proves the theorem.

Corollary. *The functions $\psi_k(x)$ satisfy the generalized Cauchy equation* (1.11).

Proof. Let $S=(1, 2, \ldots, g+1)$. By the theorem, the functions $\partial_-\psi_0(x)\psi_0(y), \ldots, \partial_-\psi_g(x)\psi_g(y)$ are linearly dependent with coefficients constant with respect to the operator ∂_-, i.e., there exist functions $b_k(x)$ such that

$$\sum_{k=0}^{g} b_k(x+y)\partial_-\psi_k(x)\psi_k(y) = 0. \tag{3.10}$$

Therefore

$$\sum_{k=0}^{g} b_k(x+y)\psi_k\psi_k(y) = b_*(x+y). \tag{3.11}$$

Denoting $c_k = b_k/b_*$, we obtain (1.11).

Remark. Equation (1.11) not only follows from Eqs. (3.5) but is equivalent to them.

Indeed, applying to Eq. (1.11) the operators $\partial_-^{n_0}, \ldots, \partial_-^{n_g}$, we obtain a system of linear equations for $c_k(x+y)$. The existence of a solution of this system has the consequence that the determinant of the matrix of coefficients is equal to zero, i.e., Eq. (3.5) holds. At the same time

$$c_k(x+y) = \frac{\det M_k(x, y)}{\det M(x, y)}, \tag{3.12}$$

where the elements $M_{i,j}$, $i, j \leq 0, \ldots, g-1$, of the matrix M are

$$M_{i,j} = \partial_-^i \psi_j(x)\psi_j(y), \tag{3.13}$$

and the matrix M_k is obtained from M by replacing column k by the vector $(1, 0, \ldots, 0)^T$.

4. EXPLICIT EXPRESSIONS AND THE NUMBER OF PARAMETERS

The results of the previous section, in conjunction with formula (2.13), enable us to give explicit expressions for the solutions of the vector analog of the Cauchy equation.

We consider the function $\phi(x, Q)$ given by (2.13), in which the vectors U and Z are arbitrary; then the complete family of algebraic solutions of the functional equation (1.11) constructed above is given by

$$\psi_k = \phi(x, Q_k)\exp(a_k x), \tag{4.1}$$

where a_k are arbitrary constants. The fact that the vector Z is arbitrary follows from the possibility of varying the divisor of the poles $\gamma_1, \ldots, \gamma_g$. The arbitrariness of the vector U and the constants a_k is related to the arbitrariness in the choice of the polynomials q_α.

Important Remark. At the first glance, there is a further arbitrary set of parameters. The functions ψ_k can be multiplied by constants:

$$\psi_k \leftarrow \psi_k \exp(b_k). \tag{4.2}$$

However, one can show that such a transformation is equivalent to a shift of the vector Z, i.e., the complete family of algebro—geometric solutions can be represented in the form

$$\psi_k = \frac{\theta\big(A(Q_k) + Ux\big)}{\theta\big(A(P_0) + Ux\big)} \cdot \exp\big(a_k x + b_k\big), \qquad k = 0, \ldots, g. \tag{4.3}$$

The dimension of the moduli space of the curves of genus $g > 1$ is $3g-3$. Therefore, the total number of parameters (curve+vector U+constants a_k, b_k+points P_0, Q_0, $\ldots$, Q_g) is

$$R_- = (3g-3) + g + 2(g+1) + 1 + (g+1) = 7g+1 \tag{4.4}$$

[for genus $g=1$, the number of parameters is given by the same formula (4.4)].

We show that in the case of genus $g=2$ formula (4.3) gives general solutions of Eq. (1.11) for $N=2$.

As we have already said above, Eqs. (3.5) follow from (1.11). We consider these equations for the sets $S_1=(2, 3, \ldots, g+2)$, $S_2=(1, 3, \ldots, g+2)$, $S_{g+1}=(1, 2, 3, \ldots, g-1, g, g+2)$, and in them we then set $y=0$. The corresponding equations

$$W_{S_i}(\psi_0, \ldots, \psi_g)\big|_{y=0} = 0 \tag{4.5}$$

give a system of $g+1$ equations of degree $g+2$ for the unknown function $\psi_0, \ldots, \psi_g$. The coefficients of the system depend as on parameters on the values of the derivatives $\partial_x^j \psi_i(0)$, $j=0, \ldots, g+2$, which also give the initial data for the required solutions. Therefore, the number of parameters R_C on which the general solution of the generalized Cauchy equation can depend does not exceed

$$R_C \leq R_+ = (g+1)(g+3).$$

For $g=2$,

$$R_- = (7g+1)|_{g=2} = R_+ = (g+1)(g+3)|_{g=2} = 15,$$

and this proves the generality of the solutions constructed above.

5. ADDITION FORMULAS FOR GENERAL THETA FUNCTIONS

As we have shown in the previous sections, the formulas

$$\psi_k(x) = \frac{\theta(A_k + Ux)}{\theta(A_* + Ux)} \exp\left(a_k x + b_k\right) \tag{5.1}$$

give solutions of Eq. (1.11) with $N=g$ if the theta function $\theta(z)=\theta(z|B)$ is constructed from the matrix of b periods of normalized holomorphic differentials of a nonsingular algebraic curve Γ of genus g and the vectors A_k are the images under the Abel mapping of certain points of this curve $A_k=A(Q_k)$, $A_*=A(P_0)$ [i.e., $A_k \in \operatorname{Im} A$: $\Gamma \to J(\Gamma)$].

We now consider the case of a theta function (2.5) constructed using an arbitrary matrix B with positive-definite imaginary part. The function $\phi(x, A)$ defined by

$$\phi(x, A) = \frac{\theta(A + Ux)\theta(A_*)}{\theta(A)\theta(A_* + Ux)} \tag{5.2}$$

is uniquely determined by the following analytic properties.

1. For fixed x, U, the function ϕ as a function of the variable A possesses the following translational properties:

$$\phi(x, A + e_k) = \phi(x, A), \tag{5.3}$$

$$\phi(x, A + B_k) = \phi(x, A) \cdot \exp\left(-U_k x\right). \tag{5.4}$$

2. The divisor of the poles $\phi(x, A)$ is identical to the theta divisor:

$$\theta(A) = 0. \tag{5.5}$$

3. The function $\phi(x, A)$ is normalized by the condition

$$\phi(x, A_*) \equiv 1. \tag{5.6}$$

THEOREM 5.1. *For any set $A_0, \ldots, A_N$, $N=2^g$, of points of generic position the generalized Wronskian $W_S(\phi_0, \ldots, \phi_N)$ for the functions $\phi_k(x)=\phi(x, A_k)$ is zero:*

$$W_S(\phi_0, \ldots, \phi_N) = 0 \tag{5.7}$$

for sets $S=(0<n_0<\ldots<n_N)$.

The proof of the theorem is almost a literal repetition of the proof of Theorem 3.1. We consider the analytic properties of $W_S(x, y, A_0)=W_S(\phi_0, \ldots, \phi_N)$ as functions of the variable A_0 for fixed $A_1, \ldots, A_N$. They satisfy the translational properties

$$W_S(x, y, A_0 + e_k) = W_S(x, y, A_0), \qquad W_S(x, y, A_0 + B_k) = \exp\left(-U_k(x + y)\right) W_S(x, y, A_0),$$

and have a twofold pole on the theta divisor. The dimension of the space of such functions is 2^g+1. On the other hand,

$$W_S(x, y, A_k) = 0, \qquad k = 1, \ldots, N = 2^g.$$

In addition, $W_S(x, y, A_*)=0$ by virtue of the normalization condition (5.6). Therefore, $W_S(x, y, A_0)$ is identically equal to zero.

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