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FORMAL GROUPS AND THE ATIYAH-HIRZEBRUCH FORMULA

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Abstract. In this article, manifolds with actions of compact Lie groups are considered. For each rational Hirzebruch genus $h: \Omega_* \to Q$, an "equivariant genus" h^G, a homomorphism from the bordism ring of G-manifolds to the ring $K(BG) \otimes Q$, is constructed. With the aid of the language of formal groups, for some genera it is proved that for a connected compact Lie group G, the image of h^G belongs to the subring $Q \subset K(BG) \otimes Q$. As a consequence, extremely simple relations between the values of these genera on bordism classes of S^1-manifolds and submanifolds of its fixed points are found. In particular, a new proof of the Atiyah-Hirzebruch formula is obtained.

Bibliography: 10 items.

In [1] it was proved that the signature of every S^1-manifold X is equal to the sum of the signatures of the submanifolds F_s of its fixed points:

$$\mathrm{Sign}([X]) = \sum_s \mathrm{Sign}([F_s]).$$

As a unitary variant of this formula, there is a relation between the values of the classical T_y-genus of an S^1-manifold, i.e. an almost complex manifold on which the action of S^1 preserves the complex structure in the stable tangent bundle, and its fixed submanifolds:

$$T_y([X]) = \sum_s (-y)^{\epsilon_s^-} T_y([F_s]).$$

Here ϵ_s^- is the number of summands in the decomposition of the representation of S^1 in the fibers of the normal bundle over the submanifold F_s into irreducible representations $\eta^{j_{si}}$ (the action of $z \in S^1$ in η^k is multiplication by z^k) with $j_{si} < 0$. We shall denote by ϵ_s^+ the number of the remaining summands.

The previous proof of both formulas in [1] is based on the index theorem of Atiyah and Singer. In this article they are obtained as a consequence of a fundamentally different approach, whose essence is reduced to the study of the analytic properties of the Conner-Floyd expressions.

We recall these expressions for actions with isolated fixed points (see [2], [3] and

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[4]; for an arbitrary action they were first obtained in [5], whose fomulas were improved in [6]).

Assume that the action of S^1 on X has only isolated fixed points p_s and that the representation of the group in the fibers of the tangent bundle over them is $\sum_{i=1}^n \eta^{j_{si}}$.

If $[u]_j$ is the jth power of u in the formal group of "geometric cobordisms"

$$f(u, v) = g^{-1}(g(u) + g(v)),\ g(u) = \sum_{n=0}^{\infty} \frac{[CP^n]}{n+1} u^{n+1},$$

i.e. $[u]_j = g^{-1}(jg(u))$, then the equalities of Conner and Floyd assert that the Laurent series

$$\Phi(u) = \sum_s \prod_{i=1}^n \frac{1}{[u]_{j_{si}}},$$

with coefficients in $U^* \otimes Q$, contains only the right part, and the independent tem is the bordism class of X.

To each rational Hirzebruch genus h: $U^* \to Q$ there corresponds a numerical realization $\Phi(u)$, a series in $1 - \eta$ with rational coefficients:

$$\Phi_h(\eta) = \sum_s \prod_{i=1}^n \frac{1}{g_h^{-1}(\ln \eta^{j_{si}})},$$

where $g_h^{-1}(t)$ is functionally inverse to the logarithm

$$g_h(t) = \sum_{n=0}^{\infty} \frac{h([CP^n])}{n+1} t^{n+1}$$

of the fomal group $f_h(u, v)$ corresponding to the homomorphism h. We note that $\Phi_h(\eta)$ is the image of the series $\Phi(u)$, to which there corresponds a cobordism class in $U^*(CP^\infty) \otimes Q = U^*[[u]] \otimes Q$, under the homomorphism $\widetilde{h}$: $U^*(CP^\infty) \otimes Q \to K(CP^\infty) \otimes Q$ induced by the genus h. The existence of the transformation of functors $\widetilde{h}$ follows from [7].

We shall assume that the series $g_h^{-1}(\ln \eta)$ is the expansion about 1 of an analytic function in some neighborhood of that point; then from the Conner-Floyd equalities it follows that $\Phi_h(\eta)$ is analytic in some neighborhood of 1, and that $\Phi_h(1) = h([X])$.

Our task is to prove that if the function $g_h^{-1}(\ln \eta)$ is analytic in the disk $|\eta| < 2$ and does not have zeros there, except at 1, then $\Phi_h(\eta)$, which could have poles at the roots of 1, is also analytic in the disk $|\eta| < 2$. From this it follows that, for a genus h such that $g_h^{-1}(\ln \eta)$ is a rational function with one simple zero at 1, i.e. $g_h^{-1}(\ln \eta) = (\eta - 1)/(a\eta + b)$, $a + b = 1$, $\Phi_h(\eta)$ is analytic everywhere and hence is a constant. Then its value at 1, equal to $h([X])$, coincides with

$$\lim_{\eta \to \infty} \Phi_h(\eta) = \sum_s a^{\varepsilon_s^+} (-b)^{\varepsilon_s^-}.$$

Precisely in this way, a new proof of the Atiyah-Hirzebruch formula will be obtained; for the two-parameter genus $T_{x,y}$ (the value of $T_{x,y}$ on the bordism class

$[CP^n]$ is equal to $\Sigma_{i=0}^{n} x^{n-i}(-y)^i$; we note that $T_{1,y}$ coincides with the T_y-genus) we shall obtain the relation

$$T_{x,y}([X]) = \sum_s x_s^{\varepsilon_s^+} (-y)^{\varepsilon_s^-} T_{x,y}([F_s]).$$

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§1. "Characteristic" homomorphisms for G-bundles

To each characteristic class $\chi \in U^i(BU)$ of complex vector bundles in unitary cobordism there corresponds a "characteristic" homomorphism

$$\chi^*: U_n(BU(k)) \longrightarrow U^{-n+i} = U^{-n+i}(pt),$$

which associates to a bundle ξ over the manifold X the image of the cobordism class $\chi(\xi)$ under the composite

$$U^i(X) \xrightarrow{D} U_{n-i}(X) \to U_{n-i} \cong U^{-n+i},$$

where D is the duality homomorphism.

In this section, an analogous homomorphism for complex G-bundles (here and in what follows G is a compact Lie group) will be constructed and studied.

1. Consider the category of complex G-bundles over unitary G-manifolds. Two such bundles ξ_1 and ξ_2 over G-manifolds X_1 and X_2 are bordant if there exists a G-bundle ζ over W such that $\partial W = X_1 \cup -X_2$ and the restriction of ζ to X_i, $i = 1, 2$, coincides with ξ_i. The bordism groups obtained in this way will be denoted by $U^G_{n,k}$, where $k = \dim_C \xi$ and $n = \dim_R X$. In the case when $G = \{e\}$ is trivial, $U^e_{n,k}$ coincides with $U_n(BU(k))$. Then $U^G_{*,*}$ becomes a ring and a U_*-module in the usual way. The submodule $U^G_{*,0}$ is identified with the bordism module U^G_* of unitary G-manifolds.

We shall denote by X_G the space $(X \times EG)/G$, and by ξ_G the image of the G-bundle ξ under the homomorphism

$$\mathrm{Vect}_G(X) \longrightarrow \mathrm{Vect}(X_G).$$

If $p_!: U^*(X_G) \to U^*(BG)$ is the Gysin homomorphism induced by the projection $p: X_G \to BG$, then the formula

$$\chi^G([\xi]) = p_!(\chi(\xi_G))$$

defines an "equivariant characteristic" homomorphism

$$\chi^G: U^G_{n,k} \to U^{-n+i}(BG).$$

Its relation with χ is given by the following lemma.

Lemma 1.1. *Let $U^G_{n,k} \to U^e_{n,k}$ be the homomorphism which "forgets" the G-action; then the diagram*

$$\begin{array}{ccc} U^G_{n,k} & \xrightarrow{\chi^G} & U^{-n+l}(BG) \\ \downarrow & & \downarrow \\ U^e_{n,k} & \xrightarrow{\chi^e} & U^{-n+l} \end{array}$$

is commutative.

The proof of the lemma follows immediately from the definition of the Gysin homomorphism and from the fact that the restriction to the fiber X of the fibering $X_G \to BG$ of ξ_G coincides with ξ.

In what follows, it will be convenient to denote the "characteristic" homomorphism 1^G corresponding to $1 \in U^0(BU)$ by

$$\chi_0^G:\ U_*^G \to U^*(BG).$$

2. Let H be a normal subgroup of G. The set of fixed points under the action of H on a unitary G-manifold X is the disjoint union of almost complex submanifolds F_s (not necessarily connected). The normal bundles ν_s over the submanifolds F_s have a natural complex G-bundle structure.

As is known, there exists an equivariant embedding of X into the space of a unitary representation $\widetilde{\Delta}$ of the group G. (To avoid repeating conditions, we shall agree that in this section and in the next one, only unitary manifolds, bundles, representations, etc., will be considered.) Denote the restriction of the normal G-bundle over X in the space of the representation $\widetilde{\Delta}$ to the submanifold F_s by $(-\widetilde{\nu}_s)$. It is evident that the sum $\nu_s \oplus (-\widetilde{\nu}_s)$ is a trivial G-bundle. Let Δ be a maximal direct summand of $\widetilde{\Delta}$ whose restriction to the subgroup H does not contain trivial representations of H. In an analogous way we select a direct summand $(-\nu_s)$ in the G-bundle $(-\widetilde{\nu}_s)$.

Theorem 1.1. *Let ξ_s be the restriction of a complex G-bundle ξ over a G-manifold X to a submanifold F_s. Then, with the notation introduced above,*

$$e(\Delta_G)\,\chi^G([\xi]) = \sum_s p_{s!}\left(e\left((-\nu_s)_G\right)\cdot \chi(\xi_{sG})\right),$$

where $p_{s!}: U^*(F_{sG}) \to U^*(BG)$ is the Gysin homomorphism and, for an arbitrary bundle ζ, $e(\zeta)$ is the Euler class of ζ.

Proof. The composite of the embedding of X in the space of the representation $\widetilde{\Delta}$ and the projection on the direct summand Δ defines an equivariant map $h: X \to \Delta$.

For each bundle ζ we shall denote by $E\zeta$ its total space and by $S\zeta$ its sphere bundle. $(\Delta \times EG)/G$ coincides with $E\Delta_G$ by definition. Here and in what follows, Δ denotes both the representation and its space.

Let $h \times \mathrm{id}: X \times EG \to \Delta \times EG$; then the corresponding quotient map

$$\widetilde{h}: X_G \to (\Delta \times EG)/G$$

induces a Gysin homomorphism

$$\widetilde{h}_!: U^*(X_G) \to U^*(E\Delta_G,\ S\Delta_G).$$

Lemma 1.2. *Let $i^*: U^*(E\Delta_G, S\Delta_G) \to U^*(BG)$ be the homomorphism from the* *exact sequence of the pair; then for every* $x \in U^*(X_G)$ *the following equality is satisfied*:

$$e(\Delta_G)\, p_!(x) = i^* \widetilde{h}_!(x).$$

Proof. From the definition of the Gysin homomorphism it follows immediately that

$$h_!(x) = t(\Delta_G)\, p_!(x),$$

where $t(\Delta_G)$ is the Thom class of Δ_G. Apply to both sides of this equality the homomorphism i^*. The lemma follows from the fact that $i^* t(\Delta_G) = e(\Delta_G)$.

We shall state a simple corollary of Lemma 1.2. We denote by $I^*(G)$ the ideal of $U^*(BG)$ consisting of those cobordism classes which are annihilated by multiplication by the Euler classes of bundles associated with representations of G.

Corollary. *If the action of the group* G *on the manifold* X *has no fixed points, then the image of* $p_!$: $U^*(X_G) \to U^*(BG)$ *belongs to the ideal* $I^*(G)$.

Proof. If H coincides with G, then, by definition of $\widetilde{h}$, the nonexistence of fixed points implies that the image of X_G belongs to $S\Delta_G$. This means that $i^*\widetilde{h}_!$ is a trivial homomorphism. By Lemma 1.2, the image of $p_!$ is annihilated by multiplication by $e(\Delta_G)$.

If we return to the proof of the theorem, we note that for an arbitrary action of G on X, $\widetilde{h}$ maps the pair (X_G, N_G) to the pair $(E\Delta_G, S\Delta_G)$. Here N is the complement of tubular neighborhoods of the fixed points under the action of H. The restriction of $\widetilde{h}$ to a closed tubular neighborhood of F_s defines a map of pairs $\widetilde{h}_s$: $(E\nu_{sG}, S\nu_{sG}) \to (E\Delta_G, S\Delta_G)$ which induces a Gysin homomorphism:

$$\widetilde{h}_{s!} : U^*(E\nu_{sG}) = U^*(F_{sG}) \to U^*(BG).$$

Lemma 1.3. *If* f_s: $E\nu_{sG} \to X_G$ *is the inclusion, then*

$$\widetilde{h}_{s!} \circ f_s^* : U^*(X_G) \to U^*(BG) \quad \textit{and} \quad \sum_s \widetilde{h}_{s!} \circ f_s^* = i^* \circ \widetilde{h}_!.$$

Proof. The Gysin homomorphisms induced by the maps of the commutative diagram

$$\begin{array}{ccc} E\Delta_G & \xrightarrow{i} & (E\Delta_G, S\Delta_G) \\ \uparrow \widetilde{h} & & \uparrow \\ X_G & \to & (X_G, N_G) \end{array}$$

also form a commutative diagram:

$$\begin{array}{ccc} U^*(E\Delta_G, S\Delta_G) & \xrightarrow{i^*} & U^*(BG) \\ \uparrow \widetilde{h}_! & & \uparrow \\ U^*(X_G) & \longrightarrow & U^*(X_G \setminus N_G) \end{array}$$

The natural identification of $X_G \setminus N_G$ with the disjoint union of the $E\nu_{sG}$ will yield the lemma.

Lemma 1.4. *For* $x \in U^*(F_{sG})$ *the following equality holds*:

$$\widetilde{h}_{s!}(x) = p_{s!}(x\, e((-\nu_s)_G)).$$

Proof. The map $\tilde{h}_s$ can be factored as the composite

$$\begin{array}{ccc} & (E(p_s^*\Delta_G), S(p^*\Delta_G)) & \\ {}^{g}\nearrow & & \searrow^{p_s} \\ (E\nu_{sG}, S\nu_{sG}) & \xrightarrow{\tilde{h}_s} & (E\Delta_G, S\Delta_G) \end{array}$$

where g is the quotient map of the equivariant map

$$E\nu_s \times EG \to F_s \times \Delta \times EG,$$

obtained from the projection $E\nu_s \to F_s$, the equivariant inclusion of $E\nu_s$ in Δ and the identity map of EG. This means that $\tilde{h}_{s!}(x) = p_{s!}g_!(x)$. The map g is the identity on the base of the bundles; therefore $g^*(x) = x$ and $g_!(x) = g_!(g^*(x)) = xg_!(1)$. It remains to show that $g_!(1) = e((-\nu_s)_G)$.

According to the definition of $\tilde{h}$, g is the embedding with normal bundle $(-\nu_s)_G$. However, for any bundles ζ_1 and ζ_2 over the common base, the "diagonal" section of $\pi^*\zeta_2$ (π: $E(\zeta_1 \oplus \zeta_2) \to Y$ is the projection on the base) is transversal to the zero section and their intersection is the image of the embedding i: $(E\zeta_1, S\zeta_1) \to (E(\zeta_1 \oplus \zeta_2), S(\zeta_1 \oplus \zeta_2))$. From the definition of the Euler class and of the homomorphism $i_!$ it follows that $i_! = e(\zeta_2)$, which concludes the proof of the lemma.

The theorem follows immediately from the preceding lemmas and from the fact that $f_s^*(\chi(\xi_G)) = \chi(\xi_{sG})$.

Remark. In the case when the subgroup H coincides with the group G, Theorem 1.1 gives a relation between the value of the homomorphism χ^G on the bordism class of the G-bundle ξ and invariants in the cobordisms of the fixed submanifolds.

3. Along with the expression for $\chi_0^{S^1}$ given by Theorem 1.1, in what follows we shall need a modification of it, which will be obtained precisely in this subsection.

Let F_s be a connected component of the set of fixed points under the action of S^1 on an S^1-manifold X. The normal bundle ν_s, like every complex S^1-bundle over a trivial S^1-manifold, will be represented in the form $\Sigma_{j\neq 0}\nu_{sj} \otimes \eta^j$, where η^j is the jth tensor power of the standard representation of $S^1\eta$, as in the Introduction (see [8]).

The collection of complex bundles ν_{sj}, of which only a finite number are different from zero, defines a bordism class belonging to the group

$$R_n = \Sigma U_l\left(\prod_{j\neq 0} BU(n_j)\right).$$

The summation is taken over all collections of nonnegative integers n_j and l such that $2\Sigma n_j + l = n$.

The sum over all the connected components of these classes gives the image of the bordism class of the S^1-manifold X, $[X, S^1] \in U_n^{S^1}$, under the homomorphism β: $U_*^{S^1} \to R_*$.

We choose as generators of the U_*-module $U_*(CP^\infty) = U_*(BU(1))$ the bordism classes $(CP^n) \in U_{2n}(CP^\infty)$ corresponding to the inclusion of CP^n in CP^∞ or, what is the same, the canonical bundle $\eta_{(n)}$ over CP^n. The standard multiplication in R_* allows us to choose as generators of the U_*-module R_*, in this case, the monomials

$$(CP_{j_1}^{l_1}) \times \ldots \times (CP_{j_r}^{l_r}).$$

It will be convenient to denote by η not only the canonical representation of S^1 but also the corresponding canonical bundle over CP^∞; that is, $\eta_{S^1} = \eta$. Then for the S^1-bundle $\eta_{(n)}$ over CP^n, the bundle $(\eta_{(n)} \otimes \eta)_{S^1}$ over $CP^n \times CP^\infty$ is equal to $\eta_{(n)} \otimes \eta$. The Euler class of $(-\eta_{(n)}) \otimes \eta$, where $(-\eta_{(n)})$ is the n-dimensional bundle complementary to $\eta_{(n)}$, is defined by

$$e((-\eta_{(n)}) \otimes \eta) f(u, v) = u^{n+1},$$

where $f(u, v) = e(\eta \otimes \eta) = u + v + \sum \alpha_{ij} u^i v^j$, the formal group of "geometric" cobordisms. Hence, if

$$A_n(u, v) = e((\eta_{(n)}) \otimes \eta) \in U^*(CP^n \times CP^\infty) = U^*[[u, v]]/v^{n+1} = 0,$$

then

$$A_n(u, v) = \frac{u^n}{\frac{1}{u} f(u, v)}.$$

Let $B^n(u)$ be the image of $A_n(u, v)$ under the Gysin homomorphism $U^*(CP^n \times CP^\infty) \to U^*(CP^\infty)$ induced by the projection. We note that for $A_n(u, v)$ this homomorphism corresponds to the substitution of $[CP^{n-k}]$ for v^k.

Theorem 1.1 immediately yields the following assertion.

Theorem 1.2. *There exists a U_*-module homomorphism $\Psi: R_* \to U^*[[u]] \otimes Q[u^{-1}]$ such that $\Psi \circ \beta$ coincides with the composition of $\chi_0^{S^1}$ and the inclusion $U^*(CP^\infty) \to U^*[[u]] \otimes Q[u^{-1}]$. The values of Ψ on the generators of the U_*-module are given by the formula*

$$\Psi\left(\prod_{m=1}^{r} (CP_{j_m}^{l_m})\right) = \prod_{m=1}^{r} \left(\frac{1}{[u]_{j_m}}\right)^{l_m+1} B_{l_m}([u]_{j_m}), \qquad [u]_i = e(\eta^i).$$

4. Consider an arbitrary homomorphism $\alpha: G_1 \to G$ of Lie groups. It induces a map $\alpha_*: BG_1 \to BG$ of universal classifying spaces and hence a homomorphism $\alpha^*: U^*(BG) \to U^*(BG_1)$.

On the other hand, by means of α each G-bundle becomes a G_1-bundle, i.e. there exists a homomorphism

$$\alpha^{\#}: U_{*,*}^{G} \to U_{*,*}^{G_1}.$$

The commutative diagram

$$\begin{array}{ccc} X_{G_1} & \to & X_G \\ \downarrow & & \downarrow \\ BG_1 & \to & BG \end{array}$$

where X is an arbitrary G-manifold, easily yields

Theorem 1.3. *For every characteristic class χ, the diagram*

$$\begin{array}{ccc} U_{**}^{G} & \xrightarrow{\chi^G} & U^*(BG) \\ \alpha^{\#}\downarrow & & \alpha^*\downarrow \\ U_{**}^{G_1} & \xrightarrow{\chi^{G_1}} & U^*(BG_1) \end{array}$$

is commutative.

§2. Equivariant Hirzebruch genera. Statement and proof of the main theorem

1. From the viewpoint of characteristic classes, a rational Hirzebruch genus, i.e. a homomorphism $h\colon U_* \to Q$, is given by a series $t/h(t)$ with $h(t) = t + \sum_{i>1}\lambda_i t^i$, $\lambda_i \in Q$. The "action" of such a series on the bordism class $[CP^n]$ is given by the formula

$$h([CP^n]) = \left[\left[\frac{t}{h(t)}\right]^{n+1}\right]_n,$$

where $[r(u)]_n$ denotes the nth coefficient of the series $r(u)$. In [2], S. P. Novikov proved that $h(t)$ coincides with the series $g_h^{-1}(t)$ functionally inverse to the logarithm

$$g_h(t) = \sum_{n=0}^{\infty} \frac{h([CP^n])}{n+1} t^{n+1}$$

of the formal group $f_h(u, v)$ which is the image of the formal group of "geometric" cobordisms under the homomorphism.

By a theorem of Dold [7], to each rational Hirzebruch genus h there corresponds a transformation of functors $\tilde{h}\colon U^*(Y) \to K^{\#}(Y) \otimes Q$ (where $K^{\#}$ is the Z_2-graded K-functor) such that $\tilde{h}\colon U^* \to Q$ coincides with the composite $U^* \simeq U_* \xrightarrow{h} Q$.

The proof of the following lemma is analogous to the proofs of Theorem 6.4 and Corollary 6.5 in [9]:

Lemma 2.1. *The value of the homomorphism h at the generator $u \in U^2(CP^\infty)$ is equal to* $\mathrm{ch}^{-1}(g_h^{-1}(t))$, *where* ch *is the Chern character; that is,*

$$\tilde{h}(u) = g_h^{-1}(\ln \eta) \in K(CP^\infty) \otimes Q = Q[[1-\eta]].$$

Definition. *An equivariant Hirzebruch genus corresponding to a rational genus* h: $U_* \to Q$ is a homomorphism $h^G = \tilde{h} \circ \chi_0^G\colon U_{ev}^G \to K(BG) \otimes Q$.

Since Lemma 1.1 implies the commutativity of the diagram

$$\begin{array}{ccccc} U_*^G & \to & U^*(BG) & \to & K(BG) \otimes Q \\ \downarrow & & \downarrow & & \downarrow \\ U_* & \to & U^* & \longrightarrow & Q \end{array}$$

where $\epsilon\colon K^{\#}(Y) \otimes Q \to Q$ is the "augmentation", we have

Lemma 2.2. *The value of a genus on the bordism class of a G-manifold X is equal to* $\epsilon(h^G([X, G]))$.

2. Now we proceed to prove the main result.

Theorem 2.1. *For a connected compact Lie group G, the image of the homomorphism* $T_{x,y}^G\colon U_{ev}^G \to K(BG) \otimes Q$ *belongs to the subring* $Q \subset K(BG) \otimes Q$. *Moreover, for an* S^1-*manifold* X,

$$T_{x,y}([X]) = \sum_s x^{\epsilon_s^+}(-y)^{\epsilon_s^-} T_{x,y}([F_s]).$$

The Hirzebruch genus $T_{x,y}$ and the nonnegative integers ϵ_s^+ and ϵ_s^- appearing in the statement are the same as in the Introduction.

Proof. First of all we shall show that the first part of the theorem ($\mathrm{Im}\, T^G_{x,y} \subset Q$) is a simple consequence of Theorem 1.3 and Lemma 2.3.

Lemma 2.3. *The image of* $T^{S^1}_{x,y}$ *belongs to* $Q \subset K(CP^\infty) \otimes Q$.

Indeed, for a connected compact Lie group G the homomorphism $\alpha^*: K(BG) \otimes Q \to K(BH) \otimes Q$ induced by the inclusion of a maximal torus H in G is a monomorphism. Therefore, if there exists a G-manifold X such that $T^G_{x,y}([X, G]) \notin Q$, then also $\alpha^*(T^G_{x,y}([X, G])) \notin Q \subset K(BH) \otimes Q$. Evidently, there is an embedding of S^1 in H, $\alpha_1: S^1 \to H$, such that $\alpha_1^*(\alpha^*(T^G_{x,y}([X, G])))$ does not belong to Q either. However, this contradicts Lemma 2.3 because by Theorem 1.3

$$\alpha_1^*(\alpha^*(T^G_{x,y}([X, G]))) = T^{S^1}_{x,y}([X, S^1]).$$

Proof of Lemma 2.3. Consider an S^1-manifold X. Let

$$\beta([X, S^1]) = \sum_l [M_l] \prod_m (CP^{i_{ml}}_{j_{ml}});$$

then by Theorem 1.2

$$T^{S^1}_{x,y}([X, S^1]) = \sum_l T_{x,y}([M_l]) \prod_m (\tilde{T}_{x,y}([u]_{j_{ml}}))^{(i_{ml}+1)} \tilde{T}_{x,y}(B_{i_{ml}}([u]_{j_{ml}})). \qquad (1)$$

We shall calculate $\tilde{T}_{x,y}([u]_j)$ and $\tilde{T}_{x,y}(B_N(u))$. Since

$$T_{x,y}([CP^n]) = \frac{x^{n+1} - (-y)^{n+1}}{x+y},$$

we have

$$g_{T_{x,y}}(t) = \sum_{n=0}^{\infty} \frac{x^{n+1} - (-y)^{n+1}}{(x+y)(n+1)} t^{n+1} = \frac{1}{x+y} \ln\left(\frac{1+yt}{1-xt}\right).$$

Therefore

$$g^{-1}_{T_{x,y}}(t) = \frac{e^{(x+y)t} - 1}{xe^{(x+y)t} + y}$$

and hence

$$\tilde{T}_{x,y}([u]_j) = g^{-1}_{x,y}(j \ln \eta) = \frac{\eta^{j(x+y)} - 1}{x\eta^{j(x+y)} + y}.$$

By definition of $B_N(u)$, to find $\tilde{T}_{x,y}(B_N(u))$ we have to apply $T_{x,y}$ to the coefficients of the series $A_N(u, v)$ and replace v^k by $T_{x,y}([CP^{N-k}])$ in the resulting series $A_{NT_{x,y}}(u, v)$. Since

$$f_{T_{x,y}}(u, v) = g^{-1}_{T_{x,y}}(g_{T_{x,y}}(u) + g_{T_{x,y}}(v)) = \frac{u + v + (y-x)uv}{1 + yxuv},$$

we have

$$A_{NT_{x,y}}(u, v) \equiv \frac{u^N(1 + yxuv)}{1 + \frac{v}{u} + (y-x)v} \pmod{v^{N+1}}.$$

Therefore

$$A_{NT_{x,y}}(u, v) = \sum_{k=0}^{N} (-1)^k v^k u^{N-k} (1 + (y-x)u)^k$$
$$+ \sum_{k=0}^{N-1} (-1)^k v^{k+1} u^{N-k+1} xy (1 + (y-x)uv)^k.$$

Thus we obtain

$$\tilde{T}_{x,y}(B_N(u)) = \sum_{k=0}^{N} (-1)^k \frac{x^{N-k+1} - (-y)^{N-k+1}}{x+y} \left(\frac{\eta^{x+y}-1}{x\eta^{x+y}+y}\right)^{N-k} \left(\frac{x+y\eta^{x+y}}{x\eta^{x+y}+y}\right)^k$$
$$+ \sum_{k=0}^{N-1} (-1)^k xy \frac{x^{N-k} - (-y)^{N-k}}{x+y} \left(\frac{\eta^{x+y}-1}{x\eta^{x+y}+y}\right)^{N-k+1} \left(\frac{x+y\eta^{x+y}}{x\eta^{x+y}+y}\right)^k.$$

We denote by $r^{(N)}_{x,y}(\eta)$ the function of η given by the right side of this equality. With the preceding formulas, the equality (1) takes the form

$$T^{S^1}_{x,y}([X, S^1]) = \sum_i T_{x,y}([M_i]) \prod_m \left(\frac{x\eta^{j_{mi}(x+y)} + y}{\eta^{j_{mi}(x+y)} - y}\right)^{l_{mi}+1} \tau^{(l_{mi})}_{x,y}(\eta^{j_{mi}}). \tag{2}$$

Let us pause to consider in detail the meaning of the latter equality.

Let $\Phi_{x,y}(\eta)$ be the function of the complex variable η given by the right side of (2). It is easy to see that in a deleted neighborhood of 1 it is analytic; hence it has a Laurent series expansion in the variable $1-\eta$ there. By (2), this series coincides with $T^{S^1}_{x,y}([X, S^1]) \in Q[[1-\eta]]$. This implies that $\Phi_{x,y}(\eta)$ is analytic not only in a deleted neighborhood, but also at 1 itself.

Our immediate task will be to prove that there are no poles at roots of 1 and, as a consequence, that $\Phi_{x,y}(\eta)$ is analytic in the whole plane.

Lemma 2.4. *Let $\tilde{x} = x/(x+y)$ and $\tilde{y} = y/(x+y)$. Then*

$$(x+y)^N \Phi_{\tilde{x},\tilde{y}}(\eta^{x+y}) = \Phi_{x,y}(\eta), \quad N = \dim_{\mathbf{C}} X.$$

The proof of the lemma can easily be obtained from the fact that

$$\chi^{S^1}_0([X, S^1]) \in U^{-2N}(CP^\infty), \qquad \tilde{T}_{\tilde{x},\tilde{y}}(u) = \frac{\eta-1}{\tilde{x}\eta+\tilde{y}} = (x+y)\frac{\eta-1}{x\eta+y},$$
$$(x+y)^n T_{\tilde{x},\tilde{y}}([CP^n]) = T_{x,y}([CP^n]).$$

By this lemma it is enough to consider the case when $x + y = 1$, what will be assumed till the conclusion of Lemma 2.3.

Assume that H (the normal subgroup appearing in §1.2) is a cyclic subgroup of S^1 of order n. In the notation of Theorem 1.1,

$$e(\Delta_{S^1})\chi^{S^1}_0([X, S^1]) = \sum_s p_{s^1}(e(-\nu_s)_{S^1}).$$

Since by definition of the representation Δ of S^1 its restriction to the subgroup Z_n does not contain trivial summands, we have $\Delta = \Sigma_m \eta^{j_m}$, where none of the j_m is divisible by n. Hence

$$\left[\prod_m \left(\frac{\eta^{j_m}-1}{x\eta^{j_m}+y}\right)\right] T^{S^1}_{x,y}([X, S^1]) = \sum_s \tilde{T}_{x,y}\,[p_{s!}\,(e(-\nu_s)_{S^1})]. \tag{3}$$

Now we consider an arbitrary S^1-bundle ζ over an S^1-manifold F such that the action of Z_n is trivial on F. ζ can be represented as a sum of S^1-bundles ζ_r, $0 \leq r \leq n-1$. The generator of Z_n acts on a fiber of ζ_r by multiplication by $\exp(2\pi i r/n)$. Hence, if the S^1-bundle $\tilde{\zeta}_r$ is $\zeta_r \otimes \eta^{-r}$, then Z_n acts trivially on $\tilde{\zeta}_r$. Since $\zeta_r = \tilde{\zeta}_r \otimes \eta^r$, we have

$$e(\zeta_{S^1}) = \prod_{r=0}^{n-1} e(\tilde{\zeta}_{rS^1} \otimes p^*(\eta^r)),$$

where $p\colon F_{S^1} \to CP^\infty$.

Let $\mu_{r,k}$ be the Wu generators of $\tilde{\zeta}_{rS^1}$. Then

$$\tilde{T}_{x,y}(p_!(e(\zeta_{S^1}))) = \tilde{T}_{x,y}\left[p_!\left(\prod_{r,k} f(\mu_{r,k}, p^*([u]_r))\right)\right].$$

The coefficient of $p^*([u]_r)^i$ in the series

$$\prod_k f_{T_{x,y}}(\mu_{r,k}, p^*([u]_r)) = \prod_k \frac{\mu_{r,k} + p^*([u]_r) + (y-x)\,\mu_{r,k}p^*([u]_r)}{1 + yx\mu_{r,k}p^*([u]_r)}$$

is a symmetric polynomial in $\mu_{r,k}$. We denote the corresponding polynomial in the Chern classes of $\tilde{\zeta}_{rS^1}$ by $P_{i,k}$. The dimension of its lowest term is not smaller than $i - \dim \zeta_r$.

Thus

$$\tilde{T}_{x,y}(p_!(e(\zeta_{S^1}))) = \sum_\omega \tilde{T}_{x,y}\left(\prod_{r=0}^{n-1} ([u]_r)^{i_r}\right) \tilde{T}_{x,y}\left(p_!\left(\prod_{r=0}^{n-1} P_{i_r,r}\right)\right), \quad \omega = (i_1, \ldots, i_{n-1}). \tag{4}$$

The projection $\alpha\colon S^1 \to S^1/Z_n = S^1$ of S^1 onto the quotient group induces a map of classifying spaces

$$\alpha_*\colon CP^\infty \to CP^\infty,$$

with $\alpha^*(u) = [u]_n$. Since the S^1-bundle $\tilde{\zeta}_r$ is the inverse image under $\alpha^\#$ of some S^1-bundle $\tilde{\zeta}'_r$ (we recall that the subgroup Z_n acts trivially on the fibration space of $\tilde{\zeta}_r$), it follows from Theorem 1.3 that $p_!(\Pi_{r=0}^{n-1} P_{i_r,r}) \in \operatorname{Im} \alpha^*$.

Since the diagram

$$\begin{array}{ccc} U^*(CP^\infty) & \xrightarrow{\tilde{T}_{x,y}} & K(CP^\infty) \otimes Q \\ \alpha^*\downarrow & & \downarrow \alpha^* \\ U^*(CP^\infty) & \longrightarrow & K(CP^\infty) \otimes Q \end{array}$$

is commutative and $\alpha^*(\eta) = \eta^n$, we have that

$$\tilde{T}_{x,y}\left(p_!\left(\prod_{r=0}^{n-1} P_{i_r,r}\right)\right) \in \operatorname{Im} \alpha^* = Q[[1-\eta^n]].$$

From this and from (3) and (4) it follows that

$$T^{S^1}_{x,y}([X, S^1]) = \prod_m \left(\frac{x\eta^{j_m}+y}{\eta^{j_m}-1}\right)\left(\sum_k P_k\cdot(1-\eta^n)^k\right), \tag{5}$$

where P_k is a polynomial in $(\eta^r - 1)/(x\eta^r + y)$.

Let η_1 be the closest point to 1 at which there might be a pole of $\Phi_{x,y}(\eta)$; that is, the closest point to 1 of the form $\exp(2\pi ir/n)$, $r < n$ and $(r, n) = 1$, for which there is a j_{mi} divisible by n. The function $\Phi_{x,y}(\eta)$ is analytic in the disc $|\eta - 1| < |\eta_1 - 1|$; therefore the series $T^{S^1}_{x,y}([X, S^1])$ converges uniformly to it on every compact subset of this disc. From (5) it easily follows that the limit of $\Phi_{x,y}(\eta)$ for $\eta \to \eta_1$ exists. Hence $\Phi_{x,y}(\eta)$ is analytic in the disc $|\eta - 1| < |\eta_2 - 1|$, and the series $T^{S^1}_{x,y}([X, S^1])$ converges uniformly on every compact subset of that disc. Here η_2 is the next point at which there can be a pole of $\Phi_{x,y}(\eta)$. If we continue this process we obtain that $\Phi_{x,y}(\eta)$ is analytic in the whole closed complex plane. Therefore it is a constant. This concludes the proof of Lemma 2.3.

Now we pass to the second part of the theorem. By Lemma 2.2, $T_{x,y}([X]) = \Phi_{x,y}(1)$. Since, by the previous part of the proof, $\Phi_{x,y}(\eta)$ is constant, we have that $\Phi_{x,y}(1) = \lim_{\eta\to\infty}\Phi_{x,y}(\eta)$, and

$$\lim_{\eta\to\infty}\Phi_{x,y}(\eta) = \sum_i T_{x,y}([M_i])\prod_m \lim_{\eta\to\infty}\left(\frac{x\eta^{j_{mi}(x+y)}+y}{\eta^{j_{mi}(x+y)}-1}\right)^{l_{mi}+1}\tau^{(l_{mi})}_{x,y}(\eta^{j_{mi}}).$$

Remark. In what follows, all the limits are found under the assumption that $x + y > 0$. In the other case all the formulas are valid if we replace $\eta \to \infty$ by $\eta \to 0$.

Assume that $j_{mi} > 0$. Then

$$\lim_{\eta\to\infty}\left(\frac{x\eta^{j_{mi}(x+y)}+y}{\eta^{j_{mi}(x+y)}-1}\right)^{l_{mi}+1}\tau^{(l_{mi})}_{x,y}(\eta^{j_{mi}}) = x^{l_{mi}+1}\lim_{\eta\to\infty}\tau^{(l_{mi})}_{x,y}(\eta^{j_{mi}}).$$

If we remember the definition of $\tau^N_{x,y}(\eta)$, we obtain

$$\lim_{\eta\to\infty}\tau^{(N)}_{x,y}(\eta^{j_{mi}}) = \sum_{k=0}^{N}(-1)^k\frac{x^{N-k+1}-(-y)^{N-k+1}}{x+y}\frac{1}{x^{N-k}}\left(\frac{y}{x}\right)^k + \sum_{k=0}^{N-1}(-1)^k xy\frac{x^{N-k}-(-y)^{N-k}}{x+y}\frac{1}{x^{N-k+1}}\left(\frac{y}{x}\right)^k$$

$$= \frac{1}{x^N(x+y)}\left[\sum_{k=0}^{N}(-y)^k(x^{N-k+1}-(-y)^{N-k+1}) - \sum_{k=0}^{N-1}(-y)^{k+1}(x^{N-k}-(-y)^{N-k})\right] = \frac{1}{x^N(x+y)}(x^{N+1}-(-y)^{N+1}).$$

In an analogous way, for $j_{mi} < 0$ we find that

$$\lim_{\eta\to\infty}\left(\frac{x\eta^{j_{mi}(x+y)}+y}{\eta^{j_{mi}(x+y)}-1}\right)^{l_{mi}+1}\tau_{x,y}^{(l_{mi})}(\eta^{j_{mi}})=(-y)\frac{x^{l_{mi}+1}-(-y)^{l_{mi}+1}}{x+y}.$$

Thus

$$\lim_{\eta\to\infty}\Phi_{x,y}(\eta)=\sum_i T_{x,y}([M_i])\,x^{\epsilon_i^+}(-y)^{\epsilon_i^-}\prod_m T_{x,y}([CP^{l_{mi}}]),$$

where ϵ_i^+ is the number of positive integers among the j_{mi} and ϵ_i^- is the number of negative ones, respectively.

Let $\Sigma_{i_k}[M_{i_k}]\Pi_m(CP_{j_{mi_k}}^{l_{mi_k}})$ be the part of $\beta([X, S^1])=\Sigma_i[M_i]\Pi_m(CP_{j_{mi}}^{l_{mi}})$ equal to the bordism class in R_* of the S^1-bundle ν_s over a fixed submanifold F_s. Then for all the i_k we have $\epsilon_{i_k}^+=\epsilon_s^+$ and $\epsilon_{i_k}^-=\epsilon_s^-$. Since $[F_s]=\Sigma_{i_k}[M_{i_k}]\Pi_m[CP^{l_{mi_k}}]$, the proof of Theorem 2.1 is complete.

§3. The orientable case

We shall consider orientation-preserving actions of compact Lie groups on manifolds and vector bundles. All the constructions and results of the preceding sections for unitary actions automatically carry over to the present case; for this reason we shall restrict ourselves to making statements only, with minimal explanations when necessary.

To each characteristic class $\chi\in\Omega^i(BSO)$ in the oriented cobordism of vector bundles there corresponds a homomorphism of the Ω_*-module of bordisms of oriented G-bundles over oriented G-manifolds to the cobordism ring of the universal classifying space BG:

$$\chi^G:\Omega_{n,k}^G\to\Omega^{-n+i}(BG).$$

Theorem 3.1. *For every characteristic class* χ *and every* G-*bundle* ξ, *the following equality holds*:

$$e(\Delta_G)\,\chi^G([\xi])=\sum_s p_{s!}\,(e(-\nu_s)_G)\cdot\chi(\xi_{sG})).$$

The notation is the same as in Theorem 1.1, with the substitution of "orientable" for "unitary" bundles (representations).

Let χ_0^G be, as before, the "equivariant characteristic homomorphism" corresponding to the characteristic class $1\in\Omega^0(BSO)$.

Let us consider an arbitrary orientable S^1-manifold X. As we know, the structure group of the normal S^1-bundle ν_s over a connected component F_s of the set of fixed points under the action of S^1 on X can be reduced to the unitary group and ν_s becomes a complex S^1-bundle (see [10], §38). We choose the complex structure in ν_s in such a way that the representation of S^1 in the fibers has the form $\Sigma_i\eta^{j_{si}}$, $j_{si}>0$. As before, we define a homomorphism of Ω_*-modules

$$\beta':\Omega_n^{S^1}\to R_n'=\sum\Omega_l\Big(\prod_{j>0}BU(n_j)\Big),$$

where the summation is taken over all the collections of nonnegative integers n_j and l such that $2\Sigma_{j>0}n_j + l = n$.

Theorem 3.2. *There exists a homomorphism* $\Psi: R'_* \to \Omega^*[[u]] \otimes Q[u^{-1}]$ *of* Ω_**-modules such that* $\Psi \circ \beta'$ *coincides with the composite of the homomorphism* $\chi_0^{S^1}$ *with the homomorphism* $\Omega^*[[u]] \to \Omega^*[[u]] \otimes Q[u^{-1}]$. *The values of* Ψ *on the generators of the* Ω_**-module are given by the formula*

$$\Psi\left(\prod_m \left(CP_m^{l_m}\right)\right) = \prod_n \left(\frac{1}{[u]_{l_m}}\right)^{l_m+1} B_{l_m}([u]_{l_m}).$$

Theorem 3.3. *If* $\alpha: G_1 \to G$ *is a homomorphism of Lie groups, then the diagram*

$$\begin{array}{ccc} \Omega^G_{*,*} & \xrightarrow{\chi^G} & \Omega^*(BG) \\ \downarrow \alpha^\# & & \downarrow \alpha^* \\ \Omega^{G_1}_{*,*} & \xrightarrow{\chi^{G_1}} & \Omega^*(BG_1) \end{array}$$

is commutative.

As in §2, for each rational Hirzebruch genus $h: \Omega_* \to Q$ we construct an equivariant Hirzebruch genus $h^G: \Omega_*^G \to K(BG) \otimes Q$.

The values of the classical T_y-genus for $y = 1$ on almost complex manifolds coincide with the signature of these manifolds. Therefore, exactly as for Theorem 2.1, one can prove

Theorem 3.4. *For a connected compact Lie group* G, *the image of the homomorphism* $\mathrm{Sign}^G: \Omega_*^G \to K(BG) \otimes Q$ *belongs to the subring* $Q \subset K(BG) \otimes Q$. *For every oriented* S^1*-manifold* X *we have*

$$\mathrm{Sign}([X]) = \sum_s \mathrm{sign}([F_s]).$$

Addendum. In a subsequent article, the proof of the following theorem will appear:

Theorem. *If on a manifold* X *whose first Chern class* $c_1(X) \in H^2(X, Z)$ *is divisible by* k *there exists a nontrivial action of* S^1, *then* $A_k([x]) = 0$.

The proof is based on "analyticity" arguments connected with the equivariant series corresponding to the Hirzebruch genus A_k, $k = 2, 3, \ldots$, given by the series $kt \cdot e^t/(e^{kt} - 1)$.

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