COMMUTATIVE RINGS OF ORDINARY LINEAR DIFFERENTIAL OPERATORS

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In the modern theory of exact solutions of the Zakharov—Shabat equations, which include a number of fundamental equations of mathematical physics, commutative rings of ordinary linear differential operators play an important role.

The coefficients of the operators of such a ring form an invariant finite-dimensional space on which the constraints of the original operators can be integrated using methods of algebraic geometry (see [10]).

In [11-13] a classification was obtained of commutative rings of differential operators in one variable containing a pair of operators of mutually prime orders. The remarkable papers [2-4] have recently become well-known, although they were forgotten for a long time; the classification just mentioned was already (locally with respect to x) obtained in these papers. However, the method for establishing the coefficients of commuting operators was not sufficiently effective. For instance, the key result of the modern theory which asserts that the commutativity equations are completely integrable Hamiltonian systems with a set of polynomial integrals, and that the coefficients of rings of operators in general position are almost-periodic functions, was unknown. The relation between commutative algebras and the spectral theory of operators and the theory of linear Floquet equations with periodic coefficients is also an achievement of the modern theory.

An abstract algebraic exposition of the construction of the author proposed by Drinfel'd in [5] has made it possible to obtain interesting but, unfortunately, noneffective results in the problem of the classification of commutative rings of differential operators with orders which are not mutually prime, the geometric meaning of which was pointed out by Mumford [14].

In [10, Sec. 2, pp. 191-193] an idea was stated for effective analytic construction of commutative ordinary differential operators of arbitrary orders. The ideal itself is correct, but its implementation in [10, p. 192] contains some serious errors. In this paper these errors are corrected, and the solution of the problem of classifying commutative rings of differential operators in one variable corresponding to nonsingular Riemann surfaces ("rings in general position") is thereby completed.

In Sec. 1 we discuss in more detail a comparison of two approaches to describing commutative algebras, the approach of the author and that of [5, 14].

The methods of this paper have other applications as well. We recall that the construction of exact solutions of nonlinear partial differential equations does not require solving the classification problem of commutative rings of ordinary differential operators of orders which are multiples of l, but requires rather the construction of the $(l \times l)$ matrix analog of the multiparameter Baker—Akhiezer functions (see [10, Sec. 1]).

The corresponding constructions were undertaken by S. P. Novikov and the author. They give a large class of solutions of the Zakharov—Shabat equations depending on functional parameters, and in particular, they give solutions of the Kadomtsev—Petviashvili equation

$$\left[\frac{\partial}{\partial y} - L,\ \frac{\partial}{\partial t} - A\right] = 0,$$

where

$$L = \frac{d^2}{dx^2} + u(x, y, t), \qquad A = \frac{d^3}{dx^3} + \frac{3}{2} u \frac{d}{dx} + w(x, y, t).$$

This work of Novikov and the author will be published shortly in the journal "Funktsional'nyi Analiz i Ego Prilozheniya."

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1. Algebraic "Spectral Data" for Commutative Rings of Differential Operators

We consider a system of Novikov equations, i.e., a system of equations for the coefficients of the operators

$$L_1 = \sum_{i=0}^{n} u_i(x) \frac{d^i}{dx^i}, \quad L_2 = \sum_{i=0}^{m} v_i(x) \frac{d^i}{dx^i},$$

which is equivalent to the commutativity condition $[L_1, L_2] = 0$.

For the sake of definiteness, we agree that the coefficients of operators are scalar functions. Moreover, let $v_m = u_n = 1$, $u_{n-1} = 0$. The last restrictions are unimportant since they can always be made to hold by a change of the variable x and a suitable conjugation $\tilde{L}_1 = u(x) L_1 u^{-1}(x)$, $\tilde{L}_2 = u(x) L_2 u^{-1}(x)$.

The following result lies at the heart of the application of the methods of algebraic geometry in solving the Novikov equations.

THEOREM 1.1 (Burchnall and Chaundy [4]). There exists a polynomial $Q(w, E)$ in two variables such that $Q(L_2, L_1) = 0$.

Proof. The operator L_2 on the space $\mathcal{L}(E)$ of solutions of the equation $L_1 y = Ey$ defines a linear operator $L_2(E)$. Its matrix coefficients $L_2^{ij}(E)$ in the canonical basis $c_j(x, E; x_0)$, $\frac{d^i}{dx^i} c_j(x, E; x_0)\big|_{x=x_0} = \delta_{ij}$, $0 \leqslant i, j \leqslant n-1$, are polynomials in E. Let $Q(w, E) = \det(w \cdot 1 - L_2^{ij}(E))$ be the characteristic polynomial of $L_2(E)$. The kernel of the operator $Q(L_2, L_1)$ contains $\mathcal{L}(E)$ for all E and is therefore infinite dimensional. Hence the operator itself is zero.

In order to study questions associated with the compactification of an affine curve defined by the equation $Q(w, E)$ and the behavior of the common eigenfunctions of the operators L_1 and L_2 at infinity, we introduce the term of a formal Bloch function for each operator.

LEMMA 1.2. There exists a unique solution of the equation

$$L_1 \psi(x, k) = k^n \psi(x, k) \tag{1.1}$$

in the space of formal power series of the form

$$\psi(x, k) = e^{k(x-x_0)} \Big(\sum_{s=N}^{\infty} \xi_s(x) k^{-s} \Big)$$

(N an integer) with the "normalization" condition $\xi_s = 0$, $s < 0$, $\xi_0(x) = 1$. $\xi_s(x_0) = 0$, $s > 1$. We denote this solution by $\psi(x, k; x_0)$. Any other solution of this type is equal to $\psi(x, k) = A(k)\psi(x, k; x_0)$, $A(k) = \sum_{s=N}^{\infty} A_s k^{-s}$.

A proof of this proposition and many important corollaries which we omit is contained in [13].

The operator L_2 leaves the space of solutions of Eq. (1.1) invariant. Hence by the above lemma, $L_2 \psi(x, k; x_0) = A(k) \psi(x, k; x_0)$, where $A(k) = k^m + \sum_{s=-m+1}^{\infty} A_s k^{-s}$.

The functions $\psi(x, k_j; x_0)$, $k_j^n = E$, form a basis of the space $\tilde{\mathcal{L}}(E)$, which is an eigenbasis for $L_2(E)$. The space $\tilde{\mathcal{L}}(E)$ is generated by $\psi(x, k_j; x_0)$ over the field of Laurent series in the variable k^{-1}. In this space, the matrix elements in the corresponding canonical basis are the same as in $\mathcal{L}(E)$. Hence $Q(w, E) = \prod_{j=0}^{n-1} (w - A(k_j))$.

If the values of the series A(k) are different for distinct n-th roots of E the curve $\mathfrak{R}$ is irreducible and can be completed at infinity by a single point P_0 in a neighborhood of which a local parameter is given by $E^{-1/n}(P)$. In addition, this means that for large, and hence for almost all E, the eigenvalues of the operator $L_2(E)$ are distinct. Since this case has been discussed in detail in the previous papers [10, 13], we turn at once to the general case.

The series A(k) is such that its values at certain n-th roots of E coincide if and only if there exists a series $\check{A}(k)$ such that $A(k) = \check{A}(k^l)$. Since the leading term in A(k) is equal to k^m, l is a common divisor of n and m.

In this case,

$$Q(w, E) = \prod_{j=0}^{n-1} (w - A(k_j)) = \prod_{j=0}^{n'-1} (w - \check{A}(\widetilde{k}_j))^l = \widetilde{Q}^l(w, E),$$

where $k_j^n = E, \widetilde{k}_j^{n'} = E, n'l = n$.

We keep the notation $\mathfrak{R}$ for the curve defined by the equation

$$\widetilde{Q}(w, E) = \prod_{j=0}^{n'-1} (w - \check{A}(\widetilde{k}_j)) = 0.$$

which is irreducible. At infinity $\mathfrak{R}$ is completed by the single point P_0 in a neighborhood of which $E^{-\frac{1}{n'}}(P)$ is a local parameter.

Each point P of the curve $\mathfrak{R}$, i.e., each pair P = (w, E), $\widetilde{Q}(w, E) = 0$, corresponds to an l-dimensional subspace of eigenvectors of $L_2(E)$ with eigenvalue $w = w(P)$. We choose in this subspace a basis with the normalization conditions

$$\frac{d^i}{dx^i} \psi_j(x, P; x_0)\big|_{x=x_0} = \delta_{ij}, \qquad 0 \leqslant i,\ j \leqslant l-1.$$

All the remaining coordinates of these vectors in the canonical basis of $\mathscr{L}(E)$ are meromorphic functions $\chi_j^i(P; x_0)$ on the curve $\mathfrak{R}$. Their poles coincide with P_0 and the zeros of the determinant of the diagonal minor of the matrix $w \cdot 1 - L_2(E)$ formed by elements with indices $l \leqslant i, j \leqslant n-1$.

The vector functions $\chi_j(P; x_0)$ with coordinates $\chi_j^i(P; x_0)$ define an algebraic subbundle $\eta(x_0)$ of dimension l in the trivial l-dimensional bundle over $\mathfrak{R}$. This subbundle is in essence the starting point of the abstract algebraic approach [5]. How can the dependence of $\eta(x_0)$ be determined? For $l = 1$, it has been determined by differential equations, and its properties play a large role in the papers [6-7, 16, 10-11]. For $l > 1$, as is pointed out in [14], the situation is more complicated. The "possible" translations of η turn out to be covered by a nonintegral l-distribution on the space of moduli of l-dimensional sheaves over $\mathfrak{R}$ with a fixed flag at the point P_0. Variation of the point x_0 defines a path tangent to this distribution. At this point the studies in [5, 14] are concluded.

The method in [10] does not consist in describing x_0 for variations of the sheaf, but finding the eigenfunctions $\psi_j(x, P; x_0)$, x_0 = const, themselves which generalize the Baker–Akhiezer functions (see [1-2, 6, 10, 12]).

Since the basis functions $c_i(x, E; x_0)$ are entire functions of E,

$$\psi_j(x, P; x_0) = \sum_{i=0}^{n-1} \chi_j^i(P; x_0)\, c_i(x, E; x_0)$$

is meromorphic away from P_0 with pole divisor D not depending on x. (In general, D depends on x_0.)

We consider the Wronskian matrix $\Psi(x, P; x_0)$ for the functions $\psi_j(x, P; x_0)$. The matrix function $\left(\frac{d}{dx}\Psi\right)\Psi^{-1}$ does not depend on the choice of the basis ψ_j. Therefore, in order to find its behavior in a neighborhood of P_0, we can make use of the formal series $\psi(x, \widetilde{k}_j; x_0)$, $\widetilde{k}_j^l = k$, $k^{-1} = k^{-1}(P)$ the local parameter near P_0.

Since $\psi^{(l)}(x, \widetilde{k}_j; x_0)$ can be written uniquely in the form

$$\psi^{(l)}(x, \widetilde{k}_j; x_0) = \sum_{s=0}^{l-1} \left(\frac{d^s}{dx^s} \psi(x, \widetilde{k}_j; x_0)\right) \lambda_s(x, k),$$

where $\lambda_s(x, k)$ can be written uniquely in the form k^{-1}, $\lambda_0 = k + \widetilde{u}_0(x) + O(k^{-1})$, $\lambda_s = \widetilde{u}_s(x) + O(k^{-1})$, $1 \leqslant s \leqslant l-2$, $\lambda_{l-1} = O(k^{-1})$, then

$$\left(\frac{d}{dx}\Psi\right)\Psi^{-1}=\begin{pmatrix} 0 & 1 & 0 & \ldots & 0 & 0\\ 0 & 0 & 1 & \ldots & 0 & 0\\ \cdot & \cdot & \cdot & \ldots & \cdot & \cdot\\ \cdot & \cdot & & \ldots & \cdot & \cdot\\ \cdot & \cdot & & \ldots & \cdot & \cdot\\ 0 & 0 & \cdot & \ldots & 0 & 1\\ k+\tilde{u}_0, & \tilde{u}_1 & \cdot & \ldots & \tilde{u}_{l-2} & 0 \end{pmatrix}+O(k^{-1}). \tag{1.2}$$

The functions $\tilde{u}_s(x)$ are differential polynomials in the coefficients of the initial operator L_1.

We return again to the study of the divisor D of the poles of the $\psi_j(x, P; x_0)$. For almost all solutions of the Novikov equations, i.e., for almost all commutative rings of differential operators, the curve $\mathfrak{R}$ is nonsingular. Let $P_1, \ldots, P_{n'}$ be the inverse images of the point E under the projection $E:\mathfrak{R}\to C, E(P_i)=E$. They are distinct for almost all E. We construct the Wronskian matrix $F(x, E; x_0)$ in the functions $\psi_j(x, P_i; x_0)$. The function $g(x, E; x_0) = (\det F)^2$ does not depend on the order of enumeration of the points P_i and is therefore well defined as a function of the variable E.

By hypothesis, the penultimate coefficient of the operator L_1 is zero, $u_{n-1} = 0$. Thus, the Wronskian for any basis in $\mathscr{L}(E)$ does not depend on x, so that $g(E; x_0) = g(x, E; x_0)$ also does not depend on x. The values of all the derivatives of $\psi_j(x, P; x_0)$ at $x = x_0$ are rational functions on $\mathfrak{R}$. Hence $g(E; x_0)$ is a rational function of the variable E. Its zeros coincide with the points E for which the eigenvalues of $L_2(E)$ coalesce. Moreover, the order of a zero is equal to $l\nu$, where ν is the multiplicity of a ramification point of the curve $\mathfrak{R}$. (The multiplicity of a ramification point is the number of sheets of $\mathfrak{R}$, coalescing at that point, minus 1; for curves in general position, $\nu = 1$.)

We will assume that all the poles of $\psi_j(x, P; x_0)$ are simple, i.e., D is a set of distinct points $\gamma_1, \ldots, \gamma_N$. We let $\varphi_{i,j}(x)$ denote the residue of the function $\psi_j(x, P; x_0)$ at the point γ_i. Then the poles of $g(E; x_0)$ coincide with the images of the poles γ_i and the point at infinity. Moreover, the multiplicity of the pole of $g(E; x_0)$ at the image of γ_i equals $2\varkappa_i$, twice the number of linearly independent functions $\varphi_{i,j}(x)$. The equality of the number of zeros and poles of $g(E; x_0)$ gives the relation

$$l\sum\nu = 2\sum_{i=1}^{N}\varkappa_i + N_\infty.$$

Using (1.2) we find the multiplicity N_∞ of the pole of $g(E; x_0)$ at infinity. It equals $l(n' - 1)$, where n' is the number of sheets of $\mathfrak{R}$ over C.

The usual expression for the genus of a curve in terms of the multiplicity of the ramification points of $\mathfrak{R}$ gives the equality $\sum_{i=1}^{N}\varkappa_i = lg$, where g is the genus of $\mathfrak{R}$.

In what follows we consider the case of "general position," which corresponds to the so-called "stable fibrations," so that all $\varkappa_i = 1$, i.e., the degree of the divisor D is lg, and among the functions $\varphi_{i,j}(x)$ there is only one linearly independent function for each fixed i. Thus, for each point γ_i there exist $l - 1$ constants $\alpha_{i,j}$ such that $\varphi_{i,j}(x) = \alpha_{i,j}\varphi_{i,l-1}(x)$, $0 \leqslant j \leqslant l-2$ (we may assume that the complement of this closed set is $\varphi_{i,l-1}(x) \neq 0$). Here "general position" means that his component has maximal dimension $l^2 g$. For arbitrary sets $\varkappa_i$, the solution of the inverse problem can be obtained in exact analogy to our construction which follows. We show that the dimension of the corresponding component is smaller. The corresponding parameters are the coefficients of the expansion of $l - \varkappa_i$ functions with respect to $\varkappa_i$, which give a basis among the $\varphi_{i,j}$. The dimension is equal to

$$\sum_{i=1}^{N}(l-\varkappa_i)\varkappa_i + N = l^2 g - \sum_{i=1}^{N}\varkappa_i^2 + N < l^2 g,$$

provided at least one of the numbers $\varkappa_i \neq 1$.

The set of points γ_i with associated vector multiplicities $\vec{\alpha}_i = (\alpha_{i,j})$ is a characteristic of the so-called "matrix divisors" defined by generic sheaves with a fixed "normalization," i.e., set of basis sections [17].

LEMMA 1.3. The matrix divisor $D_M = (\gamma_i, \alpha_{i,j})$ and the functions $\tilde{u}_0(x), \tilde{u}_1(x), \ldots, \tilde{u}_{l-2}(x)$ uniquely determine the eigenfunctions of the operators $\psi_j(x, P; x_0)$ for almost all sets $\alpha_{i,j}$ (belonging to the complement of a closed set).

Proof. It follows from the Riemann–Roch theorem that for every set of polynomials $q_0(k), \ldots, q_{l-1}(k)$ there exists a unique vector function $\chi_0(P), \ldots, \chi_{l-1}(P)$ such that all its coordinates have simple zeros at the points γ_i, and their residues $\chi_{i,j}$ are related by $\chi_{i,j} = \alpha_{i,j}, \chi_{i,l-1}$, while in a neighborhood of P_0 the congruences $\chi_j(P) \equiv q_j(k) (\mathrm{mod}\, O(k^{-1}))$ hold.. Indeed, the dimension of the space of vector functions associated with the divisor D and having a given singularity in a neighborhood of P_0 is equal to $l(l-1)g$, which equals the number of equations for the residues, which for an open set of $\alpha_{i,j}$ may be assumed to be independent.

Thus, every row vector of the matrix $\frac{d^s}{dx^s}\Psi|_{x=x_0}$ is uniquely determined by its singular parts in a neighborhood of P_0, where $\Psi(x, P; x_0)$ is the Wronskian matrix of the functions $\psi_i(x, P; x_0)$. Using Eq. (1.2), we find the desired singular parts of $\frac{d^s}{dx^s}\Psi|_{x=x_0}$ successively, which completes the proof of the lemma.

Conversely, if we start from the singular parts of the matrices $\frac{d^s}{dx^s}\Psi|_{x=x_0}$ obtained, we find the derivatives, and hence also functions $v_0(x), \ldots, v_{l-2}(x)$ such that the following statement holds.

LEMMA 1.4. In a neighborhood of P_0 the vector function $\psi(x, P; x_0) = (\psi_0(x, P; x_0), \ldots, \psi_{l-1}(x, P; x_0))$ can be written in the form

$$\psi(x, P; x_0) = \Big(\sum_{s=0}^{\infty} \xi_s(x) k^{-s}\Big) \Psi_0(x, k; x_0), \quad k = k(P),$$

where $\Psi_0(x, k; x_0)$ is a solution of the equation

$$\frac{d}{dx}\Psi_0(x, k; x_0) = \begin{pmatrix} 0 & 1 & \cdot & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & 0 & 1 & \cdot & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 1 \\ k + v_0, & \cdot & \cdot & & & & v_{l-2} & 0 \end{pmatrix} \Psi_0(x, k; x_0) \tag{1.3}$$

with the normalization $\Psi_0(x_0, k; x_0) = 1$, the identity matrix.

The values of the vectors $\xi_s(x)$ at x_0 are equal to $\xi_0(x_0) = (1, 0, \ldots, 0)$, $\xi_s(x_0) = 0$, $s > 1$.

The curve $\mathfrak{R}$ is called the spectrum of the commutative ring A containing the operators L_1, L_2. The functions $\psi_j(x. P; x_0)$ are eigenfunctions for all the operators in A. The set consisting of $\mathfrak{R}$, P_0, the matrix divisor $D_M = (\gamma_i, \alpha_{i,j})$, and functions $v_0(x), \ldots, v_{l-2}(x)$ is called the "algebraic spectral data" and it determines the ring A uniquely. In the next section we solve the problem of reconstructing A in terms of its data.

2. Reconstruction of Commutative Rings of Differential Operators from "Algebraic Spectral Data"

We consider the space $\mathscr{L}$ of vector functions $\varphi(x, P; x_0)$, having poles at an arbitrary set of points γ_i, $1 \leqslant i \leqslant lg$, of a nonsingular complex curve $\mathfrak{R}$ of genus g and which can be expressed in a neighborhood of a distinguished point P_0 in the form

$$\varphi(x, P; x_0) = \Big(\sum_{s=0}^{\infty} \xi_s(x) k^{-s}\Big) \Psi_0(x, k; x_0),$$

where $k^{-1} = k^{-1}(P)$ is a local parameter. Here $\Psi_0(x, k; x_0)$ is the matrix defined in the preceding paragraph in terms of the functions $v_0(x), \ldots, v_{l-2}(x)$.

LEMMA 2.1. The dimension of $\mathscr{L}$ is equal to $l(l-1)g + l$.

Proof. Let Γ be the boundary of a small neighborhood of P_0. We write $\mathfrak{R}^+$ and $\mathfrak{R}^-$ for the exterior and interior domains into which Γ separates the curve $\mathfrak{R}$. The vector functions

$\varphi^+(x, P; x_0) = \varphi(x, P; x_0)$, if $P \in \mathfrak{R}^+$, and $\varphi^-(x, P; x_0) = \varphi(x, P; x_0)\Psi_0^{-1}(x, k(P); x_0)$, if $P \in \mathfrak{R}^-$, are meromorphic functions in $\mathfrak{R}^+$ and $\mathfrak{R}^-$, respectively. Hence φ^+ and φ^- are a solution of the classical Riemann boundary problem

$$\varphi^+(x, t; x_0) = \varphi^-(x, t; x_0)\Psi_0(x, k(t); x_0), \quad t \in \Gamma, \tag{2.1}$$

$$(\varphi_j) + D \geqslant 0. \tag{2.2}$$

Condition (2.2) means that the poles of all the coordinates $\varphi_j(x, P; x_0)$ occur among the points γ_i.

The converse assertion that each solution of the boundary problem (2.1)-(2.2) gives a function $\varphi(x, P; x_0) \in \mathcal{L}$ is also true.

Following [9] (see also [18]) we discuss an algorithm for solving this boundary problem.

We consider the function f(P) with poles at γ_i and a zero of order $lg - g$ at the point P_0. Such a function exists and is unique up to a proportionality constant. Let Ω^+ and Ω^- be the functions

$$\Omega^+(x, P; x_0) = f^{-1}(P)\,\varphi^+(x, P; x_0),$$
$$\Omega^-(x, P; x_0) = k^{-1}(P)\,\varphi^-(x, P; x_0).$$

These functions are a solution of the boundary problem

$$\Omega^+(x, t; x_0) = \Omega^-(x, t; x_0)\Psi_0(x, k(t); x_0)\,k(t)\,f^{-1}(t), \qquad t \in \Gamma, \tag{2.3}$$

$$(\Omega_j) \geqslant \Delta. \tag{2.4}$$

The last condition means that the poles of the coordinates of Ω_j occur among the points $q_1, \ldots, q_g$ not equal to P_0 which are zeros of f(P), and the Ω_j vanish at P_0. The divisor $\Delta = q_1 + \ldots + q_g - P_0$.

Let A(p, q)dp denote the meromorphic analog of the Cauchy kernel on $\mathfrak{R}$, having the following properties: It is an Abelian differential with respect to the variable p, and with respect to the variable q it has poles at $q_1, \ldots, q_g$ and a zero at P_0.

As $p \to q$ we have the relation

$$A(p, q)\,dp = \frac{dp}{p-q} + \text{regular terms.} \tag{2.5}$$

In order to construct A(p, q)dp we introduce the basis $a_1, \ldots, a_g, b_1, \ldots, b_g$ of canonical cycles on $\mathfrak{R}$ with intersection matrix $a_i \circ b_j = \delta_{ij}$, $a_i \circ a_j = b_i \circ b_j = 0$. Let $d\omega_{qq_0}(p)$ be a third-order differential with zero a-periods, $\int_{a_i} d\omega_{qq_0}(p) = 0$, and two simple poles at the points p = q and p = q_0 with residues +1 and −1, respectively. This differential is a multivalued analytic function of q. We fix some branch on $\mathfrak{R}$, cut along the cycles a_i. If dw_i is a basis of holomorphic differentials on $\mathfrak{R}$, normalized by the condition $\int_{a_i} d\omega_k = \delta_{ik}$, then the desired differential A(p, q)dp is given by the formula (see [8])

$$\frac{\begin{vmatrix} d\omega_{qq_0}(p) & d\omega_1(p) & \ldots & d\omega_g(p) \\ d\omega_{qq_0}(q_1) & d\omega_1(q_1) & \ldots & d\omega_g(q_1) \\ \vdots & \vdots & \ldots & \vdots \\ d\omega_{qq_0}(q_g) & d\omega_1(q_g) & \ldots & d\omega_g(q_g) \end{vmatrix}}{\begin{vmatrix} d\omega_1(q_1) & \ldots & d\omega_g(q_1) \\ \vdots & \ldots & \vdots \\ d\omega_1(q_g) & \ldots & d\omega_g(q_g) \end{vmatrix}}. \tag{2.6}$$

The Sokhotskii–Plemelj formulas for the limit values of integrals of Cauchy type:

$$\Phi(q) = \frac{1}{2\pi i}\int_\Gamma \varphi(t)\,A(t, q)\,dt, \quad \Phi^+(t) + \Phi^-(t) = \frac{1}{\pi i}\int_\Gamma \varphi(\tau)\,A(\tau, t)\,d\tau, \tag{2.7}$$

$$\Phi^+(t) - \Phi^-(t) = \varphi(t) \tag{2.8}$$

follow from (2.5).

Let $\Omega(x, P; x_0)$ be a solution of the boundary problem (2.3)-(2.4). Then

$$\Omega(x, P; x_0) = \frac{1}{2\pi i}\int_\Gamma \varphi(x, t; x_0) A(t, P)\, dt, \tag{2.9}$$

where $\varphi(x, t; x_0) = \Omega^+(x, t; x_0) - \Omega^-(x, t; x_0)$. Equations (2.7), (2.8), and boundary condition (2.3) imply that $\varphi(x, t; x_0)$ is a solution of the system of singular equations

$$\varphi(x, t; x_0)\left[\frac{G+1}{2}\right] + \left[\frac{1}{2\pi i}\int_\Gamma \varphi(x, \tau; x_0) A(\tau, t)\, d\tau\right]\frac{G-1}{2} = 0, \tag{2.10}$$

$$G = \Psi_0(x, t; x_0) f^{-1}(t) k(t).$$

Conversely, to every solution of system (2.10) there corresponds by Eq. (2.8) a solution of the boundary problem (2.3)-(2.4). We show that this solution is unique. If $\Omega_1(x, P; x_0)$ is another solution with the same jump $\Omega_1^+(x, t; x_0) - \Omega_1^-(x, t; x_0) = \varphi(x, t; x_0)$, then the vector $\Omega(x, P; x_0) - \Omega_1(x, P; x_0)$ is already continuous on the contour Γ, and therefore it is a meromorphic function on $\mathfrak{R}$, and each component has g poles and vanishes at the point P_0. It follows from the Riemann–Roch theorem that each component of this vector function must be zero, i.e., $\Omega(x, P; x_0) = \Omega_1(x, P; x_0)$.

Thus, the number of linearly independent solutions of the boundary problem (2.3)-(2.4) is equal to the number of linearly independent solutions of system (2.10). Since the initial divisors are nonspecial, this number is equal to the index of the system of equations

$$\varkappa = [\arg \det G]_\Gamma,$$

i.e., to the increase in the argument of det G in going around the curve $\mathfrak{R}$.

Since $\det \Psi_0 = 1$, we have $\varkappa = l[\arg k(t) - \arg f(t)]_\Gamma$. Each term of the sum is equal to the difference of the number of zeros and poles of k and f^{-1}, i.e., $\varkappa = l(l-1)g + l$.

Methods for solving a system of singular equations are discussed in [15].

COROLLARY. There exists a unique vector function $\psi(x, P; x_0) \in \mathcal{L}$, such that the residues of its coordinates $\varphi_{i,j}(x)$, $0 \leqslant j \leqslant l-1$, at the points γ_i are related by $\varphi_{i,j} = \alpha_{i,j}\varphi_{i,l-1}$, $0 \leqslant j \leqslant l-2$, and

$$\psi(x, P; x_0)\Psi_0^{-1}(x, k(P); x_0)|_{P=P_0} = (1, 0, \ldots, 0).$$

Here $\alpha_{i,j}$ is a set of complex numbers in general position.

We let $\mathfrak{A}(\mathfrak{R}, P_0)$ be the ring of meromorphic functions on $\mathfrak{R}$, having their only pole at P_0.

LEMMA 2.2. If $E(P) \in \mathfrak{A}(\mathfrak{R}, P_0)$ is any function, there exists a unique operator L of degree ln, where n is the order of the pole of E(P) at P_0, such that $L\psi_i(x, P; x_0) = E(P)\psi_i(x, P; x_0)$.

Proof. Let $\Psi(x, P; x_0)$ be the Wronskian matrix for the functions $\psi_i(x, P; x_0)$. As follows from the definition of $\psi_i(x, P; x_0)$, **in a neighborhood of P_0 it can be written in the form**

$$\Psi(x, P; x_0) = \left(\sum_{s=0}^{\infty} \xi_s(x) k^{-s}\right)\Psi_0(x, k; x_0).$$

There exists a unique operator $\bar{L}$ with matrix coefficients

$$\bar{L} = \sum_{\alpha=0}^{n} w_\alpha(x)\frac{d^{\alpha l}}{dx^{\alpha l}}$$

such that

$$(\bar{L}\Psi)\Psi^{-1} = E(P)\cdot 1 \qquad (\text{mod } O(k^{-1})). \tag{2.11}$$

If the matrix functions $\chi_{m,j}(x)$ are defined by the equality

$$\frac{d^j}{dx^j}\Psi_0(x, k) = \left[\sum_{m=0}^{N(j)} \chi_{m,j}(x) k^m\right]\Psi_0(x, k), \quad N(j) = [jl^{-1}],$$

the coefficients of the operator $\bar{L}$ can be found from the system of equations for $s = -n, \ldots, 0$

$$\sum_{\alpha=0}^{n} w_\alpha \sum_{j=0}^{\alpha l} \sum_{m=0}^{N(j)} C_{\alpha l}^{j} \left(\frac{d^{\alpha l-j}}{dx^{\alpha l-j}} \xi_{s+m}\right) \chi_{m,j} = \sum_{\alpha=0}^{n} \varepsilon_\alpha \xi_{s+\alpha},$$

$$E(P) = \sum_{\alpha=0}^{n} \varepsilon_\alpha k^\alpha \,(\mathrm{mod}\, O(k^{-1})).$$

We consider the operator

$$L = \sum_{\alpha=0}^{n} \sum_{j=1}^{l} w_\alpha^{1j}(x) \frac{d^{\alpha l+j-1}}{dx^{\alpha l+j-1}}$$

with scalar coefficients. By the construction of L the functions $L\psi_i - E\psi_i$ satisfy all the requirements determining ψ_i except for one condition. The expansion in a neighborhood of P_0 of the regular vector $[(L - E(P))\psi(x, P; x_0)]\psi_0^{-1}(x, k(P; x_0)$ starts with terms of order $O(k^{-1})$. It follows from the uniqueness of $\psi(x, P; x_0)$ that the congruence (2.11) is an exact solution, i.e., $L\psi_i(x, P; x_0) = E(P)\psi_i(x, P)$.

By the lemma just proved, each set of functions $v_0(x), \ldots, v_{l-2}(x)$ and matrix divisor $D_M = (\gamma_i, \alpha_{i,j})$ defines via the functions $\psi_j(x, P; x_0)$ to which they correspond a homomorphism λ of the ring $\mathfrak{A}(\mathfrak{R}, P_0)$ into the ring of linear differential operators.

Summarizing these results, we obtain the following theorem.

THEOREM 2.3. For any commutative ring A of differential operators there exists a curve $\mathfrak{R}$ with distinguished point P_0 such that $\mathfrak{A}(\mathfrak{R}, P_0)$ is isomorphic to A. For almost all rings A the curve $\mathfrak{R}$ is nonsingular. Moreover, there exists a matrix divisor $(\gamma_i, \alpha_{i,j})$, $1 \leqslant i \leqslant lg$, $0 \leqslant j \leqslant l-2$, where g is the genus of the curve $\mathfrak{R}$, and a set of functions $v_0(x), \ldots, v_{l-2}(x)$ such that the image of the homomorphism λ determined by them coincides with A up to a change of variable $x = f(x')$ and conjugation by some function, $A = u(x) \mathrm{Im}\, \lambda\, u^{-1}(x)$. The number l is the greatest common divisor of the orders of the operators in A.

3. Induced Deformations of Vector Sheaves over Algebraic Curves

It was already mentioned above how in contrast to the method of [5, 14] our approach to the classification of commutative rings of differential operators does not require calculation of the deformations in x_0 of the vector sheaf $\eta(x_0)$ defined by the coordinates of the common eigenfunctions of the operators L_1 and L_2 in the canonical basis $c_i(x, E; x_0)$ of the space of solutions of the equations $L_1 y = Ey$ (see Sec. 1). Nevertheless, dynamical systems with "control parameters" $u_0(x), \ldots, u_{l-2}(x)$ are of interest.

In [5, 14] a nonintegrable fibration (for $l > 1$) is found on the space of moduli of stable sheaves of rank l with a fixed flag at a point. It should be noted that the construction of the sheaf $\eta(x_0)$ itself does not fix a flag, but rather a normalization, i.e., basis of sections. There exists a simple parametrization in the space of "stable" sheaves of rank l with a fixed normalization obtained with the aid of matrix divisors. That is, in general position, this parametrization is given by the set of points γ_i with associated "vector multiplicities" $\alpha_{i,j} \in \mathbf{C}$, $0 \leqslant j \leqslant l-2$ (see [17]). In this parametrization, we find a fibration covering the fibration on the space of moduli of sheaves with flags which was constructed in [5, 14].

Let $\psi_s(x, P; x_0)$, $0 \leqslant s \leqslant l-1$, be as before, common eigenfunctions of the operators L_1 and L_2 corresponding to a nonsingular curve $\mathfrak{R}$. These functions are meromorphic away from P_0 with constant poles at the points $\gamma_i(x_0)$, $1 \leqslant i \leqslant lg$.

It follows from (1.2) that there exists a set of functions $\chi_j(x, P)$ rational on $\mathfrak{R}$ such that

$$\frac{d^l}{dx^l}\psi_s(x, P; x_0) = \sum_{j=0}^{l-1} \chi_j(x, P) \frac{d^j}{dx^j}\psi_s(x, P; x_0). \tag{3.1}$$

In a neighborhood of P_0 these functions have the form

$$\chi_0(x, P) = k(P) + u_0(x) + O(k^{-1}),$$
$$\chi_s(x, P) = u_s(x) + O(k^{-1}),\ 1 \leqslant s \leqslant l-2, \tag{3.2}$$
$$\chi_{l-1}(x, P), = O(k^{-1}).$$

Away from P_0 the poles of $\chi_j(x, P)$ coincide with the zeros $\gamma_i(x)$ of the determinant of the Wronskian matrix $\Psi(x, P; x_0)$.

We remark that by Lemma 1.4 and the fact that for $l > 1$, $\det \Psi_0(x, k; x_0) = 1$, it follows that $\det \Psi(x, P; x_0)$ is a rational function with poles at $\gamma_i(x_0)$ and zeros at $\gamma_i(x)$. Thus the divisors $D(x_0)=\sum\limits_i \gamma_i(x_0)$ and $D(x)=\sum\limits_i \gamma_i(x)$ are equivalent. The divisor $D(x)$ defines a one-dimensional sheaf, viz., the determinant of the sheaf $\eta(x)$. Thus, $\det \eta(x)$ does not depend on x.

We denote by $\alpha_{ij}(x)$ the ratios of the residues of the functions $\chi_j(x, P)$ at the points $\gamma_i(x)$, i.e.,

$$c_{i,j} = \alpha_{i,j}\, c_{i,l-1}, \qquad 0 \leqslant j \leqslant l-2,$$
$$\chi_j(x, k) = \frac{c_{i,j}(x)}{k-\gamma_i(x)} + d_{i,j}(x) + O(k-\gamma_i(x)), \tag{3.3}$$

where $k - \gamma_i(x)$ is a local parameter in a neighborhood of $\gamma_i(x)$.

Since $\Psi(x, P; x_1) = \Psi(x, P; x_0)\Psi^{-1}(x_1, P; x_0)$, the desired dependence of the sheaf $\eta(x_0)$ on x_0 is given by the dependence on x of the sets $\gamma_i(x), \alpha_{i,j}\, x)$, if we put $x = x_0$.

It follows from Eq. (3.1) that $\chi_{l-1}(x, P) = (\det \Psi)'/\det \Psi$, and therefore the corresponding residues $c_{i,l-1}(x)$ are equal to $c_{i,l-1}(x) = -\gamma_i'(x)$. Since the left-hand side of Eq. (3.1) has no singularities for $P = \gamma_i(x)$, the $\alpha_{i,j}(x)$ are solutions of the system of equations

$$\sum_{j=0}^{l-2} \alpha_{i,j}(x) \frac{\partial^j}{\partial x^j} \psi_s(x, P; x_0) + \frac{\partial^{l-1}}{\partial x^{l-1}} \psi_s(x, P; x_0) = 0, \tag{3.4}$$

$s = 0, \ldots, l-1$.

The symbol $\partial/\partial x$ indicates that $P = \gamma_i(x)$ is taken constant in the differentiation.

We differentiate these equalities with respect to x:

$$\sum_{j=0}^{l-2} \alpha_{i,j}(x) \frac{\partial^{j+1}}{\partial x^{j+1}} \psi_s(x, P; x_0) + \frac{\partial^l}{\partial x^l} \psi_s(x, P, x_0) +$$
$$+ \gamma_i'(x) \Big(\sum_{j=0}^{l-2} \alpha_{i,j}(x) \frac{\partial^{j+1}}{\partial x^j \partial k} \psi_s(x, k(P); x_0\Big) + \frac{\partial^l}{\partial x^{l-1} \partial k} \psi_s(x, k(P); x_0)) +$$
$$+ \sum_{j=0}^{l-2} \frac{d}{dx} \alpha_{i,j}(x) \frac{\partial^j}{\partial x^j} \psi_s(x, P; x_0) = 0;\ \gamma_i' = \frac{\partial k(P)}{\partial x}. \tag{3.5}$$

Equalities (3.1), (3.3) give

$$\frac{\partial^l}{\partial x^l} \psi_s(x, P; x_0) = \sum_{j=0}^{l-1} d_{i,j}(x) \frac{\partial^j}{\partial x^j} \psi_s(x, P; x_0) + \sum_{j=0}^{l-1} c_{i,j}(x) \frac{\partial^{j+1}}{\partial x^j \partial k} \psi_s(x, k(P); x_0).$$

Substituting this expression into (3.5) and using the fact that $c_{i,j} = -\gamma_i' \alpha_{i,j}$, $0 \leqslant j \leqslant l-2$, and $c_{i,l-1} = -\gamma_i'(x)$, we have

$$\Big[\sum_{j=0}^{l-2} \alpha_{i,j}(x) \frac{\partial^{j+1}}{\partial x^{j+1}} \psi_s + \frac{d}{dx} \alpha_{i,j} \frac{\partial^j}{\partial x^j} \psi_s + d_{i,j} \frac{\partial^j}{\partial x^j} \psi_s\Big] + d_{i,l-1} \frac{\partial^{l-1}}{\partial x^{l-1}} \psi_s = 0.$$

Hence we get from the fact that the solutions of the last system are proportional to the original solutions $\alpha_{i,j}(x)$ of the system of equations (3.4) that

$$\alpha_{i,0}(d_{i,l-1}+\alpha_{i,l-2})=d_{i,0}+\frac{d}{dx}\alpha_{i,0},$$
$$\alpha_{i,j}(d_{i,l-1}+\alpha_{i,l-2})=d_{i,j}+\frac{d}{dx}\alpha_{i,j}+\alpha_{i,j-1}, \quad (3.6)$$
$$1\leqslant j\leqslant l-2.$$

These equations permit construction of an $(l-1)$-dimensional distribution on an open set of the space of collections γ_i, $\alpha_{i,j}$, i.e., on an open subset of the product $S^{lg}\mathfrak{R}\times C^{l-1}$ of a symmetric power of a curve and a linear space C^{l-1}.

By the Riemann–Roch theorem, each such collection uniquely determines a set of rational functions $\chi_j(P)$ with poles at γ_i having the form

$$\chi_0(P)=k(P)+u_0+O(k^{-1}),\ \chi_s(P)=u_s+O(k^{-1}),\ 1\leqslant s\leqslant l-2,$$
$$\chi_{l-1}(P)=O(k^{-1})$$

in a neighborhood of P_0, and such that the ratio of the residues $c_{i,j}$ at the points γ_i is equal to $\alpha_{i,j}$ ($\alpha_{i,j}$ in general position),

$$\alpha_{i,j}c_{i,l-1}=c_{i,j},\ 0\leqslant j\leqslant l-2.$$

Here $u_0,\ldots,u_{l-2}$ are arbitrary numbers which parametrize the fibration. For fixed values of $u_0,\ldots,u_{l-2}$ we define a vector with coordinates $\gamma_i=-c_{i,l-1}$ and $\alpha_{i,j}$, satisfying Eqs. (3.6).

The set of functions $u_0(x),\ldots,u_{l-2}(x)$ determines a path tangent to the fibration constructed. Conversely, each such path with initial point $\gamma_i;\alpha_{i,j}$ makes it possible to reconstruct the commutative ring of differential operators. Indeed, using these data the functions $\chi_j(x, P)$ are constructed, followed by the functions $\psi_s(x, P; x_0)$ which are solutions of Eq. (3.1) with normalization conditions $\frac{d^i}{dx^i}\psi_s(x,P;x_0)\big|_{x=x_0}=\delta_{is}$.

These functions are common eigenfunctions of the original operators. We will not give a detailed discussion of the construction in terms of the $\psi_s(x, P; x_0)$ of the operators themselves in the framework of this section.

LITERATURE CITED

1. N. I. Akhiezer, "A continuous analog of orthogonal polynomials on a system of intervals," Dokl. Akad. Nauk SSSR, 141, No. 2, 263-266 (1961).
2. H. F. Baker, "Note on the foregoing paper 'Commutative ordinary differential operators'," Proc. R. Soc. London, 118, 584-593 (1928).
3. J. L. Borchnall and T. W. Chaundy, "Commutative ordinary differential operators. I," Proc. London Math. Soc., 21, 420-440 (1922).
4. J. L. Burchnall and T. W. Chaundy, "Commutative ordinary differential operators. II," Proc. R. Soc. London, 118, 557-583 (1928).
5. V. G. Drinfel'd, "On commutative subrings of some noncommutative rings," Funkts. Anal. Prilozhen., 11, No. 1, 15-31 (1977).
6. B. A. Dubrovin, "Periodic problem for the Korteweg–de Vries equation in the class of finite-zone potentials," Funkts. Anal. Prilozhen., 9, No. 3, 41-51 (1975).
7. B. A. Dubrovin, V. B. Matveev, and S. P. Novikov, "Nonlinear equations of Korteweg–de Vries type, finite-zone linear operators, and Abelian varieties," Usp. Mat. Nauk, 31, No. 1, 55-136 (1976).
8. É. I. Zverovich, "Boundary problems in the theory of analytic functions," Usp. Mat. Nauk, 26, No. 1, 113-181 (1971).
9. W. Koppelman, "Singular integral equations, boundary-value problems, and the Riemann–Roch theorem," J. Math. Mech., 10, No. 2, 247-277 (1961).
10. I. M. Krichever, "Methods of algebraic geometry in the theory of nonlinear equations," Usp. Mat. Nauk, 32, No. 6, 183-208 (1977).
11. I. M. Krichever, "An algebrogeometric construction of the Zakharov–Shabat equations and their periodic solutions," Dokl. Akad. Nauk SSSR, 227, No. 2, 291-294 (1976).
12. I. M. Krichever, "Algebraic curves and commuting matrix differential operators," Funkts. Anal. Prilozhen., 10, No. 2, 75-77 (1976).
13. I. M. Krichever, "Integration of nonlinear equations by methods of algebraic geometry," Funkts. Anal. Prilozhen., 11, No. 1, 15-31 (1977).

14. D. Mumford, "An algebrogeometric construction of commuting operators and of solutions to the Toda lattice equation, KdV equation and related nonlinear equations," Harvard Univ. Preprint (1977).
15. N. I. Muskhelishvili, Singular Integral Equations [Russian translation], Fizmatgiz, Moscow (1962).
16. S. P. Novikov, "The periodic Korteweg—de Vries problem. I," Funkts. Anal. Prilozhen., 8, No. 3, 54-66 (1974).
17. A. N. Tyurin, "Classification of vector bundles over an algebraic curve," Izv. Akad. Nauk SSSR, Ser. Mat., 29, 658-680 (1965).
18. Yu. L. Rodin, "Riemann boundary problem for differentials on Riemann surfaces," Uch. Zap. Perm. Univ., 17, No. 2, 83-85 (1960).

UNITARY REPRESENTATIONS OF THE INFINITE-DIMENSIONAL CLASSICAL GROUPS $U(p, \infty)$, $SO_0(p, \infty)$, $Sp(p, \infty)$ AND THE CORRESPONDING MOTION GROUPS

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0. Introduction

0.1. We denote by $U(\infty)$, $SO(\infty)$, $Sp(\infty)$ the completions with respect to the operator norm of the groups $\bigcup_n U(n)$, $\bigcup_n SO(n)$, $\bigcup_n Sp(n)$. They are infinite-dimensional Banach Lie groups. Kirillov [1] discovered the following remarkable fact: He found that it is possible to classify all their irreducible unitary representations, and gave a heuristic explanation why these groups lack the pathological features which seemed to be inevitable in the infinite-dimensional situation. It is natural to try to take the next step and study the representations of the infinite-dimensional analogs of the noncompact classical groups. The purpose of this paper is to point out that there exists for the groups indicated in the title a quite substantial theory which exhibits many analogies with the finite-dimensional case. In particular, all these groups are of type I in the sense of von Neumann and it is possible to construct many of their irreducible unitary representations.

0.2. We describe the contents of this paper (for notation, see Sec. 0.3). In Sec. 1 a proof of the main result of [1] is given. It follows in its entirety the scheme indicated in [1], with the exception of the derivation of Kirillov's theorem from his Lemma 3, which differs from the original proof (which remains unpublished). The approach proposed here (Secs. 1.6-1.8) is based on a mapping into a certain semigroup, the "unitary trick," and a theorem of Nelson [2]. This approach is then generalized in Sec. 4. It is moreover shown that every unitary representation of $K(\infty)$ decomposes into a discrete sum of irreducibles. In Sec. 2 "Laplace operators" are constructed on $K(\infty)$ which separate the irreducible representations. It is proved in Sec. 3 that $G(p, \infty)$ and $M(p, \infty)$ are of type I in the sense of von Neumann, that their reducible representations can be disintegrated, and that the reduction of any irreducible representation to $K(p) \times K(\infty)$ has a spectrum of finite multiplicity (sometimes of multiplicity one). Thus, $K(p) \times K(\infty)$ plays the role of a maximal compact subgroup. In Sec. 4 an exposition is given of the construction of the irreducible unitary representations.

The results of this paper were announced in [3].

0.3. Notation. F denotes any of the fields **C**, **R** or the division ring of quaternions H; $Mat_{m,n}(F)$ is the space of matrices over F with m rows and n columns; L is a separable (right) Hilbert space over F with a fixed basis $l_1, l_2, \ldots$; $K(n)$ (= $U(n)$, $SO(n)$ or $Sp(n)$), the connected component of the group of isometries of the quadratic form $\bar{x}^1x^1 + \ldots + \bar{x}^nx^n$ in $L_n = l_1F + \ldots + l_nF \subset L$, viewed also as a group of operators in L; $K(\infty)$ is the closure of the group $K^0(\infty) = \bigcup K(n)$ in the operator norm; $K_m(n)$, $K_m^0(\infty)$, and $K_m(\infty)$ are the stabilizers of $\{l_1, \ldots, l_m\}$ in $K(n)$, $K^0(\infty)$, and $K(\infty)$, respectively; $G(p, q)$ (= $U(p, q)$, $SO_0(p, q)$, or $SP(p, q)$) is the connected component of the identity of the group of isometries of the quadratic

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