RATIONAL SOLUTIONS OF THE ZAKHAROV–SHABAT EQUATIONS AND COMPLETELY INTEGRABLE SYSTEMS OF N PARTICLES ON A LINE

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One constructs all the decreasing rational solutions of the Kadomtsev–Petviashvili equations. The presented method allows us to identify the motion of the poles of the obtained functions with the motion of a system of N particles on a line with a Hamiltonian of the Calogero–Moser type. Thus, this Hamiltonian system is imbedded in the theory of the algebraic–geometric solutions of the Zakharov–Shabat equations.

The fundamental purpose of the present paper is the construction of all rational solutions, decreasing for $x \to \infty$, of the Kadomtsev–Petviashvili (KP) equations

$$\frac{3}{4}\frac{\partial^2 u}{\partial y^2} + \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t} + \frac{1}{4}\left(6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3}\right)\right) = 0 .$$

This equation, describing the quasi-one-dimensional waves in a weakly dispersing medium, has been obtained for the first time in [1]. It belongs to the type of so-called Zakharov–Shabat equations i.e., equations upon the coefficients of the operators

$$L_1 = \sum_{i=0}^{n} u_i(x,y,t)\frac{\partial^i}{\partial x^i}, \quad L_2 = \sum_{i=0}^{m} v_i(x,y,t)\frac{\partial^i}{\partial x^i},$$

equivalent to the operator equation

$$\left[L_1 - \frac{\partial}{\partial y}, L_2 - \frac{\partial}{\partial t}\right] = 0 \quad ([2]).$$

In particular, if

$$L_1 = \frac{\partial^2}{\partial x^2} + u(x,y,t), \text{ and } L_2 = \frac{\partial^3}{\partial x^3} + \frac{3}{2}u\frac{\partial}{\partial x} + w(x,y,t),$$

then the corresponding Zakharov–Shabat equations have the form

$$\frac{3}{2}\frac{\partial u}{\partial y} = -\frac{3}{2}\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial w}{\partial x},$$

$$\frac{\partial w}{\partial y} - \frac{\partial u}{\partial t} = \frac{\partial^2 w}{\partial x^2} - \frac{3}{2}u\frac{\partial u}{\partial x} - \frac{\partial^3 u}{\partial x^3}.$$

Eliminating $w(x,y,t)$, from this system, we obtain the KP equation [3].

One knows a procedure for the construction of exact quasiperiodic solutions of this equation by the methods of algebraic geometry [4, 5]. It has been proved in [6] that the equations upon the coefficients of a collection of linear differential operators of n variables, equivalent to the condition of their commutativity, can be reduced to problems of

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algebraic geometry if the number of the operators in this collection is equal to $n+1$. The last statement refers to the local solutions of the indicated equations. In the present paper it is proved that the class of the exactly integrable solutions of the Zakharov—Shabat equations can be can be selected by nonlocal requirements on the functions under consideration, namely the requirement of the rationality of the obtained solutions. It should be mentioned that such a nonlocal requirement was the formulation of a problem by Novikov [7], who has proved the complete integrability of the Korteweg—de Vries equation in the class of quasi-periodic functions with a finite number of prohibited zones for the corresponding Sturm—Liouville operators.

Making use of the method of the inverse problem in scattering theory, a large class of nonsingular rational solutions of the KP equation has been found in [8].

The suggested method of integration of the KP equation in the field of the rational functions allows us to identify the motion of the poles of the obtained functions with the motion of a system of N particles on a line with the Hamiltonian $H=\frac{1}{2}\sum_{j=1}^{N} p_j^2 + 2\sum_{i<j}(x_i - x_j)^{-2}$ and with flows given by the "higher Hamiltonians," obtained by the integration of the initial system in [9]. Thus, the Moser theory of the Hamiltonian system is imbedded in the theory of algebraic—geometric solutions of the Zakharov—Shabat equations, as the theory of special solutions. The connection between the motion of the poles of the rational solutions of the Korteweg—de Vries equation and the motion of a discrete system has been discovered for the first time in [10], stimulating subsequent investigations in this domain.

1. Rational Solutions of the Kadomtsev—Petviashvili Equations

Let $u(x,y,t)$ be a solution of the KP equation, depending rationally on the variable x and decreasing for $x \to \infty$. It turns out a posteriori that in this case $u(x,y,t)$ will be a rational function of all of its arguments.

Expanding $u(x,y,t)$ in a Laurent series in the neighborhood of its pole $x_j(y,t)$ insertingit into the KP equation and comparing the two principal singular terms in all the terms of the left-hand side, we obtain easily that $u(x,y,t)=-2\sum_{j=1}^{N}(x-x_j(y,t))^{-2}$.

THEOREM 1.1. The function $u(x,y,t)$ is a rational (with respect to the variable x) solution of the KP equation, decreasing for $x \to \infty$ if and only if $u(x,y,t)=-2\sum_{j=1}^{N}(x-x_j(y,t))^{-2}$ and there exists a function $\psi(x,y,t,\kappa)$ of the form

$$\psi(x,y,t,\kappa)=\left(1+\sum_{j=1}^{N}\frac{a_j(y,t,\kappa)}{x-x_j(y,t)}\right)e^{\kappa x+\kappa^2 y+\kappa^3 t} \tag{1.1}$$

such that

$$L_1\psi=\frac{\partial}{\partial y}\psi \quad , L_2\psi=\frac{\partial}{\partial t}\psi \ ,$$

$$L_1=\frac{\partial^2}{\partial x^2}+u(x,y,t) \ , \ L_2=\frac{\partial^3}{\partial x^3}+\frac{3}{2}u\frac{\partial}{\partial x}+w(x,y,t),$$

$$\omega(x,y,t)=\sum_{j=1}^{N}\left[3(x-x_j(y,t))^{-3}+\frac{3}{2}(x-x_j(y,t))^{-2}\frac{\partial}{\partial y}x_j(y,t)\right].$$

Before we proceed to the proof of the theorem, we formulate a lemma which will allow us to identify the dynamics relative to y of the poles $x_j(y,t)$ with the motion of a Moser system of particles.

LEMMA 1.2. The nonstationary Schrödinger equation

$$\left(\frac{\partial}{\partial y}-\frac{\partial^2}{\partial x^2}+\sum_{j=1}^{N}\frac{2}{(x-x_j(y))^2}\right)\psi=0 \tag{1.2}$$

has a solution $\psi(x,y,t)$ of the form (1.1) (where t must be considered a fixed parameter) if and only if the matrices T and Λ

$$\Lambda_{jk}=p_j\delta_{jk}+\frac{2(1-\delta_{jk})}{x_j-x_k}, \quad p_j=\frac{\partial}{\partial y}x_j,$$

$$T_{jk}=\frac{2(1-\delta_{jk})}{(x_j-x_k)^2}-2\delta_{jk}\left(\sum_{s\neq j}\frac{1}{(x_j-x_s)^2}\right),$$

satisfy the matrix equation

$$\left[\frac{\partial}{\partial y}-T,\Lambda\right]=0 . \tag{1.3}$$

Proof. Inserting the expression for ψ into the left-hand side of the equality (1.2), we obtain

$$\left(\sum_{j=1}^{N}\frac{\alpha_j}{(x-x_j)^2}-\frac{\beta_j}{x-x_j}\right)e^{kx+k^2y+k^3t}=0.$$

where the vectors $\alpha=(\alpha_1,\dots,\alpha_N)$ and $\beta=(\beta_1,\dots,\beta_N)$ are given by

$$\alpha=\Lambda a+2k[a+e_0,\quad \beta=\left(\frac{\partial}{\partial y}-T\right)a,$$

$a=(a_1,\dots,a_N)$, $e_0=(2,\dots,2)$, I is the identity matrix.

The equalities $\alpha=0$, $\beta=0$ are consistent for all k if and only if (1.3) holds. Thus, the lemma is proved.

The representation (1.3) of the equations of the motion of a Hamiltonian system of particles with the Hamiltonian $H=\frac{1}{2}\sum_{j=1}^{N}p_j+2\sum_{i<j}(x_i-x_j)^{-2}$ has been found in [9]. Therefore, from Theorem 1.1 and from the proved lemma we obtain the

COROLLARY. The dynamics relative to y of the poles $x_j(y,t)$ of the rational solutions of the dynamics of a Moser system of N particles on a straight line.

Proof of Theorem 1.1. The sufficiency of the conditions of the theorem is obvious. Indeed, if $\psi(x,y,t,k)$ exists, then the operator $\left[L_1-\frac{\partial}{\partial y}, L_2-\frac{\partial}{\partial t}\right]$, containing differentiation only with respect to x, annihilates $\psi(x,y,t,k)$ for all k. Consequently, its kernel is infinite-dimensional and the operator is zero. As mentioned above, the latter means that $u(x,y,t)$ is a solution of the KP equation.

Let $u(x,y,t)$ be a rational solution of the KP equation, decreasing for $x\to\infty$. Then, $u(x,y,t)=-2\sum_{j=1}^{N}(x-x_j(y,t))^{-2}$. As in Lemma 1.1, in terms of the functions $x_j(y,t)$ one defines the matrix functions $\Lambda(y,t)$ and $T(y,t)$.

We consider the function $\psi(x,y,t,\kappa)$ of the form (1.1), where $a_j(y,t,\kappa)$ are determined from the equation $\Lambda a+2\kappa I a+e_0=0$. A direct computation shows that then

$$\left(\frac{\partial}{\partial y}-L_1\right)\psi=\left(\sum_{j=1}^{N}\frac{\beta_j}{x-x_j}\right)e^{\kappa x+\kappa^2 y+\kappa^3 t}, \tag{1.4}$$

$$\left(\frac{\partial}{\partial t}-L_2\right)\psi=\left(\sum_{j=1}^{N}\frac{\gamma_j}{x-x_j}\right)e^{\kappa x+\kappa^2 y+\kappa^3 t}, \tag{1.5}$$

where

$$\beta=\left(\frac{\partial}{\partial y}-T\right)a\ ,\quad \gamma=\left(\frac{\partial}{\partial t}-T_2+\frac{3}{2}\Lambda\frac{\partial}{\partial y}\right)a\ ,$$

$$T_{2jk}=3\delta_{jk}\left(\sum_{s\neq j}\frac{1}{(x_j-x_s)^3}\right)-\frac{3(1-\delta_{jk})}{(x_j-x_k)^3}\ .$$

Applying to the right-hand side of the equalities (1.4) and (1.5) the operators $\frac{\partial}{\partial t}-L_2$ and $\frac{\partial}{\partial y}-L_1$, respectively, we derive that the commutativity of these operators is equivalent to the fact that $\beta=0$ and $\gamma=0$. Thus, the theorem is proved. In addition to Eq. (1.3) defining the dynamics with respect to y of the poles $x_j(y,t)$, we obtain the equation on the dynamics of $x_j(y,t)$ with respect to t in the form

$$\left[\frac{\partial}{\partial t}-T_2+\frac{3}{2}\Lambda T,\ \Lambda\right]=0\ . \tag{1.6}$$

The system of the Eqs. (1.3) and (1.6), defining the functions $x_j(y,t)$, is equivalent to the KP equation in the class of rational functions.

The solution of the Kadomtsev–Petviashvili equation, having N poles, depends on $2N$ initial data: $x_j(0,0)$ and $\left(\frac{\partial}{\partial y}x_j\right)(0,0)$.

In [9] one has introduced "higher Hamiltonian systems" $H_p=\frac{1}{p}\,d\tau\,\Lambda^p$, $H_2=H$. For Hamiltonian flows corresponding to higher Hamiltonians there exist matrix commutation representations. In particular, equation (1.6) is equivalent to the Hamiltonian system of equations for the Hamiltonian H_3.

COROLLARY. The dynamics relative to t of the poles of the rational solutions of the KP equation are identical with the motion of a Hamiltonian system with the Hamiltonian H_3.

Theorem 1.1 reduces the problem of the construction of the rational solutions of the KP equation to the construction of the simultaneous eigenfunctions of linear operators. These functions, having the form (1.1), can be reduced to another form. Since the vector a is determined from the equality $\Lambda a+2\kappa I a+e_0=0$, it follows that $a_j(y,t,\kappa)$ depends rationally on the variable κ. The poles $a_j(y,t,\kappa)$ coincide with the zeros of the characteristic polynomial $q_1(\kappa)=\det(2\kappa I+\Lambda)$. By virtue of (1.3) and (1.6), this characteristic polynomial does not depend on y and t. Consequently,

$$\psi(x,y,t,\kappa)=\left(1+\frac{q(x,y,t,\kappa)}{q_1(\kappa)}\right)e^{\kappa x+\kappa^2 y+\kappa^3 t}$$

The degree of the polynomial $q(x,y,t,\kappa)$ is strictly smaller than $N=\deg q_1(\kappa)$.

In the next section it will be proved that for almost all solutions of the KP equation there exist N numbers $\varkappa_s$, $\varkappa_i \neq \varkappa_j$ such that $\frac{\partial}{\partial \kappa}\psi(x,y,t,\kappa)\big|_{\kappa=\varkappa_s}=0$. The collection of $2N$ parameters ($\varkappa_s$ and the coefficients of the polynomial $q_1(\kappa)$) determine uniquely the function $\psi(x,y,t,\kappa)$ and, therefore, also $u(x,y,t)$.

Making use of this result, we arrive at the following theorem.

THEOREM 1.3. For almost all solutions of the KP equation, depending rationally on x and decreasing for $x\to\infty$, we have the formula

$$u(x,y,t)=2\frac{\partial^2}{\partial x^2}\ln\det\Theta\ , \tag{1.7}$$

where the matrix elements Θ_{js} are given by

$$\Theta_{js}=(x+2\varkappa_s y+3\varkappa_s^2 t)\varkappa_s^j+j\varkappa_s^{j-1}-\varkappa_s^j\left(\frac{\partial}{\partial\kappa}\ln q(\kappa)\Big|_{\kappa=\varkappa_s}\right).$$

COROLLARY. The solutions of the KP equation which depend rationally on x are rational functions of all of its arguments.

2. Rational Solutions of the Zakharov–Shabat Equations

The construction of the rational solutions of the Zakharov–Shabat equations is carried out in two steps. First one constructs a class of functions $\psi(x,y,t,\kappa)$ and then one establishes operators for which $\psi(x,y,t,\kappa)$ are "proper."

Assume that the following collection of data is given: the polynomials

$$Q(\kappa)=c_n\kappa^n+\dots+c_0\ ,\quad R(\kappa)=r_m\kappa^m+\dots+r_0\ ,$$

the points $\varkappa_s$, $\varkappa_i\neq\varkappa_j$, the rectangular matrices $A^s=(a_{ij}^s)$. $1\leqslant i\leqslant l_s$, $1\leqslant j\leqslant h_s$, the ranks l_s. Then, for any polynomial

$$q_1(\kappa)=\kappa^N+b_1\kappa^{N-1}+\dots+b_N\ ,\quad N=\sum_s l_s$$

there exists a unique function

$$\psi(x,y,t,\kappa)=\left(1+\frac{q(x,y,t,\kappa)}{q_1(\kappa)}\right)e^{\kappa x+Q(\kappa)y+R(\kappa)t}$$

such that the coefficients of the expansions at the points $\varkappa_s$,

$$\psi(x,y,t,\kappa)=\sum_{j=0}^{\infty}\zeta_{j,s}(x,y,t)(\kappa-\varkappa_s)^j$$

satisfy the system of equations

$$\sum_{j=1}^{h_s}a_{ij}^s\zeta_{j,s}(x,y,t)=0. \tag{2.1}$$

Here we assume that $\deg q(x,y,t,\kappa)\leqslant N-1$. We denote by $\varphi_{\alpha j,s}(x,y,t)$ the polynomials

$$\varphi_{\alpha j,s}=\left[\frac{e^{-\kappa x-Q(\kappa)y-R(\kappa)t}}{j!}\ \frac{\partial^j}{\partial\kappa^j}\ \frac{\kappa^\alpha e^{\kappa x+Q(\kappa)y+R(\kappa)t}}{q_1(\kappa)}\right]_{\kappa=\varkappa_s}$$

for $\alpha \geqslant 1$,

$$\varphi_{0j,s} = \left[\frac{e^{-\kappa x - Q(\kappa)y - R(\kappa)t}}{j!} \frac{\partial^j}{\partial \kappa^j} e^{\kappa x + Q(\kappa) + R(\kappa)t} \right]_{\kappa = \varkappa_s}.$$

The Eqs. (2.1) can be rewritten in the form

$$\sum_{\alpha=0}^{N-1} \chi_\alpha \left(\sum_{j=1}^{h_s} a_{ij}^s \varphi_{\alpha j,s} \right) = -\sum_{j=1}^{h_s} \varphi_{0j,s} a_{ij}^s . \tag{2.2}$$

Thus, the coefficients of the polynomial $q(x,y,t,\kappa) = \sum_{\alpha=0}^{N-1} \chi_\alpha(x,y,t)\kappa^\alpha$, determined from the equations (2.2), are rational functions of their arguments. Their poles coincide with the zeros of the determinant of the matrix Θ, defined by the left-hand sides of the equations (2.2) (the rows of the matrix are indexed by α while the columns by the pairs (i,s), $1 \leqslant i \leqslant \ell_s$).

We note that from here there follows the possibility of the representation of $\psi(x,y,t,\kappa)$ in the form

$$\psi(x,y,t,\kappa) = \left(1 + \sum_{j=1} \frac{a_j(y,t,\kappa)}{x - x_j(y,t)}\right) e^{\kappa x + Q(\kappa)y + R(\kappa)t},$$

where $\prod_{j=1}^{M}(x - x_j(y,t)) = \mathrm{const} \times \det \Theta$.

<u>THEOREM 2.1.</u> There exist unique operators

$$L_1 = \sum_{i=0}^{n} u_i(x,y,t)\frac{\partial^i}{\partial x^i}, \quad L_2 = \sum_{i=0}^{m} v_i(x,y,t)\frac{\partial^i}{\partial x^i}$$

such that $(L_1 - \frac{\partial}{\partial y})\psi = (L_2 - \frac{\partial}{\partial t})\psi = 0$. Their coefficients are differentiable polynomials of the functions $\chi_\alpha(x,y,t)$ and, consequently, they depend rationally on their arguments.

<u>Proof.</u> We consider the expansion of $\psi(x,y,t,\kappa)$ in the neighborhood of infinity:

$$\psi(x,y,t,\kappa) = e^{\kappa x + Q(\kappa)y + R(\kappa)t}\left(1 + \sum_{s=1}^{\infty} \xi_s(x,y,t)\kappa^{-s}\right). \tag{2.3}$$

The functions $\xi_s(x,y,t)$ are linear combinations with constant coefficients of $\chi_\alpha(x,y,t)$; $\xi_1(x,y,t) = \chi_{N-1}(x,y,t), \ldots$

The coefficients of L_1 are obtained from the system of linear equations

$$\sum_{i=0}^{n} u_i \sum_{\ell=0}^{i} C_i^\ell \frac{\partial^{i-\ell}}{\partial x^{i-\ell}} \xi_{s+\ell} = \sum_{i=0}^{N} c_i \xi_{s+i}, \tag{2.4}$$

$$(s = -n+1, \ldots, 0;\ \xi_j = 0,\ j < 0).$$

This sytem is equivalent to the congruence

$$\left(\left(L_1-\frac{\partial}{\partial y}\right)\psi(x,y,t,\kappa)\right)e^{-\kappa x-Q(\kappa)y-R(\kappa)t}\equiv 0(\mathrm{mod}\,O(\kappa^{-1})). \tag{2.5}$$

The congruence (2.5) means that the function

$$\tilde{\psi}(x,y,t,\kappa)=\left(L_1-\frac{\partial}{\partial y}\right)\psi(x,y,t,\kappa)$$

has the form

$$\tilde{\psi}(x,y,t,\kappa)=\frac{\tilde{q}(x,y,t,\kappa)}{q_1(\kappa)}\,e^{\kappa x+Q(\kappa)y+R(\kappa)t},$$

where the degree of the polynomial $\tilde{q}(x,y,t,\kappa)$ does not exceed $N-1$.

Since the linear conditions (2.1) are invariant relative to the action of linear operators, it follows that they hold also for the coefficients of the expansion of the function $\tilde{\psi}(x,y,t,\kappa)$. An equivalent system of homogeneous equations on $\chi_\alpha(x,y,t)$ is $\tilde{q}=(x,y,t,\kappa)=\sum_{\alpha=0}^{N-1}\tilde{\chi}_\alpha(x,y,t)\kappa^\alpha$, whose left-hand sides coincide with the left-hand sides of the nonhomogeneous equations (2.2). The rank of this system is N; therefore, its solution is necessarily the zero solution. Consequently, $\tilde{\psi}(x,y,t,\kappa)=0$ or $(L_1-\frac{\partial}{\partial y})\psi=0$. The operator L_2 is constructed in a similar manner.

COROLLARY. The constructed operators satisfy the equation

$$\left[L_1-\frac{\partial}{\partial y},\ L_2-\frac{\partial}{\partial t}\right]=0\ .$$

In order to obtain the rational solutions of the Kadomtsev—Petviashvili equations it is necessary to take $Q(\kappa)=\kappa^2$, $R(\kappa)=\kappa^3$ in the scheme we have presented for the construction of the rational solutions of the general Zakharov—Shabat equations.

From Eqs. (2.4) it follows that $u(x,y,t)=-2\frac{\partial}{\partial x}\chi_{N-1}(x,y,t)$. Therefore, the poles of $u(x,y,t)$ coincide with the poles of $\chi_{N-1}(x,y,t)$ or, which is the same, with the zeros of the polynomial $\det\Theta$. By virtue of the fact that $u(x,y,t)$ is the total derivative of a rational function, the residues at these poles are equal to zero. Thus,

$$u(x,y,t)=\sum_{j=1}^{M}\frac{-2}{(x-x_j(y,t))^2}=2\frac{\partial^2}{\partial x^2}\ln\det\Theta\ . \tag{2.6}$$

For fixed y and t, the number M of the poles of the function $u(x,y,t)$ is equal to the degree of the polynomial $\det\Theta$. The degree of the polynomial $\varphi_{\alpha j,s}$ is equal to j, therefore, for example, for the matrix of general position A^s, satisfying the conditions $a^s_{ij}=0$ for $j>m(i,s)$, this number is equal to $M=\sum m(i,s)$. From here, a $2N$ parameter family of rational solutions of the KP equation, having N poles, is given by the functions $\psi(x,y,t,\kappa)$, for which condition (2.1) has the form

$$\frac{\partial}{\partial\kappa}\psi(x,y,t,\kappa)\Big|_{\kappa=\varkappa_s}=0\ . \tag{2.7}$$

In this case the matrix Θ will have the form (1.8) and, in combination with the equality (2.6), we obtain the proof of Theorem 1.3.

The solution of the KP equation and the general position correspond to the confluence of the points $\varkappa_s$. In this case the conditions (2.7) turn into the general conditions (2.1).

3. Rational Solutions of the Lax Type Equations. The Korteweg–de Vries Equations and the Boussinesq Equation

We present briefly the general scheme for the construction of the rational solutions of the equations of the Lax type. These equations describe the solutions of the Zakharov–Shabat equations, independent of the variable y and, consequently, they are equivalent to the operator equation:

$$\left[\frac{\partial}{\partial t}-L_1, L_2\right]=0. \tag{3.1}$$

The equations of Korteweg–de Vries and of Boussinesq

$$4u_t=6uu_x+u_{xxx}, \tag{3.2}$$

$$\frac{\partial^2 u}{\partial t^2}\pm\frac{\partial^2 u}{\partial x^2}+\frac{1}{4}\frac{\partial^4 u}{\partial x^4}+\frac{3}{2}\frac{\partial}{\partial x}\left(u\frac{\partial u}{\partial x}\right)=0. \tag{3.3}$$

also belong to them. We consider the function $\psi(x,t,\kappa)$ of the form

$$\psi(x,t,\kappa)=\left(1+\frac{q(x,t,\kappa)}{q_1(\kappa)}\right)e^{\kappa x+R(\kappa)t},$$

$$\deg q(x,t,\kappa)<\deg q_1(\kappa)=N.$$

We assume that the coefficients of the expansion of $\psi(x,t,\kappa)$ at zero,

$$\psi(x,t,\kappa)=\sum_{j=0}^{\infty}\zeta_j(x,t)\kappa^j$$

satisfy the conditions

$$\zeta_{j_i}(x,t)=0, \tag{3.4}$$

where j_i is the i-th number not divisible by n, $1\leqslant i\leqslant N$.

For this function, as for the proof of Theorem 2.1, we obtain

LEMMA 3.1. There exists a unique operator

$$L_1=\sum_{i=0}^{m}v_i(x,t)\frac{\partial^i}{\partial x^i}, \quad m=\deg R,$$

such that $\left(L_1-\frac{\partial}{\partial t}\right)\psi=0$.

Expanding $\psi(x,t,\kappa)$ at infinity, one can construct a unique operator L_2 such that

$$\left((L_2-\kappa^n)\psi(x,t,\kappa)\right)e^{-\kappa x-R(\kappa)t}\equiv 0(\operatorname{mod}O(\kappa^{-1})).$$

This equality means that

$$\psi(x,t,\kappa)=(L_2-\kappa^n)\psi(x,t,\kappa)$$

has the form

$$\frac{\tilde{q}(x,t,\kappa)}{q_1(\kappa)} e^{\kappa x + R(\kappa)t}, \quad \deg \tilde{q} < N.$$

Conditions 3.3 are invariant relative to the action of linear operators and relative to multiplication by κ^n and, therefore, they hold also for the coefficients of the expansion of the function $\tilde{\psi}(x,t,\kappa)$. As in Sec. 2, we find that from this we obtain the following lemma.

LEMMA 3.2. There exists a unique operator

$$L_2 = \sum_{i=0}^{n} u_i(x,t) \frac{\partial^i}{\partial x^i} \quad \text{such that} \quad L_2 \psi(x,t,\kappa) = \kappa^n \psi(x,t,\kappa).$$

COROLLARY. The constructed operators satisfy the equality (3.1).

Example 1. If $n=2$, $R(\kappa) = \kappa^3$, then each polynomial $q_1(\kappa)$ of degree N defines a pair of operators

$$L_2 = \frac{\partial^2}{\partial x^2} + u(x,t) \quad \text{and} \quad L_1, \ \deg L_1 = 3.$$

The function $u(x,t)$ is a solution of the Korteweg–de Vries equation (3.2).

As in Sec. 2, for $u(x,t)$ we obtain the formula

$$u(x,t) = 2\frac{\partial^2}{\partial x^2} \ln \det \Theta = \sum_{j=1}^{M} \frac{-2}{(x - x_j(t))^2}, \tag{3.5}$$

where $\Theta_{is} = \frac{\partial^{3N-2i-s}}{\partial x^{3N-2-s}} \frac{\partial^{2N+1}}{\partial \kappa^{2N+1}} e^{\kappa x + \kappa^3 t} \frac{1}{q_1(\kappa)}\Big|_{\kappa=0}$.

In order to obtain the last equality it is necessary to replace the expressions for $\zeta_j(x,t)$ in terms of $\chi_\alpha(x,t)$ $(\sum \chi_\alpha(x,t)\kappa^\alpha = q(x,t,\kappa)$ in Eqs. (3.4). They become

$$\theta \chi = e_0,$$

where $\chi = (\chi_0, \ldots, \chi_{N-1})$, while the s-th coordinate of the vector e_0 is equal to $\frac{1}{j_s!} \cdot \frac{\partial^{j_s}}{\partial \kappa^{j_s}} e^{\kappa x + \kappa^3 t}$. The number M of poles of these solutions is equal to $\frac{N(N+1)}{2}$.

Example 2. If $n=3$, $R(\kappa) = \kappa^2 \pm 1$, then each polynomial $q_1(\kappa)$ of degree N yields a solution $u(x,t)$ of the Boussinesq equation (3.3). Equality (3.5) holds for them, in which

$$\Theta_{is} = \frac{\partial^{4N-3i-s}}{\partial x^{4N-3i-s}} \frac{\partial^m}{\partial \kappa^m} \frac{e^{\kappa x + (\kappa^2 \pm 1)t}}{q_1(\kappa)}\Big|_{\kappa=0}, \quad m = \begin{cases} \frac{3N}{2} - 1, & N \equiv 0 \pmod 2 \\ \frac{3N}{2} - \frac{1}{2}, & N \equiv 1 \pmod 2. \end{cases}$$

The number of poles of $u(x,t)$ is equal to $\frac{N(N+2)}{4}$ if N is even and to $\frac{(N+1)^2}{4}$ if N is odd.

4. Rational Solutions of the Novikov Equations

We have already mentioned that the algebraic–geometric construction of the quasiperiodic solutions of the Zakharov–Shabat equations gives solutions of the commutativity equations of the extended algebra of operators, containing in addition to the operators $L_1 - \frac{\partial}{\partial y}$ and $L_2 - \frac{\partial}{\partial t}$

also the rings of the operators of the form $L_1=\sum_i \omega_i(x,y,t)\frac{\partial^i}{\partial x^i}$ isomorphic to the ring of functions on a nonsingular complex curve and having a unique pole at an isolated point. We show that the constructed rational solutions of the Zakharov–Shabat equations are separatrices of the family of quasiperiodic solutions in the following sense: there exists a ring of operators, commutative among themselves and with $L_1-\frac{\partial}{\partial y}$, $L_2-\frac{\partial}{\partial t}$, which is isomorphic to the ring of functions on a singular curve, birationally isomorphic to a complex line with a unique pole at "infinity."

Multiplication by any polynomial $P(k)$ gives a linear operator in the spaces of series of the form $\zeta=\sum_{j=0}^{\infty}\zeta_j(x,y,t)(k-\varkappa)^j$. We denote by $\mathfrak{A}$ the ring of the polynomials for which the corresponding homomorphisms form invariant subspaces, given by the equations (2.1).

THEOREM 4.1. If $\psi(x,t,y,k)$ is the function defined in Sec. 2, then for any polynomial $P(k)\in\mathfrak{A}$ there exists a unique operator $L_P=\sum_{i=0}^{\deg P} u_i(x,y,t)\frac{\partial^i}{\partial x^i}$ such that $L_P\psi(x,y,t,k)=P(k)\psi(x,y,t,k)$.

As a consequence of this we obtain that these operators commute among themselves and with the operators $L_1-\frac{\partial}{\partial y}$, $L_2-\frac{\partial}{\partial t}$.

The proof of the theorem is entirely similar to the proof of Lemma 3.2.

For example, if the conditions (2.1) have the form $\frac{\partial}{\partial k}\psi(x,y,t,k)\big|_{k=\varkappa_s}=0$, then $\mathfrak{A}_\varkappa$ is a ring of polynomials such that $\frac{\partial}{\partial k}P(k)$ is divisible by $\prod_s(k-\varkappa_s)$. If we set $t=0$ and $y=0$, i.e., we consider the function

$$\psi(x,k)=\left(1+\frac{q(x,k)}{q_1(k)}\right)e^{kx},$$

satisfying the conditions $\frac{\partial}{\partial k}\psi(x,k)\big|_{k=\varkappa_s}=0$, where $\deg q<\deg q_1=N$, then, by Theorem 4.1, it will give a homomorphism λ from the ring $\mathfrak{A}_\varkappa$, $\varkappa=(\varkappa_1,\ldots)$, into the ring of the ordinary differential operators.

The coefficients of these operators satisfy ordinary differential equations which are equivalent to the condition of their commutativity. These equations are called Novikov type equations.

Their general solution has been found in [4, 5]. Applying the methods of these papers, one obtains easily the following theorem.

THEOREM 4.2. For any subring A of the ring of the linear differential operators isomorphic to $\mathfrak{A}_\varkappa$, there exists a polynomial $q_1(k)$ such that the function $\psi(x,u)$ corresponding to it and to the conditions $\frac{\partial}{\partial k}\psi\big|_{k=\varkappa_s}=0$, gives a homomorphism of A and $\mathfrak{A}_\varkappa$.

The corresponding solutions of the Novikov equations are rational functions.

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LIMIT STATES FOR MODIFIED NAVIER—STOKES EQUATIONS IN THREE-DIMENSIONAL SPACE

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One gives the description of the limit set $\mathfrak{M}_R$ (when $t\to\infty$) for all the trajectories (solutions) of the system

$$\frac{\partial\vec{v}}{\partial t}-\nu\Delta\vec{v}+\sum_{K=1}^{3}v_K\frac{\partial\vec{v}}{\partial x_K}+\operatorname{grad}p=\vec{f},\quad \operatorname{div}\vec{v}=0$$

where $\nu=\mu_0+\mu_1\int_\Omega \vec{v}_x^2(x,t)\,dx$, $\mu_i=\mathrm{const}>0$, satisfying the boundary condition $\vec{v}|_{\partial\Omega}=0$ at the boundary of the bounded domain Ω and emanating at $t=0$ from the points of the sphere $K_R=\{\vec{a}(x):\vec{a}(x)\in\overset{\circ}{J}(\Omega),\|\vec{a}\|_{2,\Omega}\leqslant R\}$. In particular, it is proved that for R greater than some R_0, the semigroup $V_t, t\geqslant 0$, which corresponds to this problem can be extended to a group $V_t, t\in R^1$, possessing a series of interesting properties.

In [1], for the two-dimensional Navier—Stokes equations

$$v_t-\nu\Delta v+v_k v_{x_k}=-\nabla p+f(x),\quad \operatorname{div} v=0, \tag{1}$$

in a bounded domain $\Omega\subset E^2$, with the boundary conditions

$$v|_{\partial\Omega}=0^{*)} \tag{2}$$

we have considered the following question: what flows can be seen by an observer during the lapse of a very large (mathematically infinite) period of time, if $f(x)$ and Ω are fixed and the initial fields $v(x,0)$ take all values from the Hilbert space $\overset{\circ}{J}(\Omega)$? In the case of small Reynolds numbers, the answer is simple: he will contemplate the unique stationary solution of the problem (1), (2). However, as it is known with the increase of the Reynolds

*The statements in [1] hold also for other boundary conditions; for example, for periodic conditions with respect to one or both variables x_k.

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