ELLIPTIC SOLUTIONS OF THE KADOMTSEV–PETVIASHVILI EQUATION AND INTEGRABLE SYSTEMS OF PARTICLES

I. M. Krichever

UDC 517.9

The main purpose of this paper is to construct "action-angle" type variables for a system of particles with a pairwise interaction potential, whose Hamiltonian has the form

$$H=\frac{1}{2}\sum_{i=1}^{n} p_i^2-2\sum_{i<j}\wp(x_i-x_j), \tag{1}$$

where $\wp(x)$ is Weierstrass' $\wp$-function (see [1]). The sign of the potential, corresponding to an attractive system of particles, is chosen to simplify formulas used later on from which we can exclude the term $\sqrt{-1}$. The change of this sign occurs when we go over to imaginary time. The methods we develop allow an integration of the equations of motion of system (1) in terms of Riemann's θ-function.

It is known that the equations of motion of system (1) have the Lax-type representation (see [2]):

$$\dot{L}=[M,L], \tag{2}$$

where the matrices L and M depend on x_i and p_i. It follows from this representation that the qualities $J_k=\frac{1}{k}\operatorname{tr}L^k, k=1,\ldots,n$, are integrals of system (1). It is proved in [3] that they are independent and in involution. Then by Liouville's theorem, system (1) is completely integrable.

In [4] there was reported for the first time a remarkable connection between Hamiltonian systems of particles on a line and the dynamics of the poles of special solutions of nonlinear equations that can be integrated by a method of an inverse problem. For the Korteweg–de Vries equation it was shown that the dynamics of the poles of an elliptic solution $u(x,t)=2\sum_{i=1}^{N}\wp(x-x_i(t))$ is equivalent to restricting the equations of motion of a system with the Hamiltonian J_3 to fixed points of system (1), grad H = 0. It turns out that N has the form n(n+1)/2. A similar assertion was obtained for the Boussinesq equation. The restriction on the number of particles is connected with the need to consider equations of motion on stationary manifolds of Hamiltonian systems. The connection between Hamiltonian systems and elliptic solutions of nonlinear equations turns out to be a natural one in the case of two-dimensional systems. For example, in the case of the Kadomtsev–Petviashvili (K–P) equation:

$$\frac{3}{4}u_{yy}=\frac{\partial}{\partial x}\left\{u_t+\frac{1}{4}(6uu_x-u_{xxx})\right\}. \tag{3}$$

The elliptic solutions of this equation have the form u = const $+\ 2\sum\wp(x-x_i(y,t))$. Here the dynamics of the poles with respect to y and t is described by the commuting Hamiltonian flows corresponding to H and J_3, respectively. This assertion in the case of a degenerate Weierstrass $\wp$-function $-x^{-2}$ was obtained in [5] (see also [6]).

Subsequently, pole systems corresponding to different nonlinear equations were investigated by many workers (see the survey [7] and [8]).

The connection between completely integrable systems of particles on a line and the pole solutions of nonlinear equations can be used in two directions. For all degenerate Weierstrass functions corresponding to one or two infinite periods ($\wp(x)$ goes into $\sinh^{-2}(x)$ or x^{-2}, respectively), the integrals J_k are known, and moreover,

G. M. Krzhizhanovskii Moscow Energy Institute. Translated from Funktsional'nyi Analiz i Ego Prilozheniya, Vol. 14, No. 4, pp. 45–54, October–December, 1980. Original article submitted March 11, 1980.

 0016-2663/80/1404-0282$07.50

there is a mechanism of integrating the equations of motion. The coordinates $x_j(t)$ of particles turn out to be the eigenvalues of the matrix $x_j(0)\delta_{ij} - L(0)t$, where $L(0)$ depends on the initial coordinates and impulses (see [9] and [10]). Substituting these coordinates into the formula $u(x, y, t) = 2\sum_{i=1}^{n}(x - x_i(y, t))^{-2}$ allows one to obtain e.g., rational solutions of the K–P equation. On the other hand, having a construction for rational solutions of the K–P equation, one can obtain independently the integration of the equations of a system of particles with the potential x^{-2}. The main aim of [5] was precisely to achieve this second possibility; there we used the idea of "finite-zoned integration" to find exact formulas for the rational solutions of the K–P equation.

In contrast to the rational and trigonometric cases, for a system (1) with a nondegenerate Weierstrass $\wp$-function neither a construction of "angle" type variables corresponding to the involutive integrals J_k nor a more explicit integration of the equations of motion was known.

An exception is the solutions of the K–dV equation that are the sum of the three elliptic functions found in [18] without any connection with the theory of integrable systems on a line,

$$u = 2\wp(x - x_1(t)) + 2\wp(x - x_2(t)) + 2\wp(x - x_3(t)).$$

For two particles system (1) was integrated in [11], where it was proved that the level manifold of the integrals J_1, H is a two-dimensional Abelian manifold.

1. Linear Nonstationary Schrödinger Equation and a System of Particles on a Circle

Methods of integrating the K–P equation (3) are based on the following commutation representation (see [12] and [13]):

$$\left[\frac{\partial}{\partial y} - L, \frac{\partial}{\partial t} - M\right] = 0,$$
$$L = \frac{\partial^2}{\partial x^2} - u(x, y, t), \quad M = \frac{\partial^3}{\partial x^3} - \frac{3}{2}u\frac{\partial}{\partial x} + w(x, y, t). \tag{4}$$

It will be proved later that the Hamiltonian of a system of type (1) are connected with the existence of solutions of a specific form for linear operators with elliptic potentials.

First of all, we recall the basic definitions and properties of the classical functions of Weierstrass (see [1]). Let ω_1 and ω_2 be a pair of periods. The sigma-function of Weierstrass is the entire function defined by the product

$$\sigma(z) = z\prod_{m,\, n \neq 0}\left(1 - \frac{z}{\omega_{mn}}\right)\exp\left[\frac{z}{\omega_{mn}} + \frac{1}{2}\left(\frac{z}{\omega_{mn}}\right)^2\right], \tag{5}$$

where $\omega_{mn} = m\omega_1 + n\omega_2$. The remaining functions can be defined by the relations

$$\zeta(z) = \frac{\sigma'(z)}{\sigma(z)}, \quad \wp(z) = -\zeta'(z). \tag{6}$$

In contrast to the $\wp$-functions, the σ- and ζ-functions are not doubly periodic. Under translations of the periods they transform as follows:

$$\zeta(\alpha + \omega_l) = \zeta(\alpha) + \eta_l; \quad \eta_1\omega_2 - \eta_2\omega_1 = 2\pi i;$$
$$\sigma(\alpha + \omega_l) = -\sigma(\alpha)\exp\left[\eta_l\left(\alpha + \frac{\omega_l}{2}\right)\right]. \tag{7}$$

In a neighborhood of $\alpha = 0$, the Weierstrass functions have the form

$$\sigma(\alpha) = \alpha + O(\alpha^5); \quad \zeta(\alpha) = \alpha^{-1} + O(\alpha^3); \quad \wp(\alpha) = \alpha^{-2} + O(\alpha^2). \tag{8}$$

THEOREM 1. The equation

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} + 2\sum_{i=1}^{n}\wp(x - x_i(t))\right)\psi = 0 \tag{9}$$

has a solution ψ of the form

$$\psi = \sum_{i=1}^{n} a_i(t, k, \alpha)\Phi(x - x_i, \alpha)e^{kx + k^2 t}, \tag{10}$$

where

$$-\Phi(x, \alpha) = \frac{\sigma(x-\alpha)}{\sigma(\alpha)\sigma(x)} e^{\zeta(\alpha)x}, \tag{11}$$

if and only if the $x_i(t)$ satisfy the equations of motion of the system of particles (1):

$$\ddot{x}_i = 4 \sum_{k \neq i} \wp'(x_i - x_k). \tag{12}$$

The choice of this representation for ψ is connected with the fact that the function $\Phi(x, \alpha)$ is a solution of Lamé's equation [14]

$$\left(\frac{d^2}{dx^2} - 2\wp(x)\right) \Phi(x, \alpha) = \wp(\alpha) \Phi(x, \alpha). \tag{13}$$

From translation relations (7) it easily follows that $\Phi(x, \alpha)$ is doubly periodic in α, $\Phi(x, \alpha+\omega_l) = \Phi(x, \alpha)$.

As a function of x, $\Phi(x, \alpha)$ satisfies the relation

$$\Phi(x + \omega_l, \alpha) = \Phi(x, \alpha) \exp[\zeta(\alpha)\omega_l - \eta_l\alpha]. \tag{7'}$$

A function ψ of form (10) has simple poles at the poles $x = x_i$. By substituting it into (9) and equating to zero the coefficients of $(x-x_i)^{-2}$ and $(x-x_i)^{-1}$, we obtain the following equations:

$$\begin{gathered} a_i\dot{x}_i + 2ka_i + 2\sum_{j\neq i} a_j\Phi(x_i - x_j, \alpha) = 0, \\ \dot{a}_i - \wp(\alpha)a_i + a_i\left(\sum_{k\neq i} 2\wp(x_i - x_k)\right) + 2\sum_{j\neq i} a_j\Phi'(x_i - x_j, \alpha) = 0. \end{gathered} \tag{14}$$

If we introduce the vector $a = (a_1, \ldots, a_n)$ and the matrices

$$\begin{gathered} L_{ij}(\alpha) = \dot{x}_i\delta_{ij} + 2(1-\delta_{ij})\Phi(x_i - x_j, \alpha), \\ T_{ij}(\alpha) = \delta_{ij}\left(-\wp(\alpha) + 2\sum_{k\neq i}\wp(x_i - x_k)\right) + 2(1-\delta_{ij})\Phi'(x_i - x_j, \alpha), \end{gathered} \tag{15}$$

then Eqs. (14) can be written in the form

$$(L(\alpha) + 2k\cdot 1)a = 0, \quad \left(\frac{\partial}{\partial t} + T\right)a = 0. \tag{16}$$

For Eqs. (16) to be compatible, it is necessary and sufficient that the following relation holds:

$$\left[L, \frac{\partial}{\partial t} + T\right] = 0 \leftrightarrow \dot{L} = [L, T]. \tag{17}$$

LEMMA 1. Equations (17) hold if and only if the $x_i(t)$ satisfy (12).

The assertion of the lemma follows from a straightforward substitution of the expressions for L and T into (17). In addition, we can use the assertion, well known in the theory of system (1), that the commutation representation (17) is equivalent to (12) provided that $\Phi(x, \alpha)$ satisfies the functional equation:

$$[\Phi'(x)\Phi(y) - \Phi(x)\Phi'(y)]\Phi^{-1}(x+y) = [\wp(y) - \wp(x)] \tag{18}$$

and the relation

$$\Phi(x)\Phi(-x) = \wp(\alpha) - \wp(x). \tag{19}$$

To verify the first of these relations, notice that the left-hand side of (18) is doubly periodic in both x and y (this follows from (7')), and it has a second-order pole at $x = 0$, $y = 0$. Hence, it is equal to the right-hand side. Similarly, the left-hand side of (19) is periodic in x and α, and has a second-order pole at $x = 0$ and $\alpha = 0$.

Thus, we have found the solutions of the functional equations (18), (19), and so the commutation representations (17) for system (1); these, in contrast to all those used earlier, depend on an additional "spectral parameter" α defined on an elliptic curve Γ with periods ω_1 and ω_2. This additional parameter also allows us to proceed to the integration of (1) by using the methods of algebraic geometry.

We consider a matrix $A(t, \alpha)$ satisfying the equation

$$\left(\frac{\partial}{\partial t} + T(t, \alpha)\right)A(t, \alpha) = 0 \tag{20}$$

and normalized by the condition $A(0, \alpha) \equiv 1$.

From (17) it follows that

$$L(t,\alpha)A(t,\alpha)=A(t,\alpha)L(0,\alpha). \tag{21}$$

Hence, the function

$$R(k,\alpha)=\det(2k+L(t,\alpha)) \tag{22}$$

is independent of t. Matrix $L(t, \alpha)$, which has essential singularities at $\alpha = 0$, can be represented in the form

$$L(t,\alpha)=G(t,\alpha)\tilde{L}(t,\alpha)G^{-1}(t,\alpha), \tag{23}$$

where $\tilde{L}$ does not have an essential singularity, and G is a diagonal matrix, $G_{ij} = \delta_{ij}\exp(\zeta(\alpha)x_j)$. Consequently, the coefficients $r_i(\alpha)$ in the expression

$$R(k,\alpha)=\sum_{i=0}^{n} r_i(\alpha)k^i \tag{24}$$

are elliptic functions with poles at $\alpha = 0$. The functions $r_i(\alpha)$ are representable as a linear combination of a $\wp$-function and its derivatives. The coefficients in this expansion are integrals of system (1). Each set of fixed values of these integrals defines an algebraic curve Γ_n by the equation $R(k, \alpha) = 0$; Γ_n is an n-sheeted covering of the original curve Γ.

Example 1. Let $n = 2$, then

$$R(k,\alpha)=4k^2+2k(\dot{x}_1+\dot{x}_2)+\dot{x}_1\dot{x}_2+4\wp(x_1-x_2)-4\wp(\alpha).$$

In a neighborhood of $\alpha = 0$ the singular part of the matrix $\tilde{L}$ has the form $\tilde{L}_{ii} = O(1)$, $\tilde{L}_{ij} = -2\alpha^{-1} + O(1)$, $i \neq j$. The eigenvalue $-2\alpha^{-1}$ of this matrix is $(n-1)$-fold degenerate; in addition, there is another eigenvalue equal to $2(n-1)\alpha^{-1}$.

Thus, in a neighborhood of $\alpha = 0$ the function $R(k, \alpha)$ can be represented in the form

$$R(k,\alpha)=(k-(n-1)\alpha^{-1}+b_n(\alpha))\prod_{l=1}^{n-1}(k+\alpha^{-1}+b_l(\alpha)), \tag{25}$$

where the $b_l(\alpha)$ are regular functions of α. Hence, function k defined on Γ_n has simple poles on all the sheets at points P_i lying over $\alpha = 0$. Its expansions in terms of the local parameter α on these sheets are given by the factors on the right-hand side of (25). It follows from (25) that one of the sheets is isolated; for brevity, we call it the "upper" sheet.

LEMMA 2. The genus g of the surface Γ_n is n.

The covering elliptic curve is such that $2g-2 = \nu$, where ν is the number of branch points on the covering Γ_n over Γ. The branch points coincide with the zeros on Γ_n of $\partial R/\partial k$. By differentiating (25) with respect to k and substituting for k the corresponding expansions, we obtain that $\partial R/\partial k$ has simple poles on all sheets apart from the upper one, on which the pole is of order $n-1$. For any meromorphic function the number of zeros is equal to the number of poles. Hence, $\nu = 2(n-1)$, or $g = n$.

The Jacobi manifold $J(\Gamma_n)$ of Γ_n is an n-dimensional torus. We show below that the coordinates on this torus are "angle" type variables for system (1).

To each point P of the curve Γ_n, i.e., to each pair $(k, \alpha) = P$ connected by relation $R(k, \alpha)$, corresponds a unique eigenvector $a(0, P) = (a_1(0, P),\dots, a_n(0, P))$ of matrix $L(0, \alpha)$, normalized by the condition $a_1(0, P) \equiv 1$. All other coordinates $a_i(0, P)$ are meromorphic functions on the curves Γ_n outside the points P_j. The number of poles of $a(0, P)$ is $n-1$. To prove this, we consider a matrix $F(\alpha)$ whose columns are vectors $a(0, P_j(\alpha))$, where the $P_j(\alpha)$ are the inverse images of the point α. The function $|\det F(\alpha)|^2$ does not depend on the enumeration of the sheets, i.e., it is properly defined as a function of α. It is meromorphic, and has double poles at the images of the poles of $a(0, P)$. The zeros of this function coincide with the images of the branch points of Γ_n. If N is the number of poles of $a(0, P)$, then $2N = \nu = 2n-2$.

In a neighborhood of "infinitely distant" points P_j, $a_i(0, P)$ has the form

$$a_i(0,P)=\left(-\frac{1}{n-1}+O(\alpha)\right)\exp[\zeta(\alpha)(x_i^0-x_1^0)], \quad i>1, \quad j\neq n. \tag{26}$$

On the upper sheet, $j = n$

$$a_i(0, P) = (1 + O(\alpha)) \exp[\zeta(\alpha)(x_i^0 - x_1^0)]. \tag{27}$$

Here and in what follows $x_i^0 = x_i(0)$ is the initial position of a particle.

The fundamental matrix $A(t, \alpha)$ of system (20), $A(0, \alpha) = 1$, is an analytic function of α outside $\alpha = 0$. If $A_i(t, \alpha)$ is the i-th column of A, then the column-vector

$$a(t, P) = \sum_{i=1}^{n} a_i(0, P) A_i(t, \alpha) \tag{28}$$

is a solution of system (20) that is eigen for L:

$$(L(t, \alpha) + 2k \cdot 1) a(t, P) = 0, \quad P = (k, \alpha). \tag{29}$$

To find the form $a(t, P)$ at the inverse images of $\alpha = 0$, we turn from a pair L, T satisfying (17) to a calibrationally equivalent pair

$$\tilde{L}, \quad \tilde{T} = G^{-1}\partial G + G^{-1}TG,$$

where $\tilde{L}$ and G are as in (23). The following relation holds

$$P(\alpha)\delta_{ij} + \tilde{T}(t, \alpha) = \alpha^{-1}\tilde{L}(t, \alpha) + O(1). \tag{30}$$

Consequently, the eigenvalues of the matrix $\tilde{T}$ have the following form: for $j \neq n$

$$\mu_j(t, \alpha) = -k_j^2 + O(1) = -(\alpha^{-2} + 2b_j(0)\alpha^{-1} + O(1)). \tag{31}$$

Here the $k_j = -(\alpha^{-1} + b_j(\alpha))$ are the expansions of the eigenvalues of L on different sheets Γ_n. On the "upper" sheet we have

$$\mu_n(t, \alpha) = 2k_n\alpha^{-1} + \alpha^{-2} + O(1). \tag{32}$$

We denote by $\nu_j(\alpha) = \mu_j(t, \alpha) + O(1)$ the singular parts of the eigenvalues of $\tilde{T}$. They are independent of t, and so the solutions of the equation

$$\left(\frac{\partial}{\partial t} + \tilde{T}\right)\tilde{a}(t, P) = 0,$$

which are eigenfunctions of the matrix $\tilde{L}$, have the form

$$\tilde{a}(t, P) = \tilde{a}(0, P)(1 + O(\alpha)) \exp(\nu_j(\alpha) t).$$

The vectors a and $\tilde{a}$ are connected by the simple relation

$$a(t, P) = G(t, \alpha)\tilde{a}(t, P). \tag{33}$$

Thus, we have the following result.

LEMMA 3. The coordinates $a_i(t, P)$ of the vector-function $a(t, P)$ are meromorphic on the curve Γ_n outside the points P_j. Their poles $\gamma_1, \ldots, \gamma_{n-1}$ do not depend on t. In a neighborhood of P_j, $a_i(t, P)$ has the form

$$a_i(t, P) = c_{ij}(\alpha) \exp[\zeta(\alpha)(x_i(t) - x_1^0) + \nu_j(\alpha) t], \tag{34}$$

where the $c_{ij}(\alpha)$ are regular in a neighborhood of $\alpha = 0$, and

$$c_{1j}(0) = 1, \quad j = 1, \ldots, n; \quad c_{in}(0) = 1, \quad c_{ij}(0) = -\frac{1}{n-1}, \quad i > 1, \quad j \neq n. \tag{35}$$

We return again to the eigenfunction

$$\psi(x, t, P) = \sum_{i=1}^{n} a_i(t, P)\Phi(x - x_i, \alpha) e^{kx + k^2 t}.$$

The function $\Phi(x - x_i, \alpha)$ has essential singularities on all the sheets Γ_n. It follows from (31), (34), and (25) that $\psi(x, t, P)$ does not have essential singularities at the points P_j, $j \neq n$. From (35) it follows that ψ does not have a pole at this point.

THEOREM 2. The eigenfunction $\psi(x, t, P)$ of the nonstationary Schrödinger equation (9) is defined on the n-sheeted covering Γ_n of the original elliptic curve. $\psi(x, t, P)$ is meromorphic everywhere on Γ_n except for the one essentially singular point P_n. Its poles $\gamma_1, \ldots, \gamma_{n-1}$ do not depend on t and x. In a neighborhood of P_n, $\psi(x, t, P)$ has the form

$$\psi(x, t, P) = \left(n\alpha^{-1} + \sum_{i=0}^{\infty} \xi_i(x, t)\alpha^i\right) \exp\left[\lambda(\alpha)(x - x_1^0) + \lambda^2 t\right], \tag{36}$$

where $\lambda(\alpha) = n\alpha^{-1} + b_n(0)$

Thus, $\psi(x, t, P)$ is a classical Baker–Akhiezer function (see [15-17]). It is defined uniquely by its poles $\gamma_1, \ldots, \gamma_{n-1}$ and by the value x_1^0. In fact, by shifting a reference point we can move a pole from a specified point and so assume that ψ has n arbitrary poles $\gamma_1, \ldots, \gamma_n$.

It was proved in [15] that a function $\psi(x, t, P)$, with the properties stated in Theorem 2, is a solution of the nonstationary Schrödinger equation. A general scheme for constructing expressions for Baker–Akhiezer type functions in terms of Riemann's θ-function is given in [16], where there is also a formula for the potential u(x, t) for the nonstationary Schrödinger equation. A confrontation of results in [15] and [16], and those obtained above yields the following assertion.

THEOREM 3. Coordinates $x_i(t)$ of the system of particles (1) are defined by the equation

$$\theta(\vec{U}x + \vec{V}t + \vec{W}) = 0 = \text{const} \times \prod_{i=1}^{n} \sigma(x - x_i(t)). \tag{37}$$

To prove this, we note that the formula for the potential for a nonstationary Schrödinger equation has the form

$$u(x, t) = 2\frac{\partial^2}{\partial x^2} \ln \theta(\vec{U}x + \vec{V}t + \vec{W}) + \text{const.} \tag{38}$$

As is clear from (9), the poles of u coincide with the zeros of θ on the one hand, and with the $x_i(t)$ on the other.

Here θ is Riemann's theta-function, $\vec{U}$, $\vec{V}$ are constant vectors equal to the periods of Abelian differentials of the second kind with singularities at P_n. A more detailed definition of them can be found in [16, 17]. Below we shall only discuss the character of our answer. Both the theta-function and $\vec{U}$, $\vec{V}$ are defined uniquely by the curve Γ_n, or equivalently, by the characteristic polynomial $R(k, \alpha)$, which does not depend on t. Consequently, these parameters depend only on the integrals of system (1). Let $\gamma_1, \ldots, \gamma_{n-1}$ be the roots of the equation $\det(2k \cdot 1 + L_{ij}) = 0$, $i, j > 1$ (i.e., the roots of the lower right minor of the characteristic matrix), on the curve Γ_n. Abel's transformation maps the symmetric power $S^n\Gamma_n$ of the curve (a nonordered collection of n points) into the Jacobi manifold $J(\Gamma_n)$, $\omega: S^n\Gamma_n \to J(\Gamma_n)$. The vector $\vec{W}$ in (37) is the image of the set $\gamma_1, \ldots, \gamma_{n-1}$, P_n $(x_1^0 = 0)$ under Abel's transformation. It transforms linearly in t, which also proves that the coordinates on the Jacobi manifold are angle-type variables.

2. Elliptic Solutions of the Kadomtsev–Petviashvili Equation

Without directly substituting elliptic solutions into the K–P equation, we can obtain in a straightforward way an identification of the poles of these solutions with system (1) from the commutation representation (7).

THEOREM 4. A function u(x, y, t) is an elliptic solution of the K–P equation if and only if

$$u(x, y, t) = c + 2\sum_{j=1}^{n} \wp(x - x_j(y, t)) \tag{39}$$

and the equations

$$\left(\frac{\partial}{\partial y} - L\right)\psi = \left(\frac{\partial}{\partial t} - M\right)\psi = 0$$

have a solution of the form

$$\psi = \sum_{i=1}^{n} a_i(t, y, P)\,\Phi(x - x_i, \alpha)\, e^{kx + k^2 y + k^3 t}. \tag{40}$$

COROLLARY 1. The dynamics of the poles of the $x_i(y, t)$ with respect to y coincides with the dynamics of particles of system (1).

This assertion follows from results in the preceding section. Similarly to the equation $\partial/\partial y - L$, the availability of solutions of form (40) for the equation $\partial/\partial y - M$ is equivalent to a commutation equation of the type (17), and coincides with the equations of a Hamiltonian flow corresponding to the Hamiltonian J_3.

COROLLARY 2. The dynamics of the $x_i(y, t)$ with respect to t coincides with the third Hamiltonian flow of system (1).

COROLLARY 3. The elliptic solution $u(x, y, t)$ of the K–P equation (39) can be expressed in terms of the θ-function of the covering Γ_n of the elliptic curve Γ:

$$u = \text{const} + 2\frac{\partial^2}{\partial x^2}\ln\theta(\vec{U}x + \vec{V}y + \vec{Z}t + \vec{W}). \tag{41}$$

We use the connection between the poles of elliptic solutions of the K–P equation and systems of type (1) in yet another way to prove that under obvious restrictions, these solutions have no singularities for real x, y, t.

We consider the equation

$$\frac{3}{4}\frac{\partial^2 u}{\partial y^2} + \frac{\partial}{\partial x}\left[u_t + \frac{1}{4}\left(6uu_x - u_{xxx}\right)\right] = 0, \tag{42}$$

which differs from (3) by a sign. It has the commutation representation $\left[i\frac{\partial}{\partial y} - L, \frac{\partial}{\partial t} - M\right] = 0$ and its elliptic solutions are connected with a Hamiltonian system with the Hamiltonian

$$H = \frac{1}{2}\sum_{k=1}^{n} p_k^2 + 2\sum_{k\neq j}\wp(x_k - x_j). \tag{43}$$

This Hamiltonian differs from (1) by the sign of the potential energy.

Let ω_1, ω_2 be the periods of the $\wp$-function; they are complex conjugates. Then $\wp(\bar{z}) = \overline{\wp(z)}$. We consider real solutions of the K–P equation of form (39). They are determined by the initial coordinates $x_j(0, 0)$ and the initial impulses $x_{jy}(0, 0)$. Suppose that these data tolerate conjugation, i.e., $n = 2m$ and $x_j = \bar{x}_{j+m}$, $j = 1, \ldots, m$. Then $u(x, y, t)$ is real for all real x, y, t.

COROLLARY. If $x_j(0, 0)$ does not lie on the real axis, then the solution (39), (41) does not have a singularity for real x and y.

The existence of a singularity means that one of the particles falls onto the real axis, but then the conjugate particle must collide with it. This contradicts the law of conservation of energy since the potential is repelling and singular.

MATRIX SYSTEMS

We briefly state conditions on curves so that the constructions [16] of "finite-zoned" solutions of the commutation equations

$$\left[\frac{\partial}{\partial y} - L, \frac{\partial}{\partial t} - M\right] = 0, \tag{44}$$

where L and M are operators with matrix coefficients, lead to elliptic solutions. We follow the notation in [16].

By [16], every nonsingular algebraic curve Γ of genus g with l distinguished points $P_1, \ldots, P_l$ and fixed local parameters $z_j(P)$ in their neighborhoods, and also a set $\gamma_1, \ldots, \gamma_{g+l-1}$ of points in general position, determines the solution of Eqs. (44).

If the curve Γ_N is an N-sheeted covering of the elliptic curve Γ, i.e., it is given by the equation

$$k^N + \sum_{i=0}^{N-1} r_i(\alpha)k^i, \tag{45}$$

where the $r_i(\alpha)$ are elliptic functions with a single pole at the point $\alpha = 0$.

We suppose that Γ_N has no branches over $\alpha = 0$. This means that the function k on Γ_N has N simple poles $P_1, \ldots, P_N$ (the inverse images of the point $\alpha = 0$). We denote by ν_j the residue of k on the j-th sheet, i.e., $k - \nu_j\alpha^{-1} = O(1)$ in a neighborhood of P_j.

Assertion. We assume that $\nu_j = 1$ if $j > l$. Then if we take the local parameters $z_j(\alpha)$ in the neighborhoods of $P_1, \ldots, P_l$ to be $z_j(\alpha) = (k_j(\alpha) - \zeta(\alpha))^{-1}$, then the corresponding solutions of (44) are elliptic.

It follows easily from this assertion that to every N-sheeted covering of an elliptic curve there corresponds elliptic solutions for systems with $(N-1) \times (N-1)$ matrix coefficients.

The proof of the assertion follows from the fact that the function

$$\varphi_i(P) = \exp[k(P)\omega_i - \zeta(\alpha)\omega_i + \eta_i\alpha], \quad i = 1, 2,$$

is properly defined as a function of P. Outside the points $P_1, \ldots, P_l$ it is holomorphic, and in neighborhoods of these points it has the form

$$\varphi_i(P) = (1 + O(\alpha)) \exp(z_j^{-1}(\alpha)\omega_i).$$

It follows from the definition of Baker–Akhiezer type functions that for $\psi(x, y, t, P)$ satisfying the equations $\left(\frac{\partial}{\partial y} - L\right)\psi = \left(\frac{\partial}{\partial t} - M\right)\psi = 0$, we have

$$\psi(x + \omega_i, y, t, P) = \psi(x, y, t, P)\varphi_i(P).$$

Hence, the coefficients of the operators L and M are meromorphic and periodic with periods ω_1 and ω_2, i.e., they are elliptic functions.

In conclusion, we mention that it would be interesting to discover how to obtain, by using coverings over a curve $\mathfrak{R}$ of genus n, the solutions of nonlinear equations expressed in terms of a θ-function of high dimension, but leading to solutions with the group of periods of $\mathfrak{R}$. It is possible that these solutions are connected with new integrable systems of particles.

LITERATURE CITED

1. H. Bateman and A. Erdelyi, Higher Transcendental Functions. Elliptic and Automorphic Functions, Lamé and Mathieu Functions [Russian translation], Nauka, Moscow (1967).
2. F. Calogero, "Exactly solvable one-dimensional many-body systems," Lett. Nuovo Cimento, 13, 411-415 (1975).
3. A. M. Perelomov, "Completely integrable classical systems connected with semisimple Lie algebras," Lett. Math. Phys., 1, 531-540 (1977).
4. H. Airault, H. McKean, and J. Moser, "Rational and elliptic solutions of the KdV equation and related many-body problem," Commun. Pure Appl. Math., 30, 95-125 (1977).
5. I. M. Krichever, "Rational solutions of the Kadomtsev–Petviashvili equation and integrable systems of N particles on a line," Funkts. Anal. Prilozhen., 12, No. 1, 76-78 (1978).
6. D. V. Choodnovsky and G. V. Choodnovsky, "Pole expansions of nonlinear partial differential equations," Nuovo Cimento, 40B, 339-350 (1977).
7. F. Calogero, "Integrable many-body problem," Preprint, Univ. di Roma, No. 89 (1978).
8. K. M. Case, "The N-soliton solutions of the Bendgamine–Ono equation," Proc. Nat. Acad. Sci. USA, 75, 3562-3563 (1978).
9. M. A. Olshanetsky and A. M. Perelomov, "Explicit solutions of some completely integrable systems," Lett. Nuovo Cimento, 17, 97-133 (1976).
10. M. A. Olshanetsky and N. V. Rogov, "Bound states in completely integrable systems with two types of particles," Ann. Inst. H. Poincare, 29, 169-177 (1978).
11. D. V. Choodnovsky, "Meromorphic solutions of nonlinear partial differential equations and particle integrable systems," J. Math. Phys., 20, No. 12, 2416-2424 (1979).
12. V. S. Dryuma, "An analytic solution of the two-dimensional Korteweg–de Vries equation," Pis'ma Zh. Eksp. Teor. Fiz., 19, No. 12, 219-225 (1973).
13. V. E. Zakharov and A. B. Shabat, "A scheme for the integration of nonlinear equations of mathematical physics by an inverse problem of scattering theory. I," Funkts. Anal. Prilozhen., 8, No. 3, 43-53 (1974).
14. E. Kamke, Handbook on Ordinary Differential Equations [in German], Chelsea Publ.
15. I. M. Krichever, "An algebraic-geometric construction of the Zakharov–Shabat equations and their periodic solutions," Dokl. Akad. Nauk SSSR, 227, No. 2, 291-294 (1976).
16. I. M. Krichever, "The integration of nonlinear equations by the methods of algebraic geometry," Funkts. Anal. Prilozhen., 11, No. 1, 15-31 (1977).

17. H. Baker, "Note on the foregoing paper, 'Commutative ordinary differential equations,'" Proc. R. Soc. London, Ser. A, 118, 570-576 (1928).
18. B. A. Dubrovin and S. P. Novikov, "Periodic and conditionally periodic analogues of many-soliton solutions of the Korteweg–de Vries equation," Zh. Eksp. Teor. Fiz., 67, No. 12, 2131-2143 (1974).

EFFECTIVE CONSTRUCTION OF NONDEGENERATE HERMITIAN–POSITIVE FUNCTIONS OF SEVERAL VARIABLES

L. A. Sakhnovich

UDC 517.57

1. Let S be some set of points of n-dimensional space $\mathbf{R}^n$ and $\Delta = S - S$. A function $\Phi(x)$ is called Hermitian-positive on Δ if for any choice of points $x_1, x_2, \ldots, x_N \in S$ and numbers $\xi_1, \xi_2, \ldots, \xi_N$ one has

$$\sum_{i,j=1}^{N} \xi_i \bar{\xi}_j \Phi(x_i - x_j) \geqslant 0. \tag{1}$$

Let $n = 2$. We consider the lattice $S(N_1, N_2)$ consisting of points $M(m, l)$ with integral coordinates $0 \leqslant m \leqslant N_1$, $0 \leqslant l \leqslant N_2$. The set $\Delta(N_1, N_2)$ consists of the points $M(m, l)$ for which $|m| \leqslant N_1$, $|l| \leqslant N_2$. By $\mathfrak{P}(N_1, N_2)$ we denote the class of functions, Hermitian-positive on $\Delta(N_1, N_2)$. Calderon, Pepinsky [1], and Rudin [2] proved the following theorem.

THEOREM. In order that any function of the class $\mathfrak{P}(N_1, N_2)$ should admit an extension to a function of class $\mathfrak{P}(\infty, \infty)$, it is necessary and sufficient that any nonnegative polynomial of the form

$$f(x, y) = \sum_{0 \leqslant k \leqslant 2N_1} \sum_{0 \leqslant l \leqslant 2N_2} a_{kl} x^k y^l$$

admit a representation

$$f(x, y) = \sum_{j=1}^{r} q_j^2(x, y), \tag{2}$$

where $q_j(x, y)$ are real polynomials.

As Hilbert [3] proved, there exists a nonnegative polynomial in two variables of the sixth degree which cannot be represented in the form (2). Consequently, one has the following:

Assertion (see [1, 2]). There exist functions of class $\mathfrak{P}(3,3)$ which cannot be extended to $\mathfrak{P}(\infty, \infty)$.

However, there have not been until now concrete examples of such functions. In the present paper classes of concrete functions of $\mathfrak{P}(2, 2)$ and $\mathfrak{P}(1, 1, 1)$ which cannot be extended, respectively, to $\mathfrak{P}(2, 3)$ and $\mathfrak{P}(1, 1, 2)$ are constructed. We note that functions of the class $\mathfrak{P}(1, 2)$ can always be extended to $\mathfrak{P}(\infty, \infty)$ (see [4-6]).

2. With each function $\Phi(m, l)$ from $\mathfrak{P}(N_1, N_2)$ we associate the Toeplitz matrix

$$C_k = \begin{bmatrix} \Phi(0, k) & \Phi(1, k) & \ldots & \Phi(N_1, k) \\ \Phi(-1, k) & \Phi(0, k) & \ldots & \Phi(N_1 - 1, k) \\ \ldots & \ldots & \ldots & \ldots \\ \Phi(-N_1, k) & \Phi(-N_1 + 1, k) & \ldots & \Phi(0, k) \end{bmatrix}, \quad 0 \leqslant k \leqslant N_2. \tag{3}$$

From the matrices C_k we construct the block Teoplitz matrix:

$$A(N_1, N_2) = \begin{bmatrix} C_0 & C_1 & \ldots & C_{N_2} \\ C_1^* & C_0 & \ldots & C_{N_2-1} \\ \ldots & \ldots & \ldots & \ldots \\ C_{N_2}^* & C_{N_2-1}^* & \ldots & C_0 \end{bmatrix}. \tag{4}$$

A. S. Popov Odessa Electrical Engineering Institute. Translated from Funktsional'nyi Analiz i Ego Prilozheniya, Vol. 14, No. 4, pp. 55-60, October-December, 1980. Original article submitted April 2, 1979.